Machine elements and contact mechanics

Follow loads through shafts, fasteners, bearings and drives, then use fatigue, lubrication, contact and tolerance models to assess an assembly.

Read the subject in depth

Each volume develops its arguments in sequence, with worked problems and links to experiments.

- ### Design and analysis of machine components

  Follow forces and motion through a component, derive its limiting response and compare a design estimate with an observable test of the assembled machine.

Advanced study across the subject

Each sequence connects the course to detailed chapters in neighbouring subjects. The complete course PDF includes their derivations, examples and exercises.

1. ### Account for compliance, contact and failure

   Determine what the support actually imposes before sizing a shaft, flexure or connection. Check its static deformation, cyclic response and sensitivity to temperature.

2. ### Actuation, bearings and measurement

   Follow the coupled force, motion and dissipation in a lubricating film, a compliant positioning system and a motion sensor.

How the course fits together

1. **Chapters 1–4**

   Translate service loads into component stresses, then follow them through shafts, bolted joints and rolling contacts. The stiffness of the parts decides how a load is shared and where a local pressure develops.

2. **Chapters 5–8**

   Gear geometry transfers torque, fits transmit pressure, springs store energy, and belts use friction to transmit tension. Distinguish prescribed displacement from prescribed force when reading a component rating.

3. **Chapters 9–12**

   Brakes dissipate energy, fluid films carry loads through motion, and weld groups distribute loads in a joint. Finish by propagating variable dimensions and strengths into a probability of meeting the design requirement.

Contents

1. Chapter 1 · From machine duty to a design load
2. Chapter 2 · Shaft reactions, twist and rotating bending
3. Problem sheet 1 · Duty blocks and shaft sections
4. Chapter 3 · Preload, joint stiffness and separation
5. Chapter 4 · Bearing arrangements and rating life
6. Problem sheet 2 · Clamped interfaces and rolling contact
7. Chapter 5 · Spur gear kinematics and tooth loads
8. Chapter 6 · Tolerances, interference and torque transmission
9. Problem sheet 3 · Gear meshes and assembled fits
10. Chapter 7 · Helical springs, wire stress and stored energy
11. Chapter 8 · Belt traction, wrap and power transmission
12. Problem sheet 4 · Elastic springs and belt traction
13. Chapter 9 · Friction clutches, braking torque and heat
14. Chapter 10 · Hydrodynamic lubrication and sliding bearings
15. Problem sheet 5 · Friction work and fluid films
16. Chapter 11 · Eccentrically loaded weld groups
17. Chapter 12 · Tolerance stacks and reliability of assemblies
18. Problem sheet 6 · Weld demand, assembly yield and reliability
19. Chapter 13 · Locating a part with six contacts and predicting datum error
20. Chapter 14 · Sizing the leaves of a compliant translation guide
21. Chapter 15 · Deriving screw-drive torque, efficiency and backdrivability
22. Chapter 16 · Matching a gear ratio to load inertia and move time
23. Chapter 17 · Force transmission and compliance near a toggle position
24. Chapter 18 · How an involute gear pair keeps contact from one tooth to the next
25. Chapter 19 · Why increasing the length of a bonded overlap gives diminishing returns
26. Chapter 20 · How a polygonal sprocket imprints speed ripple on a roller chain
27. Problem sheet 7 · Closing a flexure-guided fixture with a screw drive
28. Problem sheet 8 · Three shafts and one constraint in a planetary gearbox
29. Design and analysis of machine components
30. Account for compliance, contact and failure
31. Actuation, bearings and measurement
32. Applied study
33. Notation and assumptions
34. Further reading

Chapter 1 · From machine duty to a design load

Translate a machine's operating cycle into member forces, stress amplitudes and a life estimate for a repeated duty block.

FROM FIRST PRINCIPLES

Convert a repeated force history into stress and accumulated damage

Begin by isolating the moving part and balancing its forces. For a mass m accelerated upwards at a, an ideal vertical lifting member supplies $F = m(g + a)$ if its own inertia is negligible. In a straight prismatic member of area A loaded along its centroid, the nominal axial stress is F/A. Record the largest and smallest stress during one repeated operating cycle. Their mean and half-range are

$$\sigma_m = \frac{F_{\max} + F_{\min}}{2A},$$

$$\sigma_a = \frac{F_{\max} - F_{\min}}{2A}.$$

Fatigue life is empirical: tests on the appropriate detail relate a specified stress amplitude and mean stress to a characteristic failure count N_i. Under the linear damage-summation approximation, n_i repetitions of that level consume a fraction n_i/N_i of life. For one recurring duty block,

$$D_{\text{block}} = \sum_i \frac{n_i}{N_i},$$

$$B_{\text{predicted}} = \frac{1}{D_{\text{block}}}.$$

The second relation uses a nominal failure threshold of one. It requires the same load block to repeat and assumes damage fractions add regardless of sequence. Peak stress and cumulative fatigue damage answer different questions about that operating history.

DERIVATION AND APPLICATIONS

Convert a variable-amplitude block to a damage-equivalent stress

Two repeated machine duties can have different peak stresses yet accumulate the same fatigue damage in a specified model. Consider a fixed detail, a common stress definition and constant mean-stress conditions. Use an experimentally fitted single-slope life relation $N(S) = N_r(S_r/S)^m$, where S is alternating stress amplitude, $m > 0$ is the fitted slope and the reference point is (S_r, N_r). Assume the same environment and material condition as the fit. Palmgren–Miner summation gives the predicted damage of one load block with n_i cycles at each amplitude S_i.

$$D_b = \sum_i \frac{n_i}{N(S_i)} = \frac{1}{N_r S_r^m} \sum_i n_i S_i^m.$$

Define a constant amplitude S_{eq} acting for the same total number of cycles $n = \sum_i n_i$. Equate its predicted damage to the variable-amplitude damage. The reference point cancels from this comparison because both predictions use the same life curve.

$$n S_{\mathrm{eq}}^m = \sum_i n_i S_i^m,$$

$$S_{\mathrm{eq}} = \left(\sum_i \frac{n_i}{n} S_i^m \right)^{1/m}.$$

A rare high stress can dominate this power average. For two levels the proportion of total damage due to the higher one follows by dividing its contribution by their sum. This fraction is independent of how often the entire block is repeated.

$$f_H = \frac{n_H S_H^m}{n_H S_H^m + n_L S_L^m}.$$

Changing sequence, residual stress, mean stress or a transition between slopes can invalidate the reduction; the equality here is an exact statement about the stated damage model.

WORKED EXAMPLE

Identify which cycles dominate a conveyor duty

A block contains 900 cycles at 40.0 MPa and 100 at 80.0 MPa. For $m = 4$, the equivalent amplitude is

$$40[0.9 + 0.1(2^4)]^{1/4} = 50.3\,\mathrm{MPa}.$$

The high cycles provide $1.6/(0.9 + 1.6) = 0.640$ of damage, though they occupy one tenth of the block. Simply taking the arithmetic mean would give 44.0 MPa and understate predicted damage.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Determine a permissible high-cycle fraction

Another duty must stay below a damage-equivalent amplitude of 60.0 MPa. Let the available levels be 50.0 and 75.0 MPa with slope $m = 3$. Solving $60^3 = (1 - f)50^3 + f75^3$ gives

$$f = (60^3 - 50^3)/(75^3 - 50^3) = 0.307.$$

At most about 30.7 percent of cycles can occupy the higher level under these assumptions. The result concerns cycle count; unequal shaft speeds would require converting operating durations first.

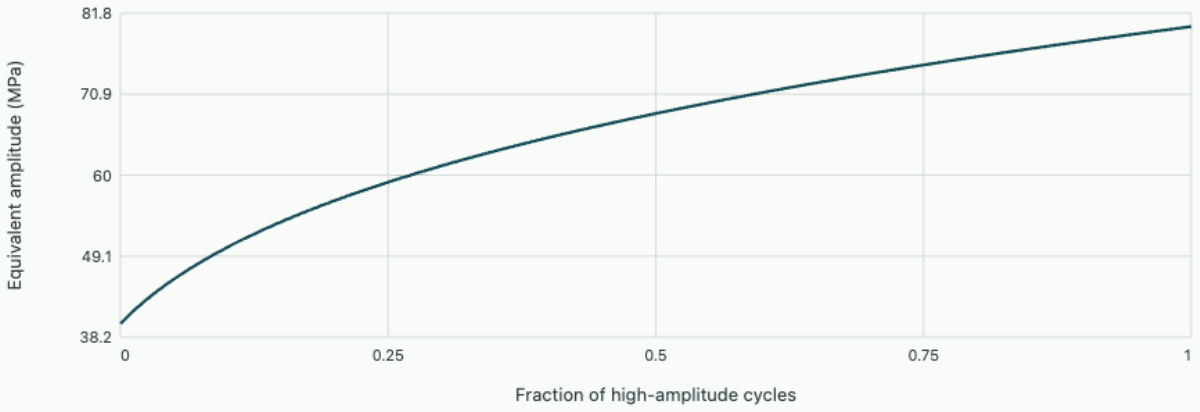
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

High / low stress amplitude	2
Life exponent	4
Low stress amplitude (MPa)	40



Damage-equivalent constant amplitude

At10% high cycles, equivalent amplitude=50.297 MPa; high cycles cause 64.0% of the damage.

Follow the force through the machine

A machine specification may give lifting capacity in kilograms or motor power in kilowatts. Neither quantity by itself gives the tension in an individual link. Begin with the moving assembly and its connections. Isolate one body, mark every contact and decide which reactions the connection can transmit. A pin carries two force components in a planar model. A flexible cable carries tension along its length. A pair of guides may transmit a couple even though each individual contact carries only a normal force. These decisions determine which part experiences each load.

Consider a carriage raised by a vertical cable. Its cable attachment transfers tension through a flat link of net area A. If upward is positive, the carriage balance reads $F - mg = ma$, hence $F = m(g + a)$. This also applies while the carriage is moving downwards and slowing down. Its upward acceleration increases tension. A speed record alone therefore conceals an important load case. Draw both the velocity and its slope before assigning signs. A cable solution with negative tension means that the cable would slacken and the assumed constraint has failed.

Separate a steady bias from an alternating load

Cut the link across its smallest section. With the load centred and bending negligible, integration of an approximately uniform axial stress gives

$$F = \sigma A.$$

The stress at the two turning points defines the mean stress and alternating amplitude for that cycle. An amplitude is measured from the mean, so a recorded peak-to-peak range must be divided by two.

$$\begin{aligned} \sigma_{\max,\min} &= \frac{F_{\max,\min}}{A}, \\ \sigma_m &= \frac{\sigma_{\max} + \sigma_{\min}}{2}, \\ \sigma_a &= \frac{\sigma_{\max} - \sigma_{\min}}{2}. \end{aligned}$$

A part loaded from zero to a maximum tensile stress has equal positive mean and alternating components. A fully reversed load with the same peak tensile stress has zero mean and twice that alternating amplitude. To assess fatigue, a material curve must describe the applicable mean stress, surface condition, stress concentration and environment. For yielding, first compare the largest local stress with the yield strength. Those two calculations use different features of the same history.

Background · Why can the surface of a steadily loaded rotating shaft see alternating stress?

A transverse force fixed in the laboratory creates a bending moment with a stationary direction. A particular material fibre at radius r rotates through the tensile and compressive sides. For linear bending of a circular shaft with second moment I, that fibre samples

$$\sigma_x(t) = \frac{Mr}{I}\cos(\Omega t + \theta_0).$$

At 1500 rpm the fibre experiences 25 complete stress cycles per second, even while a force transducer on the housing sees a steady reading. A torque also fixed in magnitude gives a steady torsional component in the rotating material frame. Trace the history at one material point before combining a bending cycle with a torsional mean.

A lift with repeated starts

A carriage and its payload have a combined mass of 800 kg. In each trip the drive accelerates upwards at 2.0 m/s² and later decelerates while travelling upwards at 3.0 m/s². The cable passes without mechanical advantage to a link with net section 80 mm². Gravity is 9.81 m/s². Assume the cable remains taut, and treat the measured accelerations as representing the load on the carriage.

The maximum tension is $800(9.81 + 2.0) = 9448\,\mathrm{N}$ and the minimum during the stated loaded cycle is

$$800(9.81 - 3.0) = 5448\,\mathrm{N}.$$

Dividing by area gives 118.1 and 68.1 MPa. Their mean is 93.1 MPa and their amplitude is 25.0 MPa. A stopped, suspended carriage still loads the link at 98.1 MPa. Unloading the carriage fully between trips would introduce another turning point. The original two-level description would no longer define every cycle experienced by the link.

Convert a repeated block into a fatigue estimate

Suppose tests at the applicable stress ratios give lives of 8.0×10^5 cycles for ordinary loaded trips and 1.2×10^5 for heavier trips. One representative block contains 200 ordinary and 30 heavy cycles. Palmgren–Miner averaging assigns the used fraction n_i/N_i to each level. If a block is repeated B times, its predicted accumulated damage is

$$D = B\sum_i n_i/N_i = B(200/800000 + 30/120000).$$

The conventional estimate $D = 1$ gives 2000 blocks. The smaller count of heavy cycles contributes half of the estimated damage. This calculation can help decide which trips to instrument. It assumes that sequence effects and interactions between levels can be neglected. Changing the order of the same cycles would leave this prediction unchanged. An overload that yields the link, changes its residual stress or damages a connection calls for evidence beyond this linear rule. Record the peak event separately when preparing the input for a component calculation.

Combine alternating stress with a tensile mean stress

A duty histogram can contain cycles with the same amplitude but different tensile mean stresses. Applying a fully reversed fatigue curve to every cycle ignores this distinction. A common preliminary engineering construction at a selected life is the modified Goodman line. Let $S_f(N)$ be a component's fully reversed fatigue strength at that life and S_u its tensile ultimate strength. Draw the line joining $(\sigma_m = 0, \sigma_a = S_f)$ and $(\sigma_m = S_u, \sigma_a = 0)$ on a mean–alternating stress diagram. Points on the line satisfy

$$\frac{\sigma_a}{S_f(N)} + \frac{\sigma_m}{S_u} = 1,$$

$$\sigma_{a,\mathrm{eq}} = \frac{\sigma_a}{1 - \sigma_m / S_u}.$$

The second expression gives the fully reversed amplitude assigned to the measured cycle by this interpolation. This is an empirical mean-stress correction, rather than a derivation of material behaviour from elasticity. Use component or specimen data appropriate to surface, size, loading mode, environment and stress concentration. Compressive mean stresses and regions near yield require their own treatment. If a local cycle produces plastic strain, an elastic stress-life calculation can be inappropriate.

If all stresses scale in proportion to an external load multiplier n_L, the intersection of that load ray with the Goodman line gives

$$n_L = 1/(\sigma_a/S_f + \sigma_m/S_u).$$

A fixed installation preload is different: it contributes to the mean but will not scale with a later service-load multiplier. Adding preload to a stress diagram and multiplying the whole state by a single factor would then misstate the available service-load increase.

Compare a raised preload with an increased cycle range

Consider a smooth tensile link with illustrative life-10^6-cycle strength $S_f = 180\,\mathrm{MPa}$, ultimate strength $S_u = 600\,\mathrm{MPa}$ and yield strength 360 MPa. Its measured stress cycles from 40 to 200 MPa. Mean and amplitude are 120 and 80 MPa; the Goodman sum is

$$80/180 + 120/600 = 0.644.$$

The model’s equivalent reversed amplitude is

$$80/(1 - 120/600) = 100\,\mathrm{MPa}.$$

The peak of 200 MPa is below yield. If the full cycle scales with service load, the fatigue-ray multiplier is 1.55, while first yield at the nominal peak gives

$$360/200 = 1.80.$$

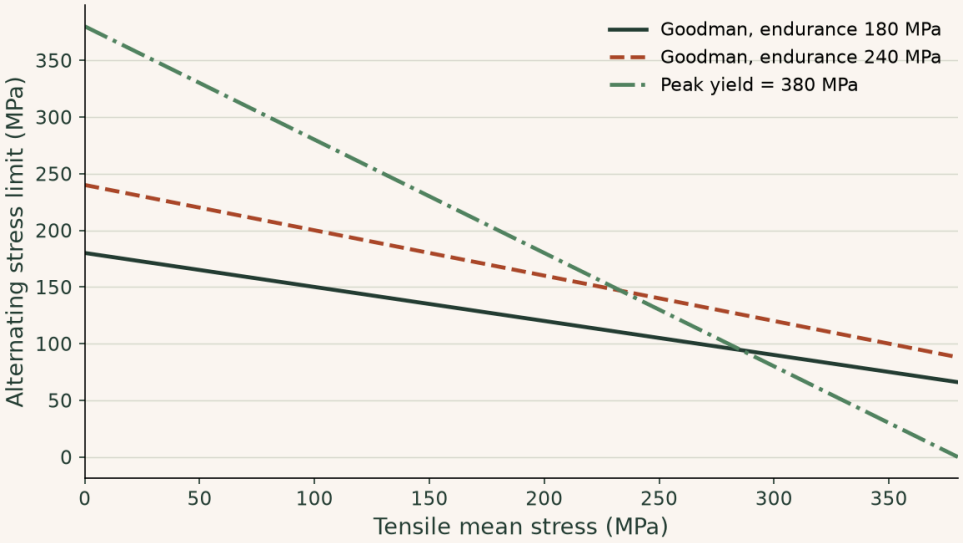
Suppose instead that a 60 MPa assembly preload remains fixed and only the remainder scales. At the original operating point the service component then has a mean of 60 MPa and an amplitude of 80 MPa. Its fatigue boundary is $n_L(80/180) + (60 + 60n_L)/600 = 1$, giving $n_L = 1.65$. The peak yield condition becomes $60 + 140n_L = 360$, or $n_L = 2.14$. Both multipliers differ from scaling everything because the preload has a different physical source. A further rise of fixed preload by 30 MPa at the original service range changes the equivalent reversed amplitude to $80/(1 - 150/600) = 106.7\,\mathrm{MPa}$.

This comparison shows which load measurement must change if the aim is reducing fatigue demand rather than merely reducing the largest force in a test record.

The same diagram can represent a measured range with uncertainty. A sensor offset changes the reconstructed mean while leaving the alternating component unchanged. A gain calibration error generally changes both. Reporting only the highest gauge reading leaves these two uncertainty sources indistinguishable when estimating fatigue demand.

Allocate fluctuating and mean stress in a fatigue screen

Use a linear Goodman estimate $\sigma_a/S_e + \sigma_m/S_u = 1$ with $S_u = 600$ MPa. Also require the peak tensile stress $\sigma_m + \sigma_a$ to remain below the 380 MPa yield scale. At each mean stress the smaller ordinate is the controlling screen.



At a 300 MPa mean stress the 240 MPa endurance line allows 120 MPa alternating stress, but the peak-yield screen allows only 80 MPa. Endurance estimates depend on surface, size and load history; neither straight line establishes a life distribution.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

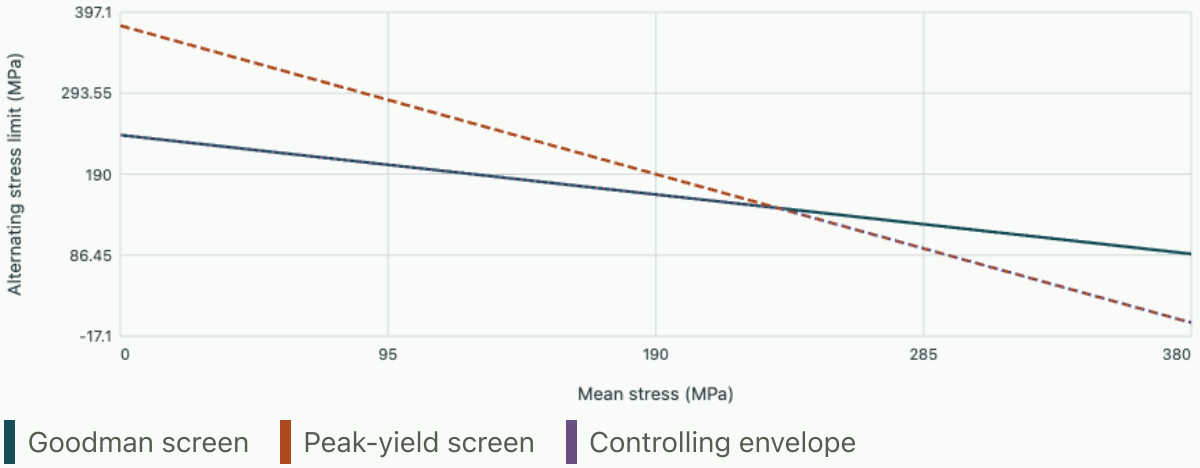
Move the endurance, ultimate and yield scales independently and identify which line controls at large tensile mean stress.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Endurance screen (MPa)	**240**
Ultimate strength (MPa)	**600**
Yield scale (MPa)	**380**



The two screens cross at mean stress 233.333 MPa. These deterministic limits are not a statistical fatigue-life distribution.

Study the full derivation

Fatigue crack growth, optical inspection and the meaning of remaining life · Design and analysis of machine components

A polished test coupon contains a measurable crack and experiences repeated tensile cycles. A stress–life curve for a polished, initially uncracked specimen cannot directly give the remaining growth interval from this measured state. Use calibrated crack-growth data to predict a laboratory observation and quantify what the inference depends on.

Chapter 2 · Shaft reactions, twist and rotating bending

Trace power and belt forces into shaft reactions, then check the shaft section for yielding, twist and cyclic bending.

FROM FIRST PRINCIPLES

Derive bending and torsional stress at a round shaft surface

A rotating shaft transmits torque through circumferential shear and can carry transverse loads through bending. Consider a slender, solid, circular shaft of diameter d under small elastic deformation. In pure bending, plane sections stay plane so longitudinal strain varies linearly with distance y from the neutral axis. In circular torsion, shear strain varies linearly with radius r. Choosing stress gradients to reproduce the applied moment and torque gives

$$M = \int_A y\sigma \, dA,$$
$$\sigma = My/I,$$
$$T = \int_A r\tau \, dA,$$
$$\tau = Tr/J.$$

Here $I = \int_A y^2 dA = \pi d^4/64$ and $J = \int_A r^2 dA = \pi d^4/32$. Taking the surface distance as $d/2$ gives the nominal maximum components.

$$|\sigma_b|_{\max} = \frac{32|M|}{\pi d^3},$$
$$|\tau_t|_{\max} = \frac{16|T|}{\pi d^3}.$$

Torque can be inferred from mechanical power using $T = P/\omega$, while bending moment requires transverse force and support equilibrium. A stationary transverse load produces alternating bending at a material point as the shaft rotates. The formulas assume a smooth section; shoulders and keyways alter local stress.

Background · Torsional stiffness

$$k_\theta = GJ/L$$

For a uniform elastic circular shaft use its polar second moment. A noncircular section generally needs a distinct torsion constant and warping analysis.

DERIVATION AND APPLICATIONS

Distribute a fixed-end shaft torque by twist compatibility

A shaft supported against rotation at both ends has an extra reaction torque that cannot be found from overall moment equilibrium alone. Consider two coaxial circular segments meeting at a collar. Apply a static torque T at the collar. The left and right outer ends are held fixed. Assume linear elastic shear, small twist, fully bonded interfaces and torque transfer without slip. Segment j has length L_j, shear modulus G_j and polar second moment J_j. Common collar angle θ twists each segment through the same magnitude.

$$\theta = \frac{T_L L_L}{G_L J_L} = \frac{T_R L_R}{G_R J_R},$$
$$T_L + T_R = T.$$

The symbols T_L, T_R are positive magnitudes of the moments resisting the applied torque. Define torsional stiffness $k_j = G_j J_j / L_j$ and substitute into equilibrium. The two segments behave as parallel torsional springs when observed from the collar.

$$\theta = \frac{T}{k_L + k_R},$$
$$T_L = \frac{k_L}{k_L + k_R} T,$$
$$T_R = \frac{k_R}{k_L + k_R} T.$$

For a solid round shaft $J_j = \pi d_j^4 / 32$. Local surface shear is proportional to torque times outer radius divided by J_j. The increase of torque carried by a stiffer segment therefore does not translate directly into the same factor of stress.

$$\tau_{j,\max} = \frac{T_j d_j}{2 J_j} = \frac{16 T_j}{\pi d_j^3}.$$

If one support becomes torsionally soft, its torque approaches zero and the stiffer side carries nearly all of T. The fixed-end assumption is especially important in using rotation measurements to identify a support reaction.

WORKED EXAMPLE

Split torque across two lengths of shaft

Two segments have the same 20.0 mm diameter and shear modulus 80.0 GPa, with lengths 0.200 and 0.400 m. The shorter segment is twice as stiff. A 90.0 N m collar torque splits into 60.0 and 30.0 N m.

The shorter segment reaches $16(60)/[\pi(0.0200)^3] = 38.2$ MPa surface shear. Its twist is $60(0.200)/(80.0 \times 10^9 J) = 0.00955$ rad, shared by the collar.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Select a diameter for equal torque share

Another shaft uses identical materials, with the right segment twice as long as the left. To carry equal torques require

$$d_R^4/L_R = d_L^4/L_L.$$

If $d_L = 25.0$ mm, $d_R = 25.0(2)^{1/4} = 29.7$ mm. Each carries half the applied torque, but the right surface stress is only $(25.0/29.7)^3 = 0.595$ of the left because its larger diameter provides greater torsional section modulus.

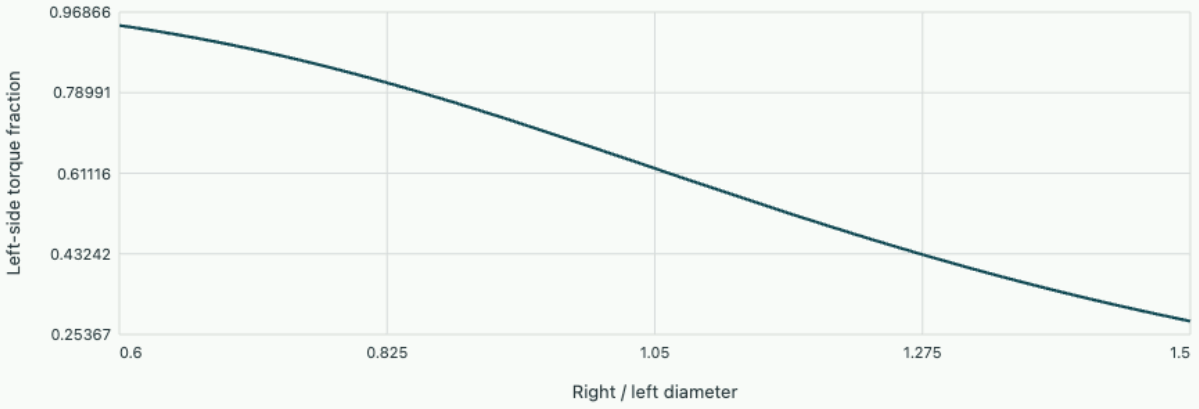
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Left length (m)	0.2
Right length (m)	0.4
Applied collar torque (N m)	90



Compatible torsion

At equal20 mm diameters, torque shares 60.000 and 30.000 N m. Equal sharing requires diameter ratio 1.189.

Rotation determines the torque, contacts determine the bending

A shaft transmits a given power through torque at a specified speed. Mechanical work in a small rotation is $dW = T\,d\theta$, so power is $T\omega$, with angular speed in radians per second. Belt tensions act at opposite sides of a pulley. Their difference gives the net driving torque. For two nearly parallel belt spans pulling in the same transverse direction their sum loads the shaft sideways. A belt with greater pretension can increase bearing loads although the transmitted power remains constant.

$$\omega = 2\pi n/60,$$
$$T = P/\omega = (F_1 - F_2)r,$$
$$F_\perp \simeq F_1 + F_2.$$

Choose bearing centres as ideal simple supports and identify where each torque enters or leaves the shaft. A section between an input coupling and an output pulley carries the transmitted torque. Beyond the output pulley it may carry transverse shear while carrying virtually zero torque. Mark the bending and torque diagrams separately so the combination at a critical cross-section represents loads that actually coexist there.

Determine the stress components from section geometry

For a solid circular shaft the bending second moment is $I = \pi d^4/64$, and its polar moment is $J = 2I$. The elastic bending stress at the outer fibre is $M(d/2)/I$; the torsional shear stress is $T(d/2)/J$. Under proportional elastic loading, the plane-stress von Mises expression then gives a static yield check for a ductile material.

$$\sigma_b = 32M/(\pi d^3),$$
$$\tau_t = 16T/(\pi d^3),$$
$$\sigma_v = \sqrt{\sigma_b^2 + 3\tau_t^2}.$$

The cubic diameter dependence makes a shoulder important even before its local stress concentration is considered. These equations describe a smooth prismatic section and nominal stresses. At a fillet, the maximum stresses may be estimated from geometric concentration factors appropriate to bending and torsion. For cyclic design a fatigue concentration factor may differ from the elastic geometric factor because sensitivity to the notch depends on material and radius.

Size the loading at the pulley seat

A 120 mm radius pulley sits at the middle of a 0.40 m span and draws 6.0 kW at 600 rpm. The two belt forces are parallel and their sum is 1800 N. The driving coupling lies immediately outside the left bearing. The shaft is 32 mm in diameter. From $\omega = 62.83\,\mathrm{rad/s}$, the transmitted torque is 95.5 N m and the tension difference is

$$95.5/0.120 = 795.8\,\mathrm{N}.$$

Combining difference and sum gives 1298 and 502 N.

Each bearing carries 900 N. Taking moments about a cut just to the left of the pulley gives

$$M = 900(0.20) = 180\,\mathrm{N\,m}.$$

There the shaft still transmits full torque. Substitution gives 56.0 MPa bending and 14.8 MPa torsional shear. Thus the nominal von Mises stress is 61.6 MPa, giving a nominal yield safety factor of 4.06 for steel with 250 MPa yield strength. This calculation has not yet accounted for a keyway or the pulley seat fillet.

Follow one material point through a revolution

A fixed vertical belt pull produces a steady bending moment in the laboratory. Mark a point on the rotating shaft surface. It passes through the tensile and compressive bending fibres once each per turn, so that its bending stress is fully reversed with amplitude 56.0 MPa. Steady driving torque supplies mean shear. A fatigue test at zero mean axial load describes only one part of this combined loading history. A shaft that spends a long time at constant speed can accumulate many stress cycles while the torque meter shows a level trace.

The twist angle is a further service requirement. Integrate $d\theta/dx = T/(GJ)$ along the torque-carrying length. With $G = 80\,\mathrm{GPa}$, $J = 1.029 \times 10^{-7}\,\mathrm{m}^4$, and a 0.20 m torque-carrying span, the relative twist is 0.00232 rad, or 0.133°. Bending stress varies as the inverse cube of diameter and twist at fixed torque as the inverse fourth power. A diameter selected from a strength limit can still give an unacceptable positioning error at the driven component.

Use the static deflection to estimate a shaft critical speed

The shaft diameter required to transmit torque need not satisfy the dynamic constraints of a rotating machine. For a first screening, model a relatively light simply supported shaft carrying a heavy disc at midspan. Treat the disc as a concentrated translating mass and neglect shaft mass, gyroscopic coupling, bearing flexibility and damping at first. Under a static transverse force F, bending theory gives midpoint deflection $\delta = FL^3/(48EI)$, hence lateral stiffness $k_s = 48EI/L^3$. With gravity loading only from the disc, $\delta_g = mg/k_s$. The ideal bending natural frequency then follows without explicitly measuring disc mass:

$$\omega_n = \sqrt{k_s/m} = \sqrt{g/\delta_g}.$$

A small rotating mass eccentricity e produces lateral forcing $me\Omega^2$ synchronous with speed. The undamped critical speed occurs when Ω equals ω_n. Gravity sag itself is a static offset in vertical operation of this horizontal shaft; its magnitude is useful because it reveals the ratio m/k_s. For a shaft with distributed mass or several mounted elements, one measured sag at a disc need not represent the relevant modal inertia. A compliant test stand also contributes to the measured deflection and can lower the assembled critical speed.

Adding simple viscous damping gives synchronous orbit-amplitude ratio $X/e = r^2/\sqrt{(1-r^2)^2 + (2\zeta r)^2}$, $r = \Omega/\omega_n$, for the ideal isotropic single-disc model. It approaches unity far above critical speed but can be much larger near it.

Check a shaft after increasing disc mass

A machine shaft carrying a balanced test disc deflects 0.40 mm at its centre when the disc's weight is applied. The fixture was tared for its own sag. The estimated natural frequency is $\sqrt{9.81/0.00040} = 156.6\,\mathrm{rad/s}$, corresponding to 1496 rpm. A production target of 1200 rpm gives $r = 0.802$. With small residual eccentricity $e = 0.050\,\mathrm{mm}$ and damping ratio 0.04, the predicted orbit amplitude is

$$0.050(0.802)^2/\sqrt{(1-0.802^2)^2 + (0.08\times 0.802)^2} = 0.089\,\mathrm{mm}.$$

A new disc doubles the dominant mass at the same shaft stiffness. Sag doubles to 0.80 mm and the estimated critical speed falls to 1057 rpm. The machine must now pass through that speed during startup and operates above it, despite the torque capability of the shaft being essentially unchanged. To restore its original critical speed under the doubled mass while retaining the span and material, stiffness must double. For a smooth solid circular shaft $I \propto d^4$, so

$$d_{\mathrm{new}}/d_{\mathrm{old}} = 2^{1/4} = 1.189.$$

A 30 mm shaft would need about 35.7 mm under this model. Restoring the natural frequency alone does not fix a bearing or coupling alignment issue, and a run-up through resonance requires transient analysis if the dwell time is significant relative to the decay time.

The static stiffness estimate assumes the midpoint force and the displacement gauge describe the same work-conjugate location and direction. An overhung disc or a gauge near a support can give a very different apparent stiffness. Recording the shaft layout alongside the sag value keeps the screening calculation reproducible when the assembly changes.

Why a single universal joint produces a twice-per-revolution speed fluctuation

A Hooke joint lets two intersecting shafts transmit rotation at a fixed misalignment angle β. Its cross carries two perpendicular trunnions. Choose the input axis as z, tilt the output axis about x, and measure input rotation θ and output rotation ϕ from aligned angular references. A unit vector along the first trunnion is $(\cos\theta, \sin\theta, 0)$. The second rotates in the tilted output plane and can be written $(-\sin\phi, \cos\beta\cos\phi, \sin\beta\cos\phi)$. Their dot product vanishes, giving $\tan\phi = \cos\beta\tan\theta$. Follow a continuous angular branch over a full turn.

Differentiate this constraint before averaging. Using the tangent relation to remove ϕ from the derivative gives

$$\frac{\omega_{\mathrm{out}}}{\omega_{\mathrm{in}}} = \frac{d\phi}{d\theta} = \frac{\cos\beta}{1-\sin^2\beta\sin^2\theta}.$$

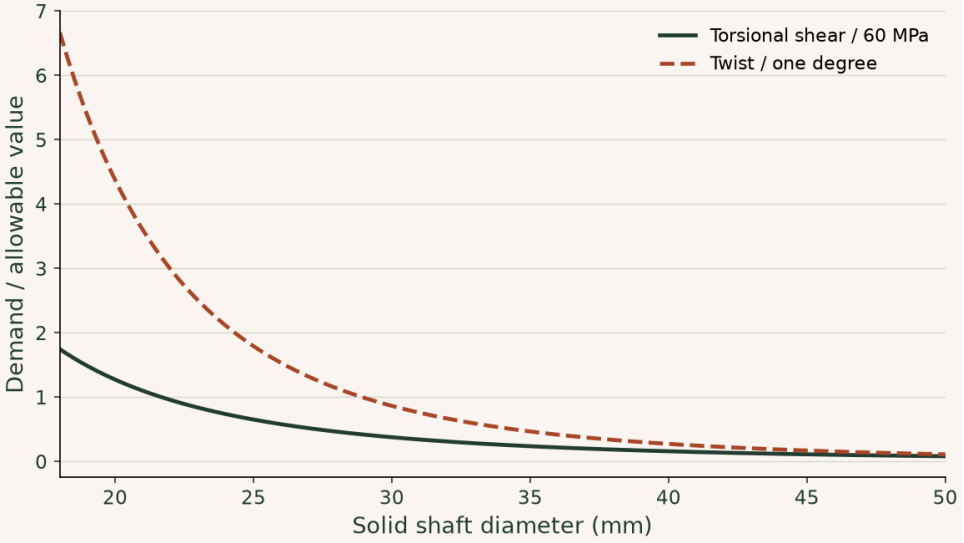
The extremes are $\cos\beta$ and $1/\cos\beta$. Both shafts complete one revolution in the same time, yet the instantaneous ratio changes twice per revolution. For small β in radians it is $1-(\beta^2/2)\cos 2\theta + O(\beta^4)$. This identifies a second-order dependence on small misalignment and a forcing frequency twice the shaft frequency.

A test rig drives a single joint at a constant 600 rpm with $\beta = 20^\circ$. The driven shaft ranges from $600\cos 20^\circ = 563.8\,\mathrm{rpm}$ to $600/\cos 20^\circ = 638.5\,\mathrm{rpm}$. Its speed ripple repeats at $2(600/60) = 20.0\,\mathrm{Hz}$. A large downstream inertia will therefore demand an alternating accelerating torque even if the external load is steady. A double-joint arrangement can cancel the kinematic ripple at the final shaft: with parallel outer shafts, equal joint angles and correctly phased forks, the second joint implements the inverse angular mapping of the first. The intermediate shaft still fluctuates, so bearing and torsional loads within the assembly merit separate treatment.

Practise on sheet 1 · Set a joint-angle limit from the driven-shaft speed tolerance. Further reading · Jönköping University · Applied Mechanics · §10.20 Shaft couplings, “The Hooke joint” and “Two Hooke joints”.

Will shaft strength or angular stiffness set the diameter?

A solid shaft transmits 120 N·m over 0.80 m with shear modulus 80 GPa. Limit nominal shear to 60 MPa and end rotation to one degree. Compare $\tau = TR/J$ and $\phi = TL/(GJ)$ only after dividing each quantity by its own allowable.



Strength requires about 21.7 mm, while the twist limit requires about 28.9 mm. The stiffness requirement controls this ideal span. The different diameter powers, three and four, also make the two margins respond differently to a small machining change.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

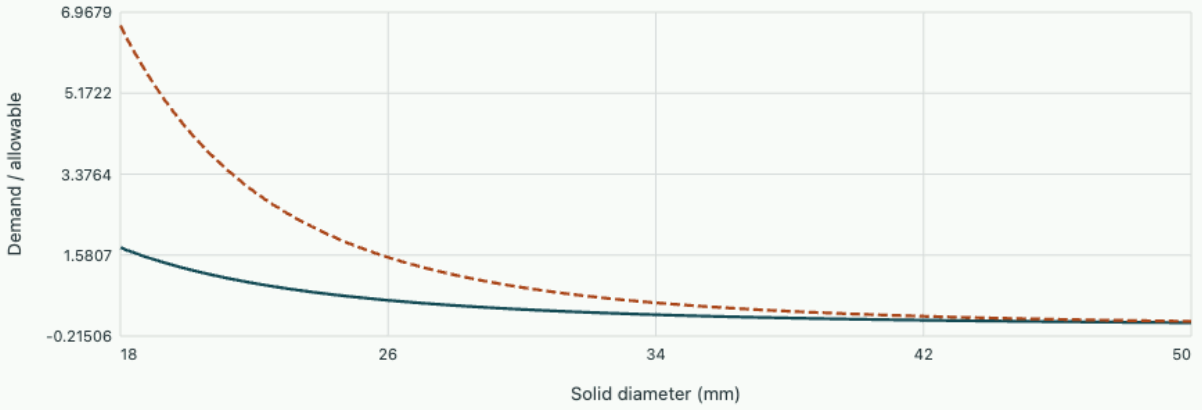
Raise torque and length separately. Compare how much diameter the strength and angular-stiffness conditions require.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Torque (N m)	120
Length (m)	0.8
Allowable twist (degrees)	1



Strength requires 21.677 mm; twist requires 28.929 mm.

Study the full derivation

Helical gears, mesh forces and bearing reactions · Design and analysis of machine components

A motor supplies a known torque to a helical pinion. Select the free body carefully: torque fixes one component of the tooth load, the helix creates thrust, and the location of that thrust can alter bearing reactions even in a parallel-shaft transmission. Build the force diagram and reverse it from measured support loads.

Problem sheet 1 · Duty blocks and shaft sections

Resolve a duty record into stresses and compare shaft bending, torsion and twist using the actual load paths.

An accelerating counterweight

A 550 kg counterweight hangs from a single vertical link of area 60 mm². Its upward acceleration ranges from −1.8 to +2.2 m/s² during each loaded cycle. Find the force extremes and the stress mean and amplitude. During commissioning the same link is fully unloaded once per trip instead. Which stress quantity changes for a simple zero-to-peak cycle?

▼ **Resolve the accelerations**

The link force is $m(g+a)$, giving 4406 and 6606 N for the stated endpoints. The corresponding stresses are 73.43 and 110.09 MPa. The mean is 91.76 MPa and the amplitude 18.33 MPa. For a single excursion from zero to the same peak, mean and amplitude are both 55.05 MPa. The peak is unchanged although the amplitude triples. Multiple turning points in an actual trip require separate cycle counting.

Use a measured duty block

A pump link sees 800 cycles per shift at a stress level with test life 2.0 million cycles, and 60 cycles at a level with test life 150,000 cycles. Find the expected shift count at unit damage under linear accumulation. What is the effect of halving the number of heavy events only?

▼ **Weight each level by its tested life**

The two contributions are $800/2000000 = 0.0004$ and $60/150000 = 0.0004$ per shift. Their sum gives 1250 shifts. With only 30 heavy events per shift, the total becomes 0.0006 and the estimate rises to 1667 shifts. The predicted improvement is one third, despite the heavy-event count falling by half.

Bending at an off-centre pulley

A smooth 30 mm shaft spans 0.45 m between bearings. A pulley 0.15 m from the left support imposes a 1200 N downward load. The shaft receives 60 N m torque at its left end and delivers it at the pulley. Find both reactions, the nominal stresses just left of the pulley and the twist between left bearing and pulley for $G = 80$ GPa.

▼ **Follow the section cut**

Moment balance about the left gives $R_B = 1200(0.15)/0.45 = 400\,\mathrm{N}$, hence $R_A = 800\,\mathrm{N}$ and $M = 120\,\mathrm{N\,m}$. Using millimetres and N mm,

$$\sigma_b = 32(120000)/(\pi 30^3) = 45.3\,\mathrm{MPa},$$

$$\tau = 16(60000)/(\pi 30^3) = 11.3\,\mathrm{MPa}.$$

With $J = \pi(0.030)^4/32$, $\theta = 60(0.15)/(80 \times 10^9 J) = 0.00141\,\mathrm{rad}$, or 0.081°. The section to the right carries bending but no drive torque.

Set a joint-angle limit from the driven-shaft speed tolerance

A single joint drives a lightly loaded inspection spindle at a mean 900 rpm. Its instantaneous speed may be at most 2.0% above and no more than 2.0% below the mean. Determine the maximum shaft angle, the minimum speed at that angle and the frequency of the speed fluctuation. Would increasing the input speed remove the fractional error?

▼ **Show the worked solution**

The upper bound needs $1/\cos\beta \le 1.02$, hence

$$\cos\beta \ge 1/1.02 = 0.980392.$$

The lower bound only requires $\cos\beta \ge 0.98$, so the upper bound controls:

$$\beta_{\max} = \arccos(1/1.02) = 11.36^\circ.$$

At this angle the minimum is $900/1.02 = 882.35\,\mathrm{rpm}$ and the maximum is $918.0\,\mathrm{rpm}$. The fluctuation frequency is

$$2(900/60) = 30.0\,\mathrm{Hz}.$$

Increasing input rpm raises both the fluctuation frequency and its absolute amplitude in rpm at fixed angle. The fractional speed ratio follows joint geometry.

Revisit chapter 2 · Why a single universal joint produces a twice-per-revolution speed fluctuation.

Explore the governing equations after solving the sheet: From machine duty to a design load; Shaft reactions, twist and rotating bending. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Chapter 3 · Preload, joint stiffness and separation

Compute how a tensile service load changes bolt force and clamp force, and determine when a closed-joint calculation ceases to apply.

FROM FIRST PRINCIPLES

Derive bolt and clamp increments from compatible deformation

A tightened bolt stretches while the clamped members compress. Before external loading their force magnitudes equal the preload F_i. Apply a separating load P and assume both components respond elastically and the interfaces stay closed. A small opening displacement δ stretches the bolt additionally and relieves the members by the same displacement. With stiffnesses k_b, k_m the increments are

$$\begin{aligned} \Delta F_b &= k_b \delta, \\ -\Delta F_m &= k_m \delta, \\ P &= \Delta F_b - \Delta F_m = (k_b + k_m)\delta. \end{aligned}$$

Here member force F_m is positive in compression. Eliminate the displacement to find both service forces and the fraction C of external load reaching the bolt.

$$\begin{aligned} C &= \frac{k_b}{k_b + k_m}, \\ F_b &= F_i + CP, \\ F_m &= F_i - (1 - C)P. \end{aligned}$$

Setting $F_m = 0$ predicts incipient separation at $P = F_i/(1 - C)$. Once the interface can no longer supply compressive reaction, the closed-joint stiffness partition ceases to describe the load path. The derivation explains why service bolt tension grows by only a fraction of applied load before separation.

Background · Joint stiffness fraction

$$C = k_b/(k_b + k_m)$$

Before separation, bolt and clamped-member stiffnesses decide what fraction of a separating external force increases the bolt load.

DERIVATION AND APPLICATIONS

Predict preload loss caused by settlement of a bolted joint

A tightened joint may lose clamp load before it sees any external service force. Model the bolt by axial stiffness k_b and the compressed members by stiffness k_m. Assume elastic incremental behaviour, closed contact, fixed temperature and a nut that does not rotate after tightening. Initial preload F_i stretches the bolt and compresses the members. A small permanent settlement s at asperities or interfaces consumes part of the initial imposed interference.

$$\Delta_{\text{installed}} = F_i \left(\frac{1}{k_b} + \frac{1}{k_m} \right),$$
$$\Delta_{\text{remaining}} = \Delta_{\text{installed}} - s.$$

The same preload passes through the bolt and compressed stack before an external tensile load is applied. Recovering their combined elastic displacement therefore reduces the preload according to their series stiffness. The result applies as long as positive clamp load remains.

$$F_r = F_i - k_{\text{eq}} s,$$
$$k_{\text{eq}} = \left(\frac{1}{k_b} + \frac{1}{k_m} \right)^{-1}.$$

Now apply an external separating force P. In a closed joint the bolt takes fraction $C = k_b/(k_b + k_m)$ of that force and the clamp decreases by $(1 - C)P$. Setting the residual clamp to zero gives the revised separation threshold.

$$F_{\text{clamp}} = F_r - (1 - C)P,$$
$$P_{\text{sep}} = \frac{F_i - k_{\text{eq}} s}{1 - C}.$$

Settlement is a displacement, so a more compliant bolt can reduce the preload lost per micrometre. Other design constraints, including stiffness, load introduction and fatigue, remain relevant when altering bolt geometry.

WORKED EXAMPLE

Audit a joint after a measured bedding-in displacement

Let $k_b = 200$ kN/mm, $k_m = 800$ kN/mm and $F_i = 24.0$ kN. The series stiffness is 160 kN/mm. A settlement of 15.0 µm reduces preload by $160(0.0150) = 2.40$ kN, leaving 21.6 kN. Since $C = 0.200$, the new separation load is 27.0 kN. Before settlement it would have been 30.0 kN.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Find an allowable settlement from a service condition

A separate joint has $k_b = 120$ kN/mm, $k_m = 360$ kN/mm and initial preload 18.0 kN. At maximum separating force 12.0 kN it must retain 5.00 kN clamp. The required residual preload is

$$5.00 + 0.750(12.0) = 14.0\,\mathrm{kN}.$$

The allowable preload loss is 4.00 kN and $k_{\mathrm{eq}} = 90.0$ kN/mm, yielding $s_{\max} = 0.0444$ mm. The stated requirement is more restrictive than merely keeping contact closed.

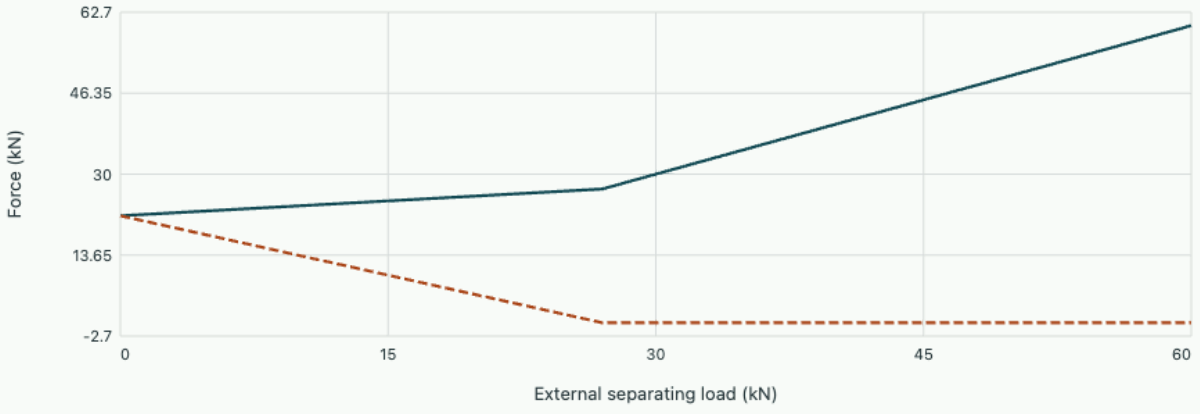
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Installed preload (kN)	24
Bolt stiffness fraction	0.2
Settlement (µm)	15



Retained preload 21.600 kN; separation at 27.000 kN. Equivalent settlement stiffness160 kN/mm is held fixed while the load fraction is explored.

Establish the state before the external load arrives

A tensioned bolt in an assembled flange has already elongated. The material clamped between head and nut has already shortened. In the ideal concentric joint these initial forces have equal magnitude F_i, as a free-body diagram of either flange shows. Tightening stores elastic energy in both parts. Calling a joint unloaded usually means that the external service load is zero. The thread root can still carry a significant tensile stress.

Represent the effective bolt grip by a tensile spring k_b and the compressed members by a spring k_m. For a uniform segment the axial stiffness is EA/L; threaded and plain segments in series have added compliances. The member stiffness depends on the path through which compressive force spreads from the head and washer. A thin flexible flange may also bend. For this derivation consider one bolt, an axial load introduced at the outside faces and no eccentricity or prying. Let clamp force denote a positive magnitude in compression.

Background · How does a small loss of grip thickness reduce preload?

Surface roughness can flatten under load after tightening. Suppose this reduces the unloaded thickness of the clamped stack by s while the nut stays on the same thread position. The recovered bolt elongation and the elastic recovery of the clamped members must together accommodate that lost thickness.

$$s = \Delta F \left(\frac{1}{k_b} + \frac{1}{k_m} \right),$$

$$\Delta F = \frac{k_b k_m}{k_b + k_m}\, s.$$

With bolt stiffness 200 kN/mm, member stiffness 800 kN/mm and settlement 0.010 mm, preload falls by 1.6 kN. This estimate assumes the stack is still in compression and both springs retain their tangent stiffness. A longer, more compliant bolt loses less preload for the same settlement because a given length change produces a smaller force change.

Apply compatibility to the load increment

An external separating force P allows the bolt to extend a little while the clamped members recover some of their initial compression. As long as the faces remain in contact, the extension and recovery have the same magnitude x. The bolt-force increase is $k_b x$ and the clamp-force decrease is $k_m x$. Force equilibrium across the joint gives

$$P = F_b - F_m = (k_b + k_m)x.$$

Eliminate x and introduce the load fraction C.

$$C = \frac{k_b}{k_b + k_m},$$
$$F_b = F_i + CP,$$
$$F_m = F_i - (1 - C)P.$$

The slopes of bolt force and clamp-force magnitude have opposite signs. The joint opens in this ideal model when F_m reaches zero, at $P_{\text{sep}} = F_i/(1 - C)$. Continuing the same line beyond that point would predict tension between unbonded faces. Once open, the bolt must carry the full external tensile force and the load slope changes to unity. Both expressions give the same bolt force at the instant of separation.

Check a service cycle and a separate overload

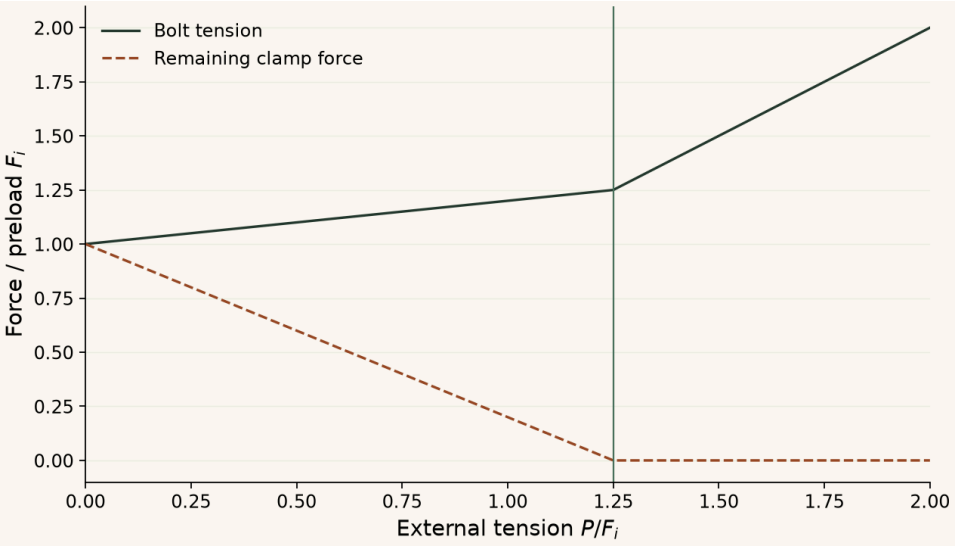
Take $k_b = 180\,\text{kN/mm}$, $k_m = 720\,\text{kN/mm}$, $F_i = 25\,\text{kN}$ and bolt tensile stress area $A_t = 58.0\,\text{mm}^2$. A pressure pulsation creates a separating force cycling between 4 and 16 kN. Here $C = 0.20$, hence bolt force ranges from 25.8 to 28.2 kN. Clamp force falls from 21.8 to 12.2 kN and remains positive. The bolt’s alternating tensile stress is the force range divided by twice its tensile stress area.

$$\sigma_a = (28.2 - 25.8)10^3/(2 \times 58.0) = 20.7\,\text{MPa}.$$

At the maximum ordinary load the calculated tensile stress is 486 MPa. If the specified proof stress is 600 MPa, the axial proof-load benchmark is $600(58.0) = 34.8\,\text{kN}$. The estimated separation threshold is

$$25/0.8 = 31.25\,\text{kN}.$$

A singular tensile overload of 34 kN opens the interface and brings the bolt to 34 kN, just below the supplied axial proof benchmark. It can nevertheless break the seal at the flange face. Different requirements can control the same event.



For a load-sharing factor of 0.20 the joint opens at 1.25 times its initial preload. The slope of bolt force against external force changes at separation.

Use attainable preload, including its scatter

Increasing preload by 3 kN raises both initial forces and delays separation by $3/(1-C) = 3.75\,\mathrm{kN}$. It also raises the peak bolt stress by $3000/58 = 51.7\,\mathrm{MPa}$ throughout the closed regime. Reducing bolt stiffness relative to the members can lower the alternating force. For example a longer effective grip can increase bolt compliance. Recheck the residual clamp and separation conditions whenever changing stiffness.

If tightening produces a range of preloads, assess separation with the low end and proof stress with the high end. An assembly tolerance of ±3 kN around the nominal value gives a minimum separation threshold of $22/0.8 = 27.5\,\mathrm{kN}$. The largest ordinary bolt force would be $28 + 0.2(16) = 31.2\,\mathrm{kN}$. The preload scatter enters these checks even though the external service force is perfectly known. In a real flange, surface settling, differential thermal expansion and off-axis loading can change the available clamp force after assembly.

Resolve tightening torque into thread lift and friction

Joint preload depends on the tightening method as well as bolt and flange stiffness. For an elementary torque estimate, unwrap one turn of the thread at pitch diameter d_2. Its lead l over the circumferential distance πd_2 defines helix angle $\lambda = \arctan[l/(\pi d_2)]$. For a square thread lifting axial load F_i, resolving an applied tangential force and Coulomb friction on the incline gives

$$F_{\tan} = F_i \tan(\lambda + \phi),$$

where $\tan\phi = \mu_t$. For a symmetric V-thread of half-angle α, a common small-helix approximation increases the effective friction to $\mu_t/\cos\alpha$ because the flanks carry a larger normal force.

Under the head or nut, sliding friction at effective radius r_b adds torque $\mu_b F_i r_b$. A simple total tightening estimate is

$$T \simeq F_i[(d_2/2)\tan(\lambda + \arctan(\mu_t/\cos\alpha)) + \mu_b r_b].$$

The first term includes ideal screw lift and thread friction; the second is bearing-face friction. In the limit of zero friction the thread term reduces to $F_i l/(2\pi)$, as verified by equating work over one revolution to axial load times lead. This lower bound also shows why most measured tightening torque can disappear into friction rather than useful bolt elongation. Effective bearing radius depends on contact geometry and pressure distribution, and the coefficients may differ between production lots.

Determine preload sensitivity to lubrication at fixed torque

Use an illustrative single-start thread with $d_2 = 9.0$ mm, $l = 1.50$ mm, flank half-angle 30° and effective bearing radius 7.0 mm. The helix angle is

$$\arctan[1.5/(\pi 9)] = 3.04^\circ.$$

With $\mu_t = \mu_b = 0.15$, the effective thread friction angle is 9.83°. The thread torque arm is $4.5\tan(12.87^\circ) = 1.03\,\mathrm{mm}$, and bearing friction contributes another 1.05 mm. A tightening torque of 40 N m therefore suggests $F_i = 40000/(1.03 + 1.05) = 19.2\,\mathrm{kN}$. The frictionless lift portion would be only $19.2(1.50)/(2\pi) = 4.59\,\mathrm{N\,m}$.

If lubrication reduces both coefficients to 0.08 at the same applied torque, the effective thread friction angle becomes 5.28°. The torque arms are now $4.5\tan(8.32^\circ) = 0.658$ mm and 0.560 mm. Implied preload rises to

$$40000/1.218 = 32.8\,\mathrm{kN}.$$

A tensile area of 58 mm² would then carry about 566 MPa before the service force arrived, compared with 331 MPa originally. An old torque setting may therefore raise preload substantially after a change of coating or lubricant. A direct elongation measurement can help distinguish this friction change from loss of clamp due to settling, although the effective bolt grip and modulus must also be known.

The axial stretch provides an independent expected signal. Suppose the effective uniform tensile length is 50 mm with $E = 210\,\mathrm{GPa}$ and area 58 mm². The two preload estimates imply elastic elongations $F_i L/(EA) = 0.079$ and 0.135 mm. An instrument resolving about 0.01 mm could distinguish the two states in this example if its gauge length includes the correctly modelled bolt segments. Thread and shank regions of different areas contribute as elements in series. A clamp-force reading taken after surfaces have settled can also differ from a tightening-time elongation record without a change in friction coefficient.

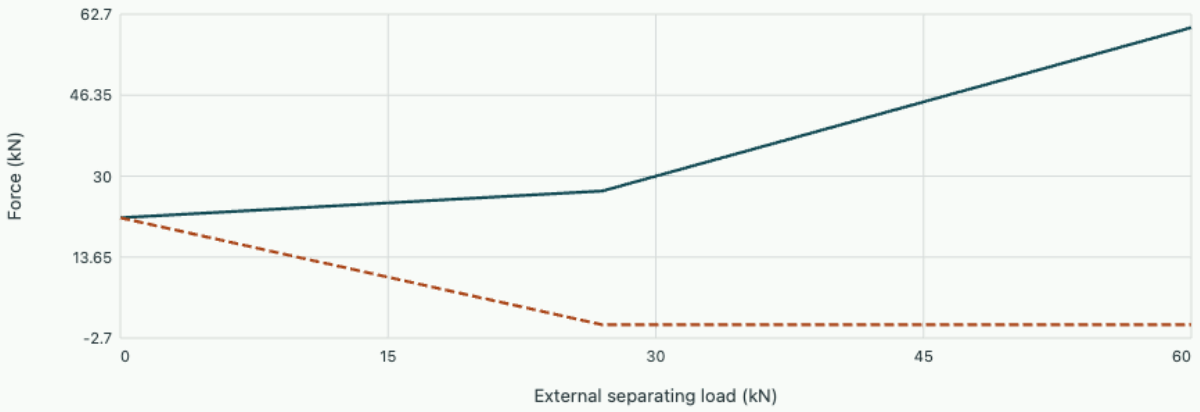
Interactive calculation · Change the inputs of this example

Change the parameters of this reference plot and compare its limiting cases. The fixed illustration remains beside the derivation; the adjustable calculation exposes the same dependence on geometry, material properties or loading.

Open this live calculation

Try the calculation

Installed preload (kN)	24
Bolt stiffness fraction	0.2
Settlement (µm)	15



Retained preload 21.600 kN; separation at 27.000 kN. Equivalent settlement stiffness160 kN/mm is held fixed while the load fraction is explored.

Chapter 4 · Bearing arrangements and rating life

Select the reactions carried by each bearing and convert a changing bearing load into an equivalent load and a basic rating life.

FROM FIRST PRINCIPLES

Derive an equivalent bearing load from damage per revolution

A bearing's basic dynamic rating C calibrates an empirical relationship between constant equivalent load P and 90 percent survival life. In the usual power-law model, p is the bearing-type exponent and L_{10} is measured in revolutions. At constant shaft speed n in revolutions per minute, division by revolutions per hour converts to operating hours.

$$L_{10} = 10^6 (C/P)^p,$$

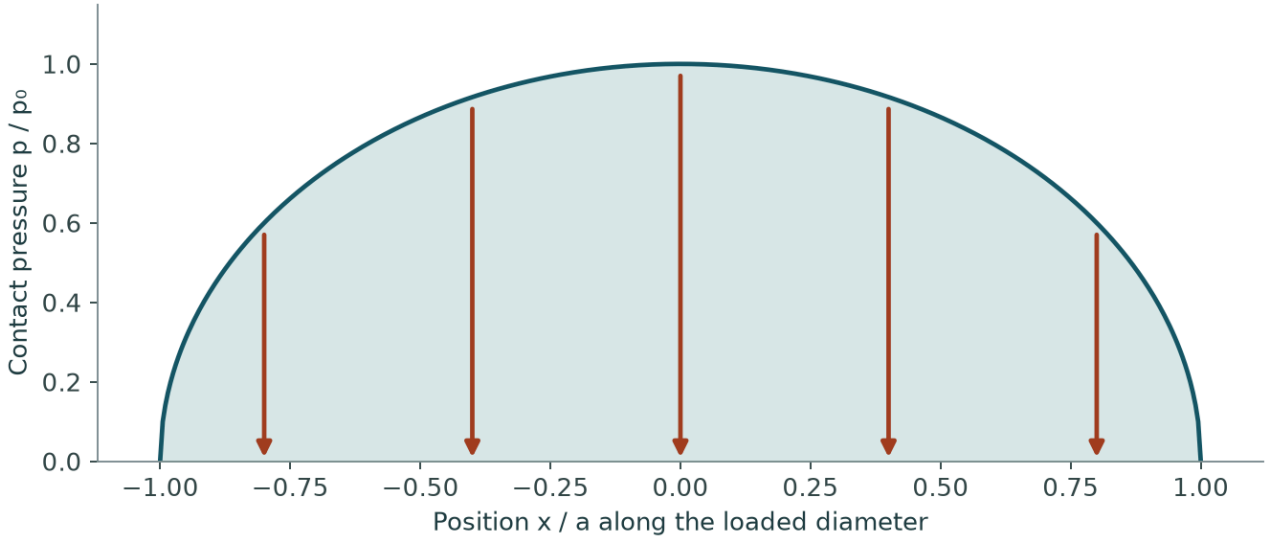
$$L_{10h} = \frac{10^6}{60n} (C/P)^p.$$

For variable duty let segment j cover N_j revolutions at load P_j. Assuming damage fractions per revolution add linearly, define the constant load P_{eq} producing the same predicted damage over $N = \sum_j N_j$ revolutions. Substitution of the rating law cancels its common factors.

$$\frac{N}{L_{10}(P_{\mathrm{eq}})} = \sum_j \frac{N_j}{L_{10}(P_j)},$$

$$P_{\mathrm{eq}} = \left(\sum_j \frac{N_j}{N} P_j^p \right)^{1/p}.$$

The load fractions must be weighted by revolutions. Equal durations at different speeds produce different numbers of contact cycles. At standstill or very low speed, permanent indentation can control an overload, requiring a separate static assessment.



From a pressure profile to the load on a bearing contact

For a frictionless sphere on a plane, rotate this diameter cut about its centre. The force on an annulus is 2πr p(r) dr. Integrating the Hertz profile out to radius a gives F = 2πa²p₀/3. The peak pressure is consequently one and a half times the area average. This factor is specific to the axisymmetric contact.

DERIVATION AND APPLICATIONS

Convert two bearing life distributions into a shaft survival curve

A shaft supported by two bearings generally requires both to work until the next service interval. Suppose life at a fixed speed and load follows a two-parameter Weibull distribution with shape β common to both bearings. Let L_j denote the life at which bearing j has survival 0.90. Assume independent fatigue failures and no wear-out mechanism shared by the pair. Write the individual survival curve with a constant a_j to be chosen from the rated survival condition.

$$S_j(t) = \exp(-a_j t^{\beta}),$$
$$a_j = \frac{-\ln 0.90}{L_j^{\beta}}.$$

The shaft survives only if both bearings survive. Independence turns that event into a product of probabilities and hence a sum of the exponents. This sum has the same Weibull shape, allowing the shaft's own 90 percent survival life to be identified.

$$S_{\text{shaft}}(t) = \exp[-(a_1 + a_2)t^{\beta}],$$
$$L_{\text{shaft}} = (L_1^{-\beta} + L_2^{-\beta})^{-1/\beta}.$$

For equal bearings this becomes

$$L_{\text{shaft}} = L\,2^{-1/\beta}.$$

To evaluate any mission time one can use the equivalent pair life in the same survival expression. Alternatively, solving for mission time at a requested survival probability S_* gives

$$t_* = L_{\mathrm{shaft}} \left(\frac{\ln S_*}{\ln 0.90} \right)^{1/\beta}.$$

A rating life is a population quantile. The distribution shape used here is an additional modelling input, and contamination affecting both bearings would undermine independence even if the individual life curves remained suitable.

WORKED EXAMPLE

Size identical bearings for a service interval

With $\beta = 2.00$ and individual $L = 20{,}000$ hours, the pair has $L_{\mathrm{shaft}} = 20{,}000/\sqrt{2} = 14{,}142$ hours. At 20,000 hours each bearing survives with probability 0.90, giving pair survival $0.90^2 = 0.810$. If the shaft itself must reach 20,000 hours at 90 percent survival, each individual life must instead be at least 28,284 hours under these assumptions.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Combine two unequal ratings

Another shaft uses bearings with 90 percent lives of 12,000 and 24,000 hours, again with $\beta = 2.00$. Pair life is $12{,}000/\sqrt{1 + 1/4} = 10{,}733$ hours. At a 6000 hour mission the pair survival is

$$0.90^{(6000/12000)^2 + (6000/24000)^2} = 0.968.$$

The shorter-life bearing contributes four times as much to the cumulative failure exponent as the longer-life bearing.

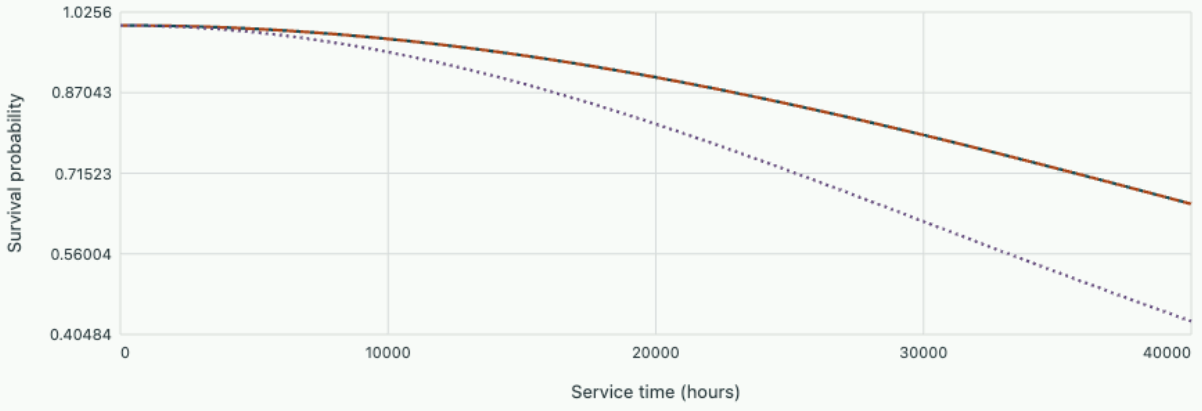
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

First individual L10 (hours)	**20000**
Second individual L10 (hours)	**20000**
Weibull shape β	**2**



Pair L10=14142 hours. Multiplication assumes independent failures and the same Weibull shape for both bearings.

Start from this chapter. Recover the Hertz exponents for load and contact radius under an imposed approach. Halve the approach, then compare both ratios with the finite-block pressure field.

OneSimulation laboratory · Test the contact law under imposed approach

Constrain motion while allowing expansion

A radial bearing supports the shaft perpendicular to its axis. Some bearing types can also transmit an axial reaction in one or both directions. A common two-bearing shaft arrangement uses one locating

bearing or bearing pair to establish axial position. A second support permits axial expansion while still providing radial support. Depending on construction, that relative movement occurs within the bearing or between a ring and its seat. Drawing a free-body diagram with two fully fixed axial supports can hide an unwanted thermal force.

For example, a 0.60 m steel shaft with thermal expansion coefficient $12 \times 10^{-6}\,\mathrm{K}^{-1}$ lengthens by 0.288 mm when it warms by 40 K relative to a housing of unchanged length. If the support arrangement can accommodate that travel, it need not create a significant thermal axial reaction. A locating pair set with insufficient operating clearance can behave very differently. First identify radial reactions in each transverse plane and how the axial thrust returns to the housing. Only then assign loads to individual bearings.

Express a survival criterion in revolutions

A loaded rolling contact subjects the ring material to repeated subsurface stresses. The basic dynamic load rating C is associated with a basic rating life of one million revolutions at a constant equivalent dynamic load equal to C. The basic rating life L_{10} describes a population under the rating assumptions. Ninety percent are expected to reach or exceed this life with respect to rolling-contact fatigue. For ball bearings use exponent $p = 3$; for roller bearings use $p = 10/3$. The rating model is empirical.

$$L_{10} = (C/P)^{p} \text{ million revolutions},$$

$$L_{10h} = \frac{10^{6}}{60 n} (C/P)^{p}.$$

Here P is the equivalent dynamic bearing load and n the speed in revolutions per minute. A purely radial load on a suitable radial ball bearing can be used directly. When radial and axial components both act, obtain the appropriate equivalent-load coefficients and applicability thresholds for that bearing. Their directions relative to the shaft matter. The hypotenuse of axial and radial loads generally fails to represent the internal rolling contact load distribution.

Weight changing loads by revolutions

Suppose interval i occupies fraction f_i of the revolutions. Under the usual linear accumulation approximation, the life fraction consumed per revolution scales with P_i^{p}/C^{p}. Equating the weighted consumption to that of a constant load gives

$$P_{\mathrm{eq}}^{p} = \sum_{i} f_i P_i^{p},$$

$$f_i = \frac{n_i t_i}{\sum_{j} n_j t_j}.$$

Consider a ball bearing rated at $C = 18\,\mathrm{kN}$. It operates for 30 minutes at 1000 rpm under 2 kN of pure radial load, followed by 10 minutes at 1500 rpm under 4 kN. The intervals contain 30,000 and 15,000 revolutions. Thus $P_{\mathrm{eq}}^3 = (2/3)2^3 + (1/3)4^3 = 26.667\,\mathrm{kN}^3$, giving 2.988 kN. The rating life is $18^3/26.667 = 218.7$ million revolutions. The cycle mean speed is $45000/40 = 1125$ rpm, giving about 3240 operating hours. The heavier interval contributes four fifths of the calculated damage even though it occupies only one quarter of the operating time.

Treat slow overloads as a separate question

While the shaft is stationary there are no accumulated revolutions in the preceding fatigue sum. A large stationary force can still leave permanent indentations at the rolling contacts. A static check compares the bearing-specific static equivalent load P_0 with its basic static rating C_0. For an example catalogue entry $C_0 = 10\,\mathrm{kN}$ and a stationary equivalent load of 6 kN, the ratio $s_0 = C_0/P_0 = 1.67$. Compare this value with the application requirement, taking account of shock and required smoothness.

A selected rolling bearing also requires sufficient lubrication and compatible fits and operating clearances. The basic life calculation quantifies one fatigue mechanism. It is especially useful for comparing changes to a known duty cycle because of the strongly nonlinear dependence on load. When a measured bearing fails well before its estimated rating life, inspect mounting marks, lubrication and contamination evidence before changing the equivalent load to fit the failure time.

The manufacturer explains the bearing ratings and exponents in its SKF rolling bearing catalogue.

Recover the size of a contact from its pressure field

A catalogue load rating describes an entire bearing. To estimate the local pressure beneath one rolling element, specify the load on that contact. Start with two smooth isotropic elastic spheres in frictionless normal contact. Their radii R_1, R_2 define $1/R^* = 1/R_1 + 1/R_2$ for external contact. A plane has zero curvature. The half-space compliances add, giving $1/E^* = (1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2$. A conforming raceway generally has two different principal curvatures, which would require an elliptical contact calculation.

The Hertz elastic solution gives an axisymmetric pressure $p(r) = p_0\sqrt{1 - r^2/a^2}$ within a contact radius a, with $p_0 = 2E^* a/(\pi R^*)$. These two statements use the half-space solution. The normal force follows by integrating pressure over rings of area $2\pi r\,dr$. The integral of $r\sqrt{1 - r^2/a^2}$ from zero to a is $a^2/3$. Substituting the peak pressure then gives

$$F = \tfrac{2}{3}\pi a^2 p_0 = \frac{4E^* a^3}{3R^*},$$

$$a = \left(\frac{3FR^*}{4E^*}\right)^{1/3},$$

$$p_0 = \frac{3F}{2\pi a^2}.$$

For a steel sphere of radius 10 mm pressing on a steel flat with 25 N, $E = 210$ GPa and $\nu = 0.30$ for each body give $E^* = 115.4$ GPa. The radius is 0.118 mm, the centre pressure 0.864 GPa and the mean pressure 0.576 GPa. The elastic approach is $a^2/R^* = 1.38$ µm. The small ratio $a/R^* \simeq 0.012$ supports the local geometry approximation. Elasticity must still be assessed using material properties and the subsurface stress state. The compressive surface pressure alone is insufficient as a uniaxial yield check.

An eightfold increase in load doubles the radius. Area rises fourfold, and the peak pressure only doubles. How can a pressure field on a discrete mesh be compared with this circular solution when the edge cuts across elements? Use the second pressure moment $M_2 = \int r^2 p \, dA$. Under Hertz loading the ratio M_2/F is the ratio of two integrals. Substitution $s = r/a$ gives $\int_0^1 s^3 \sqrt{1 - s^2} \, ds = 2/15$ in the numerator and $\int_0^1 s \sqrt{1 - s^2} \, ds = 1/3$ in the denominator. Therefore

$$M_2/F = 2a^2/5.$$

Multiplying a discrete ratio by $5/2$ before taking its square root gives a Hertz-equivalent radius. Compute both sums with the surface integration weights. A quarter-model symmetry factor cancels in the ratio, but it must be restored for the full force.

On its initial mesh the OneSimulation contact experiment returns 2060.2 N and a moment-based radius of 0.5033 mm. For that force, 10 mm curvature radius and the same steel pair, the half-space formula predicts 0.5116 mm. The radius difference is about −1.6%. A non-Hertz pressure field can have the same second moment. Agreement of the radius is one test alongside force balance, mesh refinement and inspection of the profile.

Practise on sheet 2 · Infer a contact load from an optical footprint. Further reading · Heidelberg University · Continuum Mechanics · §5.8 · Contact of two elastic spheres (Hertz solution).

Distinguish displacement control from a rating test

Contact between a rolling element and a race concentrates elastic load over a small area. An ideal sphere on an elastic half-space gives $a \propto \sqrt{R\delta}$ and $F \propto E^* \sqrt{R} \, \delta^{3/2}$ for imposed approach δ. Predict the radius and force ratios when approach falls from 25 to 12.5 micrometres. In a second test halve both Young's moduli while keeping the original approach. Which output should remain almost constant? Compare both predictions on the same mesh. These static contact outputs describe the stress-producing contact mechanism; the bearing rating life also requires repeated loading and bearing-specific rating data. Inspect the sphere-contact experiment.

Derive preload redistribution in opposing contact sets

A locating arrangement can use two opposing angular-contact bearings. An axial displacement loads one contact set more heavily while unloading the other. For a simplified comparison, represent each complete bearing by the same measured axial force law $F = k\delta^{3/2}$ over the working range. The

exponent is motivated by elastic Hertzian point contacts; actual bearing geometry, contact angle and load sharing are incorporated here into an illustrative equivalent coefficient. Assume rigid mounting seats, no clearance after installation, symmetric preload and no radial load in this calculation.

If the installed preload per bearing is $F_0 = k\delta_0^{3/2}$, the unloaded equilibrium has two equal opposing internal forces. An external axial load displaces the shaft by x, giving

$$P = k[(\delta_0 + x)^{3/2} - (\delta_0 - x)^{3/2}], \qquad |x| \le \delta_0.$$

Linearising about zero displacement gives pair stiffness $dP/dx = 3k\sqrt{\delta_0} = 3F_0/\delta_0$. At $x = \delta_0$ the relieving bearing just loses its force, while the loaded one carries $2^{3/2}F_0$. Beyond that point its companion cannot develop tensile rolling contact and must be removed from the active force balance. A constant stiffness extrapolated from the preloaded point misses this contact transition. A preloaded arrangement can provide useful stiffness at small loads while still putting appreciable internal force on both contacts when the external thrust is zero.

Check stiffness and unloading against a thrust cycle

Suppose a measured arrangement gives $F_0 = 2.0\,\mathrm{kN}$ per bearing at $\delta_0 = 20\,\mu\mathrm{m}$. Initial pair stiffness is

$$3(2000)/(20) = 300\,\mathrm{N}/\mu\mathrm{m}.$$

A 600 N thrust thus moves the shaft about 2 µm according to the small-displacement estimate. Substituting $x/\delta_0 = 0.10$ into the full relation gives $P = 2000[(1.10)^{3/2} - (0.90)^{3/2}] = 600\,\mathrm{N}$ to the shown precision. Individual bearing forces are approximately 2307 and 1708 N, so replacing each by half the applied thrust would seriously misrepresent both rolling loads.

At $x = 10\,\mu\mathrm{m}$, the full model gives

$$P = 2000[1.5^{3/2} - 0.5^{3/2}] = 2967\,\mathrm{N}.$$

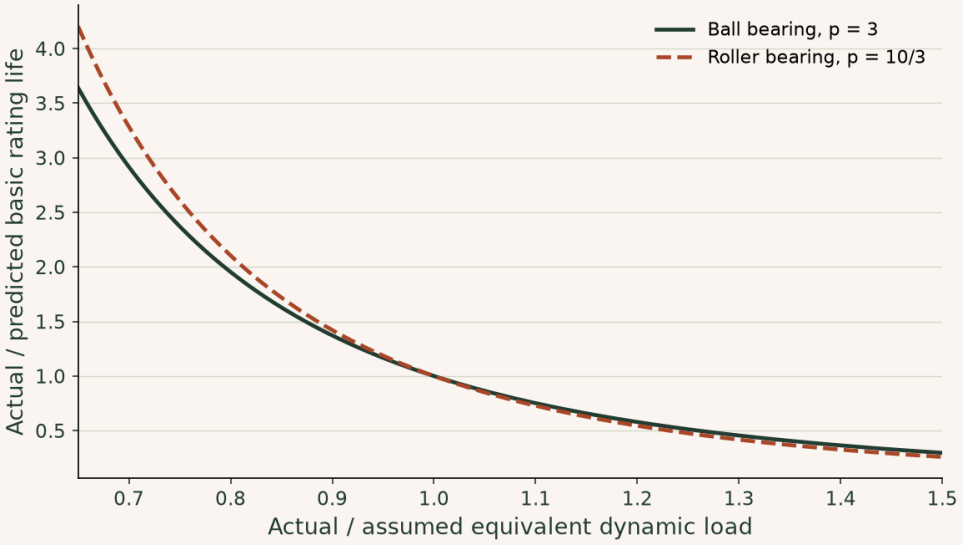
Unloading starts at $P = 2^{3/2}(2000) = 5657\,\mathrm{N}$. A peak external thrust of 6.0 kN exceeds that value. The loaded bearing alone then carries approximately 6.0 kN, with $\delta_0 + x = \delta_0(6000/2000)^{2/3} = 41.6\,\mu\mathrm{m}$, hence $x = 21.6\,\mu\mathrm{m}$. A thermal adjustment that increases initial compression raises both stiffness and the unloading threshold, but it also raises bearing load during low-thrust operation. Any catalogue life calculation must use the appropriate equivalent load for each bearing over the duty cycle; external thrust by itself does not describe its internal preload history.

The relation gives a particularly simple load-range comparison at the onset of unloading. Immediately before it, the lightly loaded bearing's tangent stiffness approaches zero as its contact deflection vanishes. The loaded bearing still has tangent stiffness proportional to $\sqrt{2\delta_0}$, so shaft displacement retains a finite slope through the transition. A visible sudden jump in measured shaft position would point instead to clearance elsewhere, seat movement or an inadequately captured connection.

Quasistatic loading and unloading measurements can also reveal hysteresis due to friction, which the purely elastic contact law used here cannot reproduce.

Quantify how a load underestimate changes bearing life

Keep dynamic load rating and speed fixed. The basic rating life varies as $(C/P)^p$, so a multiplicative error in equivalent load enters with the full life exponent. Take the ratio to the original prediction before converting revolutions to hours.



A 20% load underestimate in the sense Pactual = 1.2 Passumed leaves about 58% of predicted ball-bearing life and 54% for the roller exponent. Basic rating life is a population quantile under the rating assumptions.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

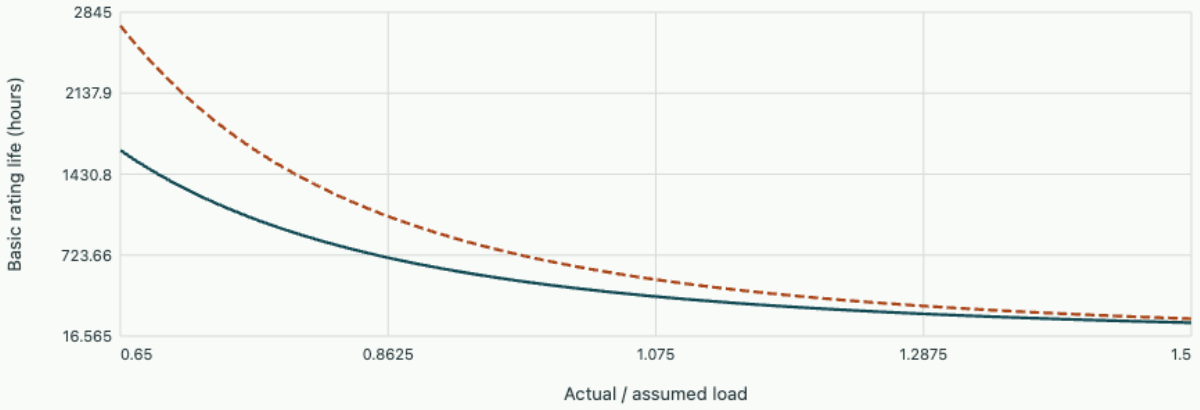
Alter the nominal load and speed, then compare the two rating exponents under the same load uncertainty.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Dynamic rating C (kN)	30
Assumed load (kN)	10
Speed (rpm)	1000



Ball-bearing reference life 450.0 h. A 20% extra load retains 57.9% of it.

Study the full derivation

Helical gears, mesh forces and bearing reactions · Design and analysis of machine components

A motor supplies a known torque to a helical pinion. Select the free body carefully: torque fixes one component of the tooth load, the helix creates thrust, and the location of that thrust can alter bearing reactions even in a parallel-shaft transmission. Build the force diagram and reverse it from measured support loads.

Problem sheet 2 · Clamped interfaces and rolling contact

Evaluate the least favourable preload and bearing duty, then distinguish thermal expansion from externally applied loading.

A tolerance band on preload

A joint has stiffnesses $k_b = 200$ and $k_m = 600$ kN/mm and is preloaded to 18 ± 2 kN. The external separating force cycles from 2 to 12 kN. Find the minimum residual clamp, maximum bolt force and alternating stress for a stress area of 36 mm². Find the smallest separation load.

▼ **Use the appropriate end of the tolerance interval**

The bolt load fraction is 0.25. The least residual clamp is

$$16 - 0.75(12) = 7\,\mathrm{kN}.$$

The largest bolt force is

$$20 + 0.25(12) = 23\,\mathrm{kN}.$$

For any fixed preload within the band, the stress amplitude is

$$0.25(12 - 2)1000/(2 \times 36) = 34.7\,\mathrm{MPa}.$$

The lowest preload separates first, at $16/0.75 = 21.33\,\mathrm{kN}$, above the ordinary maximum.

Convert a load block into bearing hours

A ball bearing turns for 15 min at 800 rpm under radial load 3 kN, then 10 min at 1200 rpm under 5 kN. Find the equivalent radial load, the basic life in hours for $C = 22\,\mathrm{kN}$, and the static ratio for a separate stationary load $P_0 = 8\,\mathrm{kN}$ and $C_0 = 12\,\mathrm{kN}$.

▼ **Count revolutions before averaging**

Each interval contains 12,000 revolutions. Thus $P_{\mathrm{eq}}^3 = (3^3 + 5^3)/2 = 76\,\mathrm{kN}^3$, hence $P_{\mathrm{eq}} = 4.236\,\mathrm{kN}$. Life is $22^3/76 = 140.1$ million revolutions at cycle-average speed $24000/25 = 960\,\mathrm{rpm}$, or about 2432 h. For the separate stationary event, $C_0/P_0 = 1.50$. A stationary overload adds no revolutions but can indent the races.

Two bearing seats warm by different amounts

A 0.50 m steel shaft warms by 55 K while its aluminium housing between the support planes warms by 20 K. Use linear expansions 12 and 23 micrometres per metre per kelvin. Determine relative growth. Does 0.08 mm axial travel accommodate it? Estimate the force after that travel is exhausted if the equivalent axial stiffness is 110 kN/mm.

▼ **Compare travel with differential growth**

The shaft lengthens by 0.330 mm and the housing by 0.230 mm. Thus one end must accommodate 0.100 mm relative to the locating end. The available travel leaves 0.020 mm incompatible. In the stated linear idealisation the extra axial force magnitude is $110(0.020) = 2.2\,\mathrm{kN}$. Changing the locating scheme or the operating clearances changes this reaction.

Infer a contact load from an optical footprint

An optical instrument resolves a circular contact radius of 0.150 mm, with a radius uncertainty of ±2%. A 10 mm radius sphere and a flat have effective modulus 115 GPa. Estimate load and peak pressure from the footprint. How do the two relative uncertainties differ if the elastic model and other inputs are exact? A numerical pressure map gives

$$M_2/F = 9.00 \times 10^{-9}\,\mathrm{m}^2.$$

Is that result consistent with the optical radius?

▼ **Worked solution**

Invert the Hertz law to obtain

$$F = 4E^{*}a^{3}/(3R^{*}) = 51.75\,\mathrm{N}.$$

The peak pressure is

$$2E^{*}a/(\pi R^{*}) = 1.098\,\mathrm{GPa}.$$

The ratio of peak to mean pressure is $3/2$, which also follows by integrating the assumed profile.

Since the inferred load scales as a^3, its first-order relative uncertainty is about 6%. The exact upper factor is

$$1.02^{3} = 1.061208.$$

Peak pressure scales linearly with a, giving 2%. Treating the reconstructed force and measured area as independent error sources would count the same radius uncertainty twice. The numerical moment

implies a radius of $\sqrt{(2.5 \times 9.00 \times 10^{-9})} = 0.150$ mm. Agreement of these independent radius estimates tests the shape assumption involved in converting a pressure moment into an edge position.

Revisit chapter 4 · Recover the size of a contact from its pressure field.

Explore the governing equations after solving the sheet: Preload, joint stiffness and separation; Bearing arrangements and rating life. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Measure contact-law exponents from two solutions

Fit power laws $F \propto d^p$ and $a \propto d^q$ locally from the two runs at 210 GPa. At $d = 25$ µm the normal force is 2060.2 N and the inferred contact radius 0.5033 mm. At 12.5 µm they are 714.5 N and 0.3481 mm. Derive both exponents, compare them with Hertz theory, and name an independent numerical check.

▼ **Work through the solution**

Divide the two power laws to eliminate their unknown prefactors. Taking logarithms gives

$$p = \ln(714.5/2060.2)/\ln(1/2) = 1.528,$$
$$q = \ln(0.3481/0.5033)/\ln(1/2) = 0.532.$$

The elastic half-space predicts $3/2$ and $1/2$. Two observations fit two power laws but cannot tell whether the exponents vary with approach. Repeat with a finer contact mesh and a third approach. Check the force integral and the radius from the pressure distribution separately.

Use the OneSimulation laboratory to check your answer

Vary the corresponding laboratory reference interactively. Compare the recalculated observable with the recorded OneSimulation runs and the approximation used in this exercise.

Chapter 5 · Spur gear kinematics and tooth loads

Use pitch-line compatibility and power balance to find gear speed, tooth forces and the bending demand at a tooth root.

FROM FIRST PRINCIPLES

Connect gear speed and torque through the pitch point

For two externally meshing ideal spur gears of pitch radii r_1, r_2, the pitch circles roll together at their contact point. Their tangential speeds there have equal magnitude and their rotations are opposite. With a common angular sign convention the speed relation and torque-producing tangential tooth force F_t follow from geometry and moment equilibrium.

$$\omega_1 r_1 = -\omega_2 r_2,$$
$$|T_1| = F_t r_1,$$
$$|T_2| = F_t r_2.$$

The two pitch circles have the same module, so radius scales with tooth count z. Combining the relations gives both the signed speed ratio and the ideal ratio of delivered to applied torque magnitudes.

$$\frac{\omega_2}{\omega_1} = -\frac{z_1}{z_2},$$
$$\frac{|T_2|}{|T_1|} = \frac{z_2}{z_1}.$$

For frictionless involute teeth the contact force acts along their common normal, at pressure angle α to the pitch tangent. Resolving it gives

$$F_r = F_t \tan\alpha,$$
$$F_n = F_t / \cos\alpha.$$

The radial separating load affects shaft and bearing reactions even though only the tangential component transmits shaft power.

DERIVATION AND APPLICATIONS

Resolve the axial bearing load of a helical gear

Helical teeth maintain an axial component of contact force while transmitting torque. Consider a parallel-axis external helical mesh with helix angle β at the pitch cylinder and normal pressure angle α_n. Assume a representative quasi-static contact force, ideal geometry and negligible friction. Define the

tangential force F_t from pitch diameter d and applied shaft torque T. To resolve the remaining components, first project the normal contact force F_n onto the tangent plane of the pitch cylinder.

$$F_t = \frac{2T}{d},$$
$$F_{\text{plane}} = F_n \cos \alpha_n,$$
$$F_t = F_{\text{plane}} \cos \beta.$$

The projection in the tangent plane is perpendicular to the tooth trace and thus makes angle β to the circumferential direction. Its other component acts axially. The radial component comes directly from the normal pressure angle. Eliminating F_{plane} and F_n gives the magnitudes

$$F_a = F_t \tan |\beta|,$$
$$F_r = \frac{F_t \tan \alpha_n}{\cos \beta},$$
$$F_n = \frac{F_t}{\cos \alpha_n \cos \beta}.$$

Work transmission can be checked with circumferential velocity $v = \omega d/2$. Ideal axial and radial components do no shaft work because the gear has no velocity along those axes. They still enter bearing and housing loads.

$$P = T\omega = F_t v,$$
$$F_n^2 = F_t^2 + F_a^2 + F_r^2.$$

The limiting spur geometry $\beta = 0$ removes axial thrust and recovers $F_r = F_t \tan \alpha_n$. Thrust direction depends on handedness, rotation and whether the gear drives or is driven, so components calculated here are magnitudes.

WORKED EXAMPLE

Determine the bearing thrust at a conveyor mesh

Take $d = 100$ mm, $T = 50.0$ N m, $\beta = 25.0^\circ$ and $\alpha_n = 20.0^\circ$. Pitch force is 1000 N. The axial thrust is $1000 \tan 25^\circ = 466$ N and radial separating force is

$$1000 \tan 20^\circ / \cos 25^\circ = 402 \,\text{N}.$$

The vector resultant is 1.17 kN. Its axial component must be reacted by a locating bearing or an opposing mesh.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Limit helix angle using a thrust allowance

A different gear transmits 80.0 N m at $d = 160$ mm, also giving $F_t = 1000$ N. Its bearing arrangement permits 300 N of steady axial load from the mesh. Thus

$$|\beta| \le \arctan(300/1000) = 16.7^\circ.$$

With normal pressure angle 20.0° the radial force at that limit is 380 N. Altering helix angle changes thrust and radial loading while the tangential force remains fixed by torque.

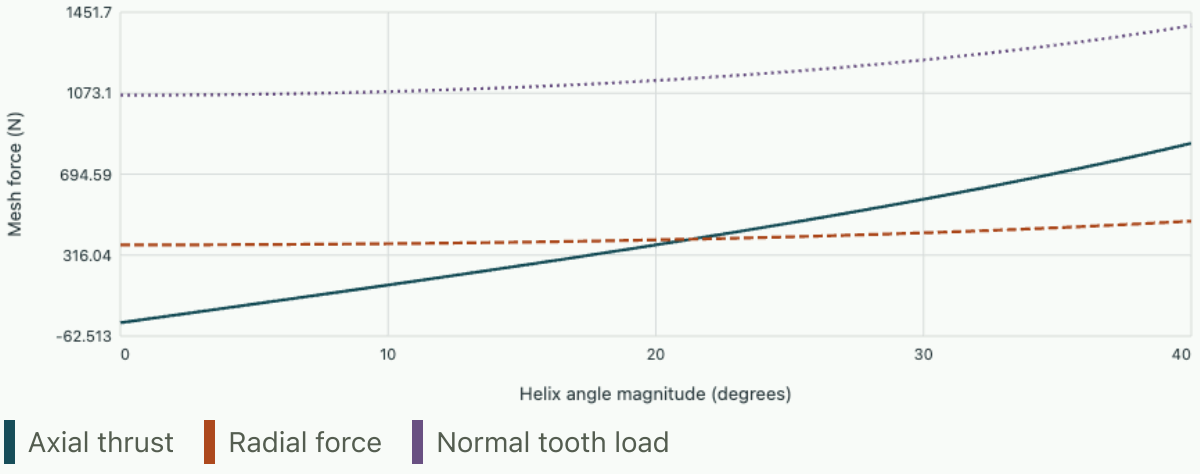
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Torque (N m)	50
Pitch diameter (mm)	100
Normal pressure angle (degrees)	20



Tangential force 1000.000 N. At25° helix, axial thrust 466.308 N; the thrust direction additionally depends on helix hand and rotation.

Replace the mesh by two rolling pitch circles

Two external spur gears have straight teeth parallel to their axes. The common normal to a pair of involute flanks passes through a fixed pitch point on the line joining the centres. At that point the pitch circles roll together at equal tangential speed. The actual contacting flanks slide relative to one another away from the pitch point. Distinguishing pitch-circle motion from local sliding explains why the simple velocity ratio can hold while the tooth surfaces still require lubrication.

For unshifted standard gears of the same module m, pitch diameter equals module times tooth count, and the design centre distance equals half the sum of the pitch diameters. With a common angular sign convention on both parallel axes, the two external gears turn in opposite directions. Writing equality of pitch speeds in magnitude and then inserting direction gives

$$d_i = m z_i,$$

$$a = (d_1 + d_2)/2,$$

$$\frac{\omega_2}{\omega_1} = -\frac{d_1}{d_2} = -\frac{z_1}{z_2}.$$

An idler changes the direction of the last gear in a simple train without changing the magnitude of the first-to-last ratio. A compound shaft carrying two distinct gears is different. Those two gears share angular speed, so the ratios of successive meshes multiply. Draw the shafts before multiplying any tooth counts.

Resolve the force along the common normal

Neglect tooth friction in the force construction. The resultant F_n acts normal to the contacting flanks. Its projection along the pitch tangent, $F_t = F_n \cos\alpha$, produces torque on each shaft. Its radial projection, $F_r = F_t \tan\alpha$, pushes the two gear centres apart. A straight spur mesh with aligned shafts produces no ideal axial component. The bearings must return both transverse force components to the housing.

$$T_1 = F_t d_1/2,$$

$$T_2 = F_t d_2/2,$$

$$P = F_t v,$$

$$v = \pi d_1 n_1/60.$$

The transmitted tooth forces have equal magnitudes on both gears. The larger gear develops more torque because that force acts through a larger radius. Multiplying output torque by the correspondingly lower output speed recovers the input power in an ideal mesh. When estimating useful output power for a real stage, multiply input power by the specified efficiency and state which shaft receives that output.

A low-speed conveyor stage

A motor drives a 20-tooth pinion at 1200 rpm. It meshes with a 60-tooth gear of module 3 mm and pressure angle 20°. The input power is 4.0 kW. Pitch diameters are 60 and 180 mm, the centre distance is 120 mm, and the output turns at 400 rpm in the opposite direction. Angular input speed is 125.66 rad/s, giving pinion torque 31.83 N m. The pitch-line speed is 3.770 m/s.

Tangential force follows from $4000/3.770 = 1061\,\mathrm{N}$. The radial separating force is

$$1061 \tan 20^\circ = 386\,\mathrm{N}.$$

The normal force is

$$1061/\cos 20^\circ = 1129\,\mathrm{N}.$$

The ideal output torque is 95.5 N m. With a measured stage efficiency of 0.96, useful output is 3.84 kW and output torque is 91.7 N m. A design based on shaft torque alone omits the transverse force that bends the shaft and loads the bearings.

Build a first tooth-root estimate

For a preliminary bending model treat the loaded tooth as a short cantilever of face width b, effective thickness s and moment arm h. The rectangular section modulus is $bs^2/6$; the bending moment is approximately $F_t h$. Hence root stress scales as $6F_t h/(bs^2)$. For geometrically similar teeth write $Y = s^2/(6hm)$, a dimensionless form factor in the module convention used here. Then $\sigma_b \simeq F_t/(bmY)$.

Taking $b = 24\,\mathrm{mm}$, $m = 3\,\mathrm{mm}$ and an illustrative $Y = 0.32$ gives

$$1061/(24 \times 3 \times 0.32) = 46.1\,\mathrm{MPa}.$$

This estimates nominal bending in the simplified geometry. The real tooth also sees local root concentration and surface contact stress. Uneven load across the face, dynamic tooth loads and changing load sharing during mesh can change the result. The shaft experiences a transverse force throughout steady running; an individual tooth is loaded only as it enters each mesh. That distinction changes its number of fatigue cycles.

Count how many tooth pairs can carry load simultaneously

The pitch circles determine the transmission ratio but do not establish whether tooth engagement is continuous. In an ideal external spur-gear pair the point of contact moves along the common line of action between intersections with the two addendum circles. A pair engages before the pitch point and disengages after it. The transverse contact ratio is that total active path length divided by base pitch,

the spacing of adjacent involutes measured along the same line. If contact ratio lies between one and two, one pair is in mesh during part of a pitch interval and two during the remainder.

For standard unshifted pitch radii r_1, r_2, base radii $r_{bi} = r_i \cos\alpha$, addendum radii r_{ai} and operating centre distance a_w, the distance from the base tangent point to the addendum intersection is $\sqrt{r_{ai}^2 - r_{bi}^2}$. Subtract the distance between the two base tangent points, $a_w \sin\alpha_w$. The operating angle satisfies

$$\cos\alpha_w = (r_{b1} + r_{b2})/a_w.$$

For module m and reference pressure angle α, base pitch is $p_b = \pi m \cos\alpha$. Therefore

$$\varepsilon_\alpha = \frac{\sqrt{r_{a1}^2 - r_{b1}^2} + \sqrt{r_{a2}^2 - r_{b2}^2} - a_w \sin\alpha_w}{p_b}.$$

This geometric expression presumes sufficient usable involute profile at both roots and no interference, profile modification or edge contact that shortens the path. It yields geometric overlap, not equal division of tooth load; stiffness and deflection evolve as a tooth crosses the mesh.

Evaluate a centre-distance change in a reducer

A 20-tooth pinion meshes with a 40-tooth gear, both module 3 mm with 20° reference pressure angle and an addendum of one module. Reference pitch radii are 30 and 60 mm, base radii 28.19 and 56.38 mm, and addendum radii 33 and 63 mm. At the reference centre distance 90 mm, the first two square-root terms are 17.155 and 28.109 mm. The tangent-point spacing is $90 \sin 20^\circ = 30.782$ mm, giving active path length 14.482 mm. Base pitch is 8.856 mm and transverse contact ratio is 1.635.

In this ideal mesh, about 63.5% of each base-pitch interval has two pairs engaged and 36.5% has one. Increase the centre distance to 90.5 mm while retaining exactly the same gears. The operating pressure angle increases to $\arccos[(28.19 + 56.38)/90.5] = 20.85^\circ$. The base pitch and addendum intersections remain fixed, but the longer tangent-point spacing lowers the contact ratio to 1.473. The single-pair portion expands to 52.7%. The tooth-count ratio remains exactly two, so a speed measurement alone cannot detect this reduction in load-sharing opportunity. Measuring backlash, centre distance or mesh vibration supplies different evidence. A target for tooth root stress should account for the time spent in single-pair mesh as well as nominal torque.

At 1200 pinion rpm a 20-tooth pinion gives a mesh frequency of $20(1200)/60 = 400$ Hz. The altered centre distance leaves this frequency unchanged while changing the fraction of each 2.50 ms mesh interval spent in single-pair contact. Dynamic excitation amplitude can therefore change without a shift in the measured mesh frequency.

Find the three speeds and housing reaction of a planetary gearset

A simple epicyclic train has a sun with N_s teeth, an internal ring with N_r teeth, and planets whose centres are carried by a rotating arm. One fixed-axis gear ratio cannot relate all three members. In a frame turning with the carrier, the planet centres are stationary and each mesh is an ordinary pitch-circle rolling constraint. The sun–planet mesh reverses the relative spin direction. The internal planet–ring mesh preserves it. Eliminating the planet speed gives the Willis relation

$$N_s(\omega_s - \omega_c) + N_r(\omega_r - \omega_c) = 0,$$

$$\omega_c = \frac{N_s\omega_s + N_r\omega_r}{N_s + N_r}.$$

All speeds use the same positive direction in the stationary frame. With ring locked, the carrier turns with the sun at the reduction ratio $1 + N_r/N_s$. If both input members turn together, all parts rotate as a rigid train and the planets have no spin relative to the carrier. These two limits help detect a sign error at an internal mesh.

Next define all torques as torques applied to the train. In ideal steady operation the external powers sum to zero. For independent admissible sun and ring speeds, substitute the carrier relation into $T_s\omega_s + T_r\omega_r + T_c\omega_c = 0$. The two coefficients must each vanish, giving $T_s = -T_cN_s/(N_s + N_r)$ and $T_r = -T_cN_r/(N_s + N_r)$. This also yields $T_s + T_r + T_c = 0$, as required by angular momentum balance at constant speed.

For a bench mixer choose $N_s = 24,\ N_r = 72$. Same-module gears then need $N_p = (72 - 24)/2 = 24$ teeth per planet. With the ring bolted to the frame and the sun driven at $1200\,\mathrm{rpm}$, the carrier turns at $300\,\mathrm{rpm}$. At $T_s = +12.0\,\mathrm{N\,m}$, the load applies $T_c = -48.0\,\mathrm{N\,m}$ and the frame applies $T_r = +36.0\,\mathrm{N\,m}$ to the gearset. Sun input and carrier output each have magnitude $1.51\,\mathrm{kW}$. The stationary ring transmits no power but its mounts still carry a reaction torque of 36.0 N m.

Practise on sheet 3 · Trace the power when both the sun and ring move. Further reading · Jönköping University · Applied Mechanics · §10.8 Planetary gears, “Seen from the carrier: the Willis equation” and “Torques and the forces on the teeth”.

Study the full derivation

Helical gears, mesh forces and bearing reactions · Design and analysis of machine components

A motor supplies a known torque to a helical pinion. Select the free body carefully: torque fixes one component of the tooth load, the helix creates thrust, and the location of that thrust can alter bearing reactions even in a parallel-shaft transmission. Build the force diagram and reverse it from measured support loads.

Chapter 6 · Tolerances, interference and torque transmission

Turn shaft and hub limits into a range of contact pressures and check whether a cylindrical interference fit can transmit torque.

FROM FIRST PRINCIPLES

Relate diametral interference to frictional holding torque

An unassembled shaft can have a larger free diameter than its mating hub bore. Define positive diametral interference $I = d_s - d_h$. Under elastic assembly the shaft surface moves inwards and the bore outwards until their radii agree. Assume axisymmetry, uniform pressure p, nominal diameter d, contact length L, and combined radial compliance c_r giving total radial displacement per unit pressure. Compatibility gives

$$I/2 = |u_s| + u_h = c_r p,$$

$$p = \frac{I}{2c_r}.$$

Under ideal Coulomb friction, the local shear traction at impending uniform circumferential slip is μp. Every area element of the cylindrical contact has moment arm $d/2$. Integrating the limiting traction yields

$$T_{\lim} = \int_A \frac{d}{2}\mu p \, dA = \frac{\pi}{2}\mu p L d^2.$$

Uniform axial sliding instead has limiting force $F_{\lim} = \mu p \pi d L$. These rely on the same available friction traction, so simultaneous axial and torsional loading must share that capacity. The factor of two in compatibility matters because manufacturing limits usually specify diameters, while surface displacements are radial.

DERIVATION AND APPLICATIONS

Predict how differential thermal expansion changes an interference fit

A shaft and hub assembled with a press fit can lose contact pressure when their temperatures change. Let d be their common nominal diameter at reference temperature and $I_0 > 0$ the initial diametral interference, defined as free shaft diameter minus free hub-bore diameter. Assume small strains, uniform temperature in each part, free axial expansion and elastic response while contact is closed. Let α_s, α_h be the linear expansion coefficients and $\Delta T_s, \Delta T_h$ the temperature changes. To first order in strain and interference-to-diameter ratio,

$$I(T) = I_0 + d(\alpha_s \Delta T_s - \alpha_h \Delta T_h).$$

If c_r is the combined radial compliance per unit pressure, the radial mismatch is $I/2$. Positive pressure requires positive interference. When the calculated interference becomes negative the unloaded circular parts have clearance; the elastic closure formula then ceases to apply.

$$p(T) = \frac{\max[0, I(T)]}{2c_r}.$$

For uniform axial contact length L and constant friction coefficient μ, the maximum tangential traction is μp. Integrating that traction over the cylindrical surface, with moment arm $d/2$, translates a temperature difference into a torque capacity.

$$T_{\lim} = \frac{\pi}{2}\mu p L d^2,$$

$$\frac{T_{\lim}(T)}{T_{\lim}(0)} = \frac{\max[0, I(T)]}{I_0}.$$

This ratio assumes the elastic moduli, friction coefficient and dimensions appearing outside the interference stay effectively constant. Thermal gradients that distort the circular geometry can invalidate the uniform-pressure assumption before mean interference reaches zero.

WORKED EXAMPLE

Estimate capacity loss after uniform warming

A 50.0 mm steel shaft in an aluminium hub has $I_0 = 40.0$ µm. Their expansion coefficients are 12.0 and 23.0 µm/(m K). A common 40.0 K temperature rise reduces interference by $0.0500(11 \times 10^{-6})(40.0) = 22.0$ µm. Remaining interference is 18.0 µm. A measured reference capacity of 120 N m scales to $120(18/40) = 54.0$ N m in this model.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Choose a temporary assembly temperature

A separate 80.0 mm bore with expansion coefficient 18.0×10^{-6} K⁻¹ has 50.0 µm reference interference. With the shaft kept at reference temperature, a desired assembly clearance of 20.0 µm requires bore growth of 70.0 µm. The uniform hub temperature rise is

$$70.0 \times 10^{-6} / [0.0800(18.0 \times 10^{-6})] = 48.6\,\mathrm{K}.$$

The nominal interference returns after both parts settle back to the reference temperature, provided the material response and surfaces remain elastic.

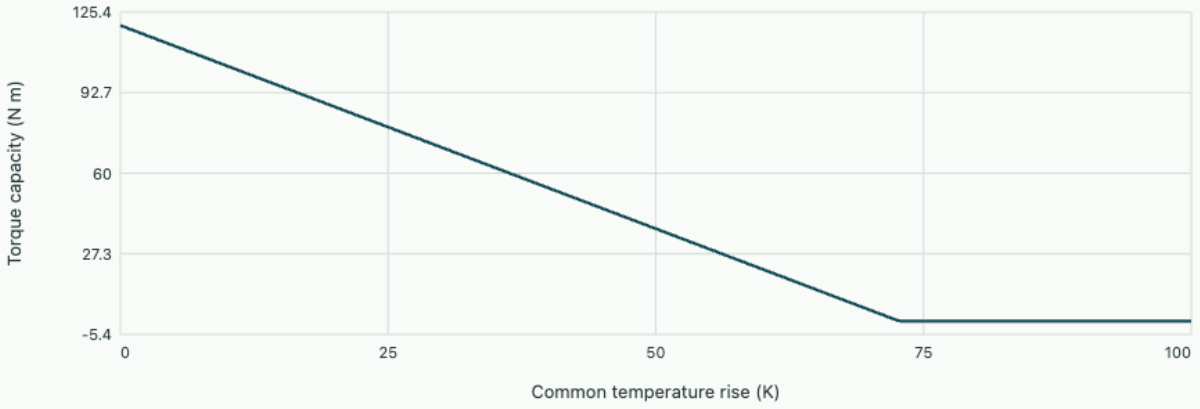
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Initial interference (µm)	40
Hub expansion coefficient (µm/m K)	23
Reference torque capacity (N m)	120



Friction capacity under common warming

Contact is lost at common warming 72.727 K in this50 mm steel-shaft/hub model. Tensile contact pressure is replaced by zero; no artificial negative torque capacity is allowed.

Start from this chapter. In a displacement-controlled elastic contact, scaling both elastic moduli changes reaction force. Use the sphere and block to test the scaling, then consider which geometric assumptions change for a cylindrical interference fit.

OneSimulation laboratory · Test the contact law under imposed approach

What thermal clearance survives the tolerance stack?

At room temperature a ring bore is slightly smaller than its mating shaft. An employee measures a temperature rise of 100 K at the ring and sees that the nominal pair should slide together. A second

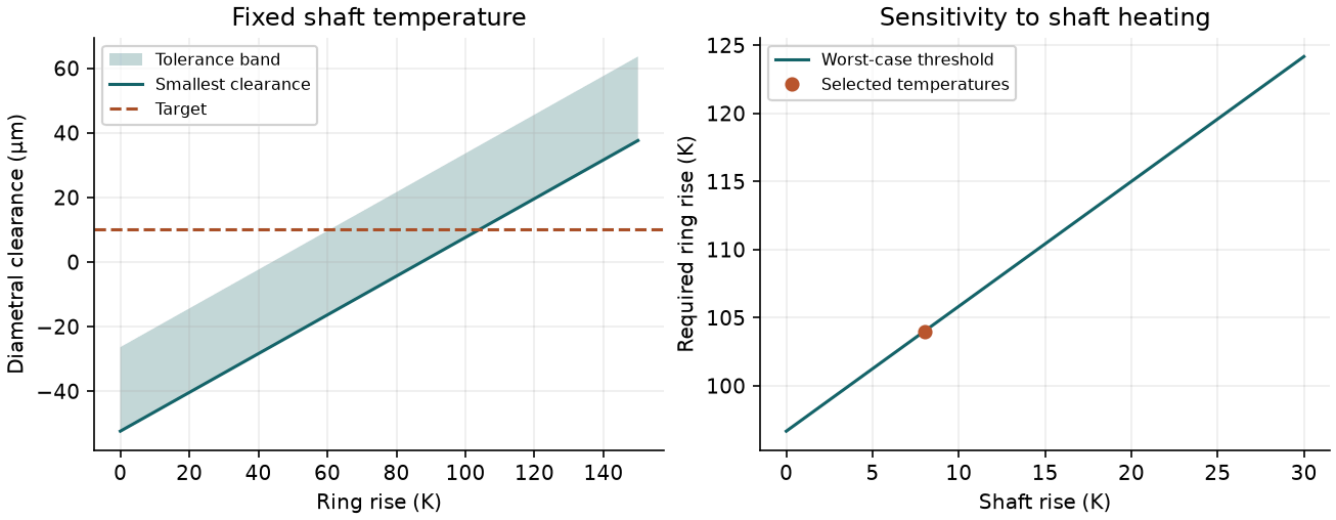
assembly made to the same drawings nevertheless has very little clearance. Resolve this apparent disagreement by carrying the diameter tolerance and the warming of the shaft through the thermal expansion calculation.

Write the initial interference as $i = d_{\mathrm{shaft},0} - d_{\mathrm{bore},0}$, positive for a press fit. In free thermal expansion a length changes by $\alpha d \Delta T$. Thus the available diametral clearance before engagement is the hot bore diameter less the current shaft diameter. Replacing their near-equal nominal diameters in the small thermal corrections by the common scale d gives the first-order relation

$$C = -i + d(\alpha_r \Delta T_r - \alpha_s \Delta T_s).$$

All terms are diametral. Using a desired radial clearance in this formula would understate the diametral target by a factor of two. The common-diameter approximation drops terms of order $i\alpha\Delta T$; products at that scale can matter for very tight metrology. Temperature rises here are relative to the same inspection reference.

Take $d = 50$ mm, interference between 22 and 48 µm, a ring expansion coefficient of 12×10^{-6} per kelvin and a shaft coefficient of 11×10^{-6} per kelvin. If the ring warms by 100 K its bore gains 60.0 µm. A shaft warming by only 8 K has already gained 4.4 µm. The remaining clearance consequently spans 7.6 to 33.6 µm. The largest assembly clears easily by eye, while the worst-case pair falls short of a specified 10 µm diametral clearance even though both nominal operating temperatures were observed.



A single temperature does not identify the available clearance. The width of the tolerance band survives a uniform shift of the entire band by thermal expansion.

Set the smallest clearance equal to the design target and solve for the ring rise. The example gives

$$(10 + 48 + 4.4)/0.60 = 104\,\mathrm{K}.$$

This calculation identifies the ideal free-diameter threshold. It does not set a permissible material or process temperature; gradients, heat loss before alignment, surface finish and mounting constraints require their own measurements. The notebook uses two independent temperature inputs so that the effect of the shaft can be examined directly.

For a population of assemblies it is tempting to quote the centre of the clearance range. With the nominal midpoint interference of 35 µm, the example predicts 20.6 µm of clearance. That value says nothing about the probability of meeting the target unless there is a measured distribution of part diameters. The two tolerance limits are geometric bounds. If each shaft is measured before matching it to a ring, substitute the measured interference into the same equation and retain only the uncertainty of that measurement. An assembly process using unmatched stock has to accommodate the limiting combination of the two drawings.

If both parts have the same thermal expansion coefficient and experience the same 100 K rise, what does this first-order calculation predict?

▼ **Work through the solution**

The two expansions cancel at the common nominal-diameter scale, leaving essentially the original negative clearance. Using exact distinct initial diameters gives a slightly *larger* interference, scaled by their common expansion factor. In the default example an error of just 10 K in the presumed shaft temperature shifts the clearance by 5.5 µm. Compare that sensitivity with the required assembly margin before reducing the model to a single ring thermometer.

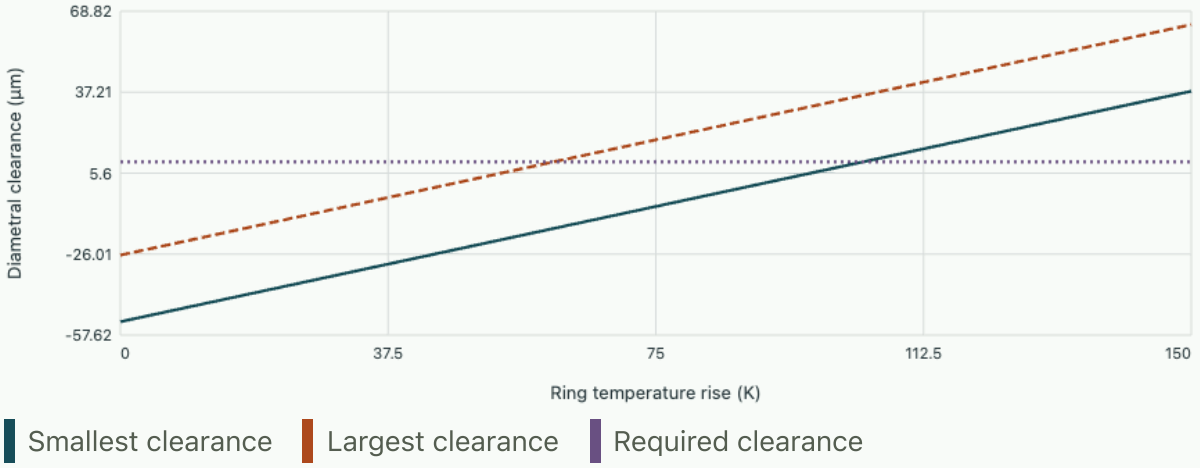
Interactive calculation · Fit clearance

Vary the notebook inputs and compare both plotted observables. The same equations generate the browser curves and the notebook reference; fixed material, geometry and model assumptions are stated in the readout.

Open this live calculation

Try the calculation

Largest interference (µm)	48
Shaft rise (K)	8
Required clearance (µm)	10



Worst-case ring rise 104.0 K. Diameter50 mm, ring expansion12×10⁻⁶/K, shaft11×10⁻⁶/K; lower interference22 µm.

Download the editable notebook for this example

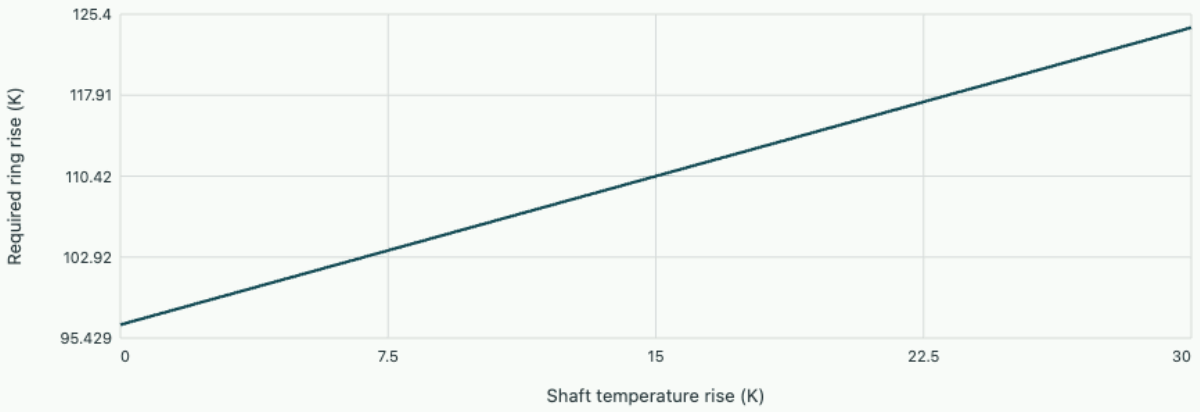
Interactive calculation · Fit threshold

Vary the notebook inputs and compare both plotted observables. The same equations generate the browser curves and the notebook reference; fixed material, geometry and model assumptions are stated in the readout.

Open this live calculation

Try the calculation

Largest interference (µm)	48
Shaft rise (K)	8
Required clearance (µm)	10



Worst-case assembly threshold

Worst-case ring rise 104.0 K. Diameter50 mm, ring expansion12×10⁻⁶/K, shaft11×10⁻⁶/K; lower interference22 µm.

Download the editable notebook for this example

Open editable notebook Download .ipynb

Convert manufacturing limits into assembly conditions

A shaft and a hub can share a nominal diameter while their manufactured diameters differ by several tens of micrometres. Name the diameters explicitly before subtracting them. For the present joint use positive diametral interference $I = d_s - d_h$, where d_s is the free shaft diameter and d_h the free bore. Assemble the smallest shaft with the largest hole to obtain the minimum interference. The opposite pair gives the maximum. Worst-case limits do not require a probability distribution for the production process.

Consider a 40 mm connection with shaft limits 40.030 to 40.045 mm and bore limits 40.000 to 40.015 mm. The minimum difference is 0.015 mm and the maximum 0.045 mm. Every permitted pair interferes. The radial mismatch is half the diametral interference because the radius changes on both sides of a diameter. Confusing these conventions introduces a factor of two in the inferred pressure and torque.

Relate geometric mismatch to elastic deformation

When the parts return to their common assembly temperature, contact pressure compresses the shaft and expands the bore. Assume cylindrical surfaces, a short open-ended elastic hub with negligible axial stress, a solid elastic shaft and approximately uniform pressure away from the ends. Define the radial compliance c_r as the sum of the inward shaft and outward hub displacements per unit interface pressure. Compatibility across the common radius then reads

$$\delta_r = I/2 = |u_s| + u_h = p\,c_r,$$

$$p = \frac{I}{2c_r}.$$

The combined compliance can be found from an elastic cylinder calculation or a verified test of geometrically identical parts. Providing it separately permits a torque calculation without assuming an infinitely stiff hub. A thinner hub generally expands further at the same pressure. Its larger compliance reduces the pressure developed by a given interference. Surface roughness that flattens during assembly can also reduce the effective interference relative to a measurement over the original asperities.

Integrate friction over the contact surface

A narrow surface patch of area dA can support a tangential friction force up to $\mu p\,dA$. All such forces act at radius $r = d/2$ and oppose impending rotation. Under uniform contact pressure and fully mobilised Coulomb friction, integrating around the circumference and along the overlap length L gives the limiting torque. For a purely axial pull the same friction forces act along the axis and their direct sum gives the limiting pull-out force.

$$T_{\lim} = \int r\mu p\,dA = \frac{\pi}{2}\mu p L d^2,$$

$$F_{\lim} = \mu p \pi d L.$$

Take $L = 35\,\mathrm{mm}$, $\mu = 0.12$, and $c_r = 0.50\,\mu\mathrm{m/MPa}$ in the 40 mm joint. The radial mismatches range from 7.5 to 22.5 micrometres, so calculated pressures range from 15 to 45 MPa. At the lower limit the tangential force capacity is $0.12(15)\pi(40)(35) = 7917\,\mathrm{N}$, and the torque capacity is $7917(0.020) = 158\,\mathrm{N\,m}$. At the upper limit pressure and capacity are three times larger. Use the lowest pressure to test transmission and the highest to evaluate hub stress.

Provide thermal clearance and account for combined service loads

Suppose only the hub is heated, so its bore grows approximately by $\alpha d \Delta T$. To slip the worst-case pair together with a further 0.020 mm of diametral assembly clearance, the bore must grow by $0.045 + 0.020 = 0.065\,\mathrm{mm}$. With $\alpha = 12 \times 10^{-6}\,\mathrm{K^{-1}}$ a uniform temperature rise of $0.065/[40(12 \times 10^{-6})] = 135\,\mathrm{K}$ is required in this ideal expansion calculation. Heat loss during transfer changes the actual temperature at insertion.

A shaft may also carry a sustained axial pull while delivering torque. Under the uniform-traction model the axial and circumferential components of the surface friction force share one bound. At the lower interference with a 4.0 kN pull the remaining circumferential capacity is $\sqrt{7917^2 - 4000^2} = 6832\,\mathrm{N}$, or about 137 N m. Checking the separate pure-load capacities would miss this interaction. If repeated microslip, an uncertain friction coefficient or stiffness near the hub edges is important, measurements of the actual assembly can determine which assumptions need closer investigation.

Derive interference pressure from thick-cylinder compliance

The radial compliance used above can be related to actual dimensions and materials. Consider a long enough cylindrical fit for a uniform central contact region, with free axial expansion so a plane-stress approximation is used for each cross-section. Let the common nominal radius be a, outer hub radius b, and positive pressure p. The solid shaft experiences uniform in-plane compression $\sigma_r = \sigma_\theta = -p$. Consequently its surface circumferential strain is $-p(1-\nu_s)/E_s$, giving radial shrinkage magnitude $pa(1-\nu_s)/E_s$.

For an elastic hub with free outside, radial equilibrium $d\sigma_r/dr + (\sigma_r - \sigma_\theta)/r = 0$ and axisymmetric compatibility give Lamé fields $\sigma_r = A - B/r^2,\ \sigma_\theta = A + B/r^2$. Setting $\sigma_r(a) = -p,\ \sigma_r(b) = 0$ gives bore hoop stress $\sigma_\theta(a) = p(b^2 + a^2)/(b^2 - a^2)$. Use $\epsilon_\theta(a) = [\sigma_\theta(a) - \nu_h \sigma_r(a)]/E_h = u_h(a)/a$. Adding hub expansion and shaft contraction leads to

$$I/2 = pa\left[\frac{1-\nu_s}{E_s} + \frac{(b^2+a^2)/(b^2-a^2) + \nu_h}{E_h}\right],$$

where I is the free diametral interference. The factor of two and the sign of radial pressure both affect the result. Significant axial restraint, plastic deformation, a hollow shaft or edge-pressure peaks require a different compliance model.

Compare thick and thin hubs at the same interference

A steel shaft and hub both have $E = 210\,\mathrm{GPa}$, $\nu = 0.30$. Their contact radius is 20 mm, hub outer radius 40 mm and measured diametral interference 30 micrometres. In mm–MPa units the combined radial compliance is

$$20[0.70 + (40^2 + 20^2)/(40^2 - 20^2) + 0.30]/210000 = 0.000254\,\mathrm{mm/MPa}.$$

Pressure is

$$0.015/0.000254 = 59.1\,\mathrm{MPa}.$$

The tensile hoop stress at the hub bore is

$$59.1(5/3) = 98.5\,\mathrm{MPa}.$$

For engagement length 25 mm and $\mu = 0.12$, torque capacity is $2\pi\mu p a^2 L = 446\,\mathrm{N\,m}$ approximately.

Now reduce the outer radius to 26 mm with the same parts at the fit surface. The hub geometry factor becomes $(26^2 + 20^2)/(26^2 - 20^2) = 3.90$, giving combined radial compliance $20(1 + 3.90)/210000 = 0.000467\,\mathrm{mm/MPa}$. Pressure falls to 32.2 MPa and torque capacity falls to about 243 N m. Yet bore hoop stress rises to $32.2(3.90) = 126\,\mathrm{MPa}$ because the thinner hub amplifies hoop demand more strongly. Reducing contact pressure alone is not evidence that the hub becomes safer. The same dimensional change can reduce torque capacity and increase peak elastic stress, so both checks should use the geometry of the actual mating parts.

For ductile elastic parts, the hub bore should also be assessed in its multiaxial state. In the thicker example its principal stresses are approximately $+98.5, 0, -59.1$ MPa under plane stress. The von Mises value is $\sqrt{98.5^2 + 59.1^2 + 98.5(59.1)} = 138\,\mathrm{MPa}$, above the tensile hoop stress by itself. For the thinner hub, using $+126, 0, -32.2$ MPa gives approximately 145 MPa. A comparison with an allowable uniaxial stress must therefore state the criterion applied. Once significant yielding occurs the linear compliance used to determine pressure is itself affected, and enlarging the nominal interference will no longer guarantee a proportional increase in transmitted torque. A pull-out or torque-to-slip test on representative surfaces checks the independent assumption about friction.

For an interference test, measuring the outside hub diameter change offers an additional observable. At the outer surface radial stress vanishes, so its hoop strain depends directly on the local tensile hoop stress.

In fact the hub’s outside hoop stress from the same Lamé field is $2pa^2/(b^2 - a^2)$. For the 40 mm outer radius it is $2(59.1)(400)/(1600 - 400) = 39.4\,\mathrm{MPa}$, so the predicted 80 mm outside diameter increases by $80(39.4)/210000 = 0.0150\,\mathrm{mm}$. For the thinner hub it is $2(32.2)(400)/(676 - 400) = 93.3\,\mathrm{MPa}$, giving an outside diameter increase $52(93.3)/210000 = 0.0231\,\mathrm{mm}$. The thinner outer wall moves more under a smaller interface pressure. Observing this change together with transmitted torque can separate an incorrect elastic pressure estimate from an incorrect friction coefficient.

For example, if the thick hub expands by the predicted 15 micrometres but slips at 330 N m under otherwise matching conditions, the implied mean friction coefficient is

$$0.12(330/446) = 0.089.$$

If the measured expansion were instead much smaller, an undersized effective interference, partial contact or a nonuniform fit would deserve investigation before calibrating friction. The micrometer reading is itself temperature sensitive: a uniform temperature difference of 10 K changes an 80 mm

steel diameter by about 9.6 micrometres at $12 \times 10^{-6}\,\mathrm{K}^{-1}$. Stabilise the temperatures before comparing the calculated elastic expansion with metrology.

Size the working flanks and shear section of a parallel key

A shaft can drive a removable hub through the flanks of a parallel key. The load path runs from the shaft keyway into one side of the key, across its shear section, and out of the opposite side into the hub. Let d be the shaft diameter. Represent the distributed tangential contact by the resultant $F = 2T/d$ at the nominal shaft radius. This model assumes a single fitted key carries the specified torque without credit for friction from a separate clamp.

There are two different projected areas. If b is the key width and L_e the effective engaged length, the shear plane has area bL_e. If h_e is the actual load-bearing radial height on the hub side, the projected bearing area on that flank is $h_e L_e$. The full key height includes material inside the shaft. Chamfers, end relief and clearance can further reduce the effective dimensions. Under a uniform-load sizing approximation,

$$\tau_{\mathrm{key}} = \frac{2T}{dbL_e},$$

$$p_{\mathrm{hub}} = \frac{2T}{dh_eL_e},$$

$$L_{e,\min} = \max\left(\frac{2T}{db\tau_{\mathrm{allow}}},\ \frac{2T}{dh_e p_{\mathrm{allow}}}\right).$$

For a conveyor pulley, $T = 180\,\mathrm{N\,m}$, $d = 40.0\,\mathrm{mm}$, $b = 12.0\,\mathrm{mm}$, $h_e = 3.30\,\mathrm{mm}$ and $L_e = 36.0\,\mathrm{mm}$. Work in N and mm, giving $F = 2(180000)/40 = 9000\,\mathrm{N}$. Mean shear is $9000/(12 \times 36) = 20.8\,\mathrm{MPa}$, whereas hub bearing pressure is

$$9000/(3.30 \times 36) = 75.8\,\mathrm{MPa}.$$

For design allowables of 45 and 90 MPa respectively, required engaged lengths are 16.7 and 30.3 mm. The proposed engagement passes these two static checks, with hub bearing pressure setting the longer minimum. A softer hub can therefore govern even when a much stronger key material is chosen. This calculation does not establish the shaft's fatigue strength at its keyway root. That requires the local shaft stress, particularly for rotating bending.

Practise on sheet 3 · Can the available hub length carry the specified steady torque?. Further reading · Jönköping University · Applied Mechanics · §10.18 Shaft–hub joints, “The parallel key”.

Problem sheet 3 · Gear meshes and assembled fits

Link tooth forces to shaft loads and use manufacturing limits to check a transmitting interface.

Resolve gear forces

An 18-tooth pinion of module 4 mm drives a 54-tooth external gear. Input torque is 90 N m at 900 rpm. Use a pressure angle of 20°. Find output speed, ideal output torque, both pitch diameters and the radial force on either shaft. What part of the bearing load would a torque-only analysis omit?

▼ **Draw the pitch circles and the force normal**

Pitch diameters are 72 and 216 mm. The output turns at 300 rpm in the opposing direction, with ideal torque 270 N m. The tangential force is

$$90/0.036 = 2500\,\mathrm{N}.$$

The radial force is $2500\tan 20^\circ = 910\,\mathrm{N}$, and the normal resultant is 2660 N. A drawing showing only torque misses both transverse forces feeding shaft bending and bearing reactions.

Find the least favourable fit

A shaft spans 50.028–50.040 mm, and its bore spans 50.005–50.016 mm. The combined radial compliance is 0.40 micrometres/MPa. The overlap is 30 mm and friction coefficient 0.10. Find pressure limits and minimum torque capacity. Is a demand of 160 N m supported by every allowed pair in this model?

▼ **Keep track of radius and diameter**

Diametral interference ranges from $28 - 16 = 12$ to $40 - 5 = 35$ micrometres. Radial mismatches are 6 and 17.5 micrometres and pressures 15 and 43.75 MPa. With N and mm the limiting torque at minimum pressure is

$$\pi(0.10)(15)(30)(50^2)/2 = 176715\,\mathrm{N\,mm} = 176.7\,\mathrm{N\,m}.$$

The smallest nominal capacity exceeds the demand by about 10%. This margin is sensitive to error in the friction coefficient.

Connect a gear stage to a friction joint

A compound reducer uses tooth pairs 20:50 and 24:48, with both meshes external, each 97% efficient. Its input runs at 1500 rpm and supplies 3 kW. Find output speed, direction and useful output torque. If

a fit of radius 20 mm supports at most 6000 N resultant surface friction and also transmits a 4000 N axial force, can it transmit that torque?

▼ **Compare drive output with remaining surface capacity**

The total reduction is $2.5 \times 2 = 5$; the output turns at 300 rpm in the input direction. Useful output power is $3000(0.97)^2 = 2823\,\mathrm{W}$, hence

$$T = 2823/[2\pi(300)/60] = 89.85\,\mathrm{N\,m}.$$

Remaining tangential force is $\sqrt{6000^2 - 4000^2} = 4472\,\mathrm{N}$, giving 89.44 N m. The joint falls just short even before an allowance for manufacturing and friction scatter.

Trace the power when both the sun and ring move

A lossless test gearbox has $N_s = 30,\ N_r = 90$. The sun runs at $+900\,\mathrm{rpm}$ with $T_s = +8.00\,\mathrm{N\,m}$, and a speed-controlled ring runs at $-120\,\mathrm{rpm}$. Determine the carrier speed, the other two torques and which ports supply or receive power.

▼ **Show the worked solution**

The carrier speed is

$$[30(900) + 90(-120)]/120 = 135\,\mathrm{rpm}.$$

The torque proportions are $T_c = -8(120/30) = -32.0\,\mathrm{N\,m}$ and $T_r = +24.0\,\mathrm{N\,m}$. Multiplying each torque by its signed speed in rad/s gives

$$P_s = +754.0\,\mathrm{W},\ P_r = -301.6\,\mathrm{W},\ P_c = -452.4\,\mathrm{W}.$$

The sun supplies all the power, and both the ring drive and carrier load receive it. A positive torque on a member rotating negatively represents negative power to the gearset.

Revisit chapter 5 · Find the three speeds and housing reaction of a planetary gearset.

Can the available hub length carry the specified steady torque?

A pump hub must transmit 240 N m on a 50.0 mm shaft. Its parallel key has width 14.0 mm and usable hub-side flank height 3.80 mm. The mean shear allowable is 45 MPa and the bearing allowable 80 MPa. Find the minimum effective length and the maximum torque if only 30.0 mm of effective engagement fits.

▼ **Show the worked solution**

The tangential force is

$$2(240000)/50 = 9600\,\mathrm{N}.$$

The shear requirement is

$$9600/(14 \times 45) = 15.24\,\mathrm{mm}.$$

The bearing requirement is

$$9600/(3.80 \times 80) = 31.58\,\mathrm{mm}.$$

The available 30.0 mm falls short. Inverting both checks at that length gives shear capacity $45(14)(30)(25) = 472500\,\mathrm{N\,mm}$ and bearing capacity $80(3.80)(30)(25) = 228000\,\mathrm{N\,mm}$. The limiting torque is 228 N m. A longer nominal key only helps if it increases effective flank engagement.

Revisit chapter 6 · Size the working flanks and shear section of a parallel key.

Explore the governing equations after solving the sheet: Spur gear kinematics and tooth loads; Tolerances, interference and torque transmission. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Notebook practice · What thermal clearance survives the tolerance stack?

Chapter 7 · Helical springs, wire stress and stored energy

Derive the rate of a close-coiled compression spring and check its working stress and available travel.

FROM FIRST PRINCIPLES

Find the spring rate from torsional strain energy in the wire

A close-coiled spring of mean coil diameter D and round wire diameter d converts an axial force F into wire torque about $FD/2$. Assume a small helix angle, slender wire, N_a active turns and linear elastic shear modulus G. The active wire length is approximately $\pi D N_a$ and its polar section moment is

$$J = \pi d^4/32.$$

The torsional energy of a straight wire element of length ds under torque T is $T^2 ds/(2GJ)$.

$$U = \int_{\text{active}} \frac{T^2}{2GJ}\, ds \simeq \frac{(FD/2)^2 \pi D N_a}{2GJ} = \frac{4F^2 D^3 N_a}{G d^4}.$$

For a linear spring loaded gradually from zero, energy also equals $F\delta/2$. Equating the two expressions gives axial deflection and rate.

$$\delta = \frac{8FD^3 N_a}{G d^4},$$

$$k = \frac{F}{\delta} = \frac{G d^4}{8D^3 N_a}.$$

More active wire stores more energy at fixed force and therefore gives greater travel. Curvature corrections to local wire stress, inactive end turns and solid-height limits require separate checks before applying this compliance across the full working stroke.

DERIVATION AND APPLICATIONS

Derive the stiffness change as one compression spring reaches a stop

Two springs in series can provide a soft initial rate and a stiffer response later in the stroke. Consider linear spring rates k_1, k_2 and a rigid stop limiting the deflection of spring one to $s > 0$. Initially both springs are unloaded, their masses are negligible and contact remains axial. Loading is quasistatic. Before stop engagement the same force passes through both springs, so their elastic displacements add. Write total travel as x.

$$x = \frac{F}{k_1} + \frac{F}{k_2},$$

$$k_{\text{series}} = \frac{k_1 k_2}{k_1 + k_2}.$$

The stop first engages when spring one has shortened by s. Its force then is $F_* = k_1 s$ and spring two has already shortened by F_*/k_2. Beyond this position the stop carries any extra force passing around spring one, whose deflection stays fixed. All further travel goes into spring two.

$$x_* = s + \frac{F_*}{k_2}, \qquad F(x) = \begin{cases} k_{\text{series}} x, & 0 \le x \le x_*, \\ F_* + k_2 (x - x_*), & x > x_*. \end{cases}$$

Force is continuous at engagement while tangent stiffness changes. Integration of the force over the total travel gives stored elastic energy. Past engagement it contains the energy already stored, the work against the existing force and the quadratic incremental contribution.

$$U(x) = \tfrac{1}{2} F_* x_* + F_* (x - x_*) + \tfrac{1}{2} k_2 (x - x_*)^2, \qquad x \ge x_*.$$

A real stop has finite compliance and impact can excite spring mass. This static law describes the slowly measured transition and does not determine the transient peak force at engagement.

WORKED EXAMPLE

Predict an end-of-stroke force

A return assembly has $k_1 = 30.0$ N/mm, $k_2 = 60.0$ N/mm and $s = 4.00$ mm. Its initial rate is 20.0 N/mm; engagement occurs at 120 N and 6.00 mm total travel. At 8.00 mm the force is

$$120 + 60(2) = 240\,\text{N}.$$

Stored energy is $360 + 240 + 120 = 720$ N mm or 0.720 J. Spring one still supports 120 N and the stop supports the extra 120 N.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Recover the hidden spring rate from a force trace

A different mechanism has measured slopes of 15.0 N/mm before contact and 40.0 N/mm afterwards. The second slope identifies k_2. From $1/k_1 = 1/15 - 1/40$ one finds $k_1 = 24.0$ N/mm. If the kink occurs at 3.20 mm total travel, its force is 48.0 N and the internal stop gap is

$$48.0/24.0 = 2.00\,\text{mm}.$$

The other spring accounts for the remaining 1.20 mm at engagement.

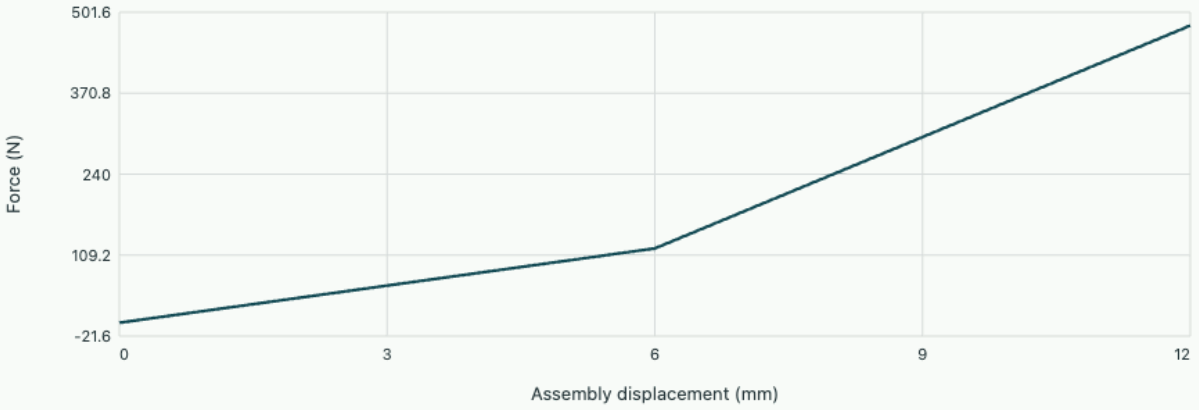
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

First spring (N/mm)	30
Second spring (N/mm)	60
First-spring available stroke (mm)	4



Spring with a stroke stop

Stop engagement at 6.000 mm and 120.000 N; initial slope 20.000 N/mm changes to 60.000 N/mm. Force stays continuous.

An axial spring force twists the wire

A helical spring appears to shorten along its axis under a compressive force. Follow one turn of its wire, however, and the main elastic deformation is torsion. For a close-coiled spring the helix angle is small and the force line along the spring axis lies a mean radius $D/2$ from the wire centre. A cut through a wire turn therefore transmits a torque $T = FD/2$ as well as direct shear. Use mean coil diameter D, wire diameter d and N_a active turns. End turns that rest against a seat may add solid height without contributing equally to the spring rate.

Approximate wire length as $\pi D N_a$ and wire polar moment as $\pi d^4/32$. Under small elastic deformation the torsional strain energy is

$$U = T^2 \ell / (2GJ).$$

Differentiating this energy with respect to axial load gives axial deflection, since an incremental load is work-conjugate to that displacement.

$$U = \frac{4F^2 D^3 N_a}{G d^4},$$

$$\delta = \frac{\partial U}{\partial F} = \frac{8 F D^3 N_a}{G d^4},$$

$$k = F/\delta = \frac{G d^4}{8 D^3 N_a}.$$

Doubling the number of active turns halves the rate at fixed wire and coil diameter. Increasing wire diameter has a much stronger fourth-power effect. Changing wire diameter at fixed outside diameter also changes the mean coil diameter, so both terms must be recomputed.

Account for the curvature of the wire

Nominal torsional surface stress in straight wire is

$$16T/(\pi d^3) = 8FD/(\pi d^3).$$

Curvature and direct shear raise the maximum stress on the inside of a real coil. One common elastic estimate for round wire is the Wahl factor $K_w = (4C-1)/(4C-4) + 0.615/C$, with spring index $C = D/d$. Here it multiplies the nominal torsional expression to estimate peak shear. A small spring index makes the curvature contribution more important. Use a consistent stress convention with the material or fatigue data used for assessment.

Compare working travel with available space

A fixture spring uses 4 mm steel wire, 28 mm mean diameter, six active turns and $G = 79\,\mathrm{GPa}$. Its rate is

$$79000(4^4)/[8(28^3)(6)] = 19.2\,\mathrm{N/mm}.$$

At 120 N it compresses 6.25 mm and stores $F\delta/2 = 0.375\,\mathrm{J}$. The spring index is seven, giving $K_w = 1.213$ and corrected shear $1.213[8(120)(28)/(\pi 4^3)] = 162\,\mathrm{MPa}$.

With eight total turns and an ideal closed and ground construction, an approximate solid height is $8(4) = 32\,\mathrm{mm}$. A 44 mm free length leaves about 12 mm before full contact. At the calculated working

deflection the remaining travel is 5.75 mm. Travel to solid height sets a geometric limit even if a stress calculation permits a larger load. When coils start contacting sequentially, the active turn count falls and the spring stiffens; the preceding constant-rate relation then no longer describes its full travel.

Check the active turn count

Two measured loads of 40 and 100 N at compressions of 2 and 5 mm give an incremental rate of $60/3 = 20\,\mathrm{N/mm}$. Using a difference between readings removes a constant force offset. A change of slope near full compression can indicate that additional coils have made contact.

Derive the contribution of a spring’s own mass

A spring designed from static force and travel also has moving material. Near an operating resonance the extra inertia changes response and can produce dynamic coil loading. As a first approximation take a uniform active helical spring of total moving mass m_s, lower end fixed, upper end attached to a translating mass M. Let s be a dimensionless material coordinate running from zero to one along active turns. For slow axial deformation the extension and axial velocity of a point grow nearly linearly with position:

$$u(s) = s x,\; \dot{u} = s \dot{x}.$$

Direct support movement, coil contact and lateral buckling are excluded.

Integrating the spring’s translational kinetic energy gives

$$T_s = \tfrac{1}{2} m_s \dot{x}^2 \int_0^1 s^2 ds = \tfrac{1}{2}(m_s/3)\dot{x}^2.$$

With spring potential $k x^2/2$, the lowest frequency estimate is $\sqrt{k/(M + m_s/3)}$. The one-third factor arises from the assumed distribution of speed; treating every turn as moving at the end velocity would overestimate inertia. Dead-end turns attached rigidly at the fixed seat have essentially zero axial speed and should not all be included at this weight.

One can refine the axial continuum idealisation by letting displacement be $u(s,t)$ and using elastic energy $(k/2)\int_0^1 u_s^2 ds$. The field equation becomes

$$m_s u_{tt} = k u_{ss},$$

with $u(0) = 0$ and $M u_{tt}(1) = -k u_s(1)$. A mode $\sin(\beta s)$ gives

$$\omega = \beta \sqrt{k/m_s},$$

$$\beta \tan \beta = m_s / M.$$

This also predicts internal surge modes, beyond the single effective-mass approximation.

Estimate the frequency reserve of a return mechanism

An actuator return spring has measured rate 4000 N/m, active spring mass 0.20 kg and an attached moving assembly of 0.50 kg. Ignoring spring mass gives

$$f_n = \sqrt{4000/0.50}/(2\pi) = 14.24\,\mathrm{Hz}.$$

Including one third of the active spring mass gives

$$f_n = \sqrt{4000/(0.50 + 0.20/3)}/(2\pi) = 13.37\,\mathrm{Hz}.$$

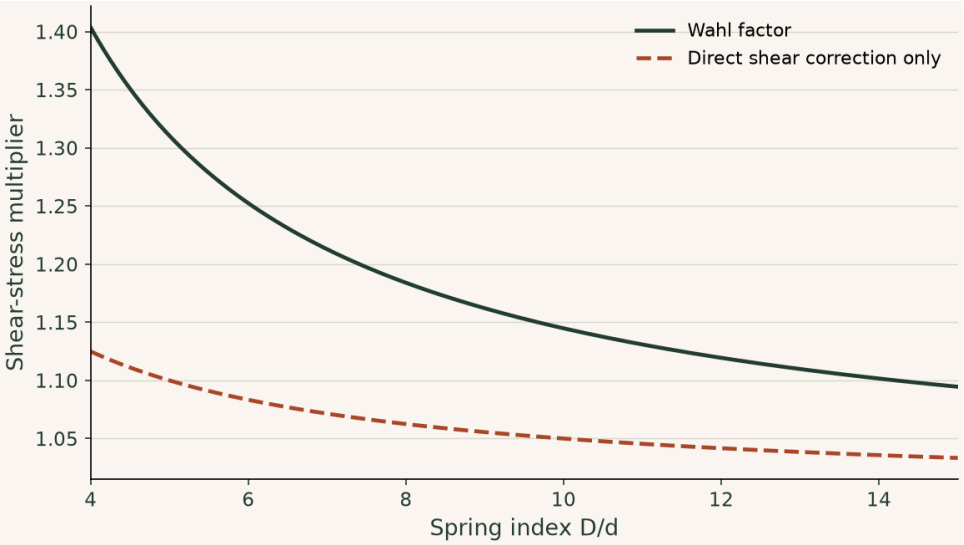
The distributed relation has $\beta \tan \beta = 0.40$, giving $\beta = 0.5932$ and $f_n = 13.35\,\mathrm{Hz}$. The quick effective-mass estimate happens to be close for this mass ratio since the first mode shape is nearly linear.

If drive motion has a strong third harmonic at 13.5 Hz, the massless estimate places resonance 0.74 Hz above it while the distributed estimate places it only 0.15 Hz below. A static travel check would miss this change completely. Reducing attached mass shifts the frequency upwards but also makes the spring’s distributed character more important. In the limiting case of no attached mass, the first root tends to $\pi/2$; the exact continuum frequency then differs from the $m_s/3$ estimate. Real helical springs also exhibit rotational motion of the wire, damping and nonuniform end conditions. Measuring one low-frequency mode can validate the effective-mass estimate while leaving higher-frequency coil surge unverified.

A frequency test with an added known tip mass can help estimate the effective spring mass empirically. Within the low-mode approximation plot $1/\omega^2$ against attached mass M: the slope is $1/k$, and the intercept is $m_s/(3k)$. Curvature as the tip mass becomes small reveals the limits of the linear assumed mode shape.

Separate wire curvature from nominal spring stress

A close-coiled helical spring carries axial load F. Nominal wire torsion gives $8FD/(\pi d^3)$. Compare a direct-shear correction with the Wahl factor, which also accounts for curvature of the wound wire.



At spring index six the two multipliers are about 1.253 and 1.083. Using only direct shear understates the estimated local stress. Increasing index lowers this multiplier, although the nominal stress also changes if D or d is changed.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

Keep wire diameter fixed while changing coil index. The correction factor falls, but the nominal stress rises with mean coil diameter.

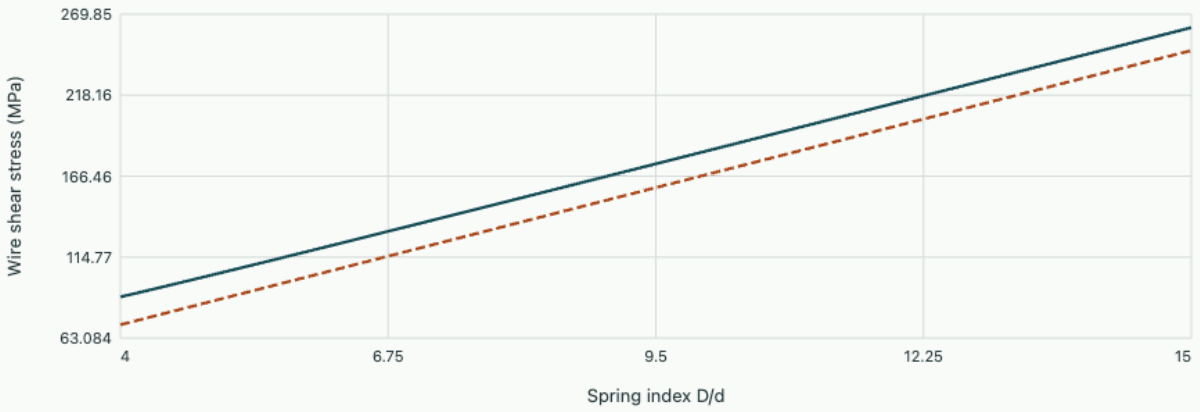
The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Axial force (N) **100**

Wire diameter (mm) **4**



Wahl correction Direct shear only

At index six the Wahl factor is 1.252. The plotted demand also includes the nominal stress change when coil diameter grows at fixed wire diameter.

Chapter 8 · Belt traction, wrap and power transmission

Derive the tension-ratio bound for a flat belt and determine the pretension and bearing load needed to transmit a specified power.

FROM FIRST PRINCIPLES

Derive the capstan bound from a small belt arc

A flexible flat belt transmits force by friction as it wraps around a pulley. Assume a thin belt, small bending stiffness, steady low-speed operation and constant coefficient of static friction μ. Consider an arc element subtending $d\theta$ radians with tension T at one end and $T + dT$ at the other. The inward components of the adjoining tensions produce a normal contact force approximately $dN = T\,d\theta$. Tangential equilibrium requires friction to carry the tension change.

$$|dT| \le \mu\, dN = \mu T\, d\theta.$$

Integrate from the slack-side tension T_2 to tight-side tension T_1 over total contact angle β. Equality describes impending gross slip in this ideal limit.

$$\ln(T_1/T_2) \le \mu\beta, \qquad T_1/T_2 \le e^{\mu\beta}.$$

The net tangential force is the tension difference. At belt speed v the transmitted mechanical power is

$$P = (T_1 - T_2)v.$$

Raising both tensions together can increase the available difference but also raises shaft and bearing loads. At high belt speed the inward acceleration of each moving element changes the normal-force balance.

DERIVATION AND APPLICATIONS

Calculate belt wrap and length from pulley geometry

A small pulley loses contact angle when the two pulley radii differ. Consider an open, crossed-free belt around circular pitch radii $R > r$ separated by centre distance $C > R - r$. Treat the belt as thin, taut and inextensible and ignore sag. The radii to the ends of one straight span are perpendicular to its tangent. Translating that tangent parallel to itself produces a right triangle with hypotenuse C and short side $R - r$. The resulting angle α and span length ℓ obey

$$\sin\alpha = \frac{R - r}{C},$$

$$\ell = \sqrt{C^2 - (R - r)^2} = C\cos\alpha.$$

The small pulley loses angle α at each side relative to a semicircle, while the larger one gains the same amount. Summing the two straight spans and the two arc lengths gives the exact pitch length in the ideal geometry.

$$\beta_s = \pi - 2\alpha,$$

$$\beta_L = \pi + 2\alpha,$$

$$L = 2\ell + \pi(R + r) + 2(R - r)\alpha.$$

For a long centre distance expand the square root and inverse sine in $(R - r)/C$. The leading correction to the simple sum of two centre distances and two semicircles is positive. The same geometry sets the small pulley's friction-limited tension ratio for a stationary or low-speed flat belt.

$$L \simeq 2C + \pi(R + r) + \frac{(R - r)^2}{C}, \qquad T_1/T_2 \le e^{\mu\beta_s}.$$

Angle in the exponential must be in radians. Increasing centre distance brings the two wraps closer to half a circle but also changes required belt length. A tensioning device must accommodate both geometry and elastic stretch.

WORKED EXAMPLE

Determine a small pulley's available traction

For $R = 100$ mm, $r = 40.0$ mm and $C = 300$ mm, $\alpha = \arcsin 0.200 = 0.2014$ rad. Small-pulley wrap is 2.739 rad or 156.9°. Exact pitch length is

$$2(293.94) + \pi(140) + 2(60)(0.2014) = 1052\,\text{mm}.$$

With $\mu = 0.300$ the limiting tension ratio is $e^{0.300(2.739)} = 2.27$, lower than the 2.57 obtained from a half-circle wrap.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Find the minimum shaft spacing for a specified wrap

Another drive uses pitch radii of 75.0 and 25.0 mm and requires at least 160° contact on the smaller pulley. This limits α to 10.0°, so

$$C \geq 50.0/\sin 10.0^\circ = 288\,\mathrm{mm}.$$

At this spacing the straight span is

$$C\cos 10.0^\circ = 284\,\mathrm{mm}.$$

A proposed spacing of 250 mm would give only $180^\circ - 2\arcsin(50/250) = 156.9^\circ$.

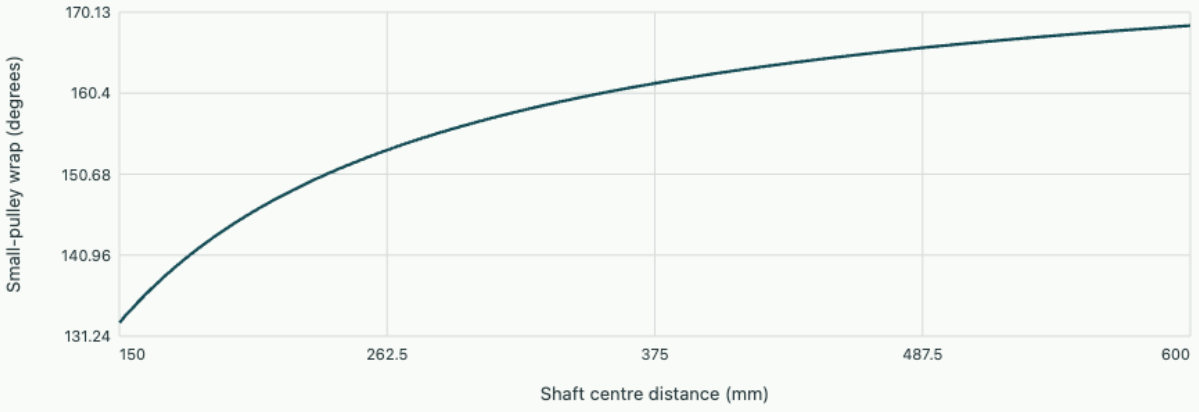
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation’s balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Large pulley radius (mm)	**100**
Small pulley radius (mm)	**40**
Friction coefficient	**0.3**



Open-belt wrap

At300 mm centres, pitch length=1051.9 mm and limiting tension ratio=2.274. Small-pulley contact controls this ideal traction calculation.

Resolve the forces on a small arc

A flat belt transmits torque through friction distributed over the part of the pulley it touches. Cut out a short belt element subtending $d\theta$. Let tension entering the element be T and tension leaving be $T + dT$. For a flexible belt in quasistatic normal equilibrium the two tension vectors have a small inward radial resultant $T\,d\theta$. The pulley supplies an outward normal force of the same magnitude. Neglect belt bending stiffness and centrifugal tension in the following low-speed model.

At impending gross slip, available tangential friction is $\mu\,dN$. Tangential equilibrium gives $dT = \mu T\,d\theta$ along the direction of increasing belt tension. Integrate from the slack to the tight end of a contact arc of angle β in radians. Below impending slip the resulting ratio is an inequality.

$$\ln(T_1/T_2) \le \mu\beta, \qquad T_1/T_2 \le e^{\mu\beta}, \qquad P = (T_1 - T_2)v.$$

The final power expression comes from transmitted torque $(T_1 - T_2)r$ times angular velocity v/r. It is the tension difference that drives the machine. Both tensions contribute to radial loading on the shafts. In an open belt drive the smaller pulley often has the smaller contact angle and controls traction first.

Compute the least operating tensions

A flat belt must transmit 2.0 kW at 8.0 m/s. It wraps through 160° on the limiting pulley, with effective friction coefficient 0.30. The required tension difference is 250 N. Converting 160° to 2.793 rad gives a maximum ratio $r_T = e^{0.30(2.793)} = 2.31$. At the limiting ratio $T_2 = 250/(r_T - 1) = 191\,\mathrm{N}$ and $T_1 = 441\,\mathrm{N}$. Lower slack-side tension would require a ratio beyond the assumed friction limit.

For negligible centrifugal effect and approximately equal span compliances, an initial pretension estimate is $T_0 = (T_1 + T_2)/2 = 316\,\mathrm{N}$ per span. If the two straight spans pull nearly parallel, the resultant transverse shaft force is approximately $T_1 + T_2 = 632\,\mathrm{N}$. If their directions differ appreciably, add the two tension vectors rather than their magnitudes. A higher pretension improves the available traction margin while increasing load on bearings and belt cords.

Decide what a loss of friction does

If oil contamination reduces the friction coefficient to 0.20 at the same wrap, the limiting tension ratio falls to

$$e^{0.20(2.793)} = 1.75.$$

If the tension sum remains 632 N, the largest transferable difference is $632(r_T - 1)/(r_T + 1) \simeq 172\,\mathrm{N}$, giving at most 1.38 kW at 8 m/s. The original power demand now exceeds available traction. A larger contact angle, different belt design or altered surface condition may address the shortfall; increasing motor power alone does not change the traction bound. A toothed timing belt has a different transmission mechanism and requires its tooth engagement and cord loading to be evaluated separately.

Interpret belt speed

The pitch-line speed in the no-slip approximation is $\pi d n/60$. A 200 mm pulley carrying the original 8 m/s belt turns at approximately 764 rpm. Speed should be measured on both shafts if slipping contact is suspected.

Include the tension needed to turn a moving belt

The quasistatic capstan law assumes the belt has negligible inertia as it changes direction round a pulley. At higher speed, each arc element of linear mass density λ needs inward centripetal force. Consider a segment subtending $d\theta$ at radius r. Its mass is $\lambda r\, d\theta$ and required inward force is $\lambda v^2 d\theta$. Radial equilibrium of the segment gives

$$T\, d\theta - dN = \lambda v^2 d\theta,$$

where dN is the outward reaction from the pulley. Available traction thus depends on the effective tension $T - T_c$, with $T_c = \lambda v^2$.

At incipient sliding $dT = \mu (T - T_c) d\theta$ for constant belt speed. Integrating gives

$$(T_1 - T_c)/(T_2 - T_c) = e^{\mu \beta}.$$

If the allowable total tight-side tension is $T_{\max}$ and wrap and friction are fixed, the maximum tension difference is $(T_{\max} - \lambda v^2)(1 - e^{-\mu\beta})$. Multiplying by v yields a cubic in speed for ideal power capacity:

$$P_{\max}(v) = (1 - e^{-\mu\beta})(T_{\max} v - \lambda v^3).$$

The derivative vanishes at $v_* = \sqrt{T_{\max}/(3\lambda)}$, where centrifugal tension consumes one third of the allowed total tight-side tension. This optimum presumes adequate pretension, fixed allowable tension independent of speed, and acceptable bending and thermal losses. It should not be applied without checking the belt and pulley operating limits.

Choose between increasing shaft speed and increasing wrap

For an illustrative belt $T_{\max} = 700\,\mathrm{N},\ \lambda = 0.25\,\mathrm{kg/m},\ \mu = 0.30,\ \beta = \pi$ rad. The traction factor is

$$1 - e^{-0.30\pi} = 0.610.$$

At 20 m/s centrifugal tension is $0.25(20)^2 = 100\,\mathrm{N}$, giving limiting tension difference $0.610(700 - 100) = 366\,\mathrm{N}$ and deliverable power 7.32 kW. At the ideal optimum $v_* = \sqrt{700/0.75} = 30.55\,\mathrm{m/s}$, centrifugal tension is 233.3 N, difference 284.7 N and power 8.70 kW. Extra speed initially helps even as the available tension difference falls.

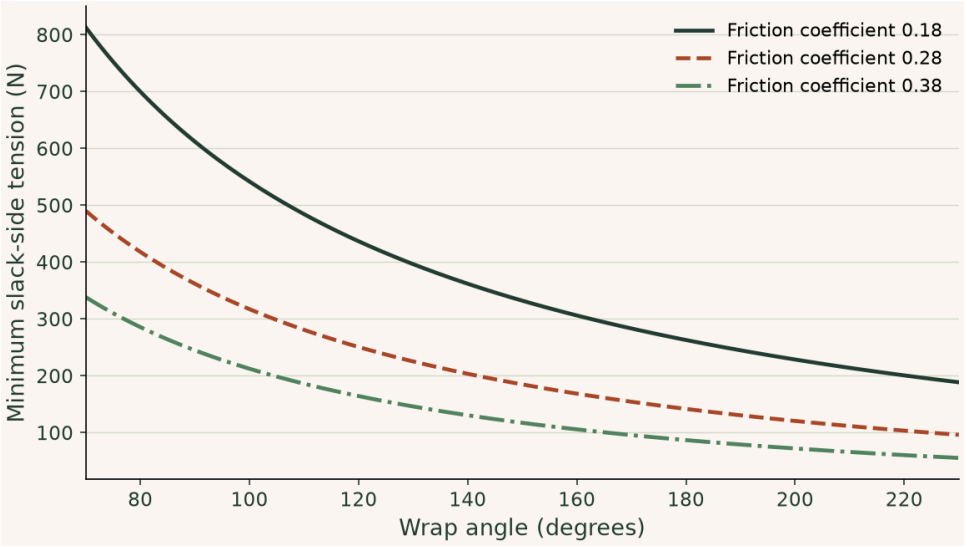
At 40 m/s centrifugal tension becomes 400 N and capacity falls back to $0.610(300)(40) = 7.32\,\mathrm{kW}$. Thus two pulley speeds can deliver the same calculated limiting power with very different running tensions and bending-cycle rates. Increasing wrap from 180° to 210° at 20 m/s instead changes the traction factor to $1 - e^{-0.30(3.665)} = 0.667$, raising calculated capacity to 8.00 kW without higher speed. An added idler can supply this wrap but also introduces another bend and bearing load. The choice should follow the limiting failure mechanism: speed-dependent belt fatigue, inadequate traction margin and shaft overloading respond differently to the two modifications.

The effect of an allowable tension change is also nonlinear when speed is chosen optimally. Substituting v_* gives optimum ideal power proportional to $T_{\max}^{3/2}/\sqrt{\lambda}$ at fixed wrap and friction. A reinforcement that increases both allowable tension and mass density must be assessed using both factors. For example, raising $T_{\max}$ by 20% while increasing λ by 44% gives only $1.2^{3/2}/\sqrt{1.44} = 1.095$, or about 9.5% more ideal peak power. A heavier belt can also impose greater centrifugal tension at an existing pulley speed, so the actual duty point may benefit less than this re-optimised maximum.

The characteristic speed should consequently be recalculated for the proposed belt construction at its actual working radius.

Find the slack-side tension needed to transmit a force

A flat belt must transmit a 200 N difference between tight- and slack-side tensions. At impending slip $T_t/T_s = e^{\mu\theta}$. Eliminate T_t to find the minimum positive slack-side tension for the required difference.



For μ = 0.28 and a 180° wrap the minimum is about 142 N, giving a tight-side tension of 342 N. Greater wrap can reduce required preload. The equality is a slip limit for this idealised contact, not a general identity for every operating belt.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

Reduce friction and find how much extra slack-side tension the same transmitted force requires.

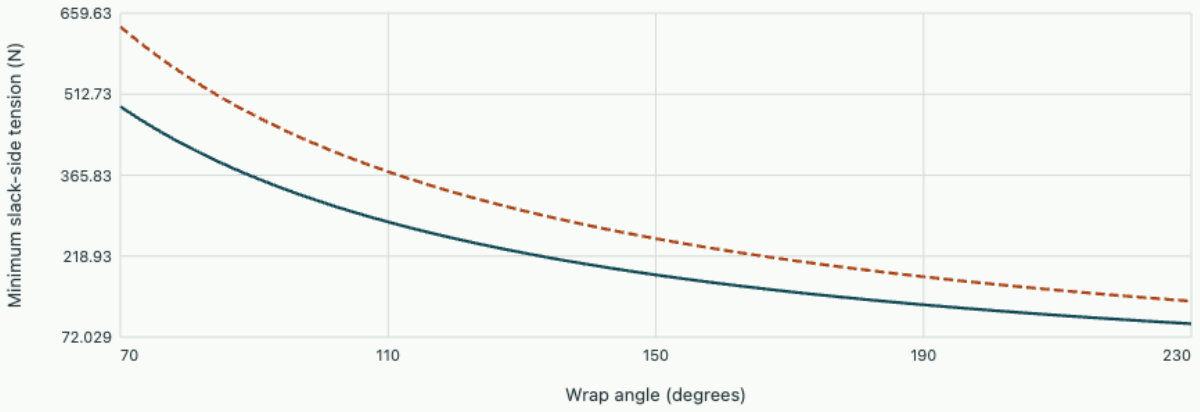
The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Required tension difference (N) **200**

Friction coefficient **0.28**



Selected friction 20% lower friction

At 180° wrap, slack-side tension must reach 141.839 N at impending slip. Tight-side tension is 341.839 N.

Problem sheet 4 · Elastic springs and belt traction

Estimate stiffness, stress and belt tensions and identify the geometric assumptions controlling each calculation.

A return spring at its working point

A close-coiled steel spring has wire diameter 3 mm, mean coil diameter 21 mm, seven active turns and $G = 79\,\mathrm{GPa}$. Find its rate, deflection and stored energy at 50 N. Calculate peak shear using the Wahl factor.

▼ **Reveal the worked solution**

The rate is $79000(3^4)/[8(21^3)(7)] = 12.34\,\mathrm{N/mm}$, giving 4.05 mm deflection and $50(0.00405)/2 = 0.101\,\mathrm{J}$. With $C = 7$, $K_w = 1.213$. Corrected shear is

$$1.213[8(50)(21)/(\pi 3^3)] = 120\,\mathrm{MPa}.$$

Combine two different springs

Springs with rates 18 and 30 N/mm are installed with negligible preload. Determine equivalent rate and travel of each at total force 90 N when in series. Then put them in parallel and find force carried by each at a 3 mm common compression.

▼ **Reveal the worked solution**

In series the same force acts in both. Their compressions are $90/18 = 5.0\,\mathrm{mm}$ and $90/30 = 3.0\,\mathrm{mm}$; effective rate is

$$90/8 = 11.25\,\mathrm{N/mm}.$$

In parallel the common deflection gives 54 and 90 N, a total 144 N and equivalent rate 48 N/mm. The active constraint is equal force in the first layout and equal travel in the second.

A reduction in wrap

A flat belt operates at 6 m/s with total span tension 600 N and friction coefficient 0.28. Estimate maximum power at a 180° wrap and again at a 120° wrap. Assume fixed sum of span tensions and negligible centrifugal tension.

▼ **Reveal the worked solution**

At limiting ratio $r = e^{\mu\beta}$, the tension difference is $600(r-1)/(r+1)$. The respective ratios are 2.410 and 1.798, giving differences 248.1 and 171.1 N. The corresponding maximum powers are 1.49 and 1.03 kW. The force pressing the shaft can remain similar even as the available power limit falls.

Explore the governing equations after solving the sheet: Helical springs, wire stress and stored energy; Belt traction, wrap and power transmission. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Chapter 9 · Friction clutches, braking torque and heat

Integrate annular contact traction and connect the torque needed to synchronise rotating bodies with their frictional heat release.

FROM FIRST PRINCIPLES

Obtain friction torque by integrating the pressure over an annulus

A dry clutch or disc brake transmits torque through normal pressure and frictional traction over its contact faces. For one axisymmetric annular face spanning radii r_i to r_o, a ring of radial width dr has area $2\pi r\,dr$. Let $p(r)$ be its pressure and take constant friction coefficient μ during slip. The ring carries normal force $dW = p\,dA$ and contributes moment $\mu r\,dW$. For z equally loaded active faces the total is

$$W = 2\pi \int_{r_i}^{r_o} p(r) r\,dr,$$

$$T = 2\pi z\mu \int_{r_i}^{r_o} p(r) r^2\,dr = z\mu W r_e.$$

The effective radius is a pressure-weighted mean lever arm. For uniform pressure, integrate the powers of r and cancel the pressure between numerator and denominator.

$$r_e = \frac{2}{3}\frac{r_o^3 - r_i^3}{r_o^2 - r_i^2}.$$

For a uniform-wear model, local sliding speed scales with radius and the pressure model is $p(r)r = \text{constant}$; integration gives

$$r_e = (r_o + r_i)/2.$$

During relative rotation the frictional heating rate is $T|\omega_1 - \omega_2|$. It changes as the two faces synchronise.

Background · Slip power

$$P_{\text{slip}} = \tau|\omega_1 - \omega_2|$$

Friction power during engagement depends on relative speed. Integrating it over engagement gives energy entering the thermal model.

DERIVATION AND APPLICATIONS

Predict stopping angle when a brake also sees a driving torque

A brake on a driven indexing shaft may oppose both stored rotational energy and continued motor work. Let the combined inertia referred to the braking shaft be J and its initial angular speed $\omega_0 > 0$. During the stopping interval assume constant brake torque magnitude T_b, constant driving torque T_d in the original direction, and negligible other losses. For a stop to occur the opposing torque must exceed the drive. Angular momentum balance gives the deceleration and stopping time.

$$J\dot{\omega} = T_d - T_b,$$

$$t_s = \frac{J\omega_0}{T_b - T_d}.$$

During uniform deceleration the average speed is $\omega_0/2$. Integrating speed over the stop yields angular travel. Multiply the brake torque by that angle to obtain the heat generated at the brake interface, assuming the brake member attached to ground is stationary.

$$\theta_s = \frac{J\omega_0^2}{2(T_b - T_d)},$$

$$Q_b = T_b\theta_s.$$

The energy balance exposes the extra term hidden when heat is identified with the initial rotor energy alone. Positive work from the still-active drive increases both the angle and brake heat as the torque margin shrinks.

$$Q_b = \frac{1}{2}J\omega_0^2 + T_d\theta_s = \frac{1}{2}J\omega_0^2\frac{T_b}{T_b - T_d}.$$

These results apply only until speed reaches zero. Whether the shaft remains stationary afterwards depends on the brake's static torque capacity and the drive command. Temperature-dependent friction during the stop would also alter the assumed constant torque.

WORKED EXAMPLE

Quantify the heat from a conveyor stop

Take $J = 0.800$ kg m², $\omega_0 = 15.0$ rad/s, $T_b = 18.0$ N m and $T_d = 6.00$ N m. Net deceleration is 15.0 rad/s², stopping time 1.00 s and angle 7.50 rad. The brake absorbs $18.0(7.50) = 135$ J, consisting of 90.0 J initial kinetic energy and 45.0 J additional drive work. Cutting the drive before braking would avoid that extra energy.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Size a brake from allowable travel

In another indexer $J = 0.300$ kg m², $\omega_0 = 20.0$ rad/s and residual drive torque is 4.00 N m. To stop within 2.50 rad requires $T_b \geq T_d + J\omega_0^2/(2\theta_s) = 28.0$ N m. At this threshold the stop takes 0.250 s and generates 70.0 J of heat. The motor contributes 10.0 J of that amount while the shaft turns through the permitted angle.

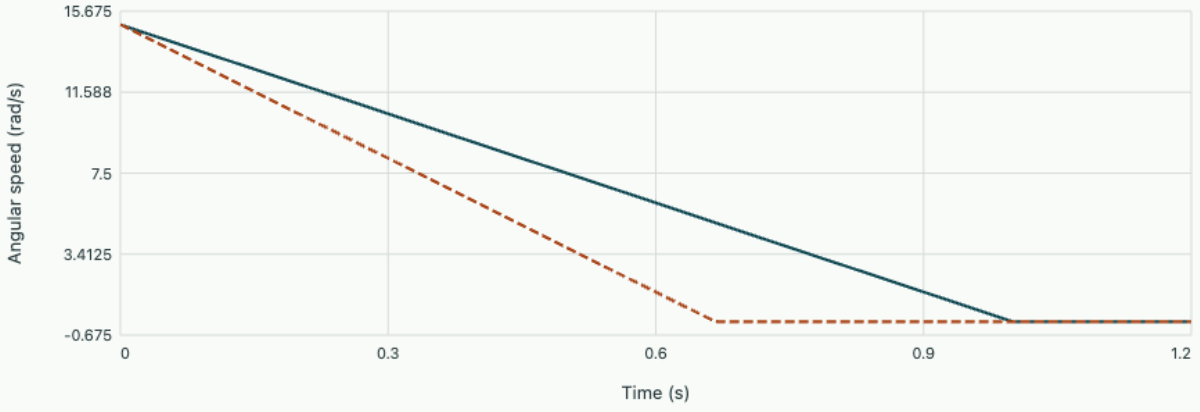
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Brake torque (N m)	18
Persistent drive torque (N m)	6
Initial angular speed (rad/s)	15



Stopping time 1.000 s; angular travel 7.500 rad; heat 135.000 J = 90.000 J initial energy + 45.000 J drive work.

Each ring acts through a different lever arm

A single-plate clutch uses an axial force to transmit torque across annular friction interfaces. Assume axisymmetric pressure $p(r)$ with inner radius r_i, outer radius r_o and friction coefficient μ. A narrow ring has normal force $dN = p(r) 2\pi r\, dr$. When slipping with fully mobilised friction, it transmits torque $dT = \mu r\, dN$. The normal force on each active interface is the axial clamp force W in an ordinary axial stack; the total torque capacity adds over the number z of active interfaces.

For uniform pressure integrate the normal force and torque separately, then eliminate pressure. For the idealised uniform-wear law the wear rate scales with pv. Since sliding speed is proportional to r, equal wear implies $p(r)r$ constant. Integration gives two different effective radii.

$$T = z\mu W r_e,$$

$$r_{e,p} = \frac{2}{3}\frac{r_o^3 - r_i^3}{r_o^2 - r_i^2},$$

$$r_{e,w} = (r_o + r_i)/2.$$

Uniform pressure is a useful model for a new ideal face; uniform wear models a different radial distribution. Both depend on alignment and the ability of the backing plates to distribute load. The distinction is greater for a wide annulus than for a narrow ring.

Determine the relative speed decay

During engagement two rotors initially have angular speeds ω_1, ω_2 and inertias I_1, I_2. If external torques are negligible during a short engagement, angular momentum conservation gives the common final speed. Friction does equal and opposite angular work on the two rotors, but at different instantaneous speeds while they slip. Their net mechanical energy decreases. Eliminating the final speed from the energy difference gives

$$\omega_f = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2},$$

$$E_{\mathrm{heat}} = \tfrac{1}{2} I_{\mathrm{red}} (\omega_1 - \omega_2)^2,$$

$$I_{\mathrm{red}} = \frac{I_1 I_2}{I_1 + I_2}.$$

For a constant slipping torque T, relative speed decreases at T/I_{red} until synchronisation. Increasing clamp force shortens the engagement but does not reduce this energy at the same initial speeds in the isolated model. It increases the heat generation rate and can alter peak surface temperature.

Check one start and repeated operation

Let $I_1 = 0.20,\ I_2 = 0.30\,\mathrm{kg\,m^2}$ and starting speeds be 100 and 0 rad/s. Then $\omega_f = 40\,\mathrm{rad/s}$, $I_{\mathrm{red}} = 0.12\,\mathrm{kg\,m^2}$ and heat release is 600 J. For $r_i = 40,\ r_o = 80\,\mathrm{mm}$, $z = 2$, $\mu = 0.30$,

$W = 1000\,\mathrm{N}$ and uniform wear, the torque is $2(0.30)(1000)(0.060) = 36\,\mathrm{N\,m}$. Synchronisation takes $0.12(100)/36 = 0.333\,\mathrm{s}$. The mean heat generation over that interval is 1.80 kW. Six identical starts per minute give an average of 60 W over a complete operating cycle. These two rates govern different aspects of cooling and surface temperature; an application with an actively driven motor during engagement requires its external work to be included.

Check the pressure at the inner radius

Under uniform wear $pr = C$ and $W = 2\pi C(r_o - r_i)$. At the stated load and radii the maximum pressure occurs at the inner edge and equals

$$1000/[2\pi(0.080 - 0.040)(0.040)] = 99.5\,\mathrm{kPa}.$$

An average face pressure alone would hide this radial concentration.

OneSimulation · Which annulus carries the most torque at fixed peak pressure?

Preserve outer radius b, friction coefficient μ and the maximum allowable lining pressure $p_{\max}$. In the uniform-wear model the inner radius a carries the peak pressure and $p(r)r = p_{\max}a$. A ring contributes tangential force $\mu p(r) 2\pi r\,dr$ with lever arm r. Integrating over one face gives

$$T = \int_a^b 2\pi\mu p(r) r^2\,dr = \pi\mu p_{\max} a(b^2 - a^2).$$

Differentiating with respect to a gives the optimum $a = b/\sqrt{3}$. The OneSimulation Gaussian rule evaluates the contact moment as the inner radius varies. At a 100 mm outer radius, 0.80 MPa peak pressure and friction coefficient 0.30, the optimum lies at 57.7 mm and gives 290 N m per rubbing face. This comparison allows the clamping force to change with the annulus. A fixed actuator force would define a different design problem.

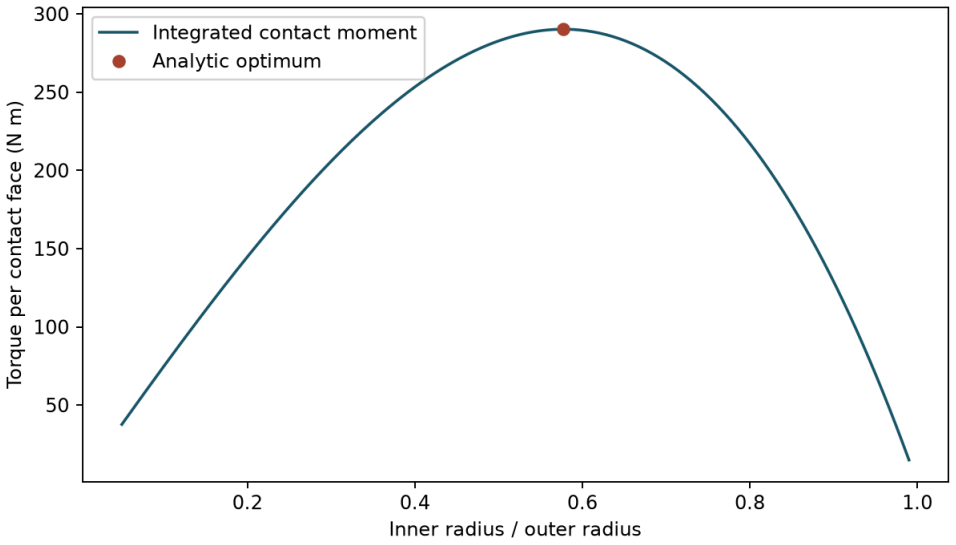
```
"""Torque from one annular rubbing face under a fixed peak pressure."""
import matplotlib.pyplot as plt
import numpy as np
from onesim.numerics.discretization.geometry.carriers.kernels.quadrature.rules.reference_quadrature
import gauss_legendre

outer = 0.10             # outer radius, m
peak, mu = 8.0e5, 0.30   # allowable contact pressure (Pa), friction coefficient
nodes, weights = np.asarray(gauss_legendre(3)).T

def torque(inner):
    r = (inner + outer)/2 + (outer - inner)*nodes/2
    pressure = peak*inner/r  # uniform wear: p*r is constant
    integrand = 2*np.pi*mu*pressure*r**2
    return (outer - inner)/2 * (weights @ integrand)

ratio = np.linspace(0.05, 0.99, 240)
values = np.array([torque(x*outer) for x in ratio])
best = 1/np.sqrt(3)
```

```
assert abs(ratio[np.argmax(values)] - best) < 0.005
print(f"Best inner radius {1000*best*outer:.2f} mm; torque {torque(best*outer):.1f} N m")
fig, ax = plt.subplots(figsize=(7, 4), constrained_layout=True)
ax.plot(ratio, values, color="#19596a", label="Integrated contact moment")
ax.plot(best, torque(best*outer), "o", color="#a5442e", label="Analytic optimum")
ax.set(xlabel="Inner radius / outer radius", ylabel="Torque per contact face (N m)")
ax.legend()
fig.savefig("machine-elements.png", dpi=170)
```



Reducing the bore adds contact area but lowers the pressure allowed across the rest of a uniformly worn face. Both effects enter the integrated torque.

Download the Python example

Vary the parameters of this exact example in the chapter's interactive calculation.

Derive the peak temperature under repeated braking

A mean heat load sets the long-time cooling requirement, while peak friction temperature may control a brake lining. A simple intermediate model treats the thermally active rotor and lining as a uniform heat capacity C connected to ambient by conductance H. Let $\theta = T - T_\infty$. For the heat rate $P(t)$ actually entering this lump, $C\dot{\theta} = P - H\theta$ with thermal time $\tau = C/H$. The generated friction heat may be shared between two bodies; use the fraction entering the modelled part or include both bodies in the same effective lump only when their temperatures justify that choice.

For a repeat duty with constant heat rate P over braking time t_b, then zero heat input over cooling time t_c, integrate each constant-input segment. Writing $a = e^{-t_b/\tau}$, $b = e^{-t_c/\tau}$ gives

$$\theta_{\mathrm{peak}} = a\theta_{\mathrm{start}} + (P/H)(1 - a),$$

$$\theta_{\mathrm{next}} = b\theta_{\mathrm{peak}}.$$

At cyclic equilibrium next equals start, so

$$\theta_{\mathrm{peak}} = \frac{P}{H}\,\frac{1-a}{1-ab},$$
$$\theta_{\mathrm{min}} = b\,\theta_{\mathrm{peak}}.$$

Integrating the balance over one full cycle independently gives mean rise $\bar{\theta} = (P/H)t_b/(t_b + t_c)$, because stored energy returns to its starting value. The peak depends on capacity C, although the mean is fixed by H and time-averaged power. A high-conductivity rotor with ample time for diffusion may fit a lumped model reasonably; very short contact events can create local flash temperatures far above the bulk estimate.

Distinguish larger thermal mass from better cooling

A test rotor receives 6000 J every minute, distributed approximately uniformly over a 10 s braking interval. Use effective rotor heat capacity $C = 800\,\mathrm{J/K}$, conductance $H = 4.0\,\mathrm{W/K}$ and ambient temperature 25 °C. During braking $P = 600\,\mathrm{W}$; during the next 50 s it is zero. The time constant is 200 s. For periodic operation $a = e^{-0.05} = 0.9512,\ b = e^{-0.25} = 0.7788$, giving peak rise $150(1-0.9512)/(1-e^{-0.30}) = 28.23$ K and minimum rise $0.7788(28.23) = 21.98$ K. The rotor cycles between approximately 47.0 and 53.2 °C, while the mean is 50 °C.

The first stop from ambient raises the rotor by only $150(1-e^{-0.05}) = 7.32$ K. A single-stop test is therefore poor evidence for repeated-duty thermal margin. Doubling rotor heat capacity while keeping cooling unchanged changes τ to 400 s. The periodic peak rise becomes $150(1-e^{-0.025})/(1-e^{-0.15}) = 26.59$ K, but the mean rise stays 25 K. By contrast, doubling conductance would halve the mean rise under the same duty. Added thermal mass mainly reduces the ripple and slows warmup; stronger heat removal lowers the long-term mean as well. Actual fans and convection coefficients can vary with rotor speed during a stop and should be represented if the measured cooldown deviates systematically from one exponential.

Thermal approach to periodic operation follows directly from the update. The difference from the steady-cycle temperature at the same point of each cycle is multiplied by $ab = e^{-60/200} = 0.741$ every minute. Reaching within 5% of that periodic offset therefore takes about ten cycles. A test of only two or three stops will still be dominated by warmup.

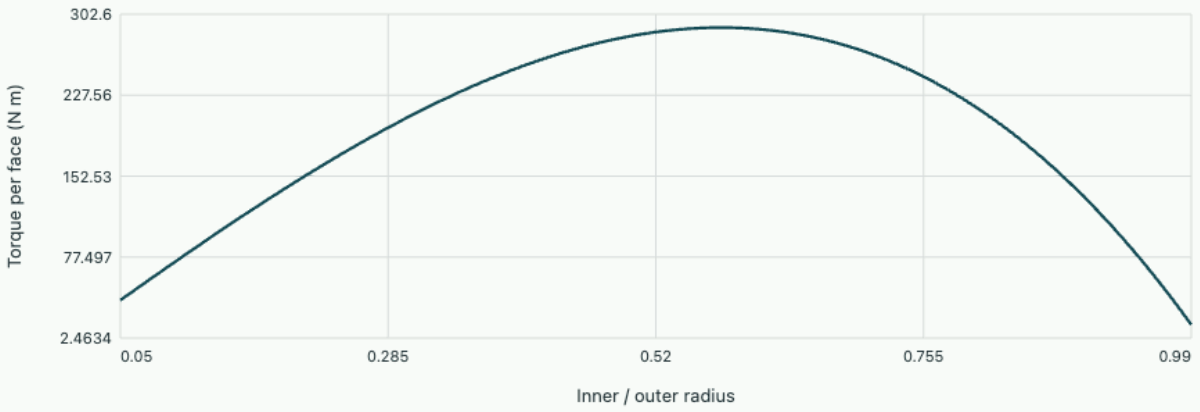
Interactive calculation · Change the inputs of this example

Change the parameters of this reference plot and compare its limiting cases. The fixed illustration remains beside the derivation; the adjustable calculation exposes the same dependence on geometry, material properties or loading.

Open this live calculation

Try the calculation

Outer radius (mm)	100
Peak pressure (MPa)	0.8
Friction coefficient	0.3



Uniform-wear pressure law

Optimum inner radius 57.74 mm; torque 290.2 N m. Peak pressure is held at the inner edge; changing that control condition changes the optimum.

Study the full derivation

Sliding wear, real contact area and life inference from a coupon · Design and analysis of machine components

A sliding carriage develops measurable clearance while its normal force remains almost constant. Derive a practical volume-loss rule from microscopic contacts, estimate its coefficient from a mass trace, and test whether a single fitted value can be transferred to another load history. Geometry, material condition and instrument resolution enter at different stages.

Chapter 10 · Hydrodynamic lubrication and sliding bearings

Derive the flow balance in a converging lubricant film and determine why the sliding surface can support a load without solid contact.

FROM FIRST PRINCIPLES

Derive the flux through a moving lubricant film

Consider two broad bearing surfaces separated by a thin gap h. Let the lower surface move at speed U and the upper one remain still. Assume incompressible Newtonian lubricant, no slip, negligible inertia and a gap changing only slowly in the streamwise direction. Pressure is nearly constant through the thickness. Locally, the streamwise momentum balance reduces to a pressure gradient balanced by viscous shear curvature.

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{dp}{dx},$$
$$u(0) = U,$$
$$u(h) = 0.$$

Integrating twice gives a linear velocity contribution from sliding and a parabolic contribution from the pressure gradient. Integrating that velocity across the gap gives the discharge per unit width.

$$u(y) = U(1 - y/h) - \frac{1}{2\mu}\frac{dp}{dx} y(h - y),$$
$$q = \frac{Uh}{2} - \frac{h^3}{12\mu}\frac{dp}{dx}.$$

In a steady film without leakage or normal surface motion the same q crosses every section. A varying gap changes the sliding contribution, forcing a pressure gradient to preserve that flow. The integral of gauge pressure over the bearing pad gives its normal load capacity.

DERIVATION AND APPLICATIONS

Relate recess pressure to restrictor resistance in a hydrostatic pad

A hydrostatic bearing can sustain load at zero sliding speed because an external supply drives lubricant through a restriction. Consider a flat circular thrust pad with a central recess of radius a and outer land radius b. Let the parallel land clearance h be uniform and much smaller than its radial extent. Use incompressible Newtonian oil of viscosity μ, steady laminar radial outflow, no squeeze motion, ambient

gauge pressure at b, and negligible pressure drop inside the recess. The pressure-driven flow per circumferential width follows the parallel-plate law.

$$Q = -2\pi r \frac{h^3}{12\mu}\frac{dp}{dr},$$

$$p(r) = \frac{6\mu Q}{\pi h^3}\ln\frac{b}{r}.$$

At the recess edge $p(a) = p_r$. The land therefore behaves as a flow resistance R_L proportional to the inverse cube of clearance. A linear feed restrictor R_f drops supply pressure p_s to p_r. Equal flow passes through the feed and land, giving a hydraulic pressure divider.

$$R_L = \frac{6\mu}{\pi h^3}\ln(b/a),$$

$$p_r = QR_L = p_s\frac{R_L}{R_f + R_L}.$$

Integrate pressure over the uniform recess and logarithmic land. The logarithm integral contains an inner-boundary term that cancels the associated part of the recess load, leaving a simple effective area.

$$W = \pi a^2 p_r + 2\pi\int_a^b rp(r)\,dr = \frac{\pi(b^2 - a^2)}{2\ln(b/a)}p_r.$$

Closing the pad increases land resistance, raises recess pressure and increases force opposing closure. When the land resistance is already very large, recess pressure approaches supply pressure and the available load saturates.

WORKED EXAMPLE

Predict load at a balanced restriction

For $a = 10.0$ mm and $b = 20.0$ mm, effective area is

$$\pi(0.0200^2 - 0.0100^2)/(2\ln 2) = 6.80 \times 10^{-4}\,\mathrm{m}^2.$$

If $R_L = R_f$ at the operating clearance and $p_s = 0.800$ MPa, recess pressure is 0.400 MPa and supported load is 272 N. This load is present without any relative sliding of the two faces.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Infer a clearance change from pressure

A different bearing has $p_r/p_s = 0.600$ initially and 0.750 after a load change. Resistance ratios $R_L/R_f = p_r/(p_s - p_r)$ therefore increase from 1.50 to 3.00. Since land resistance varies as h^{-3}, the final gap is $2^{-1/3} = 0.794$ of its initial value. An initial 25.0 µm film becomes 19.8 µm, assuming unchanged oil viscosity and feed resistance.

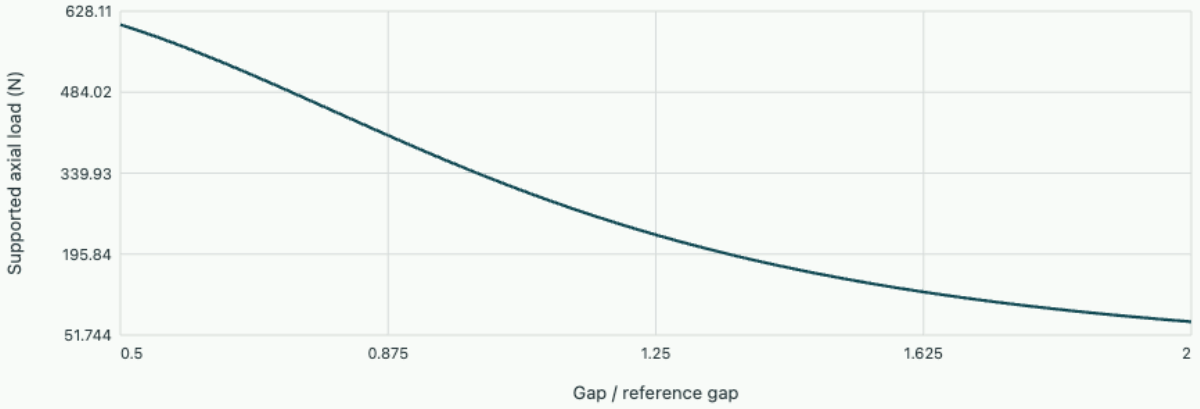
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Supply pressure (MPa)	1
Feed/land resistance at reference gap	1
Reference gap (µm)	20



Fed circular bearing

At reference gap 20 µm, supported force 339.9 N and recess pressure 0.500 MPa. The land hydraulic resistance varies as gap⁻³.

Where lubricant pressure comes from

A shaft turning inside a plain bearing drags lubricant around with its surface. A load displaces the journal from the bore centre, creating a film whose gap narrows in one direction. Wall motion carries oil into this wedge, and a pressure gradient redistributes the flow so the same fluid rate passes successive sections. Pressure integrated over the bearing can then support a load. Surface motion alone in a uniform gap between surfaces with equal end pressures cannot produce this load in the ideal steady model.

Consider a broad slider bearing rather than the full journal geometry. Let a lower wall move at speed U; a fixed upper wall forms a slowly varying small gap $h(x)$. Assume incompressible Newtonian lubricant, constant viscosity, negligible inertia and negligible leakage through the sides. Pressure is nearly uniform across the thickness. The axial momentum equation reduces to $\mu d^2u/dy^2 = dp/dx$, and no slip gives

$$u(0) = U,\; u(h) = 0.$$

$$u = U(1 - y/h) - \frac{1}{2\mu}\frac{dp}{dx}\, y(h - y),$$

$$q = \frac{Uh}{2} - \frac{h^3}{12\mu}\frac{dp}{dx}.$$

Mass conservation makes q per unit width constant along the bearing. Where the local dragged flow $Uh/2$ exceeds q, pressure rises downstream. Where it falls short, pressure falls and supplies a forward pressure-driven contribution. The transition locates a pressure maximum.

Solve a two-depth ideal bearing

Consider a stepped pad with two equal 10 mm long regions, gaps $h_1 = 0.20$ and $h_2 = 0.10\,\mathrm{mm}$ in the sliding direction. Both ends have ambient pressure, $U = 1.0\,\mathrm{m/s}$, and $\mu = 0.05\,\mathrm{Pa\,s}$. Ignore the local loss at the abrupt step and treat each region as fully developed. Integration of $dp/dx = 12\mu(Uh/2 - q)/h^3$ across both regions gives zero net pressure change. With equal lengths this fixes

$$q = \frac{U}{2}\,\frac{h_1^{-2} + h_2^{-2}}{h_1^{-3} + h_2^{-3}} = 5.56 \times 10^{-5}\,\mathrm{m^2/s}.$$

In the first region $Uh_1/2 = 1.00 \times 10^{-4}\,\mathrm{m^2/s}$. The positive pressure gradient is $3.33 \times 10^{6}\,\mathrm{Pa/m}$, giving an interface gauge pressure of 33.3 kPa. The second region has an equal opposite gradient. The resulting triangular pressure distribution integrates to a force per unit width of $0.5(0.020)(33333) = 333\,\mathrm{N/m}$. A nominal 50 mm width would thus support approximately 16.7 N under the negligible-side-leakage assumption.

Understand failure at low speed

With zero sliding speed, equal end pressures imply zero steady gauge pressure everywhere in this model, so the load capacity vanishes. Start-up and shutdown therefore differ from established hydrodynamic running. A separately pressurised hydrostatic bearing can carry load at zero shaft speed. If a solution predicts large negative absolute pressures in a diverging region, actual oil may cavitate and the full-film boundary conditions must change. Viscous heating can also lower the local viscosity. Film thickness, thermal balance and supply conditions all matter when transferring the simple slider result to a journal in service.

Predict a viscosity change

For the same slider geometry and speed with equal prescribed end pressures, the calculated flux is independent of viscosity but all gauge pressure gradients are proportional to viscosity. Halving viscosity in the ideal stepped example therefore halves its load capacity at unchanged film thickness.

Derive load carried by a closing lubricant film

Sliding motion is not the only way to develop pressure in a thin film. Two parallel circular plates moving towards each other expel viscous oil radially, producing a squeeze-film pressure. Let plate radius be R, uniform gap $h(t)$, and approach speed $v = -dh/dt > 0$. Assume incompressible Newtonian oil, no slip, negligible inertia, rigid parallel plates, negligible edge resistance and ambient pressure at the rim. The gap is much smaller than the radius. Neither plate moves radially, so depth-integrated radial flux per unit circumference is $q_r = -h^3(dp/dr)/(12\mu)$, the stationary-wall pressure-driven slit relation.

Inside radius r, incompressibility requires outward flow through its circumference to equal the volume displaced by plate motion: $2\pi r q_r = \pi r^2 v$. Thus $q_r = rv/2$ and $dp/dr = -6\mu r v/h^3$. Integrating inward from ambient at the edge gives

$$p(r) - p_a = \frac{3\mu v}{h^3}(R^2 - r^2),$$

$$W = \int_0^R (p - p_a) 2\pi r \, dr = \frac{3\pi\mu R^4 v}{2h^3}.$$

The maximum pressure is at the centre, twice the area average because the pressure shape is parabolic. The load grows very steeply as the gap narrows at imposed speed. At imposed load, however, the approach slows. This distinction mirrors the need to separate displacement-controlled and force-controlled contact tests. A stationary gap with no relative normal motion produces no squeeze-generated pressure under the stated ambient boundary condition.

Estimate the time for a thrust pad to approach its stop

A circular pad of radius 25 mm settles under a constant additional normal force of 100 N on oil of dynamic viscosity $0.15\,\mathrm{Pa\,s}$. Ignore inertia after the short initial transient. Rearranging the load law gives

$$-dh/dt = 2Wh^3/(3\pi\mu R^4).$$

Between initial and final gaps, integration gives

$$t = [3\pi\mu R^4/(4W)](h_f^{-2} - h_i^{-2}).$$

For $h_i = 150\,\mu\mathrm{m}$ and $h_f = 50\,\mu\mathrm{m}$ this yields 0.491 s.

At an intermediate 100 micrometre gap the instantaneous speed is

$$2(100)(10^{-4})^3/[3\pi(0.15)(0.025)^4] = 0.000362\,\mathrm{m/s}.$$

The centre gauge pressure is $2W/(\pi R^2) = 102\,\mathrm{kPa}$, independent of gap here because the load is prescribed and the approach speed adapts. Halving the gap at unchanged load reduces approach speed eightfold. The ideal solution takes infinite time to reach zero thickness, which should not be interpreted as assurance against surface contact: roughness, oil leakage paths, tilt and elasticity enter before an arbitrarily small gap is reached. On separation the sign of the calculated pressure reverses and cavitation or air ingress may prevent the predicted tensile liquid pressure from developing.

A load-cell test can independently discriminate squeeze action from simple dry sliding resistance. At fixed small gap and sufficiently slow normal motion the predicted force is proportional to approach speed, with a slope scaling as h^{-3}. Closing the gap from 100 to 80 micrometres should multiply the force slope by $(100/80)^3 = 1.95$. The transient test must either measure instantaneous gap or be run with an imposed motion, since holding load constant causes speed and gap to evolve together. Comparing forces at the same clock time on two different tests would not isolate the gap dependence.

Problem sheet 5 · Friction work and fluid films

Relate an engagement event to thermal loading and use film-flow continuity to infer pressure gradients.

A stopped flywheel

A flywheel of inertia 0.80 kg m² initially turns at 60 rad/s and receives a constant braking torque of 12 N m. Find stopping time, heat and total revolutions during braking. If the same stop is repeated every minute, what is the cycle-average heat load?

▼ **Reveal the worked solution**

Angular deceleration is $12/0.80 = 15\,\mathrm{rad/s^2}$, so stopping takes 4.0 s. Heat equals the initial kinetic energy $0.5(0.80)(60)^2 = 1440\,\mathrm{J}$. Mean speed is 30 rad/s and travel is 120 rad, or 19.1 revolutions. One stop every 60 s averages 24 W despite the much larger power during a stop.

Two faces, two pressure assumptions

A clutch has two active faces, 30 and 90 mm inner and outer radii, axial clamp 800 N, and friction coefficient 0.25. Find torque capacities for uniform wear and uniform pressure. A drive requires 25 N m. How does the choice of model affect the result?

▼ **Reveal the worked solution**

Uniform wear gives $r_e = 60\,\mathrm{mm}$ and torque $2(0.25)(800)(0.060) = 24.0\,\mathrm{N\,m}$. Uniform pressure gives $r_e = (2/3)(90^3 - 30^3)/(90^2 - 30^2) = 65\,\mathrm{mm}$, yielding 26.0 N m. Only the pressure model nominally reaches the required torque. Wear distribution and tolerances matter for such a small margin.

Infer a pressure gradient from a flow

A slider locally has oil-film thickness 0.15 mm, viscosity 0.040 Pa s, lower-surface speed 2.0 m/s and measured flow per width $1.20 \times 10^{-4}\,\mathrm{m^2/s}$. Find the pressure gradient and its sign. In a second region the gap is 0.10 mm at the same flow. Is the pressure still rising?

▼ **Reveal the worked solution**

Use $dp/dx = 12\mu(Uh/2 - q)/h^3$. The first dragged flow is $1.50 \times 10^{-4}\,\mathrm{m^2/s}$, giving $+4.27 \times 10^6\,\mathrm{Pa/m}$. At the smaller gap the dragged flow is 1.00×10^{-4}, giving $-9.60 \times 10^6\,\mathrm{Pa/m}$. The pressure maximum lies between the two regimes in a smooth converging film.

Explore the governing equations after solving the sheet: Friction clutches, braking torque and heat; Hydrodynamic lubrication and sliding bearings. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Chapter 11 · Eccentrically loaded weld groups

Resolve an eccentric in-plane force into direct shear and torsional shear and find the combined nominal throat demand.

FROM FIRST PRINCIPLES

Distribute an eccentric force across a weld group's throat

Idealise each thin weld by its effective throat times its length. For constant throat t and total weld length L, the load-carrying area is $A = tL$. Suppose a planar force $\mathbf{F}$ acts at offset $\mathbf{e}$ from the throat-area centroid. Resolve it into the same force through the centroid and an additional torque $M_z = (\mathbf{e} \times \mathbf{F})_z$. In a nominal elastic weld-group model the direct force produces uniform shear.

$$\boldsymbol{\tau}_d = \frac{\mathbf{F}}{tL},$$
$$M_z = (\mathbf{e} \times \mathbf{F})_z.$$

Small rigid rotation of the attached plate motivates a torsional shear distribution proportional to distance r from the centroid and tangential to the radius. Its scale follows from requiring the integral of moment to equal M_z.

$$J_t = t \int r^2 \, ds,$$
$$\boldsymbol{\tau}_M = \frac{M_z}{J_t} \, \mathbf{e}_z \times \mathbf{r},$$
$$\boldsymbol{\tau} = \boldsymbol{\tau}_d + \boldsymbol{\tau}_M.$$

The direct and torsional contributions add as vectors. The largest nominal magnitude may therefore occur where their directions most nearly align. The calculation estimates throat demand from the overall load path; local notch stresses require a more detailed model.

DERIVATION AND APPLICATIONS

Locate the elastic centre of an unequal weld group

An eccentric force produces a torsional shear flow proportional to distance from the weld-group centre. When throat thickness varies, the centre must account for the differing load-carrying areas. Consider narrow weld lines in one plane, with local effective throat $t(s)$ measured normal to the nominal load-bearing section. Assume identical material properties and linear elastic shear, uniform through each throat. The area element is $dA = t(s)ds$. Define centroid $\mathbf{c}$ so the first area moment about it vanishes.

$$A_t = \int t \, ds,$$

$$\mathbf{c} = \frac{\int \mathbf{r}\, t \, ds}{A_t},$$

$$\int (\mathbf{r} - \mathbf{c}) t \, ds = 0.$$

An in-plane force $\mathbf{F}$ through the centroid gives uniform direct shear $\mathbf{F}/A_t$. For torque M about the centroid take a shear field proportional to the local tangential displacement of a small rigid rotation. Its integral is zero by the centroid condition. Equating its moment to the applied torque fixes its scale.

$$J_t = \int |\mathbf{r} - \mathbf{c}|^2 t \, ds,$$

$$\boldsymbol{\tau}_M = \frac{M}{J_t} \, \mathbf{e}_z \times (\mathbf{r} - \mathbf{c}).$$

For a straight weld of length ℓ and constant throat t, its contribution to the polar area moment contains a local term and a centroid-offset term. Here d is the distance from that weld's midpoint to the group centroid.

$$J_{t,j} = t \left(\frac{\ell^3}{12} + \ell d^2 \right).$$

The assumptions describe nominal group demand. Local toe geometry and fabrication details can govern fatigue even when this nominal stress is low.

WORKED EXAMPLE

Find the centre and torsional resistance of two seams

Two horizontal 100 mm welds lie at $y = 0$ and 60 mm with throats 4.00 and 2.00 mm and both centred at $x = 0$. Their areas are 400 and 200 mm². The centroid is at $y = 20.0$ mm. The local moment terms total $6(100^3)/12 = 500{,}000$ mm⁴; the offset terms total $400(20^2) + 200(40^2) = 480{,}000$ mm⁴. Thus $J_t = 980{,}000$ mm⁴. The thinner upper seam lies farther from the torsional centre.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Recover an unknown throat from a measured centroid

Two equal-length parallel seams are 50.0 mm apart. The lower throat is 3.00 mm, and the centroid is known to be 20.0 mm above it. Writing $20 = 50t/(3 + t)$ gives upper throat $t = 2.00$ mm. For lengths 80.0 mm each, total throat area is 400 mm². A 4.00 kN load whose line passes through the inferred centroid then gives uniform nominal direct shear of 10.0 MPa.

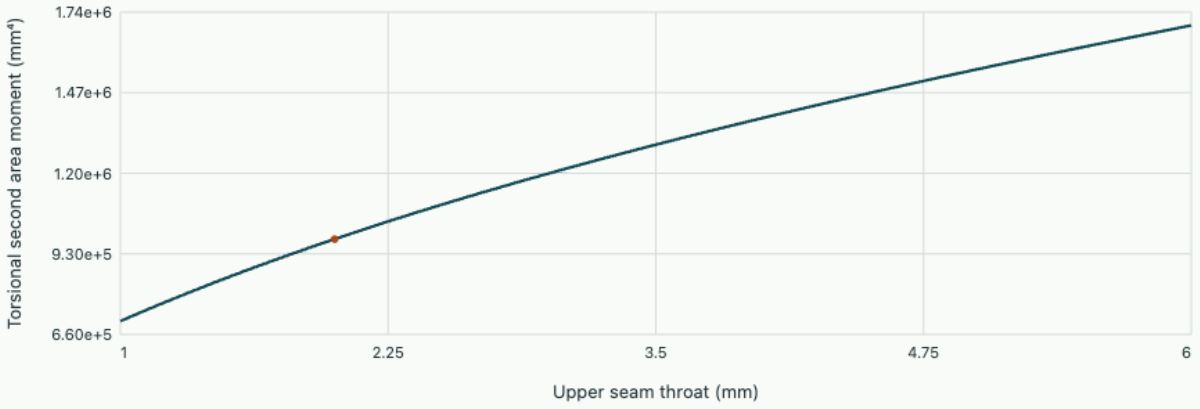
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Lower throat (mm)	4
Upper throat (mm)	2
Seam separation (mm)	60



Two100 mm seams Chosen upper throat

Centroid 20.000 mm above the lower seam; Jt=980000 mm⁴. Throat-weighted area, rather than bare line length, sets the centre.

Reduce a weld to a carrying area

A fillet weld joins two plates along a line, but its specified leg length is not the effective area transmitting force. For an ideal equal-leg 45° fillet, the shortest distance from the root to the theoretical weld face is the throat $t = s/\sqrt{2}$, where s is leg length. Effective throat area is t times effective weld length under the simplified model. End details and local weld defects require separate assessment.

Consider a bracket carrying an in-plane force whose line of action misses the weld group centroid. Move the force to that centroid and retain its moment $M_z = Fe$. The direct force gives a uniform elastic shear flow vector $q_d = F/\sum L$ under the line-weld approximation. The moment demands

another shear flow, tangent about the centroid. For a rigid attachment and uniform throat and elastic response, assume its magnitude is proportional to radius r, as expected from a small relative rotation.

$$J_L = \int_{\text{welds}} r^2 \, ds,$$

$$q_m(r) = \frac{M_z r}{J_L},$$

$$\boldsymbol{\tau} = (\mathbf{q}_d + \mathbf{q}_m)/t.$$

The line property J_L has units of length cubed, since weld length rather than throat area is integrated. For an unequal throat one must weight the centroid and polar property with the appropriate area or stiffness. Add direct and moment contributions as vectors at each point before locating maximum stress.

Calculate a corner stress

Take two vertical parallel welds of length 80 mm each, centred at $x = \pm 30\,\text{mm}$. Their endpoints lie at $y = \pm 40\,\text{mm}$. An upward force $F = 4.0\,\text{kN}$ acts at $x = +100\,\text{mm}$. Its equivalent counterclockwise moment is 400,000 N mm. The total line length is 160 mm, so the direct flow is 25 N/mm upwards. The group line polar property is

$$J_L = 2[80(30)^2 + 80^3/12] = 229333\,\text{mm}^3.$$

At the upper right corner the moment-induced flow vector is

$$M_z(-y, x)/J_L = (-69.8, 52.3)\,\text{N/mm}.$$

Adding the direct contribution gives $(-69.8, 77.3)\,\text{N/mm}$, magnitude 104.2 N/mm. By symmetry the lower right corner has the same magnitude; the left endpoints experience partial cancellation of the vertical components. For an effective throat 4 mm, the maximum nominal combined shear is 26.0 MPa.

Decide what changes the joint demand

If the force passes through the centroid the moment vanishes and the throat stress is only $25/4 = 6.25\,\text{MPa}$. Increasing throat lowers nominal stress but leaves the line shear flow required by equilibrium unchanged. Moving the force closer to the attachment or spreading the welds farther apart can reduce the moment-induced flow. A transverse load normal to the plate introduces bending or prying and would require a different weld-group analysis. Fatigue at the weld toe depends on local geometry and the applicable welded-detail category, so the present nominal elastic calculation alone cannot predict a fatigue life.

Recover a leg dimension

A 4 mm ideal throat corresponds to a theoretical equal leg of $4\sqrt{2} = 5.66\,\mathrm{mm}$. A drawing that specifies a 4 mm leg instead leaves only a 2.83 mm theoretical throat, raising the calculated nominal maximum stress from 26.0 to about 36.8 MPa.

Turn a weld-group force history into a detail stress range

The static line-weld model produces a shear-flow vector for a given applied load. A fluctuating load introduces a separate fatigue question. In common nominal-stress assessment methods, experimental S–N curves for a specific welded detail can already incorporate local weld geometry and residual stresses. Use the exact nominal stress component, range convention, detail classification and life range associated with the test data. Multiplying such a detail stress again by an arbitrary notch factor can count the same local effect twice. Conversely, a smooth-base-metal fatigue curve generally does not represent a welded detail.

For an illustrative single-slope detail curve let Δs_{ref} be the tested stress range at N_{ref} cycles and exponent m. Constant-amplitude life is $N = N_{\mathrm{ref}}(\Delta s_{\mathrm{ref}}/\Delta s)^{m}$. Under an assumed linear accumulation of damage, a repeated block containing n_i cycles of each range consumes

$$D_{\mathrm{block}} = \sum_i \frac{n_i}{N_{\mathrm{ref}}} \left(\frac{\Delta s_i}{\Delta s_{\mathrm{ref}}} \right)^{m}.$$

The stress range is maximum minus minimum stress in the designated component, not its alternating half-amplitude. For a strictly proportional cycle, each local shear component in the preceding elastic model scales with the applied force. If force direction or line of action changes, the most highly stressed location can shift; recompute the vector histories before choosing the assessment component. A mesh-refined local notch stress and a nominal throat stress are different definitions and require corresponding fatigue data.

Compare a static throat check with a cyclic duty decision

Assume the applicable nominal-stress data for an illustrative connection give $\Delta s_{\mathrm{ref}} = 70\,\mathrm{MPa}$ at $N_{\mathrm{ref}} = 2.0 \times 10^{6}$ cycles with $m = 3$ throughout the ranges considered. These are supplied exercise data. A measured local nominal stress range is 35 MPa for 9000 ordinary cycles and 80 MPa for 300 heavy cycles per operating block. Their respective lives are $2.0 \times 10^{6}(70/35)^{3} = 16.0 \times 10^{6}$ and $2.0 \times 10^{6}(70/80)^{3} = 1.34 \times 10^{6}$ cycles.

Ordinary cycles consume $9000/(16.0 \times 10^{6}) = 0.0005625$ per block; heavy cycles consume $300/(1.34 \times 10^{6}) = 0.000224$. The sum predicts approximately $1/0.0007865 = 1271$ blocks at unit damage. If an unchanged force path permits throat to grow by 20%, the nominal stress ranges scale by $1/1.2$, and the calculated block life grows by $1.2^{3} = 1.728$ to about 2196 blocks. That improvement assumes the revised detail retains the same applicable S–N curve and geometry-independent loading

distribution. If instead the hot spot shifts into the base-plate toe or a short stiff attachment raises local restraint, throat increase alone will not reproduce this scaling. Testing a representative weld or checking a matching detail category is more informative than increasing nominal strength without tracing the new failure location.

Under the supplied single-slope assumption the equivalent constant range for the same 9300 cycles is $[(9000 \times 35^3 + 300 \times 80^3)/9300]^{1/3} = 38.7\,\mathrm{MPa}$ approximately. The arithmetic average range is only 36.45 MPa and underweights the heavy events. Quoting the equivalent range together with its associated cycle count makes the fatigue calculation reproducible.

Chapter 12 · Tolerance stacks and reliability of assemblies

Determine functional clearance by worst-case and statistical stacking, and connect component life probabilities to assembly reliability.

FROM FIRST PRINCIPLES

Propagate dimensional variation into an assembly clearance

Start with the physical closing dimension of an assembly. Suppose the functional gap is $g = A - B - C$ and the measured dimensions have stable means and variances. The signed relation comes from geometry, so taking expectations gives the mean gap regardless of independence. For a conservative interval check, the smallest gap pairs the smallest A with the largest B and C.

$$\bar{g} = \bar{A} - \bar{B} - \bar{C},$$
$$g_{\min} = A_{\min} - B_{\max} - C_{\max}.$$

For a statistical estimate, subtract the means and square the sum of deviations. When component deviations are independent, all mixed expectations vanish. The signs then disappear from the variance because each individual coefficient is squared.

$$\sigma_g^2 = \sigma_A^2 + \sigma_B^2 + \sigma_C^2.$$

If the independent dimensions are normal, g is normal and the interference probability is

$$P(g < 0) = \Phi(-\bar{g}/\sigma_g),$$

with Φ the standard normal cumulative distribution. A tolerance interval alone does not supply a standard deviation or distribution. Correlated machining errors leave mixed covariance terms in the squared sum and can increase or reduce the assembly spread.

DERIVATION AND APPLICATIONS

Propagate a shared manufacturing drift through a clearance stack

Two dimensions made on the same fixture can vary together. A clearance depends on their difference, so ignoring covariance can substantially misstate the expected spread. Let H be an internal dimension and S the mating external dimension; assembled clearance is $G = H - S$. Model a shared fixture drift by a zero-mean random variable Z with standard deviation σ_Z. Let e_H, e_S be independent local deviations, also independent of Z. The sensitivities a_H, a_S convert drift into dimensional change.

$$H = \bar{H} + a_H Z + e_H,$$
$$S = \bar{S} + a_S Z + e_S.$$

Subtract the dimensions before calculating variance. Only the difference between their sensitivities multiplies the shared term. Squaring and averaging eliminates products of independent zero-mean deviations.

$$\bar{G} = \bar{H} - \bar{S},$$
$$\sigma_G^2 = (a_H - a_S)^2 \sigma_Z^2 + \sigma_{e_H}^2 + \sigma_{e_S}^2.$$

Equivalently use the variance of each measured dimension and their covariance. Matched positive sensitivities cancel common fixture drift, while sensitivities of opposite sign amplify it. These relations require finite variances and the stated correlations; no particular distribution has yet been assumed.

$$\sigma_G^2 = \sigma_H^2 + \sigma_S^2 - 2\,\mathrm{Cov}(H, S),$$
$$\mathrm{Cov}(H, S) = a_H a_S \sigma_Z^2.$$

For normally distributed deviations the ratio of mean clearance to its standard deviation also gives the normal-tail estimate of interference probability. Pairing components from unrelated fixtures may remove the very covariance measured on matched components.

WORKED EXAMPLE

Compare matched and randomly remixed parts

Take $a_H = a_S = 1$, shared drift spread 20.0 µm, and local independent spreads of 6.00 and 8.00 µm. A matched pair has clearance spread $\sqrt{6^2 + 8^2} = 10.0$ µm. If components are randomly paired from independent fixture drifts, the spread becomes

$$\sqrt{6^2 + 8^2 + 2(20)^2} = 30.0\,\mu\mathrm{m}.$$

For mean clearance 30.0 µm, normally distributed matched pairs have interference probability about 0.135 percent; independent remixed pairs give about 15.9 percent.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Infer manufacturing correlation from assembly records

Another process measures standard deviations of 12.0 µm for the bore, 10.0 µm for the shaft and 8.00 µm for their assembled clearance. Rearranging the variance identity gives covariance $(12^2 + 10^2 - 8^2)/2 = 90.0$ µm². The inferred correlation coefficient is

$$90/(12 \times 10) = 0.750.$$

A reconstruction using independent component measurements would give 15.6 µm clearance spread, nearly twice the measured value.

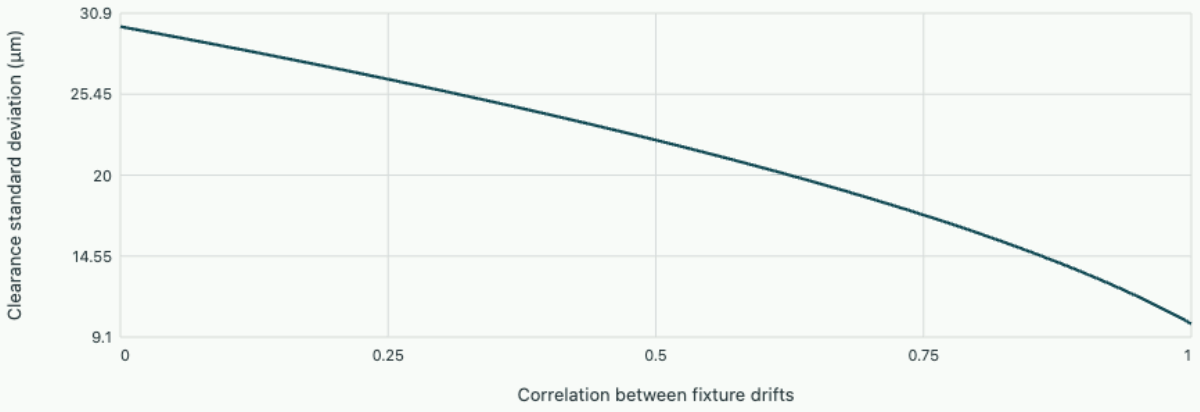
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Shared fixture spread (µm)	20
Hole local spread (µm)	6
Shaft local spread (µm)	8



Shared-process contribution

Matched fixture spread=10.000 µm; independently remixed spread=30.000 µm. Correlation can cancel common drift in a difference.

Begin with a physical closing dimension

A drawing is useful only if the manufactured pieces can meet in the intended order and provide the required clearance. Choose a positive direction along the assembly and write the closing dimension from a single datum. For a spacer fitted between two shoulders with a washer, suppose the available

nominal length is H, the spacer length is S and the washer thickness is W. The clearance is $g = H - S - W$. Housing expansion or coating thickness can be included as additional signed terms if they occur in the operating condition being assessed.

With specified independent admissible intervals, the worst-case minimum is $H_{\min} - S_{\max} - W_{\max}$. Independence is unnecessary for this conservative bound, although correlated dimensions may prevent its most extreme combination from occurring. A statistical treatment asks a different question: given a supported production distribution, what fraction is likely to achieve a required clearance? Manufacturing limits alone do not determine such a distribution.

Propagate measured spreads

Write each dimension as its mean plus a deviation. The mean clearance is the same signed sum of means. For independent dimensions the variance is the sum of their variances because squaring the coefficients removes their signs. More generally, for a dimension vector with covariance matrix Σ and coefficient vector a,

$$\sigma_g^2 = a^T \Sigma a.$$

A common machining datum can introduce correlations that either reduce or increase clearance variation.

Assume $H = 50.00 \pm 0.15,\ S = 48.00 \pm 0.12,\ W = 1.70 \pm 0.09\,\mathrm{mm}$. The nominal clearance is 0.30 mm and limiting clearance interval is $[0.30 - 0.36, 0.30 + 0.36] = [-0.06, 0.66]\,\mathrm{mm}$. Thus the stated tolerances do not guarantee a positive gap. Suppose separately that production measurements support independent, centred normal models with standard deviations 0.050, 0.040 and 0.030 mm, without further sorting. The resulting standard deviation is

$$\sigma_g = \sqrt{0.050^2 + 0.040^2 + 0.030^2} = 0.0707\,\mathrm{mm}.$$

Interference requires clearance below zero, a deviation of $0.30/0.0707 = 4.24$ standard deviations. The normal model estimates about 1.1×10^{-5} of pairs would interfere, roughly eleven per million. This small estimated fraction and the possibility in a worst-case limit calculation address different production questions. An unmeasured mean shift or sorting parts to their tolerance limits can substantially change the probability.

Combine component survival probabilities

Define reliability at an explicit mission duration as the probability that a component performs its required function through that duration. If a machine requires both of two independent components to survive, its reliability is the product. Two bearings each assessed at 0.98 mission survival give $0.98^2 = 0.9604$ for both. If two independent drives are genuinely redundant and either can carry the full demand, failure requires both to fail, giving $1 - (1 - 0.98)^2 = 0.9996$.

A shared lubricating supply or housing defect can defeat independence in either calculation. Similarly, a clearance that is satisfactory on the day of assembly need not remain satisfactory after thermal growth or wear. Carry the measured operating temperature range into the same dimension chain, and carry the relevant mission duration and shared causes into the reliability calculation. Both methods depend as strongly on the physical definition of success as on the arithmetic.

Estimate a life scale without discarding run-outs

An endurance test may end with some assemblies still functioning. Their exact failure times are unknown, yet they have demonstrated survival up to the stop time. Treating each as an observed failure at that time biases a life estimate downward, while discarding them throws away survival information. For illustration, assume independent, identically produced parts under a common load and a two-parameter Weibull survival law $R(t) = \exp[-(t/\eta)^{\beta}]$. Its scale η has units of time and is the lifetime at which survival is e^{-1}, while β sets the hazard shape. Assume β has been established independently for this exercise.

Differentiating survival gives the failure density $f(t) = (\beta/\eta)(t/\eta)^{\beta-1}R(t)$. The likelihood contribution of a failed part is $f(t_i)$; that of a right-censored surviving part is $R(t_j)$. If d failures occur in the test, collecting all terms depending on η gives

$$\ln L = \text{constant} - \beta d \ln \eta - \sum_{\text{all}} (t_i/\eta)^{\beta}.$$

Differentiating with respect to η gives the maximum-likelihood scale

$$\hat{\eta} = \left(\frac{\sum_{\text{all}} t_i^{\beta}}{d} \right)^{1/\beta}.$$

The numerator includes tested times of both failed and unfailed specimens, while the denominator counts only failures. With no failures this expression has no finite scale estimate; the data can support a lower bound under additional statistical assumptions. A Weibull model also presumes the tested population and failure mechanism remain relevant to the intended use.

Interpret a six-unit endurance test for a specified mission

Six identical drives are tested under the same continuous load. Three fail at 100, 150 and 200 h; the other three are stopped in working condition at 220 h. Use $\beta = 2$. The sum of exposure times squared is

$$100^2 + 150^2 + 200^2 + 3(220^2) = 217700\,\text{h}^2.$$

Dividing by three failures and taking the square root gives

$$\hat{\eta} = 269.4\,\mathrm{h}.$$

The fitted single-drive survival at 150 h is

$$\exp[-(150/269.4)^2] = 0.733.$$

The estimated time at 90% survival is

$$\hat{\eta}\sqrt{-\ln 0.90} = 87.4\,\mathrm{h}.$$

A proposed mission should therefore specify both duration and required probability.

If a machine needs two independently manufactured drives both functioning for the 150 h mission, fitted machine survival is $0.733^2 = 0.538$; if either one can take the entire load and failure of one does not affect its companion, it becomes

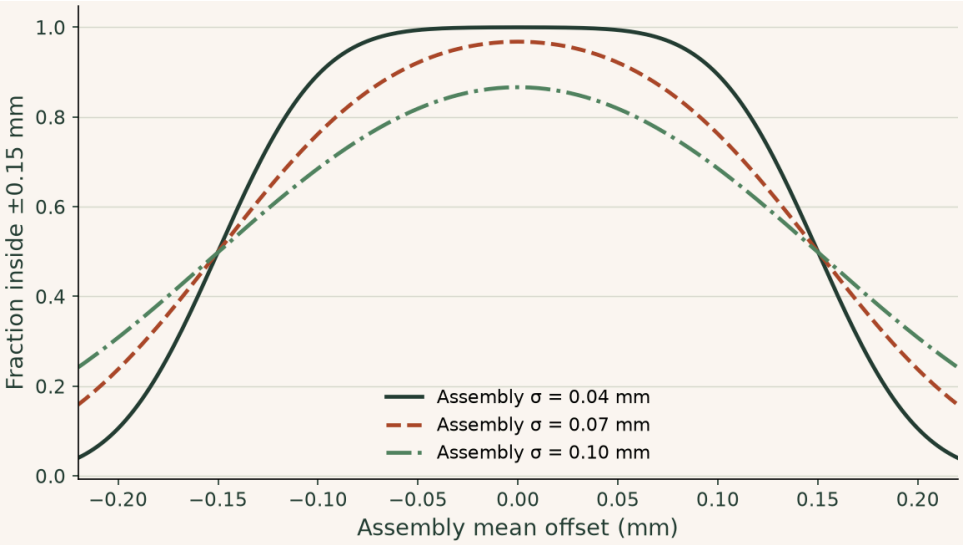
$$1 - (1 - 0.733)^2 = 0.929.$$

Neither system calculation includes uncertainty in the fitted scale or the assumed shape, which is substantial in a six-unit sample. If all drives share a contaminant or one common failure mode, independence can also fail. The three run-outs increase the scale relative to using only observed failures, but they do not certify service beyond their observed duration at an arbitrary reliability target.

The hazard inferred under the assumed shape is $h(t) = 2t/\eta^2$, so its value doubles from 100 to 200 h. Replacing the fitted distribution by a constant failure rate calibrated at one time would alter the conditional survival of an already-aged drive. This matters when a maintenance interval concerns parts that have already completed one mission.

Separate a shifted process mean from random assembly scatter

An assembled dimension must fall within 0.15 mm of nominal. Model its error as normal with mean b and standard deviation s. The yield is the difference of two normal cumulative probabilities at the tolerance limits.



For s = 0.07 mm a centred process has about 96.8% yield. A mean shift of 0.10 mm brings it to about 76.2%. Root-sum-square tolerancing describes the random spread under its independence assumptions; it cannot remove a systematic offset.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

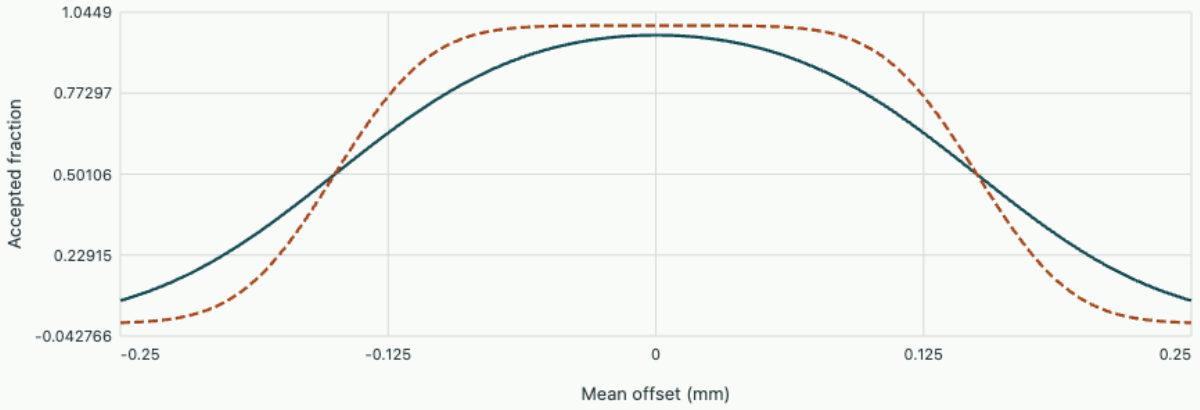
Narrow the random spread and compare that improvement with a mean shift. Both tails outside the tolerance count as rejects.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Assembly standard deviation (mm)	**0.07**
Symmetric tolerance (mm)	**0.15**



Centred acceptance 96.79%. Independent normal errors and a stable process mean are assumptions of this calculation.

Study the full derivation

Fatigue crack growth, optical inspection and the meaning of remaining life · Design and analysis of machine components

A polished test coupon contains a measurable crack and experiences repeated tensile cycles. A stress–life curve for a polished, initially uncracked specimen cannot directly give the remaining growth interval from this measured state. Use calibrated crack-growth data to predict a laboratory observation and quantify what the inference depends on.

Problem sheet 6 · Weld demand, assembly yield and reliability

Identify how eccentric loading, process variation and common causes change a machine assembly's limiting condition.

One circular weld

A circular line weld of radius 25 mm and effective throat 3 mm carries an in-plane force of 2.0 kN whose line is offset 60 mm from its centre. Determine direct shear flow, moment-induced shear flow and largest resulting throat stress. Locate a point at which the components add.

▼ **Reveal the worked solution**

Direct flow is

$$2000/(2\pi 25) = 12.73\,\mathrm{N/mm}.$$

With $J_L = 2\pi r^3$, moment flow is

$$M/(2\pi r^2) = 2000(60)/(2\pi 25^2) = 30.56\,\mathrm{N/mm}.$$

At the point where the moment-induced tangent has the same direction as the applied force, the maximum is $(12.73 + 30.56)/3 = 14.43\,\mathrm{MPa}$; at the diametrically opposite point the flow components oppose.

Link process drift to yield

A clearance has mean 0.18 mm and independent contributing standard deviations 0.030 and 0.040 mm. Assuming a normal distribution, find its standard deviation and estimated fraction below zero. One process drifts so the mean clearance falls to 0.12 mm. What is the new fraction?

▼ **Reveal the worked solution**

The combined standard deviation is

$$\sqrt{0.03^2 + 0.04^2} = 0.050\,\mathrm{mm}.$$

Initially zero lies 3.6 standard deviations below the mean, giving probability about 0.000159 or 0.0159%. After the shift it is only 2.4 standard deviations below the mean, giving about 0.00820 or 0.820%. An unchanged process spread can conceal a large change in assembly yield.

A shared supply

Two independent pumps each have mission survival probability 0.95 conditional on an operating shared power supply. Either pump alone handles the full demand. The supply survives the mission with probability 0.98. Find system survival with one and with two pumps. What upper bound holds even with many such pumps?

▼ **Reveal the worked solution**

With one pump the survival probability is $0.98(0.95) = 0.931$. With both, survival conditional on the supply is $1 - (1 - 0.95)^2 = 0.9975$, giving total $0.98(0.9975) = 0.97755$. Increasing the number of independent pumps can push their conditional survival towards unity, but the system probability remains bounded by the single supply’s 0.98.

Follow power through a belt and its supporting shaft

A pulley of pitch radius 0.10 m transmits 4.00 kW at 1200 rpm. Its belt has 180° wrap and coefficient of friction 0.30. Treat the two spans as parallel and the pulley as overhung 60 mm outside a bearing. Find the minimum pretension, approximate root bending moment, and von Mises stress in a solid shaft of diameter 25 mm. Neglect keyway concentration and shaft weight.

▼ **Worked solution**

The shaft speed is 125.66 rad/s. The torque is 31.83 N m, requiring $T_1 – T_2$ = 318.3 N. At impending slip $T_1/T_2 = \exp(0.30\pi)$ = 2.566, so $T_2 \approx$ 203.3 N and $T_1 \approx$ 521.6 N. The mean span tension is about 362 N. This is the minimum common pretension in the elementary equal-extension belt model.

The net radial pull in this parallel-span geometry is their sum, approximately 725 N. Its 0.060 m lever arm gives 43.5 N m bending moment. Taking the difference of tensions here would confuse the torque-producing tangential resultant about the pulley axis with the transverse force on the shaft.

At the surface the nominal bending stress $32M/(\pi d^3)$ is 28.4 MPa and the torsional shear $16T/(\pi d^3)$ is 10.4 MPa. The combined equivalent stress is $\sqrt{(\sigma^2+3\tau^2)} \approx$ 33.6 MPa. A shaft rotating under a stationary transverse load experiences alternating bending even when this static equivalent value is well below yield. A fatigue decision requires endurance, surface finish and notch information.

Explore the governing equations after solving the sheet: Eccentrically loaded weld groups; Tolerance stacks and reliability of assemblies. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Chapter 13 · Locating a part with six contacts and predicting datum error

A dimensionally accurate part can still be assembled in the wrong place. Use small rigid-body motions to work out what six contacts actually locate, then predict how dirt or a tilted guide appears in a measurement.

One frictionless contact supplies one scalar condition

Take the infinitesimal rigid motion of a body to consist of a translation $\mathbf{t}$ and a rotation vector $\boldsymbol{\varphi}$ in radians. A point initially at $\mathbf{r}_i$ moves by $\mathbf{t} + \boldsymbol{\varphi} \times \mathbf{r}_i$. A rigid locating pad at that point controls the component along its inward normal $\mathbf{n}_i$. If its normal height error is e_i, the displacement condition is

$$\mathbf{n}_i \cdot \mathbf{t} + (\mathbf{r}_i \times \mathbf{n}_i) \cdot \boldsymbol{\varphi} = e_i.$$

Stack the contact rows into a six-column matrix. Six independent rows locate a rigid part in three dimensions for small changes while the contacts remain seated. Three contacts in a plane support translation normal to the plane and two tilts. Two contacts on a second face locate one further translation and the remaining rotation. One on a third face sets the last translation. A preload must keep the unilateral contacts engaged. Six pads clustered along a single line would leave rotations undetermined despite the pad count.

Compute a tilted datum from three measured pad heights

Place the first three pads at coordinates $(0,0), (a,0), (0,b)$ on the nominal support plane. Normal displacement of a point (x,y) is $w = t_z + \varphi_x y - \varphi_y x$. Evaluate at the pads and subtract the origin row. The inverse relation gives

$$t_z = e_1,$$
$$\varphi_x = (e_3 - e_1)/b,$$
$$\varphi_y = -(e_2 - e_1)/a.$$

The predicted height of any feature is therefore an affine interpolation of the three heights. Outside their triangle it is an extrapolation and can magnify a small discrepancy. An additional hard pad away from the plane defined by the first three requires elastic deformation, a gap or loss of another contact. Its load sharing needs a stiffness model.

A particle under a pad and an encoder offset

Use $a = 120$ mm and $b = 80$ mm. A 12 µm particle raises only pad 2. Then $\varphi_y = -100$ µrad and $\varphi_x = t_z = 0$. A feature 180 mm from the origin in the x direction rises by 18 µm. Doubling the spacing a halves this sensitivity while leaving the particle height unchanged. The feature error varies with position and cannot be removed at every point by a single vertical zero adjustment.

Consider a horizontal translation stage whose encoder sits a vertical distance $d = 75$ mm from a tool point. A pitch change $\varphi_y = 80$ µrad produces a longitudinal difference $d\varphi_y = 6.0$ µm between the two lines of action. This is the first-order Abbe error derived from the same rigid-body rotation law. Encoder scale calibration alone cannot distinguish this error from translation at a single feature. Compare the extrapolated feature trace with the heights inside the support footprint.

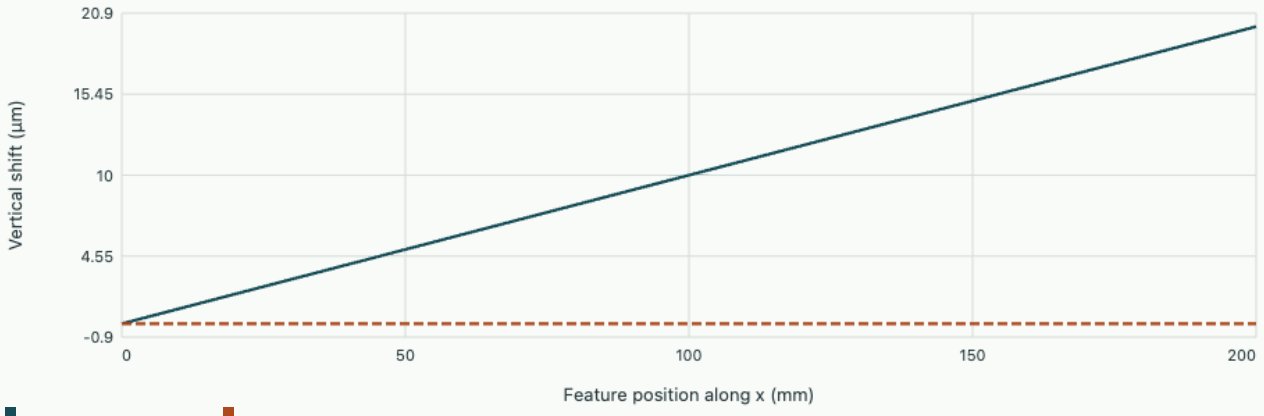
Fix one pad-height error and move pad 2 farther from the origin. Predict the shift halfway between the original pad positions versus the shift halfway between the new positions. Label the point whose physical location changes between those comparisons.

Open this experiment online

Try the calculation

Height error at pad 2 (µm) **12**

Pad spacing (mm) **120**



Pitch is -100.0 µrad. At 180 mm the feature shifts by 18.00 µm; at the second support it shifts by 12.0 µm.

Tutorial · reconstruct an entire datum plane

Pad heights at the three locations are 3, −3 and 11 µm. Find the two tilts and the vertical shift of a feature at (60,40) mm. What will a second probe at (120,80) mm read relative to nominal?

▼ **Show the plane reconstruction**

The differences are −6 µm over 120 mm and +8 µm over 80 mm, giving $\varphi_y = +50$ µrad and $\varphi_x = +100$ µrad. At the first feature the height is $3 - 3 + 4 = 4$ µm. At the second it is $3 - 6 + 8 = 5$ µm. The first point lies on the line joining pads 2 and 3, so its height is also their arithmetic mean. That geometric check detects a reversed rotation sign.

Further reading · MIT 2.72 · lecture 6 on constraint-based design

Study the full derivation

Cam displacement laws, contact geometry and follower acceleration · Design and analysis of machine components

An inspection station uses a roller to lift a specimen for imaging, hold it and return it. Specifying a ten-millimetre rise determines neither the force on the guide nor whether the follower can follow the prescribed geometry. Construct two different motions through the same endpoints and separate their geometric and dynamic consequences.

Chapter 14 · Sizing the leaves of a compliant translation guide

A pair of thin leaves can guide a carriage with very little sliding friction. Their stiffness, stress and parasitic motion stem from the same bent shape. Derive those links before choosing a thickness or stroke.

Enforce the root and platform boundary conditions

Consider two identical rectangular leaves connecting a fixed base to a rigid guided platform. Each leaf has length L, width b, thickness t and bending stiffness $EI = Ebt^3/12$. Let the platform translate sideways by a small amount δ without rotating. Ignore shear deformation, axial tension and clamp compliance in the linear bending solution. Along each unloaded span $EIw'''' = 0$, so the transverse shape is a cubic polynomial. At x=0 displacement and slope vanish. At x=L impose $w = \delta$ and $w' = 0$. With $s = x/L$ the resulting shape and curvature are

$$w = \delta(3s^2 - 2s^3),$$

$$w'' = \frac{6\delta}{L^2}(1 - 2s).$$

The curvature changes sign at midspan. Each leaf has opposite end moments and maximal bending strain at both ends. A root modeled as a free pin would remove an essential slope condition and give the wrong stiffness.

Find guide force, surface stress and secondary motion

The end shear magnitude is $EI|w'''| = 12EI|\delta|/L^3$ per leaf. Because both leaves impose the same displacement their forces add. At the outer fibre the strain magnitude is $(t/2)|w''|$. Hooke's law gives

$$F = \frac{24EI}{L^3}\delta,$$

$$|\sigma|_{\max} = \frac{3Et|\delta|}{L^2}.$$

For inextensible leaves free to shorten their projected span, the arclength exceeds the projection by approximately $\frac{1}{2}\int_0^L (w')^2 dx$. Substituting the cubic shape gives $3\delta^2/(5L)$. This parasitic travel is second order and has the same direction for positive or negative displacement. Constraining that axial motion instead introduces an axial force and nonlinearity. At large stroke use an appropriate geometric model for both stress and stiffness.

A stiffness target can conflict with a stress target

Choose $L = 40$ mm, $b = 10$ mm, $t = 0.30$ mm and $E = 200$ GPa. For each leaf $I = 2.25 \times 10^{-14}$ m⁴. The two-leaf stiffness is 1688 N/m. At 1 mm stroke the required force is 1.69 N, peak stress is 112.5

MPa and the free projected shortening is 15 µm. This small axial travel can matter when the stage is used for optical alignment.

Doubling the leaf thickness to 0.60 mm multiplies stiffness by eight, so 13.5 N is needed at the same stroke. Peak stress doubles to 225 MPa because surface fibres are farther from the neutral axis. In contrast, doubling width doubles stiffness without changing surface stress at a specified displacement. At a specified force the comparison changes because the stiffer guide travels less. Always identify which quantity is prescribed before drawing a design conclusion.

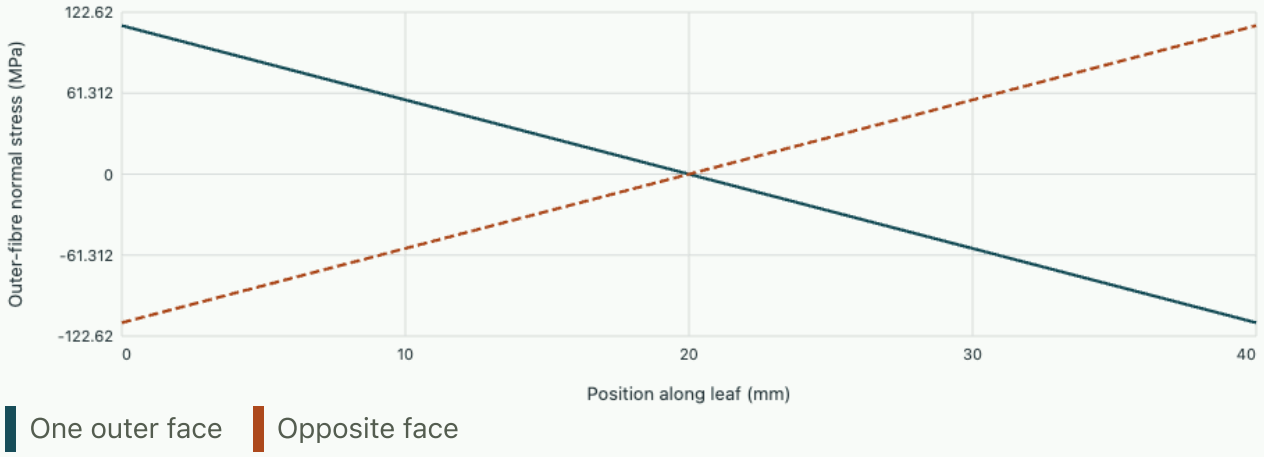
Hold the requested stroke fixed and double the leaf thickness. Check the ratios of force, peak stress and shortening against their different powers of thickness. Then reduce stroke until the original peak stress is recovered.

Open this experiment online

Try the calculation

Leaf thickness (mm) **0.3**

Stage stroke (mm) **1**



The guide force is 1.69 N, peak nominal stress is 112.5 MPa and projected shortening is 15.0 µm.

Tutorial · select stroke and compare an end moment measurement

Return to the original guide and restrict nominal peak bending stress to 90 MPa. Find the corresponding stroke, force and end-moment magnitude in one leaf. Estimate the projected shortening at that stroke.

▼ Show the displacement-limited design

Solving the stress expression gives $\delta = 0.80$ mm. Force is $1687.5(0.0008) = 1.35$ N. The end moment magnitude is $EI(6\delta/L^2) = 0.0135$ N m. The leaf end shear is 0.675 N; multiplying it by half the span gives the same end-moment magnitude. The predicted projected shortening is $3(0.0008)^2/[5(0.040)] = 9.6$ µm. Local clamp stress concentrations require the actual corner geometry.

Further reading · MIT 2.72 · lecture 8 on flexure bearings

Chapter 15 · Deriving screw-drive torque, efficiency and backdrivability

A screw trades angular travel for axial force. Unwrap one turn into an inclined plane and determine why two nominally similar threads can behave differently when the load tries to drive backward.

Convert one revolution into an inclined-plane problem

Let d be the mean thread diameter and l the lead, namely the axial advance per complete revolution. For a multiple-start thread the lead equals pitch times the number of starts. One turn around the mean diameter travels horizontally πd in the unwrapped view and rises l. The helix angle satisfies

$$\tan \lambda = l/(\pi d).$$

Use a square thread, steady motion, Coulomb friction with coefficient μ and negligible collar and bearing losses. The circumferential effort P corresponds to applied torque $T = Pd/2$.

Background · Lead angle

$$\tan \lambda = l/(2\pi r_m)$$

Lead is axial advance per revolution and can differ from the thread pitch. Friction and this angle together determine backdrivability.

Resolve the normal reaction before inserting friction

Let the axial load W act downward. For raising the load with a horizontal circumferential force P, the contact normal magnitude is $N = W\cos\lambda + P\sin\lambda$. Equilibrium along the helix gives

$$P\cos\lambda - W\sin\lambda = \mu N.$$

Separating P terms and dividing by $\cos\lambda$ yields

$$T_{\uparrow} = \frac{Wd}{2}\frac{\tan\lambda + \mu}{1 - \mu\tan\lambda} = \frac{Wd}{2}\tan(\lambda + \phi),$$

$$\phi = \arctan\mu.$$

For steady lowering the friction reverses. Define positive lowering torque as external effort in the direction of descent. The same balance then gives

$$T_{\downarrow} = (Wd/2)\tan(\phi - \lambda).$$

Positive $T_{\downarrow}$ means the load needs external assistance to lower. Negative $T_{\downarrow}$ means a retaining torque is required to prevent acceleration under the idealized sliding law. The transition is $\tan\lambda = \mu$. Static and

sliding friction may differ in a physical test.

Compare useful lifting work with input work over one turn

For one turn Wl is useful work and $2\pi T_{\uparrow}$ is supplied work. Therefore $\eta = \tan\lambda / \tan(\lambda + \phi)$. At small lead and fixed friction, torque can be modest but many turns are needed. This distinction matters when specifying useful output speed. An additional constant collar torque simply adds to the torque budget and reduces measured efficiency.

For $W = 800$ N, $d = 16$ mm, $l = 4$ mm and $\mu = 0.12$, $\tan\lambda = 0.0796$. Raising torque is 1.29 N m and efficiency is 39.5%. Lowering requires 0.256 N m of external effort. At $l = 10$ mm with the other values held fixed, raising torque becomes 2.09 N m and efficiency reaches 60.9%. Lowering torque is now −0.494 N m. The faster lead crossed the backdriving threshold near $l = \pi d\mu = 6.03$ mm. The model assumes a square flank; a V-thread adds normal loading through its flank angle.

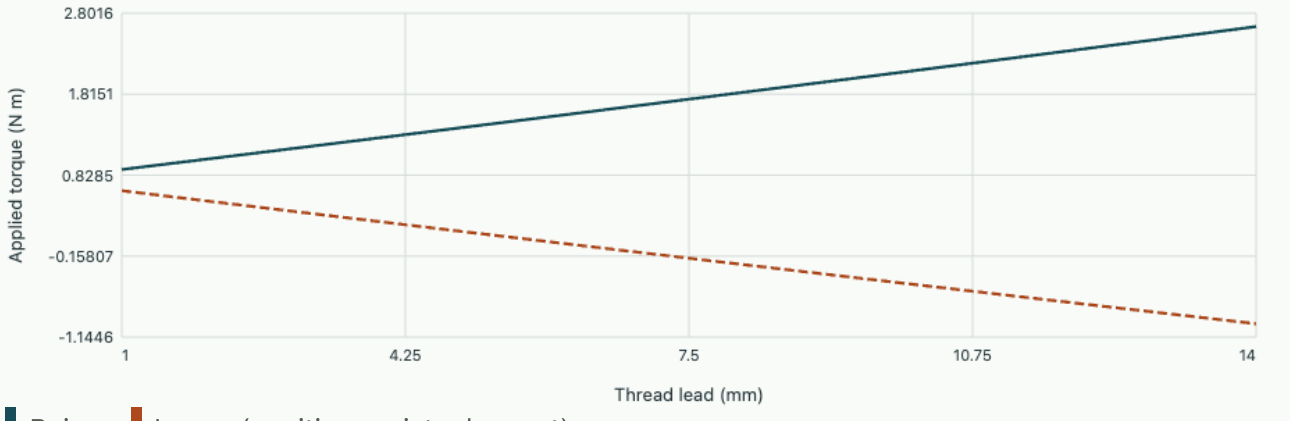
Locate the lead where lowering torque crosses zero. Double the axial load while keeping friction fixed and predict which feature of the plot should remain stationary. Then raise the friction coefficient and repeat the sign test.

Open this experiment online

Try the calculation

Sliding friction coefficient **0.12**

Axial load (N) **800**



Raise Lower (positive assists descent)

At 4 mm lead, raising takes 1.290 N m and lowering takes 0.256 N m. The ideal change of lowering-torque sign is at 6.03 mm lead.

Tutorial · identify friction from a torque measurement

The 4 mm lead screw lifts 800 N at a measured 1.50 N m thread torque after independently subtracting bearing torque. Estimate sliding friction. If the same mechanism has two thread starts of pitch 4 mm,

would the inferred friction predict self-locking?

▼ **Show the inverse force balance**

Write $y = 2T/(Wd) = 0.2344$ and $t = \tan\lambda = 0.07958$. From $y = (t+\mu)/(1-\mu t)$, obtain

$$\mu = (y - t)/(1 + yt) = 0.1520.$$

Two starts give 8 mm lead and $t = 0.1592$, just above this value. Under the fitted sliding model the torque for lowering changes sign. Temperature, lubrication and static friction affect such a near-threshold result and can be distinguished by separate raising and lowering measurements.

Further reading · MIT 2.72 · lecture 11 on screw drives

Chapter 16 · Matching a gear ratio to load inertia and move time

A larger reduction increases output torque and reduces speed. During acceleration the motor must also spin its own rotor faster. Refer the dynamics to a single shaft and find which reduction gives the shortest idealized move.

Reflect inertia using kinetic energy

Let $N = \omega_m/\omega_L$ be the reduction, with motor inertia J_m, load inertia J_L, and negligible gear inertia and losses. Kinetic energy is the sum of $\frac{1}{2}J_m\omega_m^2$ and $\frac{1}{2}J_L\omega_L^2$. Substituting $\omega_L = \omega_m/N$ gives the load inertia referred to the motor as J_L/N^2. Alternatively, express the energy in load coordinates and obtain $J_L + N^2J_m$. Both descriptions predict the same acceleration if the torque is referred to the same coordinate.

$$(J_L + N^2J_m)\alpha_L = NT_m - T_L.$$

Here T_L is a positive resisting load torque. The factor multiplying motor torque follows independently from equality of power $T_m\omega_m = (NT_m)\omega_L$ for an ideal gear. Combining reflected inertia from one side with torque from the other introduces an erroneous factor N.

Background · Gear ratio

$$i = \omega_{\text{in}}/\omega_{\text{out}}$$

Specify the chosen convention and rotation signs. Reflecting an inertia to the input of an ideal reduction divides it by the squared speed ratio.

Find the ratio for maximum acceleration at rest

First let $T_L = 0$ and let the motor deliver constant torque T_m. Differentiating the load acceleration gives

$$\frac{d\alpha_L}{dN} = T_m\,\frac{J_L - N^2J_m}{(J_L + N^2J_m)^2},$$

$$N_* = \sqrt{J_L/J_m},$$

$$\alpha_* = \frac{T_m}{2\sqrt{J_LJ_m}}.$$

At the optimum the reflected load inertia equals the motor inertia. Provided the motor can reverse torque instantly, an ideal rest-to-rest move with symmetric acceleration and braking switches at half

distance. For load travel $\Delta\theta$, $t_{\mathrm{move}} = 2\sqrt{\Delta\theta/\alpha_L}$ and the peak load speed is $\sqrt{\alpha_L \Delta\theta}$. A motor speed limit, speed-dependent torque or external load can shift the useful choice away from the inertia match.

Test a candidate against a motor speed limit

Take $J_m = 0.0002$ kg m², $J_L = 0.020$ kg m² and $T_m = 0.40$ N m. The inertia match is $N = 10$. Load acceleration is 100 rad/s². A 1 rad move takes 0.20 s in the ideal triangular speed profile and reaches 10 rad/s at the load, hence 100 rad/s at the motor. At $N = 5$ acceleration is 80 rad/s²; at $N = 20$ it is also 80 rad/s². The time penalty is the same for these two reciprocal changes around the optimum, but the motor peak speeds differ by a factor of four.

With $N = 10$ and a hard motor speed limit of 60 rad/s, the load can reach only 6 rad/s. Acceleration and deceleration each require 0.06 s and cover 0.18 rad. Traversing the remaining 0.64 rad at constant speed adds 0.107 s. The total is 0.227 s. An actuator selected purely on peak torque would miss this change in move profile.

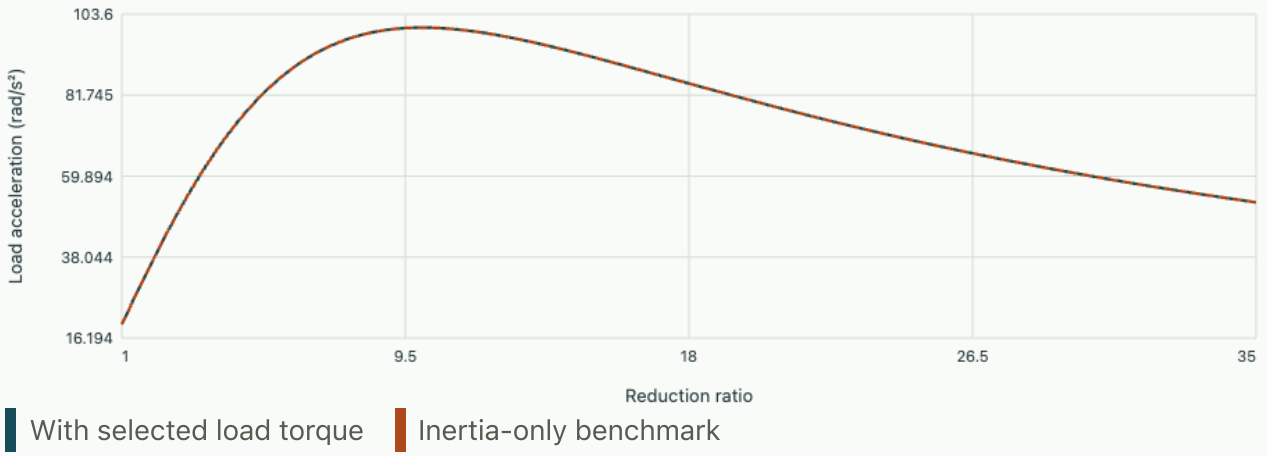
Locate the inertia-only acceleration maximum and compare ratios equally far below and above it on a multiplicative scale. Add a resisting torque and decide which direction a sensible ratio change must take to recover acceleration.

Open this experiment online

Try the calculation

Resisting torque (N m) **0**

Load inertia (kg m²) **0.02**



For constant motor torque the maximum acceleration is 100.0 rad/s² at a ratio of 10.00.

Tutorial · account for a constant resisting torque

Let $T_L = 1.0$ N m during a positive accelerating stroke with the same motor and inertias. Compare acceleration at N=10 and 12, and derive the new maximizing ratio assuming constant torque at the

motor.

▼ **Show the shifted maximum**

At N=10 the net load torque is $4 - 1 = 3$ N m and acceleration is 75 rad/s². At N=12 it is $3.8/0.0488 = 77.9$ rad/s². Differentiating the loaded acceleration and cancelling constants gives

$$N^2 - 2(T_L/T_m)N - J_L/J_m = 0.$$

The positive root is

$$2.5 + \sqrt{6.25 + 100} = 12.81.$$

During braking the external load can assist deceleration, so the symmetric switching rule used in the unforced example needs to be rederived for that operating condition.

Further reading · MIT 2.72 · lecture 14 on actuators and motion-system design

Chapter 17 · Force transmission and compliance near a toggle position

Two links approach alignment as a mechanical fixture closes. A small applied force can correspond to a large output reaction, but the ratio also makes the result sensitive to clearance and compliance. Derive it from displacement compatibility.

Choose a coordinate that remains measurable

Join two equal links of length L at an upper knee and place their outer ends on a common horizontal line. Pin the left end to the frame and allow the right end to slide horizontally. The two links form a symmetric tent with common angle θ above the base. The knee coordinates are $(L\cos\theta, L\sin\theta)$, and the slider position is $x = 2L\cos\theta$. A vertical driver presses on the knee while allowing its small horizontal motion. These kinematic conditions matter because a guide fixing the knee's horizontal coordinate would impose a different linkage.

$$y = L\sin\theta,$$

$$x = 2L\cos\theta,$$

$$\frac{dx}{dy} = \frac{-2L\sin\theta}{L\cos\theta} = -2\tan\theta.$$

A descending knee extends the horizontal span. Near alignment the horizontal motion becomes small relative to the vertical stroke. An expansion in small y gives

$$x = 2\sqrt{L^2 - y^2} \simeq 2L - y^2/L.$$

The position deficit is quadratic in knee height.

Derive force amplification and check equilibrium

Assume rigid links, ideal pins and slow movement. Let a downward driver force have magnitude P and let a horizontal resisting load at the slider have magnitude Q. For any compatible infinitesimal motion, zero net work gives

$$P|dy| = Q|dx|.$$

Hence

$$\frac{Q}{P} = \frac{1}{2\tan\theta}.$$

An independent force diagram at the knee shows that each compressed link carries $C = P/(2\sin\theta)$. Its horizontal component at the slider is $C\cos\theta$, giving the same ratio. As alignment approaches, the linearized kinematic ratio diverges. Finite stiffness, clearances, buckling of slender links and elastic

deformation at the joints then determine the actual force for a prescribed driver travel. A geometric ratio alone cannot predict the force at exact alignment.

Distinguish a force-controlled drive from a compliant load

With $L = 100$ mm and an applied $P = 60$ N, an angle of 15° gives $Q = 112$ N and link compression 116 N. At 5° the same input would balance $Q = 343$ N and link compression 344 N. Meanwhile the remaining horizontal travel to full extension shrinks from $2L(1 - \cos 15^\circ) = 6.81$ mm to 0.761 mm. The final part of the stroke therefore has high force ratio and little remaining travel.

Now replace the prescribed resisting force with a horizontal spring of stiffness 100 N/mm, first contacted at an angle of 10°. At 5° the additional extension is

$$2L(\cos 5^\circ - \cos 10^\circ) = 2.28\,\text{mm}.$$

The spring force is 228 N and the required input is only $2Q \tan 5^\circ = 39.9$ N. The ideal force-controlled estimate of 343 N answers a different question because the actual load follows the spring and the platform displacement. Compare the full angle range before deciding where maximum driver effort occurs.

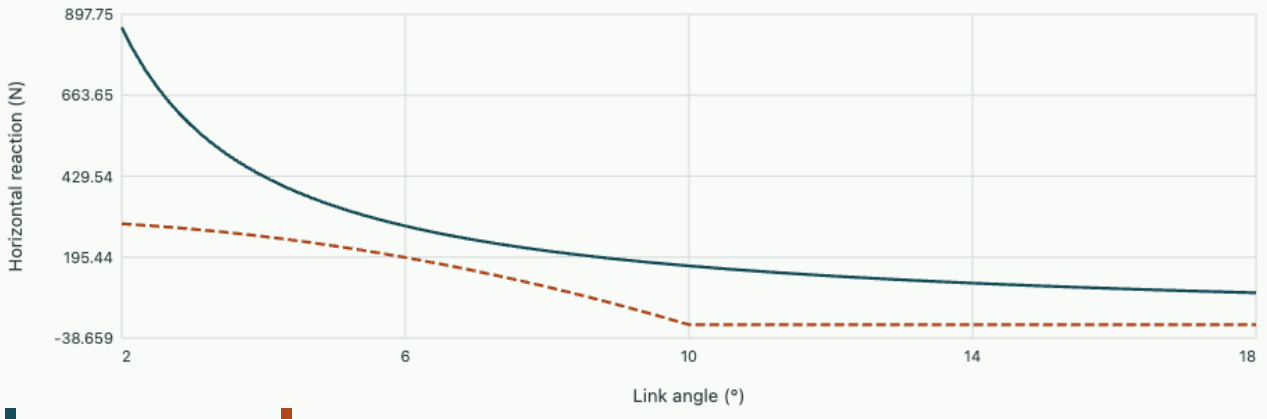
Locate the angle of first spring contact. At 5 degrees read the two different horizontal reactions. Change only the input force and explain why the displacement-controlled spring result stays put even though the force-controlled result changes.

Open this experiment online

Try the calculation

Input force (N) **60**

Spring stiffness (N/mm) **100**



At 5° a specified 60 N input would balance 342.9 N. The spring first touching at 10° instead carries 227.7 N and requires 39.8 N input.

Tutorial · recover an angle from two force readings

For a slow frictionless stroke a driver transducer reads 50 N and the horizontal reaction is 250 N. Estimate θ, link compression and residual travel for 100 mm links. What measurement could reveal pin friction independently?

▼ **Show the equilibrium and geometry**

The observed ratio gives

$$\tan\theta = 50/(2\cdot 250) = 0.10.$$

Thus $\theta = 5.71^\circ$, $C = 250/\cos\theta = 251.2$ N and residual travel is $200(1-\cos\theta) = 0.993$ mm. Compare the force needed during slow closing and opening at the same geometry and spring deflection. Ideally the elastic load gives the same relation both ways. Dry friction adds a direction-dependent component that changes that pair of readings.

Further reading · MIT 2.72 · session 17 on linkage degrees of freedom and kinematics

Study the full derivation

Cam displacement laws, contact geometry and follower acceleration · Design and analysis of machine components

An inspection station uses a roller to lift a specimen for imaging, hold it and return it. Specifying a ten-millimetre rise determines neither the force on the guide nor whether the follower can follow the prescribed geometry. Construct two different motions through the same endpoints and separate their geometric and dynamic consequences.

Chapter 18 · How an involute gear pair keeps contact from one tooth to the next

The tooth-count ratio gives speed reduction but says little about how many pairs carry load simultaneously. Derive the transverse contact ratio from the base circles and addendum circles, then check how centre-distance errors affect overlap.

Unwrap base pitch from a taut string

An involute is the curve traced by the end of a taut string as it unwinds from a base circle. At each point the string is normal to the involute and tangent to the base circle. The common normal of conjugate external involute teeth is therefore the straight line tangent to both base circles, the line of action. Successive teeth reach equivalent points on this line one base pitch apart. For module m, tooth count z and reference pressure angle α, the reference pitch radius is $r = mz/2$ and the base radius is $r_b = r\cos\alpha$. One tooth pitch on the base circle is

$$p_b = \frac{2\pi r_b}{z} = \pi m \cos\alpha.$$

The operating centre distance a can differ from the sum of the reference pitch radii. The new operating pressure angle satisfies

$$\cos\alpha_w = (r_{b1} + r_{b2})/a.$$

The two base circles and base pitch are unchanged because tooth geometry was cut before assembly. This explains how involute gears accommodate a moderate centre-distance shift while retaining their velocity ratio.

Background · Contact ratio

$$\epsilon_\alpha = L_{\text{contact}}/p_b$$

For a transverse involute pair compare the active path of contact with the base pitch. This determines how many pairs of teeth can share the load.

Compute the interval on the line of action

At first and last contact one of the addendum circles intersects the common normal. In the right triangle from its centre to a tangent-foot point on the base circle and the addendum intersection, the distance along the tangent is $\sqrt{r_a^2 - r_b^2}$. The tangent feet of both base circles are separated along the normal by $a\sin\alpha_w$. Combining the two triangle lengths gives

$$\ell = \sqrt{r_{a1}^2 - r_{b1}^2} + \sqrt{r_{a2}^2 - r_{b2}^2} - a \sin \alpha_w,$$

$$\epsilon_\alpha = \frac{\ell}{p_b}.$$

For $1 < \epsilon_\alpha < 2$, the fraction of a mesh cycle in nominal two-pair contact is $\epsilon_\alpha - 1$, and the remainder has one pair. This is a geometric sharing opportunity. Actual pair loads depend on tooth compliance, profile modifications and errors. The formula assumes the usable involute extends across the predicted path without tip interference or undercut.

Count the overlap and change centre distance

Take a spur pair with 20 and 40 teeth, module 2 mm, reference pressure angle 20°, and addendum equal to one module on both gears. The reference radii are 20 and 40 mm and addendum radii are 22 and 42 mm. At the nominal 60 mm spacing the base pitch is 5.904 mm and path length is 9.655 mm. The contact ratio is 1.635, so about 63.5% of each cycle can have two tooth pairs in contact. A 600 rpm pinion produces $20(600/60) = 200$ tooth engagements per second and a mesh period of 5.0 ms.

Increase centre distance by 0.50 mm without recutting either gear. The operating angle rises to 21.26° and the new ratio becomes 1.395. The two-pair interval falls to about 1.97 ms per mesh period from 3.18 ms. Gear ratio stays 2:1. By contrast, at nominal centres shortening both addenda to $0.8m$ changes the tangent intersection lengths and gives ratio 1.347. Measure centre distance and tooth height independently before attributing an apparent overlap reduction to a single manufacturing error.

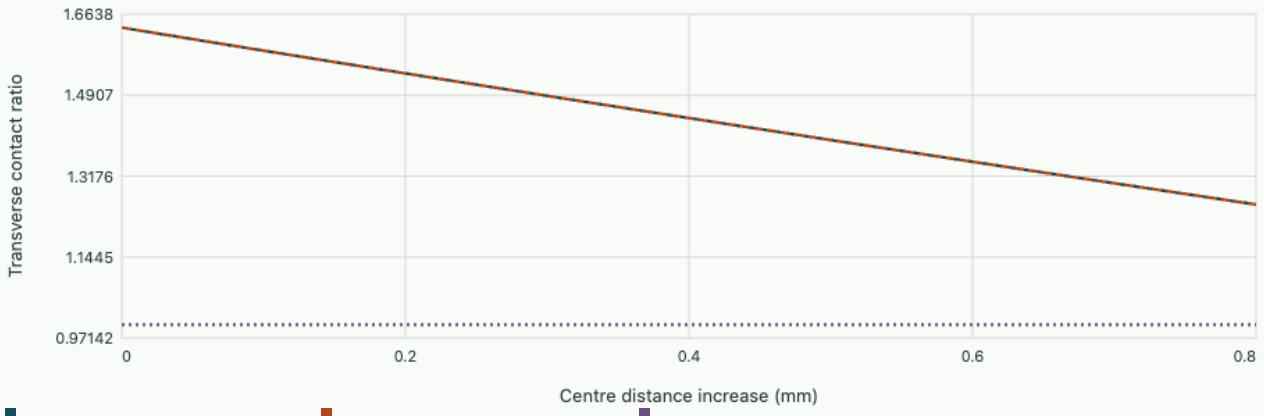
Increase the centre distance until the ratio is about 1.50. Mark that point before decreasing tooth addendum. Read the fraction of time with a single contacting pair and explain why tooth-count speed ratio stays fixed.

Open this experiment online

Try the calculation

Centre distance increase (mm) 0

Addendum coefficient 1



Selected addendum Standard addendum Continuity threshold

At the selected centre distance the operating pressure angle is 20.00° and contact ratio is 1.635. The ideal two-pair fraction is 63.5%.

A related OneSimulation calculation

The sphere-on-block contact study isolates elastic force sharing at an imposed approach. Use it to distinguish a geometrical count of engaged tooth pairs from the force that each elastic contact carries.

Explore the OneSimulation elastic contact model

Tutorial · estimate the centre-distance tolerance for overlap

For the original standard addenda, find the largest centre distance compatible with contact ratio at least 1.50, leaving interference checks and manufacturing allowances outside this geometric estimate. Compare with a first-order sensitivity estimate at nominal centres.

▼ **Isolate the operating tangent length**

Let $S = \sqrt{r_{a1}^2 - r_{b1}^2} + \sqrt{r_{a2}^2 - r_{b2}^2}$. Since $a \sin \alpha_w = \sqrt{a^2 - (r_{b1} + r_{b2})^2}$, the limiting distance is $\sqrt{(S - 1.50 p_b)^2 + (r_{b1} + r_{b2})^2}$, about 60.28 mm. Differentiating gives $d\epsilon_\alpha / da = -1/(p_b \sin \alpha_w)$, about −0.495 per mm at nominal spacing. A 0.135 loss predicts about 0.27 mm additional spacing, close because the tested change is small.

Further reading · KHK Gears · geometry of the involute path of contact

Study the full derivation

Helical gears, mesh forces and bearing reactions · Design and analysis of machine components

A motor supplies a known torque to a helical pinion. Select the free body carefully: torque fixes one component of the tooth load, the helix creates thrust, and the location of that thrust can alter bearing reactions even in a parallel-shaft transmission. Build the force diagram and reverse it from measured support loads.

Chapter 19 · Why increasing the length of a bonded overlap gives diminishing returns

Two slender straps transfer a tensile load through adhesive shear. Average stress decreases in inverse proportion to overlap length, but edge stress can approach a finite plateau. Derive the difference by coupling the axial extension of each strap to its neighbour.

Convert adhesive distortion into a distributed force

Consider identical adherends of thickness t, width b and Young's modulus E. A thin adhesive of thickness t_a and shear modulus G connects them over a length L. A tensile force P enters through strap 1 at the left and leaves through strap 2 on the right. Let n_1, n_2 be axial force per unit width, with $n_1 + n_2 = N = P/b$. Exclude bending and peel to isolate the symmetric extensional load-transfer mechanism. This idealization is more credible for an aligned or symmetrically balanced joint than for an eccentric single-lap joint.

Choose coordinates so the transmitted shear $\tau = -n_1' = n_2'$ is positive. Axial strains are $u_i' = n_i/(Et)$. Taking adhesive shear in the load-transfer direction as $\tau = (G/t_a)(u_2 - u_1)$ gives

$$\tau' = G(n_2 - n_1)/(t_a E t).$$

One more derivative eliminates the member forces:

$$\tau'' - \lambda^2 \tau = 0,$$

$$\lambda = \sqrt{\frac{2G}{t_a E t}},$$

$$\int_{-L/2}^{L/2} \tau \, dx = N.$$

The inverse transfer length $1/\lambda$ is a mechanical length scale. A stiff adhesive transfers load over a shorter distance. A stiffer or thicker adherend distributes the required relative extension over a larger region. Reversing coordinate labels reverses force derivatives consistently without changing the predicted stress magnitude.

Apply symmetry and recover the edge-to-mean ratio

Identical adherends loaded at opposite ends give a shear profile symmetric about the middle. Set $\tau = C\cosh(\lambda x)$ and choose C so its integral equals N. At the centre and edge this yields

$$\tau(x) = \frac{N\lambda \cosh(\lambda x)}{2\sinh(\lambda L/2)},$$

$$\frac{\tau_{\text{edge}}}{N/L} = \frac{\lambda L}{2}\coth\frac{\lambda L}{2}.$$

For a very short overlap the ratio approaches one because $\sinh y \sim y$. For a long overlap the edge stress approaches $N\lambda/2$ even while N/L continues to decrease. The central portion then carries little shear. The model computes a smooth nominal adhesive stress; it does not resolve singular edge detail, peel or a local failure criterion.

Compare an average-based estimate with the load-transfer profile

Take $P = 1000$ N, $b = 20$ mm, $E = 70000$ MPa, $t = 2$ mm, $t_a = 0.20$ mm and $G = 300$ MPa. The force per width is 50 N/mm and $\lambda = 0.1464$ mm^{-1}, giving a transfer length of 6.83 mm. With $L = 20$ mm the mean stress is 2.50 MPa, but the edge sees 4.07 MPa and the centre only 1.79 MPa.

At $L = 40$ mm mean stress halves to 1.25 MPa. The edge stress falls only to 3.68 MPa and the centre drops to 0.393 MPa. The long-overlap limiting edge stress is 3.66 MPa. Thus using mean stress alone would suggest substantial unused room at the ends. Doubling the width at fixed P, in contrast, halves N and every stress in this one-dimensional model. Manufacturing constraints and adhesion quality can still change which failure mechanism occurs first.

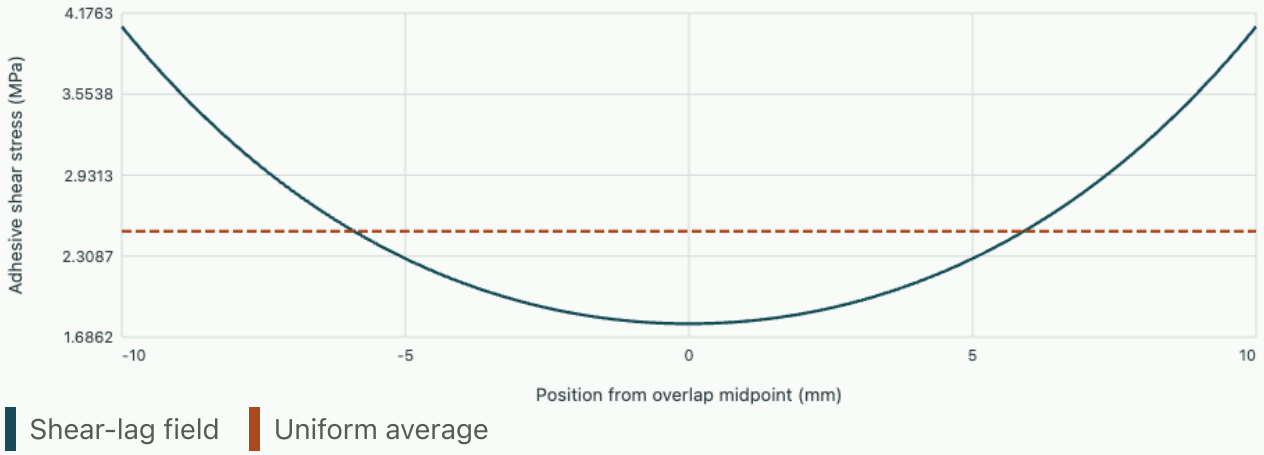
Double overlap length while keeping adhesive thickness fixed. Compare the percentage fall in centre and edge stress. Then increase adhesive thickness and predict where the newly transferred load is distributed along the overlap.

Open this experiment online

Try the calculation

Overlap length (mm) **20**

Adhesive thickness (mm) **0.2**



Transfer length is 6.83 mm. The edge stress is 4.074 MPa compared with 1.789 MPa at the centre and 2.500 MPa averaged over the overlap.

Tutorial · tune load-transfer length using adhesive thickness

At the original E, t, G and force, a preliminary model calls for a long-overlap edge stress no greater than 3.0 MPa. Find the minimum adhesive thickness implied by that asymptote. Can a finite overlap at that exact thickness satisfy the same bound?

▼ **Invert the long-overlap limit**

The limit requires $\lambda \le 2(3.0)/50 = 0.120$ mm⁻¹. Thus

$$t_a \ge 2G/(Et\lambda^2) = 0.298\,\mathrm{mm}.$$

For a finite length, $\coth(\lambda L/2) > 1$, so the edge stress at exactly that thickness stays slightly above 3.0 MPa. One must allow additional thickness, adjust the width, or evaluate a different load path. An adhesive change also alters deformation and peel, so this result is one design condition rather than a full joint qualification.

Further reading · Eindhoven University of Technology · adhesive single-lap joints and analytical shear-lag models

Chapter 20 · How a polygonal sprocket imprints speed ripple on a roller chain

A small conveyor presents glass samples to an inspection camera. The driving shaft turns at accurately regulated speed and each revolution advances exactly twelve chain pitches. A camera triggered halfway between consecutive roller engagements nevertheless sees a varying sample velocity. Predict that variation from one chord of the sprocket. Then use a displacement measurement to decide what a pitch counter or a shaft encoder could have missed.

Distinguish the pitch circle from the polygon that transports the chain

A roller chain has pin centres separated by a fixed pitch p. When two consecutive rollers are seated on a sprocket with z uniformly spaced teeth, the pin centres lie on the pitch circle of radius R. The straight link joining them is a chord. Bisect the isosceles triangle formed by these two centres and the shaft centre. Its half-angle is $a = \pi/z$, so $p = 2R\sin a$. The pitch radius follows immediately from the chord length. Using $2\pi R/z$ in place of the pitch treats the chord as an arc and introduces the very discrepancy we want to understand.

The calculation describes an ideal taut straight strand with successive rollers making a geometrically sharp handover. Neglect the mass and extension of the links, tooth clearances, roller contact details and tensioner motion. Assume a sufficiently long strand that its direction can be taken as fixed. This is a kinematic reference calculation; the vibration of an elastic chain around a finite sprocket pair would additionally require equations of motion. A useful reference should predict the repeating component even when the actual shaft has no speed ripple of its own.

Set $\theta = 0$ at the middle of one engagement, when the radius to the active roller is perpendicular to the strand. During engagement θ runs from $-a$ to $+a$ at shaft speed ω. The strand advances by the projection of the active roller's tangential motion on the strand direction. Measured from the start of the engagement, the longitudinal displacement and velocity are

$$x(\theta) = R(\sin\theta + \sin a),$$

$$u(\theta) = \omega R\cos\theta.$$

At $\theta = -a$ the displacement vanishes; at $+a$ it is precisely p. The velocity takes the same value at both ends, making the periodically repeated speed continuous. The derivative of the speed changes sign at the handover. Contact compliance and roller inertia soften this feature in a measured trace. A simulation with a rigid kinematic input and an infinitely stiff tension element can turn this geometric corner into unphysical force spikes. The predicted displacement and speed are much better observables for a first comparison.

Integrate a complete engagement before choosing a reference speed

The duration of one engagement is $2a/\omega$. Dividing one pitch by this interval gives the exact mean strand speed $\bar{u} = pz\omega/(2\pi)$. Alternatively integrate $\omega R\cos\theta$ with respect to time through the engagement and divide by the elapsed time. The two approaches agree because $p = 2R\sin a$. A third candidate, ωR, is the maximum instantaneous speed, attained at $\theta = 0$. Replacing the mean by that maximum leads to a small cumulative travel error that grows every revolution, even though the true geometric ripple resets after every pitch.

The minimum velocity occurs just as the next roller starts or ends its engagement. It is $\omega R\cos a$. Define the peak-to-peak excursion relative to the mean as $\delta = (u_{\max} - u_{\min})/\bar{u}$. Substitution and a half-angle identity give

$$\frac{u_{\max}}{\bar{u}} = \frac{a}{\sin a},$$
$$\frac{u_{\min}}{\bar{u}} = \frac{a\cos a}{\sin a},$$
$$\delta = a\tan(a/2).$$

As the tooth count becomes large, a becomes small and $\delta \simeq \pi^2/(2z^2)$. The corrective term is of order z^{-4}. Relative speed variation therefore drops by approximately a factor of four when the tooth count doubles. Holding the shaft speed fixed at the same time would double the mean strand speed. To compare motions for the same production throughput, halve the shaft speed on the larger sprocket. The engagement frequency then stays fixed, which allows a direct comparison of the phase of the observed ripple.

Open editable notebook Download .ipynb

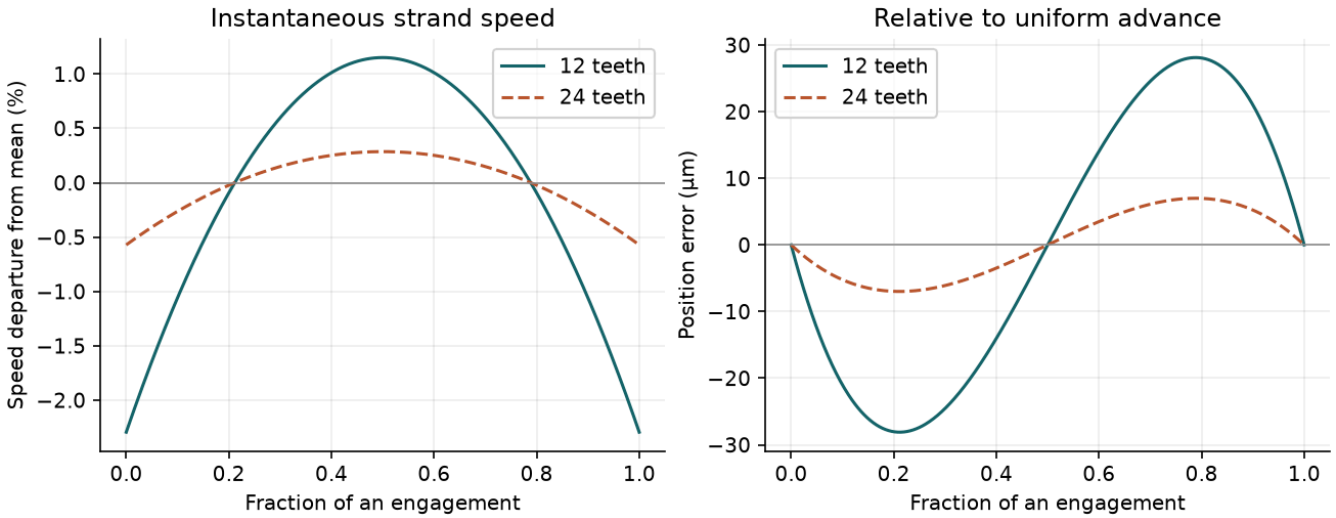
Where a time trigger sees the largest position error

Let $\phi = (\theta + a)/(2a)$ be the fraction of one engagement interval completed. The uniform-speed approximation places the sample at $p\phi$. The true position error is therefore

$$e_x = R[\sin((2\phi - 1)a) + \sin a] - p\phi.$$

It vanishes at $\phi = 0$, $1/2$ and 1. It changes sign under the transformation $\phi \mapsto 1 - \phi$. A camera triggered at just these three phases might report perfect positional accuracy even though another acquisition phase sees a repeatable shift.

Differentiate with respect to time to find where e_x has its extrema. The derivative is $u - \bar{u}$. The extremum occurs when $\cos\theta = \sin a/a$, where the instantaneous speed equals its mean. The velocity peaks at mid-engagement, while the position error there is zero because the early velocity deficit has just been recovered. This distinction is useful when comparing a speed trace obtained by differentiation with exposures of a marked link: their salient features occur at different instants.



Increasing the tooth count deforms the speed curve toward a horizontal line. The position ripple resets whenever a complete pitch has passed, so counting rollers alone misses it.

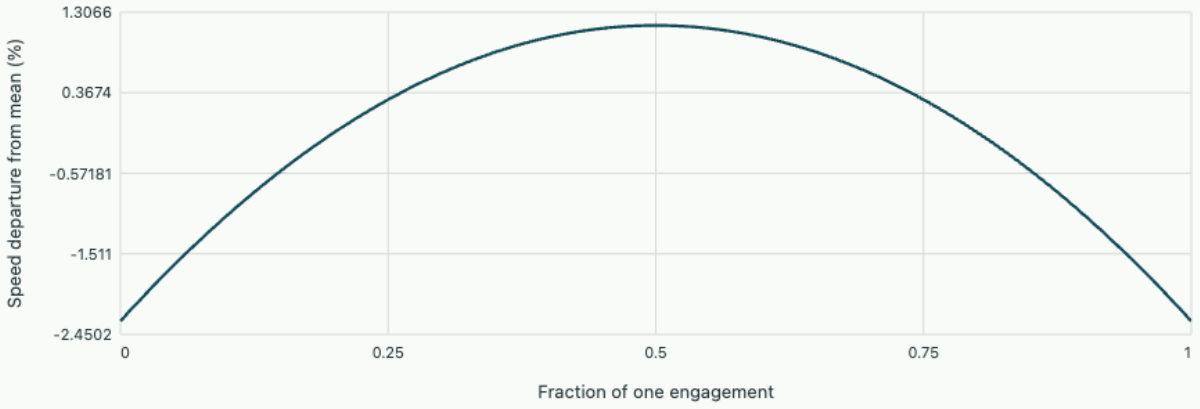
Interactive calculation · Chain speed

Change the physical and numerical inputs of the notebook model. Compare the updated curves and balance checks with the recorded notebook figures. Each plot uses the same equations and assumptions documented in that notebook; its readout names any analytical or reduced-model reference.

Open this live calculation

Try the calculation

Sprocket tooth count	**12**
Roller pitch (mm)	**12.7**
Drive speed (rpm)	**120**



Chordal velocity ripple

Mean chain speed 0.30480 m/s; tooth-pass frequency 24.000 Hz. Peak-to-peak speed ripple 3.4467%. The engagement advances exactly one pitch.

Download this example’s editable notebook

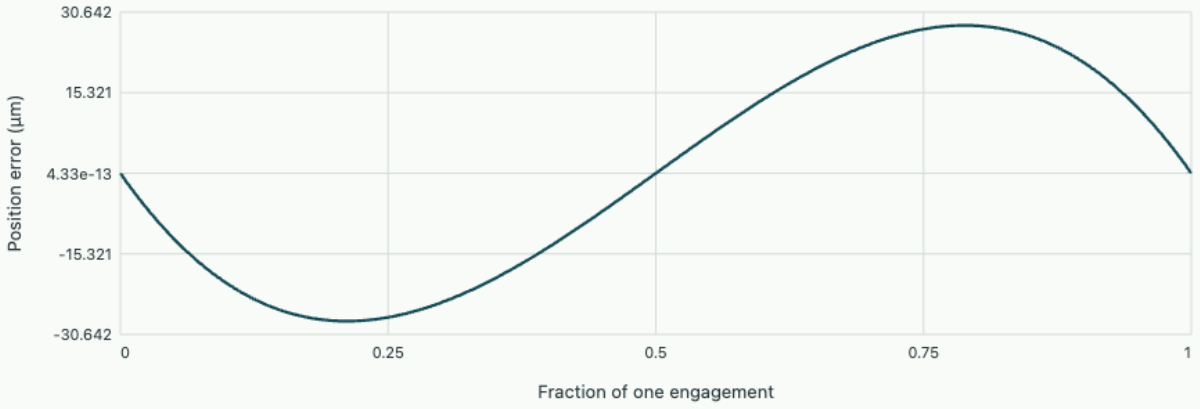
Interactive calculation · Chain position

Change the physical and numerical inputs of the notebook model. Compare the updated curves and balance checks with the recorded notebook figures. Each plot uses the same equations and assumptions documented in that notebook; its readout names any analytical or reduced-model reference.

Open this live calculation

Try the calculation

Sprocket tooth count	12
Roller pitch (mm)	12.7
Drive speed (rpm)	120



Departure from uniform advance

Mean chain speed 0.30480 m/s; tooth-pass frequency 24.000 Hz. Peak-to-peak speed ripple 3.4467%. The engagement advances exactly one pitch.

Download this example's editable notebook

Consider a chain with 12.7 mm pitch, twelve teeth and shaft speed 120 rpm. The mean speed is $0.0127 \times 12 \times 2 = 0.3048$ m/s, and rollers engage at 24 Hz. The ideal peak-to-peak speed variation is 3.45% of the mean, about 0.0105 m/s. At one quarter of an engagement the sample has moved 3.148 mm rather than the 3.175 mm predicted by uniform advance. A roughly 27 µm displacement error could matter in an optical registration task, despite perfectly correct counts over every full pitch.

This estimate also explains a sampling pitfall. A sensor sampling synchronously once every engagement sees the same phase every time. With a fixed phase offset its data look like a constant positional bias and may be removed by calibrating an offset. Sampling at several known phases within the engagement recovers the actual shape. An unsynchronised slowly drifting sampling clock can make

a high-frequency tooth-pass ripple appear as a slow fluctuation. Check the sampling schedule against the engagement frequency before attributing that signal to eccentric mounting.

Translate throughput and accuracy requirements into geometry

Suppose the speed specification allows a peak-to-peak variation of 0.5% of the mean. The large-z approximation predicts $z \simeq \pi/\sqrt{0.01} \simeq 31.4$. Only an integer tooth count can be manufactured. Evaluating the exact expression gives about 0.514% for 31 teeth and 0.482% for 32. Thus 32 is the first integer satisfying this *geometric* requirement. Bearings, packaging, minimum wrap and tension fluctuations can place other bounds on the choice. An improvement in a kinematic reference is only part of a component selection.

The notebook integrates the computed instantaneous speed over the normalised phase and verifies that its mean agrees with the independent pitch-counting formula. This provides a diagnostic for a coding error that shifts θ by one half engagement or uses degrees as radians. Try changing shaft speed over a wide range. The relative variation and position error remain the same in the ideal model; the dimensional speed variation and the repetition frequency scale linearly with rotational speed. A measured position error that increases strongly with speed suggests a dynamic effect in addition to this geometric mechanism.

Design a measurement that can resolve the ripple

At the default 24 engagements per second, one tooth advances the chain for 41.67 ms. Suppose an optical counter measures the positions of two marks separated by exactly one pitch and estimates the mean speed from the elapsed time. In the model those marks pass at an interval of one tooth period, regardless of the phase at the first mark. Their inferred speed is exactly 0.3048 m/s. Repeating the measurement more often improves its precision and still reports the same answer. An exposure shorter than a tooth period and a known angular reference are needed to recover the speed changes within that period.

A useful check is to place a high-contrast feature on one link, take successive exposures at a deliberate quarter-pitch offset in time and subtract their measured positions. Around the midpoint of one engagement, a window extending from $-\delta$ to $+\delta$ in shaft angle measures the mean speed $R\omega \sin\delta/\delta$. At the midpoint the instantaneous speed is $R\omega$. Their ratio is a sinc factor produced by the camera's finite observation window. For a full-pitch exposure, $\delta = \pi/12$, that factor is about 0.9886 and the measured midpoint velocity becomes the mean speed. Shortening the exposure to half the tooth period gives a ratio of about 0.9971. The geometry of the motion stays the same while the instrument changes the apparent size of its ripple.

The experiment also estimates the mean speed from four deliberately chosen shaft angles. A local OneSimulation installation supplies the Gauss quadrature rule. Compare its estimate with the mean from the uniformly sampled encoder signal to see how integration weights alter the measurement.

Exercises in timing and geometry

1. A 24-tooth sprocket replaces the 12-tooth one at the same production throughput. Calculate its shaft speed, tooth-pass frequency and peak-to-peak speed variation.

▼ **Work through the solution**

The chain must still advance 24 pitches each second, so the new shaft turns once per second, at 60 rpm. The tooth-pass frequency remains 24 Hz. The dimensionless angle is $\pi/24$, giving $\delta \simeq 0.00858$, approximately 0.858% of the mean speed. The absolute spread is about 0.00262 m/s at the original throughput.

---

2. A controller estimates distance as pitch radius times shaft rotation. After one revolution, by what fraction does it overestimate the chain advance for twelve teeth?

▼ **Work through the solution**

Its estimate is $2\pi R$ and the actual distance is zp. Their ratio is $a/\sin a$. For $a = \pi/12$ this is approximately 1.0115, so the accumulated estimate is high by about 1.15%. Calibrate the mean mapping by teeth times pitch per revolution. A residual within each engagement will still follow the position-error curve.

---

3. Two signals have exactly the same wavelength when expressed as a fraction of an engagement, but only one preserves its micrometre amplitude as the shaft speed changes. Which observation is consistent with the rigid-chain model?

▼ **Work through the solution**

The predicted position error depends on p, the tooth count and phase. It is independent of shaft speed. A micrometre amplitude that stays constant is consistent with this geometric contribution. A dynamic compliance response can also happen to be nearly constant over a narrow speed interval, so test additional speeds and measure tension or shaft motion before treating the observation as unique evidence.

Compare the geometric estimate with a drive selection

The manufacturer’s drive engineering catalogue discusses chordal action, chain selection and rating. Distinguish the velocity fluctuation convention used in a particular rating chart from the peak-to-peak excursion divided by mean speed defined here.

Problem sheet 7 · Closing a flexure-guided fixture with a screw drive

A bench fixture moves a specimen with a two-leaf guide and a square-thread screw. Determine the guide stroke, the actuator torque and the positioning error that a remote scale can miss. The results describe a nominal small-displacement design.

1 · Determine the allowable stroke from surface stress

Two identical fixed-guided leaves have $L = 40$ mm, $b = 10$ mm, $t = 0.30$ mm and $E = 200$ GPa. Keep peak nominal bending stress at or below 90 MPa. Find the stroke, total guide force and parasitic axial shortening. Distinguish this guide force from the specimen load.

▼ Relate leaf curvature to force and stroke

With fixed end slopes the cubic shape gives $\sigma_{\max} = 3Et\delta/L^2$, hence $\delta = 0.800$ mm. Since $I = bt^3/12 = 2.25 \times 10^{-14}$ m⁴, the two leaves have $24EI/L^3 = 1688$ N/m stiffness and require 1.35 N at that stroke. Their projected length change is $3\delta^2/(5L) = 9.6$ µm. The drive must supply the separately specified specimen load in addition to this restoring force.

Check peak bending stress at 0.80 mm stroke

2 · Size nominal torque and travel

At the end of the stroke the specimen pushes back with 150 N. The screw has mean diameter 16 mm, lead 4 mm, friction coefficient 0.12 and no bearing or collar loss. Calculate the total axial load, screw torque, number of revolutions and lowering-torque sign.

▼ Unwrap the thread at its mean diameter

Total axial demand is 151.35 N. With $t_\lambda = 4/(\pi 16) = 0.07958$, the required torque is $(151.35)(0.008)(t_\lambda + 0.12)/(1 - 0.12t_\lambda) = 0.244$ N m. The stroke requires $0.8/4 = 0.20$ revolutions or 72°. The external lowering torque is positive, about 0.0485 N m. Under the assumed friction a downward axial load alone does not maintain steady descent. The chapter study uses loads down to 200 N; use its 800 N reference torque and scale by $151.35/800$ because the torque law is linear in load.

Check the raising-to-lowering torque relation at 4 mm lead

3 · Locate error at the specimen

The position scale is mounted 60 mm above the specimen contact axis. A changing pitch of 80 µrad accompanies closure. Estimate the longitudinal scale-to-contact position difference and compare it

with a ±3 µm measurement requirement. Does the 9.6 µm flexure shortening necessarily add directly to this error?

▼ **Resolve error vectors on the measurement axis**

The Abbe difference is $60\ \mathrm{mm} \times 80 \times 10^{-6} = 4.8$ µm. A 3 µm bound demands an offset at most 37.5 mm at this pitch, or pitch at most 50 µrad at the original offset. The flexure's parasitic motion is along the leaf length while commanded translation is transverse. It adds directly only if the measurement axis has a component along that motion. A second scale or surface-tilt measurement can separate translation from rotation.

Explore how a small angular error changes a reading away from its datum

4 · Compare a two-start thread at equal pitch

Replace the screw by a two-start version at the same 4 mm pitch. Keep the mean diameter and friction. Predict revolutions, raising torque and change in backdrivability. What independent measurement would help near the transition?

▼ **Recompute the helix and friction terms**

Lead is now 8 mm, so the motion takes 0.10 revolution. The tangent of the helix angle is 0.1592, giving a nominal raising torque of about 0.345 N m. Since 0.1592 exceeds friction 0.12, the lowering torque is negative. A holding drive must oppose backdriving under the sliding model. Static breakaway friction may differ, so measuring breakaway and running torque separately at known axial load distinguishes the two conditions.

Problem sheet 8 · Three shafts and one constraint in a planetary gearbox

A teaching gearbox exposes the sun, ring and carrier on three accessible shafts. The carrier supports freely rotating planets. Recover their motion without treating the carrier as a fixed casing, infer the torque carried by a stationary ring and use two speed sensors and a power balance to diagnose a slipping fixture.

Find an admissible gear set and the speed constraint

Take a sun of 30 teeth and planets of 20 teeth. All spur gears share the same module. Determine the tooth count of an internal ring concentric with the sun, then derive the constraint among sun, ring and carrier speeds. Speeds are positive counterclockwise viewed from the same side.

▼ **Work in the rotating carrier frame**

The distance from sun centre to each planet centre is the sum of the sun and planet pitch radii. The same distance is the ring pitch radius minus the planet pitch radius. Since pitch radius is proportional to tooth count, $N_r = N_s + 2N_p = 70$. In the frame moving with the carrier every gear centre is stationary, so ordinary pitch-line kinematics applies. External sun–planet contact reverses the relative rotation and internal planet–ring contact preserves its sign. Eliminating planet spin gives

$$N_s(\omega_s - \omega_c) + N_r(\omega_r - \omega_c) = 0,$$

$$\omega_c = \frac{N_s\omega_s + N_r\omega_r}{N_s + N_r}.$$

This is one constraint on three shaft speeds. A single prescribed speed leaves another degree of freedom until another shaft is restrained, driven or coupled dynamically. It is valid for angular speeds expressed in rpm as well as radians per second, provided the same units are used for every term. Equal rotation of all three members satisfies it exactly and yields zero motion relative to the carrier.

Predict what a mark on a planet actually does

With fixed ring and sun at 1200 rpm, calculate carrier speed, planet spin relative to the carrier and planet spin measured by a stationary observer. Then determine what ring speed would hold the carrier still while the sun continues to rotate.

▼ **Restore the carrier speed after solving the mesh**

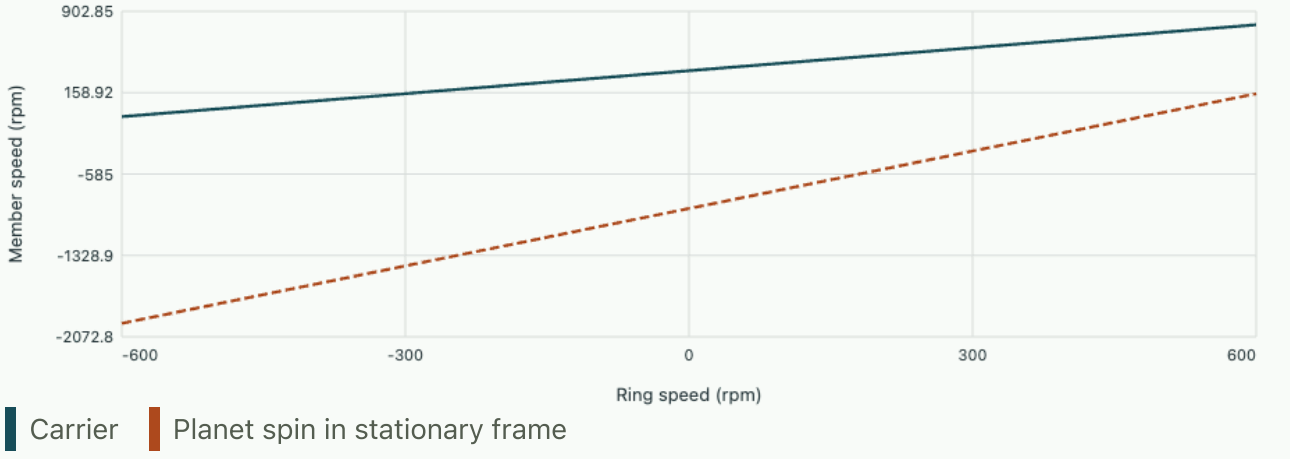
Weighted averaging gives carrier speed 360 rpm. The sun rotates relative to the carrier at 840 rpm, so the planet spins at $-(30/20)840 = -1260$ rpm relative to its support. Add the carrier rotation to get minus 900 rpm absolute spin. A marker on the planet also translates around the sun as its centre orbits; a camera measuring successive positions of a painted spot must distinguish its orientation from its path about the central shaft.

Setting the carrier speed to zero requires ring speed $-N_s\omega_s/N_r = -514.3$ rpm. At ring speed plus 1200 rpm the assembly simply co-rotates. These two checks test both contact signs. A proposed ratio that has the wrong sign for an internal mesh or neglects the carrier subtraction often fails the co-rotation test even when it happens to reproduce one fixed-member ratio.

Open this experiment online

Try the calculation

Sun teeth	30
Planet teeth	20
Sun speed (rpm)	1200



The ring has 70 teeth. With a fixed ring the carrier runs at 360.0 rpm and the planet spins at -900.0 rpm in the stationary frame.

Find the reaction at a support delivering no shaft power

In a lossless quasistatic fixed-ring test the sun receives 6.0 N m. Find the external torques applied to carrier and ring and verify both moment and power balances. What torque ratio would change first if the carrier shaft delivers only 92 percent of the incoming mechanical power?

▼ **Use two balances and keep the external-torque signs**

In steady conditions the external torques sum to zero, and their power sum $T_s\omega_s + T_r\omega_r + T_c\omega_c$ vanishes for the ideal gear set. With stationary ring, ideal carrier torque applied to the gear set is $-6(1200/360) = -20$ N m. The ring fixture applies $+14$ N m. It carries substantial reaction torque although its shaft power is zero. Incoming sun power is approximately 754 W and outgoing carrier power has the same magnitude.

At 92 percent efficiency the delivered output is about 694 W and its torque magnitude about 18.4 N m at the unchanged geometric speed. Heat generation and other parasitic torques must now enter a complete boundary balance. A measured case reaction includes any frictional torque passed through bearings and seals. It need not equal the ideal contact-only ring torque scaled by one efficiency factor. An efficiency measurement from power alone cannot locate where the heat is deposited.

Separate a slipping ring clamp from an incorrect gear ratio

A repeat run reports a sun rate of 1200 rpm and a carrier rate of 375 rpm. Estimate ring rotation implied by ideal rolling contact. Could uniform bearing loss produce that steady change with the ring still rigidly held? Propose measurements before attributing the difference to tooth slip.

▼ **Infer the unobserved shaft and check the system boundary**

Rearrange the speed constraint to obtain $\omega_r = [100(375) - 30(1200)]/70 = 21.4$ rpm. A creeping ring clamp in the positive direction could produce the observation. Dissipative tooth and bearing forces change required torque, but under ordinary engaged gearing they do not change the ratio fixed by tooth geometry for a stationary ring. Misidentified gear counts, shaft encoder scaling, backlash during a reversal or nonsteady averaged readings are alternative explanations.

Mark the ring relative to the casing and acquire all shaft signals on a common clock over complete gear cycles. Counting a planet's absolute spin instead of carrier speed produces a very different quantity, as the earlier calculation shows. If the ring does move, include its torque times angular speed in the power budget. Separately check torque sharing among multiple physical planets: the single kinematic ratio says nothing about elastic load sharing, manufacturing errors or fatigue life. Those require geometry and compliance measurements beyond the shaft-speed test.

Jönköping University Applied Mechanics · planetary gears and their relative motion

Notebook practice · How a polygonal sprocket imprints speed ripple on a roller chain

Design and analysis of machine components

Follow forces and motion through a component, derive its limiting response and compare a design estimate with an observable test of the assembled machine.

1. ## Cam displacement laws, contact geometry and follower acceleration

   An inspection station uses a roller to lift a specimen for imaging, hold it and return it. Specifying a ten-millimetre rise determines neither the force on the guide nor whether the follower can follow the prescribed geometry. Construct two different motions through the same endpoints and separate their geometric and dynamic consequences.

2. ## Helical gears, mesh forces and bearing reactions

   A motor supplies a known torque to a helical pinion. Select the free body carefully: torque fixes one component of the tooth load, the helix creates thrust, and the location of that thrust can alter bearing reactions even in a parallel-shaft transmission. Build the force diagram and reverse it from measured support loads.

3. ## Sliding wear, real contact area and life inference from a coupon

   A sliding carriage develops measurable clearance while its normal force remains almost constant. Derive a practical volume-loss rule from microscopic contacts, estimate its coefficient from a mass trace, and test whether a single fitted value can be transferred to another load history. Geometry, material condition and instrument resolution enter at different stages.

4. ## Fatigue crack growth, optical inspection and the meaning of remaining life

   A polished test coupon contains a measurable crack and experiences repeated tensile cycles. A stress–life curve for a polished, initially uncracked specimen cannot directly give the remaining growth interval from this measured state. Use calibrated crack-growth data to predict a laboratory observation and quantify what the inference depends on.

5. ## How does a converging oil film support a load?

   Two plates have a small fixed angle. The lower plate moves while the upper pad stands still. Derive the pressure from volume conservation and local shear, then change film height and speed to see why a wedge develops lift.

6. ## From a laboratory fatigue curve to a machined shaft

   A fatigue limit measured on a polished specimen does not directly describe a shoulder on a production shaft. Track where each correction enters and how the measured torque or bending history changes the inferred life.

7. ## Will tooth-root bending or flank contact govern a gear pair?

The same torque loads the tooth root and the mating flanks. Calculate two distinct demands, carry the same duty blocks through both, and see why doubling the face width gives different gains.

8. Redirecting a shaft line with bevel gears and worm drives

Two transmissions can deliver the same speed reduction around a right angle yet require very different bearings and cooling. Derive the directions of the loads and the dependence of efficiency on lead angle.

9. Can a mating pin enter every accepted hole?

A rectangular coordinate tolerance and a circular position tolerance accept different parts. Carry a hole requirement from its datum reference frame through its material condition to an actual inspection decision.

Design and analysis of machine components · 1. Cam displacement laws, contact geometry and follower acceleration

An inspection station uses a roller to lift a specimen for imaging, hold it and return it. Specifying a ten-millimetre rise determines neither the force on the guide nor whether the follower can follow the prescribed geometry. Construct two different motions through the same endpoints and separate their geometric and dynamic consequences.

Construct the pitch curve from the prescribed motion

Imagine holding the cam still and rotating the entire guide around it in the opposite direction. The centre of an inline roller traces a polar curve in the cam frame. Let the prime radius be the distance from cam axis to roller centre at zero lift. At angle θ the pitch radius is $r = R + s(\theta)$. A small rotation advances the centre tangentially by $r\,d\theta$ and radially by $s'\,d\theta$. The tangent therefore makes an angle with the circumferential direction whose tangent is the ratio of these increments. The common normal makes the same angle with the radial guide direction. For a frictionless, inline translating roller the pressure angle follows immediately:

$$\tan\alpha = \frac{s'(\theta)}{R + s(\theta)},$$

$$N\cos\alpha = F,$$

$$F_{\text{guide}} = F\tan\alpha.$$

Here the force along the follower includes inertia and any spring or specimen load. Reversing the rise changes the sign of the lateral reaction. Angular derivatives must be per radian because the tangential increment is an arc length. A calculation performed per degree systematically underestimates the slope by a factor of about fifty-seven. The actual metal profile is a normal offset of the pitch curve by the roller radius. Simply subtracting the roller radius radially works on a concentric dwell but generally gives the wrong point on the rise.

Compare motion laws at a dwell boundary

Write $u = \theta/\beta$ from zero to one on the rise. A harmonic motion is half a cosine. A cycloidal motion adds a sinusoidal correction to a linear ramp. Both end at zero velocity, making a stationary dwell possible without a jump in speed. Differentiation resolves what else happens:

$$s_H = \tfrac{h}{2}(1 - \cos\pi u),$$

$$s_C = h\left(u - \frac{\sin 2\pi u}{2\pi}\right),$$

$$s_C' = \frac{h}{\beta}(1 - \cos 2\pi u).$$

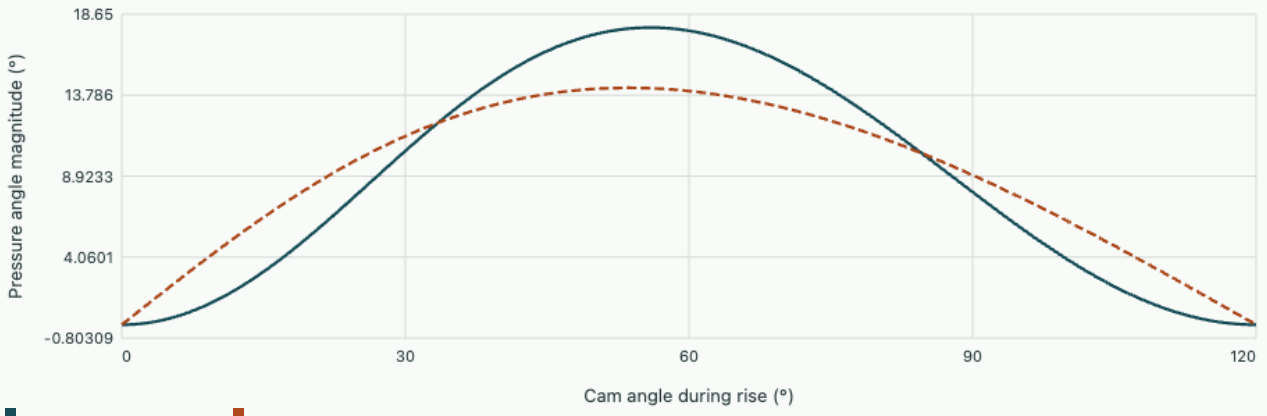
For constant angular speed the acceleration is $\Omega^2 s''$. Harmonic acceleration has a finite nonzero value just inside the rise and vanishes on the dwell. The cycloidal law has zero endpoint acceleration, although its jerk changes when entering the dwell. At mid-rise the cycloidal slope is greater. Thus an improvement in acceleration continuity can cost additional side load. At a varying spindle speed add $s'\dot{\Omega}$ by the chain rule. A measured acceleration trace during run-up can disagree with a constant-speed prediction even with an accurately cut contour.

For 10 mm rise through 120 degrees from a prime radius of 25 mm, the middle cycloidal slope is 9.55 mm per radian. The pitch radius there is 30 mm and the pressure angle is 17.66 degrees. The slope reaches its maximum there, yet the largest angle occurs slightly earlier because the denominator was still increasing. Compare both laws across the complete rise. Increase only the prime radius and predict whether the slope or the guide force changes.

Open the live experiment

Try the calculation

Lift (mm)	**10**
Rise angle (°)	**120**
Prime radius (mm)	**25**



The cycloidal peak pressure angle is 17.85°. At half rise it is 17.66°; the harmonic comparison has an acceleration jump at a dwell.

Background · How does changing the cam speed affect acceleration and jerk?

The prescribed lift $s(\theta)$ is a geometric function. At constant cam angular speed Ω, differentiating with respect to time multiplies each angular derivative by another power of speed. Angles in these derivatives are expressed in radians.

$$v = \Omega s'(\theta),$$

$$a = \Omega^2 s''(\theta),$$

$$j = \Omega^3 s'''(\theta).$$

Doubling the speed leaves the traced pitch curve and peak lift unchanged while doubling peak follower speed, quadrupling peak acceleration and multiplying peak jerk by eight. The inertial contribution to the normal contact load can therefore reach the available preload at a speed where the slow test runs quietly. During spin-up the acceleration also includes $s'\dot{\Omega}$.

How a smooth pitch curve can make a poor physical cam

The curvature depends on the second derivative as well as the pressure angle. For a polar curve, differentiate its Cartesian position twice and apply the signed planar curvature formula. With outward radial coordinates its signed radius, positive on a circular convex arc, is

$$\rho_p = \frac{(r^2 + r'^2)^{3/2}}{r^2 + 2r'^2 - rr''}.$$

On a convex segment the inward offset by roller radius has signed radius $\rho_p - r_{\mathrm{roller}}$. At zero an offset develops a cusp. A smaller roller helps in this local geometric check but can raise contact pressure at fixed normal force. Numerical differentiation of a measured cam profile twice amplifies encoder noise; estimate curvature on a documented angular window and repeat the calculation with another window. A sharp reproducible feature demands a different explanation from a peak moving with the smoothing window.

Reconstruct the motion from a camera and phase mark

Place an angular reference on the shaft and record the follower at equally spaced phase values for several turns. Average positions at equal phase to reduce uncorrelated readout noise. Differentiate the average only after expressing the phase in radians. A centred local fit yields slope and curvature with more usable precision than subtracting adjacent noisy samples twice. Inspect the first and last part of the rise separately, because smoothing across a dwell hides a real acceleration jump in the harmonic schedule.

As a cross-check, integrate the estimated derivative over the rise. It should recover the measured ten-millimetre travel. At the same time inspect the pressure angle derived from it. A fit can preserve the correct total travel while underestimating the peak local slope through excessive smoothing. Independent displacement and guide-force measurements at several phase angles help distinguish these two errors.

Tutorial · Turn a lift diagram into a measured acceleration

A cycloidal 10 mm lift takes 120 degrees at 300 rpm. Find the peak follower acceleration and the value after doubling the speed with the same cam.

▼ **Convert the time base before differentiating**

The rise lasts one third of a 0.200 s revolution, giving 0.06667 s. Differentiating the displacement twice in physical time gives peak acceleration $2\pi h/t_r^2 = 14.14$ metres per second squared. At 600 rpm the rise time halves and peak acceleration reaches 56.55 metres per second squared. The corresponding inertial contribution of a moving mass of 0.20 kg grows from 2.83 to 11.31 N. It changes sign across the rise. A spring holding the roller in contact consequently has a different margin on the two halves.

Tutorial · Interpret a guide measurement

A force transducer reads 120 N along the guide at the middle of the slower rise. Estimate contact and lateral force. Would a larger prime radius halve the axial force?

▼ **Resolve the normal with the pitch-curve slope**

Using the angle above gives a lateral force of 38.2 N and normal load of approximately 126 N. Increasing radius reduces the angle at the same kinematic point. The specified axial force comes from specimen, spring and moving mass. Reducing angle changes its resolution into normal and transverse components, while the prescribed lift history still requests the same axial acceleration. Guide friction couples the directions and would require a fresh force balance. A changing spring lever arm could change the applied axial force as well.

Tutorial · Check the roller offset independently

A convex portion of a pitch curve has local radius 8 mm; the proposed roller radius is 9 mm. The maximum pressure angle passes the design team's geometric requirement. Explain whether the nominal profile can realise that local curvature.

▼ **Inspect the normal offset**

The local offset radius would be minus one millimetre after crossing a cusp at eight. The normal-offset construction folds and the roller cannot follow the intended envelope there. Pressure angle uses the first derivative, so it cannot certify curvature. Test a smaller roller or a wider rise and recalculate normal load and contact stress. A tolerance on the measured first derivative alone leaves this failure undetected. Recheck the return as well, because its angular duration and signed curvature can differ.

Carnegie Mellon · Introduction to Mechanisms, chapter 6, cam motion and profile inversion

Design and analysis of machine components · 2. Helical gears, mesh forces and bearing reactions

A motor supplies a known torque to a helical pinion. Select the free body carefully: torque fixes one component of the tooth load, the helix creates thrust, and the location of that thrust can alter bearing reactions even in a parallel-shaft transmission. Build the force diagram and reverse it from measured support loads.

Unroll a pitch cylinder and resolve the tooth normal

Imagine cutting the pitch cylinder and unwrapping it. The tooth trace makes helix angle β with the shaft axis. The component of tooth contact force lying in the unwrapped pitch surface points perpendicular to that trace. It has circumferential projection F_t and axial projection F_a. The line of action also tilts out of the tangent plane through normal pressure angle α_n, creating a radial component. For a frictionless external mesh the magnitudes follow from two right triangles and torque balance:

$$F_t = \frac{2T}{d},$$

$$F_a = F_t \tan\beta,$$

$$F_r = \frac{F_t \tan\alpha_n}{\cos\beta},$$

$$F_n = \frac{F_t}{\cos\beta\cos\alpha_n}.$$

Projection of the transverse circular tooth pitch onto its normal section gives $p_n = p_t \cos\beta$, so the pitch diameter of a standard unshifted gear is $d = z m_n / \cos\beta$. Distinguish fixed diameter from fixed tooth count and normal module when changing helix angle. At fixed diameter and torque the tangential force stays fixed; changing the diameter changes it too. A spur gear emerges continuously as the helix angle tends to zero, because axial thrust vanishes and both pressure angles coincide.

For 100 N m, 100 mm reference diameter and 20 degrees for both helix and normal pressure angles, the tangential, axial and radial forces are respectively 2000, 728 and 775 N. Their vector magnitude is 2265 N. The direction of thrust depends on helix hand and torque direction. The two mating gears experience equal and opposite contact forces in common coordinates. Two separate diagrams drawn from opposite shaft ends can make consistent thrust arrows look identical on the paper.

Transport the mesh wrench to the shaft centreline

Let bearing A lie at coordinate zero and B at span L. A lateral force acts a distance a from A. In each independent bending plane its reaction magnitudes for a simple support model are $F(L-a)/L$ and Fa/L, with directions opposite the applied force. The axial force acts at the pitch contact, a radius away from the centreline; moving it onto the shaft centre adds a couple of magnitude $F_a d/2$. In a plane where the radius points vertically this couple acts in the vertical bending plane. Determine its sign from the cross product. The tangential force at that radius also contributes the transmitted torsional moment, balanced by the drive.

For $a = 60$ mm, $L = 150$ mm, the 2000 N tangential force gives 1200 and 800 N. Ignoring the thrust couple, the radial force would give 465 and 310 N. The 36.4 N m couple transfers 243 N between the two radial reactions without changing their sum. With the sign increasing B, the radial reactions become approximately 222 and 553 N. Dynamic load ratings use the resultant load on each bearing and a catalogue rule for its combination with axial force. Suppose A locates the shaft axially and B permits axial displacement. Then A reacts the full thrust in this idealisation. Two preloaded locating bearings require deformation compatibility, clearance and stiffness for a reliable split.

Measure a signature of mesh geometry

Make two steady tests at the same torque magnitude, reversing rotation and applied torque together. The separating radial component of an external mesh usually stays separating while thrust changes direction with the loaded flank. Tangential reactions reverse sign. Resolve measured reactions into a common coordinate frame and subtract bearing preload before comparing. A change in torque sign may also move load to a different bearing shoulder or traverse backlash. Examine displacement on each side of the reversal before attributing changes to helix geometry.

Use the involute-contact study to distinguish tooth contact ratio from the total steady mesh force.

Move the mesh along the shaft without changing the mesh forces

The experiment holds torque and pitch diameter fixed while shifting the mesh station. In each radial reaction the contribution of the lateral force varies linearly with station. The off-axis thrust contribution is instead a constant signed transfer between the supports for this geometry. Compare the two curves for support B. Their vertical separation is the thrust couple divided by span and stays fixed as the gear moves. The sum of A and B stays equal to the separating force at every station.

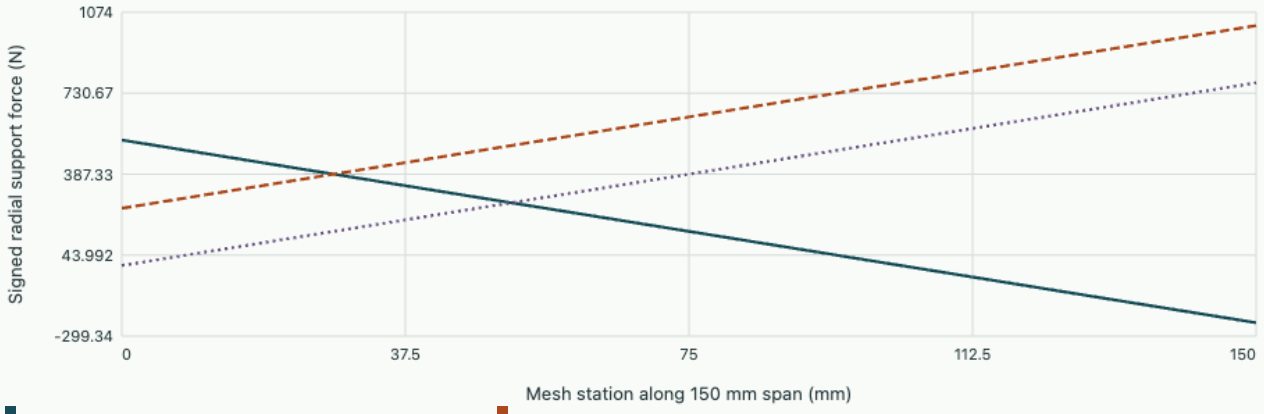
Open the live experiment

Try the calculation

Shaft torque (N m) **100**

Pitch diameter (mm) **100**

Helix angle (°) **20**



For the marked 60 mm station, radial reactions are 222.1 and 552.5 N. The 727.9 N thrust acts at pitch radius and supplies a 36.40 N m couple.

Near one end the smaller support reaction can change sign. In the signed vector convention this means the support must react in the opposite direction. A preloaded bearing can change its incremental force in this direction. A clearance support might instead unload until the opposite side takes up load. The simple-support drawing chooses neither actual contact path. Check axial location, allowable movement and measured preload before interpreting a predicted negative incremental load as a reversed material stress at one rolling contact.

Tutorial · Size from the normal module

An unshifted pinion has 30 teeth, normal module 3 mm and helix angle 25 degrees. At 90 N m find pitch diameter, tangential load and axial thrust.

▼ **Resolve geometry before applying torque**

The transverse module is $3/\cos 25^\circ = 3.31$ mm, giving diameter 99.3 mm. Consequently the tangential force is 1813 N and axial thrust 845 N. Using 90 mm as diameter would overestimate transmitted force. The torque check multiplies 1813 N by a 49.65 mm arm, recovering approximately 90 N m. The input power cannot be recovered without the rotation speed; steady torque does not by itself determine heating or time to a given number of bearing revolutions.

Tutorial · Find a missing transverse couple

Radial support reactions sum to 780 N in a 150 mm span. The mesh lies 60 mm from A, and B reacts 550 N. What signed applied couple is consistent with the departure from simple force sharing?

▼ **Subtract the force-only support moment**

A force-only prediction for B is

$$780(60/150) = 312\,\mathrm{N}.$$

The excess 238 N corresponds to a couple of magnitude 35.7 N m in the direction increasing B. With a 50 mm pitch radius that magnitude would imply 714 N axial force if off-axis thrust were its only source. Check the thrust bearing directly. Misaligned couplings or another overhung force can give the same two transverse reactions, so the inversion identifies a net moment rather than its unique origin.

Tutorial · Explain a life sensitivity

For an ideal ball bearing the reference rating model gives life in millions of revolutions as $(C/P)^3$. A catalogue combined-load calculation increases effective load from 1.00 to 1.20 kN at unchanged rating. Estimate the life ratio and explain what data the ratio omits.

▼ **Keep the bearing rating and the statistical meaning fixed**

The ratio is $(1/1.20)^3 = 0.579$, a reduction of about 42 percent in the rating life. This is a population-based rating at a chosen reliability under specified conditions. Lubrication, contamination, preload, fit, load cycles and installation can change service results. Comparing against a test measured in hours also requires shaft speed. If speed doubles at the same effective load, the reference life in revolutions stays the same while the corresponding time halves.

MIT Marine Power and Propulsion · helical gear lecture

Design and analysis of machine components · 3. Sliding wear, real contact area and life inference from a coupon

A sliding carriage develops measurable clearance while its normal force remains almost constant. Derive a practical volume-loss rule from microscopic contacts, estimate its coefficient from a mass trace, and test whether a single fitted value can be transferred to another load history. Geometry, material condition and instrument resolution enter at different stages.

Count interaction volume per travelled distance

The apparent contact area of two nominally flat surfaces is usually much larger than the sum of microscopic load-bearing patches. For an elementary model suppose these patches deform plastically at mean pressure close to a hardness H of the softer body. Their summed real area is then approximately W/H at applied normal force W. As the surfaces slide by distance ds they sweep an interaction volume proportional to this area times distance. Only a fraction of microscopic encounters leads to detached material. Absorb this probability and unknown geometry into the dimensionless empirical coefficient k. Integrating the resulting increment gives the Archard form:

$$dV = k\frac{W}{H}\,ds,$$

$$V = \frac{kWs}{H},$$

$$h_{\mathrm{mean}} = \frac{V}{A_{\mathrm{nominal}}}.$$

Force divided by pressure has units of area. The interaction-volume picture explains the dimensional structure without predicting the coefficient for a new material pair. Some tables give a dimensional specific wear rate $V/(Ws)$; it already includes the inverse hardness. Its tabulated units may be cubic millimetres per newton metre. Multiplying that number by another inverse hardness counts hardness twice. State which body loses the quoted volume. In a pair of dissimilar materials the track and the slider may have quite different loss rates.

A teaching coupon carries 100 N over 1 km at hardness 2 GPa and fitted coefficient 2×10^{-5}. The model gives 10^{-9} cubic metres, exactly one cubic millimetre. Uniform removal over 50 square millimetres would lower the surface by 20 micrometres. A profilometer on a narrow groove could report a much deeper maximum while its integrated volume still agreed. Report both track shape and volume when the function of a part depends on clearance at one point.

Separate distance, time and the running-in transition

For changing conditions the differential statement is more useful:

$$V(t) = \int_0^t \frac{k(t')W(t')}{H(t')}\,\lvert v_{\mathrm{rel}}(t')\rvert\,dt'.$$

Motion reversals contribute positive distances. A reciprocating slider returning to the starting position has zero net displacement and can have substantial wear. If force varies with speed, the average of their product is generally different from the product of their averages. A loss model calibrated during steady sliding may fail near each reversal where a lubricating film collapses. Integrating only the relative movement at the actual interface excludes contact portions undergoing nearly pure rolling.

Suppose a balance resolves 0.5 mg and the coupon density is 7.8 mg per cubic millimetre. A loss of one cubic millimetre gives 7.8 mg signal. The single-reading resolution is equivalent to about 0.064 cubic millimetres. A before/after difference contains uncertainties from both measurements. Cleaning, oil retention, corrosion and material transferred from the mating part can change mass independently of track removal. Repeated weighing after identical cleaning and a stationary witness coupon distinguish some of those effects. A single extremely precise reading cannot correct a consistent cleaning bias.

Vary follower pressure angle and examine why guide normal load changes over a stroke before integrating wear.

Examine a case in which the mean-load approximation becomes exact

In the plot the velocity increases after thirty minutes, while both half-hour loads can be set separately. Observe the change of slope at the switch. The slope measures instantaneous volume loss per time, so its jump follows the product of load and speed. With the default pair of loads the speed increase just compensates for the smaller load, leaving the integrated history straight despite the change of regime assumed by the operator. The comparison based on separate averages still runs faster.

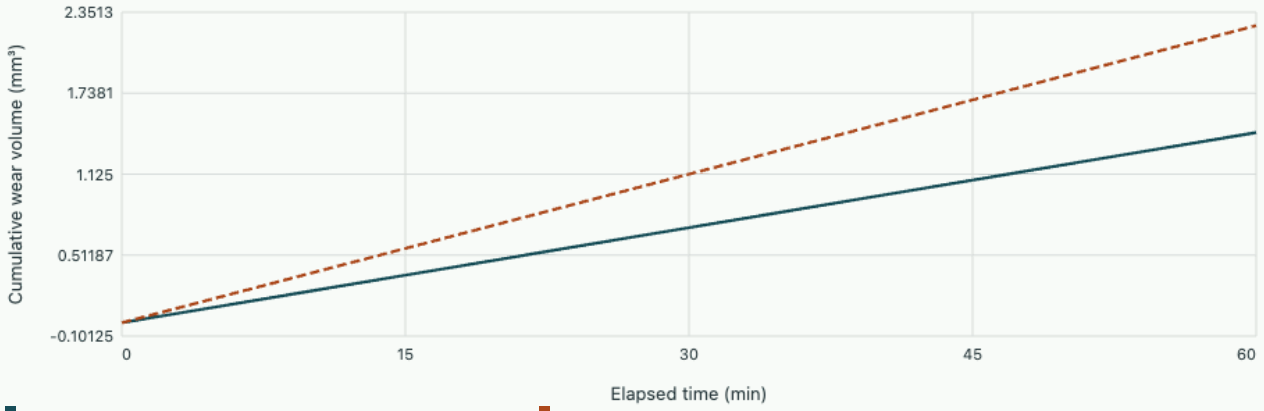
Open the live experiment

Try the calculation

First interval load (N) **200**

Second interval load (N) **50**

Hardness (GPa) **2**



Two successive operating conditions Separate average load and average speed

After one hour the integrated history predicts 1.440 mm³. Multiplying separate averages predicts 2.250 mm³ (56.3% difference).

Set the two loads equal. The accumulated trace now has two slopes, yet its final value coincides with the separate-average prediction because force is constant and integrating the velocity simply counts total distance. This counterexample helps separate agreement at one endpoint from agreement during the programme. An inspection at the changeover could discriminate two histories which yield the same final mass loss.

Tutorial · Calibrate the dimensionless coefficient

A pin loses 3.90 mg after 500 m under 100 N. Its density is 7.80 mg per cubic millimetre and hardness is 2 GPa. Infer the coefficient and predict the volume after another 1.5 km if the wear regime persists.

▼ **Convert mass into a volume before fitting**

The first loss is half a cubic millimetre, or 5.0×10^{-10} cubic metres. Solving $k = VH/(Ws)$ gives 2.0×10^{-5}. Three times the sliding distance then removes another 1.5 cubic millimetres at unchanged conditions. A straight line through multiple later measurements supports constant rate; the intercept can reflect substantial running-in wear before that linear stage. Do not force the post-running-in line through the origin unless observations justify that endpoint.

Tutorial · Compare service programmes with identical mean load and speed

Over equal half-hour intervals a guide first carries 200 N at 0.2 m/s then 50 N at 0.8 m/s. Another test uses average force and average speed throughout the same hour. Compare their modelled losses.

▼ **Integrate the load–distance product**

Each half of the first programme has force times speed equal to 40 N m/s, so its integrated exposure is 144,000 N m. The second has $125 \times 0.5 = 62.5$ N m/s throughout and reaches 225,000 N m. The second predicted volume is 1.5625 times the first if hardness and coefficient stay fixed. The first programme anticorrelates force and speed, making a product of separately averaged values inappropriate. Differences in contact temperature could change the coefficient and must be measured separately before comparing real coupons.

Tutorial · Diagnose an apparent change of mechanism

A test at fixed force and speed shows cumulative loss 0.9, 1.1 and 1.3 cubic millimetres after 100, 200 and 300 m. A colleague fits the end-to-end origin slope. Which prediction would you test at 400 m and why?

▼ **Compare local slopes and collect an independent reading**

The final two intervals both lose 0.2 cubic millimetres per 100 m. Continuing that local trend predicts 1.5 cubic millimetres at 400 m. The origin-to-300 m slope predicts about 1.73. The difference is experimentally distinguishable only with adequate repeatability. Larger initial roughness, a forming transfer layer, drift in cleaning or a change of counterface can each generate an offset. Inspect surface images and repeat new and preconditioned coupons. A single cumulative curve suggests a running-in stage but does not prove its microscopic mechanism.

École polytechnique fédérale de Lausanne · A parameter-free mechanistic model of the adhesive wear process of rough surfaces in sliding contact

Design and analysis of machine components · 4. Fatigue crack growth, optical inspection and the meaning of remaining life

A polished test coupon contains a measurable crack and experiences repeated tensile cycles. A stress–life curve for a polished, initially uncracked specimen cannot directly give the remaining growth interval from this measured state. Use calibrated crack-growth data to predict a laboratory observation and quantify what the inference depends on.

Connect load history to a crack-tip driving range

Outside a small process zone the leading elastic opening stress near an ideal crack tip scales like $K_I/\sqrt{2\pi r}$. The dimension of K_I is stress times square root of length. In a geometrically similar family under remote uniaxial tension, linearity in stress and similarity in dimensions give

$$K_I = Y\sigma\sqrt{\pi a},$$

where the dimensionless geometry factor depends on crack shape, relative crack length and loading. Define whether a denotes crack depth, half of a central crack, or an edge crack length before using a calibration.

Under constant-amplitude cycles at a fixed stress ratio, stable intermediate-rate measurements often obey the empirical Paris relation. With approximately constant Y, separation of variables explains why successive millimetres of growth take different numbers of cycles:

$$\frac{da}{dN} = C(\Delta K)^m = C(Y\Delta\sigma\sqrt{\pi})^m a^{m/2},$$

$$N = \frac{a_f^{1-m/2} - a_i^{1-m/2}}{C(Y\Delta\sigma\sqrt{\pi})^m(1-m/2)}.$$

For exponent two the integral instead gives a logarithm. The units of the coefficient depend on both exponent and stress-intensity units. A coefficient expressed with MPa and metres cannot be inserted unchanged with Pa and millimetres. Use positive growth range and verify a physically increasing final crack length gives positive cycles. Near threshold, near fast fracture, at changing load ratio or in large-scale yielding the assumptions leading to this simple expression require reconsideration.

Predict two distinctly different stages of growth

Take an illustrative constant geometry factor 1.10, range 80 MPa, coefficient 10^{-11} in metres per cycle per cubed MPa square-root metre, and exponent three. The driving range grows from 4.93 to 9.87 MPa square-root metres as the crack grows from 1 to 4 mm. Integration gives 833,350 cycles. The instantaneous growth rate at 4 mm is eight times the rate at 1 mm. Replacing the whole history by the initial rate would significantly overpredict the number of cycles. Using the final rate would err in the opposite direction.

At maximum tensile stress 120 MPa, the maximum stress intensity at 4 mm is 14.8 MPa square-root metres. A terminal criterion based on a known fracture toughness uses the maximum intensity while a growth calculation uses the range. Those quantities answer different experimental questions. Finite width can raise the geometry factor with increasing crack size. In that case integrate $1/[C\Delta K(a)^m]$ on the measured geometry instead of treating the earlier constant value as universal. Refine the length increments until the predicted cycle count is stable compared with optical measurement uncertainty.

Link an optical length to a calibrated loading history

Use the same reference marks and magnification at each inspection and record loading cycles including interruptions. Surface observations describe a surface trace, which can grow differently from a subsurface front. Locally estimate growth rate from two nearby length readings divided by their cycle separation and assign a representative crack length within the interval. Widely spaced readings average a rapidly changing rate; too narrow a spacing lets registration error dominate the difference. Repeat the fit with both larger and smaller intervals and inspect residuals.

Compare the bonded-overlap stress study with a nominal uniform-stress assumption. Its smooth transfer field supplies loading information upstream of a crack-tip analysis.

Use the inverse crack-length plot to plan a coupon observation

Plot accumulated cycles vertically and crack length horizontally. The slope gives the number of additional cycles per increment of length, which falls as the crack grows. A flatter curve near the end therefore means faster crack growth. Increasing stress range compresses the entire vertical axis because the constant-amplitude Paris equation is separable. Changing only the initial crack size removes an especially slow portion of the growth history. Try a small change at the initial end and then the same absolute change at the final end; compare how strongly each affects predicted cycles.

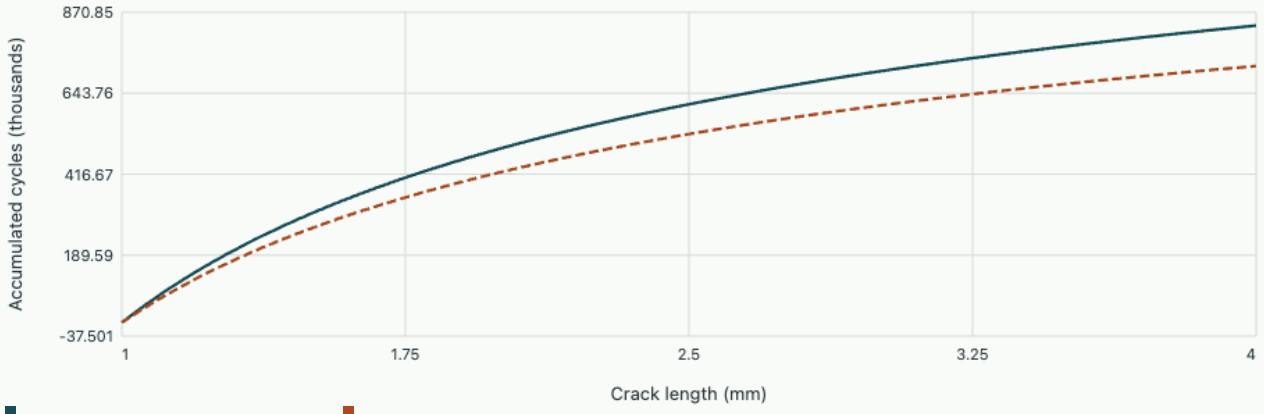
Open the live experiment

Try the calculation

Stress range (MPa) 80

Initial crack length (mm) 1

Final crack length (mm) 4



Growth over the selected length interval takes 833,349 cycles at the fitted intermediate-growth law. The endpoint growth rate increases by a factor 8.00.

On a real finite-width coupon the geometry factor can also change with length. That change alters the curvature of the graph and cannot generally be represented by rescaling the cycle axis. A lack of collapse after rescaling curves from two different loads may indicate a changing stress ratio, crack closure or a load-dependent material response. Recheck cycle counting and optical length units before selecting a more elaborate law.

Tutorial · Partition the predicted interval in half by crack length

For the coupon above estimate cycles spent between 1 and 2 mm and between 2 and 4 mm. Explain why the longer second length interval takes less time.

▼ **Compare the differences of reciprocal square roots**

For exponent three the cycle count is proportional to $a_i^{-1/2} - a_f^{-1/2}$. The first interval occupies about 58.58 percent of the total, or 488,200 cycles. The second takes about 345,200 cycles. As the stress-intensity range rises with square root of crack size, the cubed growth law accelerates faster than length itself. These fractions are independent of the common coefficient provided it truly stays the same. A rate plot offers a separate check: doubling length at constant stress increases the local rate by $2^{3/2}$.

Tutorial · Determine sensitivity to one calibration error

Load calibration overestimates the true stress range by five percent. All other quantities are independently correct. Find the resulting error in the predicted growth interval, and compare with a ten-percent coefficient error.

▼ **Use the inverse growth-law dependence**

Using the high stress gives a cycle count multiplied by $1.05^{-3} = 0.864$, about 13.6 percent low. Using a coefficient ten percent high gives a factor of $1/1.10 = 0.909$, about 9.1 percent low. A stress calibration error can also bias the fitted coefficient if the same load cell supplied calibration data. Shared errors should then be propagated jointly rather than assumed independent. Agreement obtained using one calibration frame does not guarantee transfer to a second one.

Tutorial · Distinguish no detected crack from a zero-length crack

An instrument detects a reference flaw of 0.8 mm only intermittently. A student inserts zero as the initial size after an image returns no indication and reports an infinite life for exponent three. Explain the mathematical result and design a more informative coupon test.

▼ **Identify the extrapolation and its observational limit**

The integral diverges at zero because the ideal long-crack power law predicts an arbitrarily small rate at vanishing crack length. The experiment supplied only a non-detection at finite resolution. Crack initiation, small-crack behaviour and detection probability lie outside the fit. Seed coupons with measured flaws spanning the optical threshold, take repeated coded images, and compare against a second length measurement. Report a detection curve and a conditional range of growth times for resolved starting sizes. That produces a testable inference without interpreting an undetected flaw as zero.

MIT 3.35 · fatigue crack growth lecture and laboratory resources

Design and analysis of machine components · 5. How does a converging oil film support a load?

Two plates have a small fixed angle. The lower plate moves while the upper pad stands still. Derive the pressure from volume conservation and local shear, then change film height and speed to see why a wedge develops lift.

Combine pressure flow and wall-driven flow before imposing continuity

Let x point along the moving surface from the wide inlet h_1 to the thin outlet h_2. The film has constant density and viscosity μ. It is shallow compared with its length L, and the pad is broad enough to neglect side leakage. Local steady momentum balance is

$$\mu u_{yy} = dp/dx.$$

Integrate across the gap with $u(0) = U$ and $u(h) = 0$. The linear boundary-driven velocity combines with a parabolic pressure-driven term. Integration of that velocity gives the volume flow per unit width

$$q = Uh/2 - \frac{h^3}{12\mu}\frac{dp}{dx}.$$

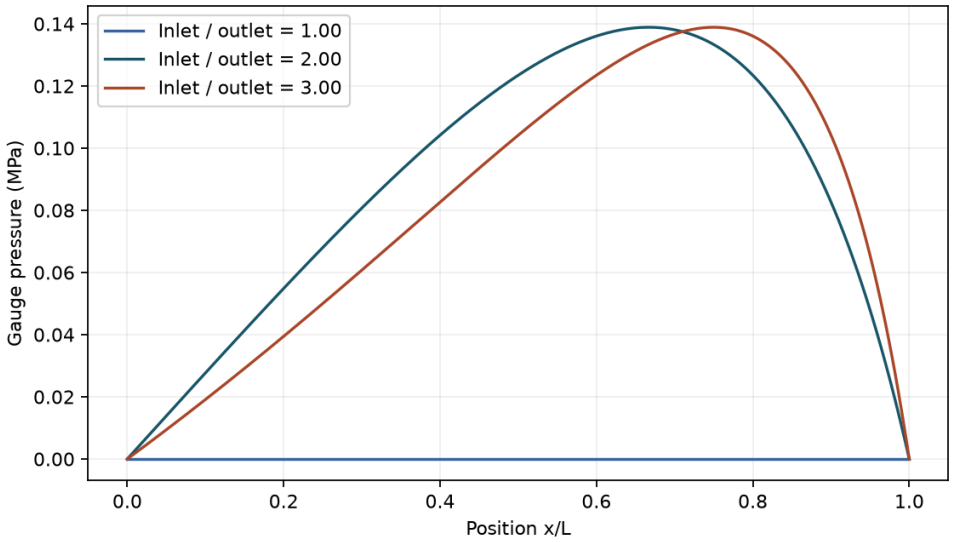
The same q passes every station. Introduce $h_* = 2q/U$, making $dp/dx = 6\mu U(h - h_*)/h^3$. Equal ambient gauge pressures at both ends require the integral of the gradient over x to vanish. For a linearly tapering film convert integration from x to h and obtain $h_* = 2h_1h_2/(h_1 + h_2)$. The pressure maximum lies where $h = h_*$. Integrating once more gives a direct pressure profile:

$$p = \frac{6\mu UL}{h_1 - h_2}\left[\frac{1}{h} - \frac{1}{h_1} - \frac{h_*}{2}\left(\frac{1}{h^2} - \frac{1}{h_1^2}\right)\right].$$

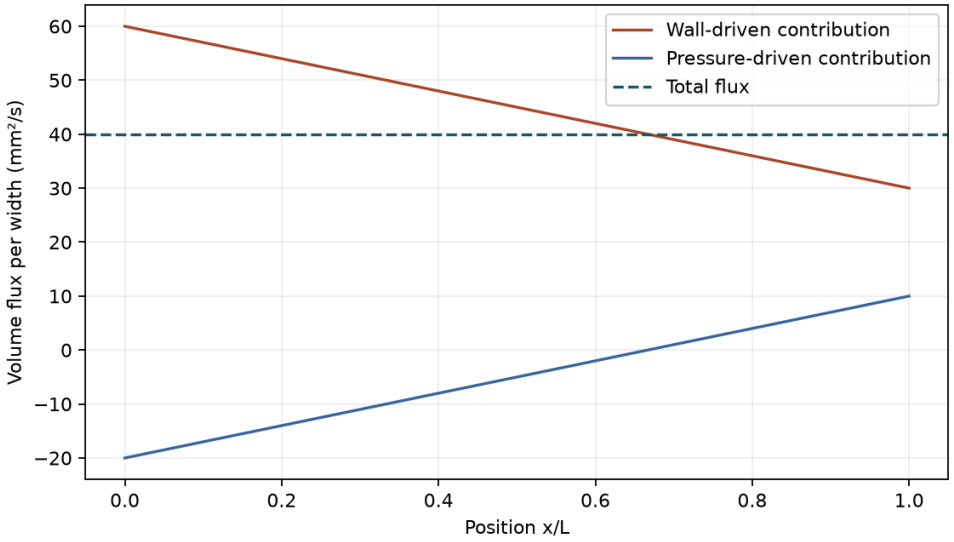
This form subtracts nearby terms in nearly parallel gaps. The notebook uses the equivalent factored result $p = 6\mu UL(h_1 - h_2)s(1 - s)/[(h_1 + h_2)h^2]$, where $s = x/L$. It remains well behaved as the slope tends to zero.

Measure the pressure gradient where the Couette flux matches the actual flow

Choose inlet and outlet heights of 120 and 60 µm, length 40 mm, sliding speed 1 m/s and viscosity 0.050 Pa s. Then $h_* = 80$ µm and the flux per unit width is 4.00×10^{-5} m²/s. The pressure maximum is two thirds of the length from the inlet. For these data it is approximately 0.139 MPa. At that point the pressure-driven part of the flow vanishes locally even though the accumulated pressure above ambient is greatest. This is the distinction between a local gradient and an absolute load-carrying pressure.



The outlet thickness is held fixed in this comparison. A parallel gap carries fluid by shear while producing zero pressure load under the stated boundary conditions.



The pressure-driven contribution opposes motion near the wide inlet and assists it nearer the outlet. Its sign changes at the pressure maximum.

Exercise · scale both clearances

Double inlet and outlet thickness while holding taper ratio, viscosity, length and speed fixed. What becomes of the pressure and the flow?

▼ **Work through the solution**

The factor $(h_1 - h_2)/(h_1 + h_2)$ is constant, while the squared local height quadruples. Pressure and load per width fall to one quarter. The harmonic-mean height h_* doubles, so the through-flow doubles.

Exercise · interpret a reversed pad

What does the algebra predict for a diverging gap with the same wall direction? Under what condition does that extrapolation fail?

▼ Work through the solution

The sign of $h_1 - h_2$ reverses, giving negative gauge pressure between the equal-pressure boundaries. Small subambient pressures can exist in a filled liquid. Once the absolute pressure reaches the cavitation threshold or gas enters the film, a full-film solution cannot represent the cavity or its mass balance. The notebook therefore keeps its default sweep in the converging regime.

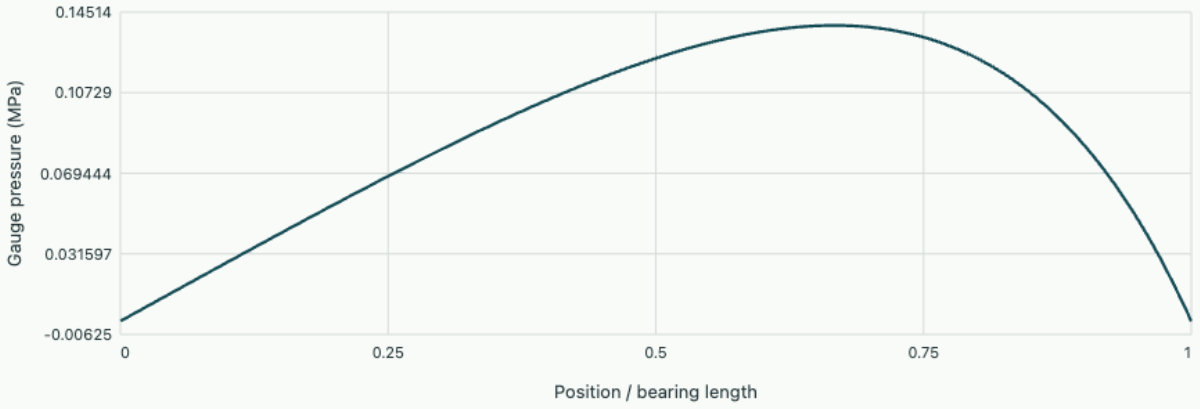
Interactive calculation · Slider pressure

Change the physical and numerical inputs of the notebook model. Compare the updated curves and balance checks with the recorded notebook figures. Each plot uses the same equations and assumptions documented in that notebook; its readout names any analytical or reduced-model reference.

Open this live calculation

Try the calculation

Inlet / outlet gap	2
Outlet gap (µm)	60
Sliding speed (m/s)	1



Reynolds pressure

Peak pressure 0.1389 MPa at x/L=0.667. Flux 40.000 mm²/s. Length40 mm, viscosity0.05 Pa s, incompressible isoviscous film, ambient-pressure ends.

Download this example’s editable notebook

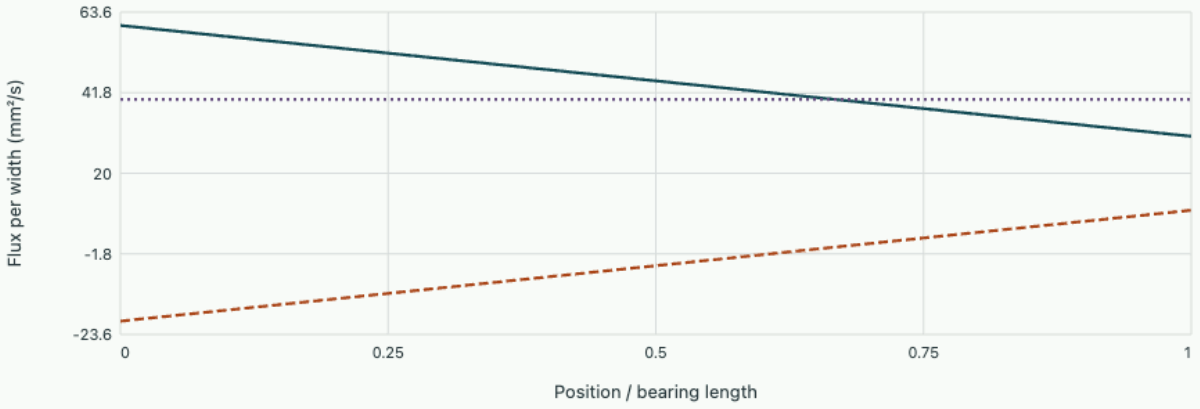
Interactive calculation · Slider flux

Change the physical and numerical inputs of the notebook model. Compare the updated curves and balance checks with the recorded notebook figures. Each plot uses the same equations and assumptions documented in that notebook; its readout names any analytical or reduced-model reference.

Open this live calculation

Try the calculation

Inlet / outlet gap	2
Outlet gap (µm)	60
Sliding speed (m/s)	1



Peak pressure 0.1389 MPa at x/L=0.667. Flux 40.000 mm²/s. Length40 mm, viscosity0.05 Pa s, incompressible isoviscous film, ambient-pressure ends.

Download this example's editable notebook

Open editable notebook Download .ipynb

Design and analysis of machine components · 6. From a laboratory fatigue curve to a machined shaft

A fatigue limit measured on a polished specimen does not directly describe a shoulder on a production shaft. Track where each correction enters and how the measured torque or bending history changes the inferred life.

Keep coupon, component and nominal stress distinct

Take a quoted fully reversed specimen strength $S'_f(N_*)$ at a stated reference life N_*. A component with a different finish, stressed volume, environment or survival requirement has different statistics of crack initiation. Here use experimentally supplied finish and size factors k_a, k_b for this process and diameter. They act on allowable amplitude. Factors calibrated at one life need supporting data before use over another regime.

$$S_f(N_*) = k_a k_b S'_f(N_*),$$

$$K_f = 1 + q(K_t - 1).$$

The elastic concentration K_t is the local-to-nominal peak from the geometry. Notch sensitivity q transfers the nominal alternating component to an effective fatigue amplitude. Its values zero and one recover no notch penalty and the full elastic concentration. Keep surface correction on the material side and stress concentration on the demand side. Applying the same notch debit to both counts it twice.

Move a load cycle to the reference mean

For a cycle with nominal maximum and minimum stresses, their half-difference is σ_a and their mean is σ_m. With elastic local response, a specified tensile-mean Goodman approximation gives a fully reversed equivalent demand. Adopt zero benefit from compressive mean for this example. A separate maximum-stress check against yield must precede this calculation.

$$a_{eq} = \frac{K_f \sigma_a}{1 - K_t \max(\sigma_m, 0)/S_u},$$

$$N_f = N_* \left(\frac{S_f(N_*)}{a_{eq}} \right)^b.$$

Take $S_u = 650$ MPa, $S'_f = 280$ MPa at a million cycles, $k_a = 0.8$, $k_b = 0.88$, $q = 0.8$, $K_t = 1.65$ and exponent $b = 5$. A 100 MPa amplitude and 30 MPa mean give $K_f = 1.52$, component strength 197.1 MPa, and equivalent demand 164.5 MPa. The illustrative power law predicts about 2.47 million repetitions. It has no endurance knee. Extrapolation to small stresses would require additional material data.

What did the heat treatment change?

A surface-hardened shoulder can have hardness and residual stress gradients across the crack-initiation zone. A coupon from the core measures neither directly. Select data from the production condition, measure the local radius and identify any residual tensile mean before assigning corrections from a handbook.

Turn a measured record into cumulative damage

Rainflow counting pairs stress reversals to recover amplitudes, means and fractional counts. Analyze an explicitly finite record and retain its half cycles. A periodic mission should instead be closed consistently across its boundary. Linear accumulation then sets $D = \sum_i n_i / N_i$ for this model. Sequence sensitivity and residual stress relaxation require independent checks.

Interactive experiment · Move a load cycle across the Goodman diagram

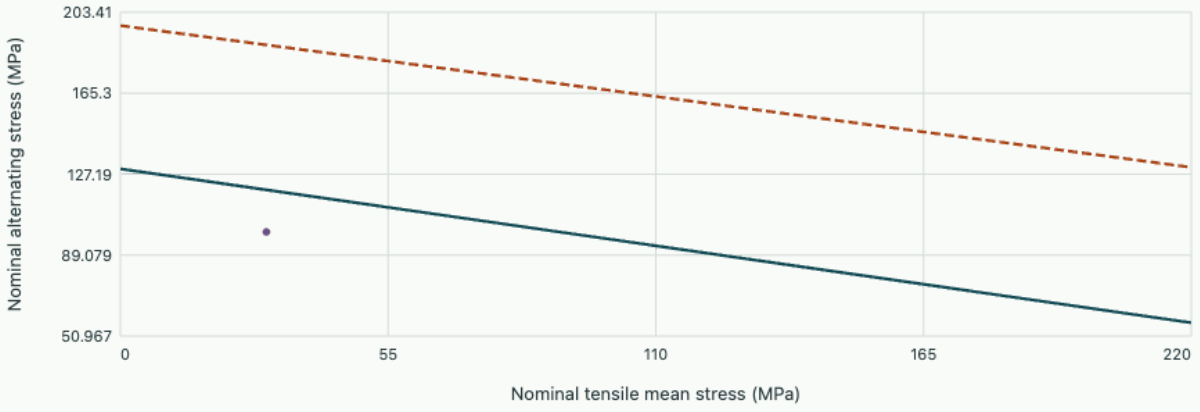
Set the mean and alternating nominal stress to place a single constant-amplitude cycle in the diagram. Its boundary corresponds to one million cycles in the illustrative material law above. Change the surface factor and elastic concentration separately to see which acts on the available strength and which on the applied demand. The unnotched comparison uses the same finish and size corrections.

Begin with a 30 MPa mean and a 100 MPa amplitude. Raising the surface factor from 0.80 to 0.90 multiplies predicted life by approximately 1.80. Increasing the mean stress lowers the admissible amplitude even when the stress range is unchanged. This interactive diagnostic treats one repeating cycle. The OneSimulation notebook below counts the distinct cycles of a finite load record before accumulating damage. The displayed law assumes an elastic local response, includes no endurance knee and needs a separate yield check.

Open this experiment online

Try the calculation

Surface factor	**0.8**
Elastic concentration Kt	**1.65**
Nominal mean stress (MPa)	**30**
Nominal alternating stress (MPa)	**100**



One-million-cycle Goodman boundary
Same component without a notch
Selected nominal cycle

Kf = 1.520, corrected strength 197.12 MPa, equivalent reversed amplitude 164.53 MPa. Illustrative constant-amplitude life: 2.469e+6 cycles; damage per cycle 4.051e-7. This finite-life model has no endurance knee and does not check yielding.

Continue in the OneSimulation notebook · Download its Python model

OneSimulation experiment · Changing the measured finish factor changes damage from the same counted load record

Run the notebook with OneSimulation 0.2.0, NumPy, SciPy and Matplotlib. Change the input data and compare your run with the recorded result.

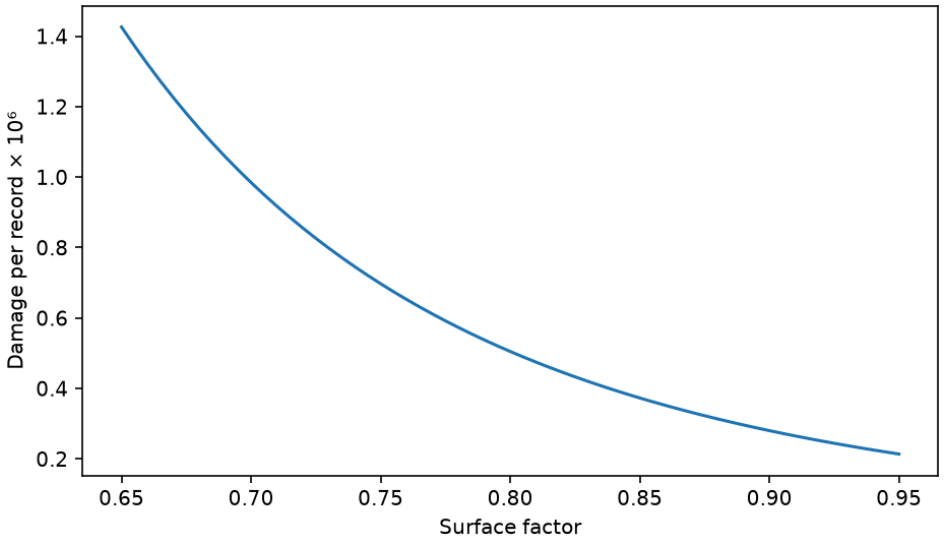
```
"""Compare the effect of measured surface factors at one shoulder under a finite record."""
import numpy as np
import matplotlib.pyplot as plt
from onesim.physics.solid_mechanics.fatigue.models.cycles.cycle_counting import fatigue_cycles

nominal = np.array([0,110,-40,85,-65,130,-55,80,0.0])  # MPa
cycles = list(fatigue_cycles(stress=nominal))
surface = np.linspace(0.65, 0.95, 61)
kt, notch_sensitivity, ultimate = 1.65, 0.8, 650.0  # ultimate in MPa
kf = 1+notch_sensitivity*(kt-1)
# Illustrative calibrated finite-life power law. Reference life: one million cycles.
def block_damage(ka):
    strength = 280*ka*0.88  # MPa at reference life, after size correction
    damage = 0.0
    for cycle in cycles:
        mean = kt*max(cycle.mean_stress, 0)
        alternating = kf*cycle.stress_amplitude/(1-mean/ultimate)
        damage += cycle.count/1e6*(alternating/strength)**5
    return damage

damage = np.array([block_damage(value) for value in surface])
print("counted cycle equivalents", sum(cycle.count for cycle in cycles))
for ka in [0.70, 0.90]:
    print(f"surface factor {ka:.2f}: damage per record = {block_damage(ka):.4e}")
```

Reference run

```
counted cycle equivalents 4.0
surface factor 0.70: damage per record = 9.8463e-07
surface factor 0.90: damage per record = 2.8025e-07
```



Changing the measured finish factor changes damage from the same counted load record

> Download editable notebook · Download Python script

Exercises on fatigue evidence

Polish the previous shaft so that k_a increases from 0.80 to 0.90. How does the predicted life change at the same cycle and exponent?

▼ **Worked solution**

At fixed mean correction and amplitude, N_f scales as k_a^5. The multiplier is $(0.90/0.80)^5 = 1.80$. Changing K_t as well would require a separate geometry update.

Two records have equal peaks at 160 MPa. The first cycles between 0 and 160, the second between –160 and 160 MPa. Which has larger corrected demand here?

▼ **Worked solution**

The first gives $1.52(80)/(1 - 1.65(80)/650) = 152.6$ MPa. The second gives $1.52(160) = 243.2$ MPa. A peak-only count would discard the dominant difference in range.

Further reading · Budynas and Nisbett, Shigley's Mechanical Engineering Design, 10th ed., Chapters 2, 5 and 6.

Design and analysis of machine components · 7. Will tooth-root bending or flank contact govern a gear pair?

The same torque loads the tooth root and the mating flanks. Calculate two distinct demands, carry the same duty blocks through both, and see why doubling the face width gives different gains.

Resolve a torque into tooth force

Take straight external spur gears with a pitch diameter d and pressure angle α. At the pitch circle, transmission of torque T fixes $F_t = 2T/d$. The nominal normal force is $F_n = F_t/\cos\alpha$. Use one conservative single-tooth load factor K for the example. Real cyclic mesh sharing, alignment, speed and surface finish require data specific to the teeth.

For root bending, replace the tooth by a cantilever section. Its bending stress scales as $6F_t\ell/(bt^2)$ for lever arm ℓ, face width b and root thickness t. A dimensionless shape factor Y collects the section geometry into a module-based estimate. For contact take elastic nonconforming line contact with reduced modulus E' and effective local radius R'. Integrating the Hertz pressure $p_0\sqrt{1-(x/a)^2}$ over width $2a$ gives $KF_n/b = \pi a p_0/2$. Compatibility with the local quadratic gap gives $a^2 = 4KF_nR'/(\pi bE')$. Eliminating a gives

$$\sigma_F = \frac{KF_t}{bmY},$$

$$p_H = \sqrt{\frac{E'KF_n}{\pi bR'}}.$$

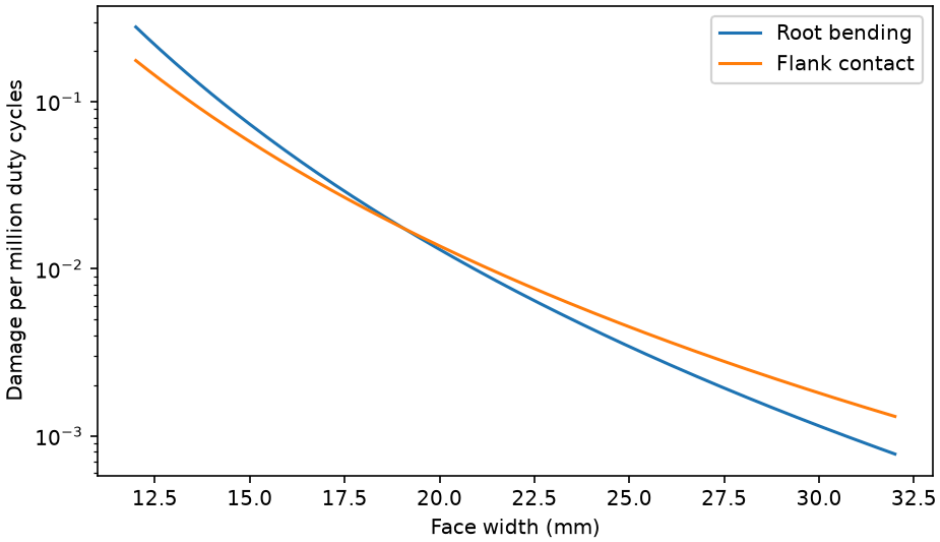
Follow one reducer through two life checks

Use $T = 80$ N m, $d = 80$ mm, $m = 2.5$ mm, $b = 16$ mm, $Y = 0.32$, $\alpha = 20^\circ$, $K = 1.30$, $E' = 115$ GPa, and $R' = 8$ mm. The nominal tangential force is 2000 N. The model gives $\sigma_F = 203.1$ MPa and $p_H = 889.5$ MPa. The contact pressure is a compressive surface quantity and its pitting limit must come from a contact-fatigue test.

For an illustrative material pair take $N_F = 10^7(220\,\mathrm{MPa}/\sigma_F)^6$ and $N_H = 10^7(950\,\mathrm{MPa}/p_H)^{10}$. Spend 80% of cycles at full torque and 20% at half. Since root stress scales with T and contact pressure with $\sqrt{T}$, damage per million cycles is $0.1(203.1/220)^6[0.8 + 0.2/64] \simeq 0.050$ for bending and $0.1(889.5/950)^{10}[0.8 + 0.2/32] \simeq 0.042$ for contact. In this assumed pair, root fatigue uses more of its life. A change of flank hardness can change which limit controls.

Where an industrial rating requires additional evidence

These two models explain scaling and identify required inputs. A standard rating requires the applicable tooth geometry, quality, dynamic and distribution factors, material condition, lubrication, life and reliability provisions. The illustrative constants are not a substitute for those inputs.



Root and flank damage have different sensitivities to face width. The worked 16 mm gear is root-governed with the example material data.

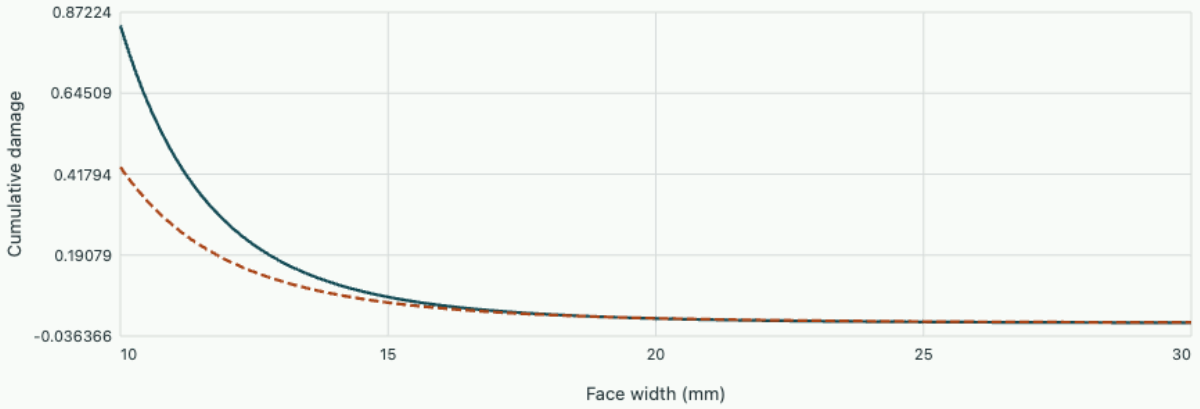
Interactive calculation · Carry a duty cycle through root and flank laws

Change torque and mission length, then compare the different face-width exponents. Half-torque cycles fill the remainder of the specified full-torque fraction.

Open this live calculation

Try the calculation

Full torque (N m)	80
Full-torque cycle fraction	0.8
Mission (million cycles)	1



At16 mm, root stress 203.125 MPa and contact pressure 889.534 MPa. Half-torque remainder; the example fatigue laws have exponents6 and10.

Size the face width for one duty and two failure mechanisms

Retain the pitch diameter, module, normal force, pressure angle and radii of the worked gear. A design brief specifies 20 million pinion load cycles, of which 80% are at 80 N m and 20% at 40 N m. It supplies an application load multiplier $K_A = 1.15$, a measured dynamic amplification $K_v = 1.10$, and face-distribution factors $K_{bF} = 1.18$ for root stress and $K_{bH} = 1.12$ for pressure. These illustrative factors replace the earlier single K and apply to load. Define $K_F = K_A K_v K_{bF} = 1.4927$ and $K_H = K_A K_v K_{bH} = 1.4168$. Root stress is proportional to K_F. Contact pressure is proportional to $\sqrt{K_H}$.

On the material side, take 10-million-cycle reference stresses of 260 MPa at the root and 1050 MPa in contact. The design data supply two independent strength multipliers for each mechanism. Root strength is $S_F = 260(0.95)(0.97) = 239.59$ MPa, and flank strength is

$S_H = 1050(0.98)(0.95) = 977.55$ MPa. Use the laboratory exponents 6 and 10 against these corrected strengths. This completely defines the teaching model from measured loads to allowable cumulative damage $D_{lim} = 0.60$. The factors are design inputs, with no claim to implement a particular industrial standard.

At $b_0 = 16$ mm the full-load root stress is 233.23 MPa and the full-load peak contact pressure is 928.64 MPa. The half-torque root stress is 116.62 MPa; the half-torque pressure is 656.64 MPa. Summing counted cycles divided by their mechanism-specific lives gives

$$D_F(b_0) = 2(0.8 + 0.2/2^6)(233.23/239.59)^6 = 1.367,$$

$$D_H(b_0) = 2(0.8 + 0.2/2^5)(928.64/977.55)^{10} = 0.965.$$

Both exceed the design limit despite the full-load stresses being below their reference strengths. At fixed distribution factors $D_F \propto b^{-6}$ and $D_H \propto b^{-5}$. The separate required widths are $16(1.367/0.60)^{1/6} = 18.35$ mm and $16(0.965/0.60)^{1/5} = 17.60$ mm. Choose a 20 mm available width. Its damages are 0.358 and 0.316. The margin is then traceable to one specified mission, one set of material data and one geometry. Check each gear separately where tooth counts and hence cycle counts differ.

A contact pattern test at 20 mm finds that the flank-distribution factor is 15% greater than assumed. Does the adopted width still meet the stated target? What width would restore it if all other inputs stayed fixed?

▼ **Worked solution**

Pressure grows by $\sqrt{1.15}$. Damage therefore grows by 1.15^5, taking the flank result from 0.316 to 0.636, just above 0.60. Required width becomes about $20(0.636/0.60)^{1/5} = 20.23$ mm. The original root result is unchanged under the problem's assumption. A new width needs a new contact-pattern check if alignment changes.

Exercises on design changes

Double the face width while keeping load sharing uniform. What happens to each computed damage?

▼ **Worked solution**

Root stress halves, reducing bending damage by $2^6 = 64$. Contact stress falls by $\sqrt{2}$, reducing contact damage by $2^5 = 32$. A widened but poorly aligned mesh can lose much of this benefit.

Why might a gear with safe tooth bending still exhibit pitting after its lubricant changes?

▼ **Worked solution**

The root check depends chiefly on tangential load and root section. Lubrication, film thickness, roughness and local sliding influence the durability of the contacting surfaces. A bending margin does not supply a contact life or a wear prediction.

Further reading · Budynas and Nisbett, Shigley’s Mechanical Engineering Design, 10th ed., Chapters 13–14 and 18.

Design and analysis of machine components · 8. Redirecting a shaft line with bevel gears and worm drives

Two transmissions can deliver the same speed reduction around a right angle yet require very different bearings and cooling. Derive the directions of the loads and the dependence of efficiency on lead angle.

Use the common pitch-cone apex

For perpendicular intersecting shafts and straight bevel teeth, matching peripheral speeds at a common cone distance R implies $d_1/d_2 = z_1/z_2$. The two cones meet along a shared generator, so $\delta_1 + \delta_2 = 90^\circ$ and $\tan\delta_1 = z_1/z_2$. Use the mean pitch diameter at the centre of the face to calculate forces. The transverse pressure-angle component $F_t \tan\alpha$ acts normal to the pitch cone. Resolve that component parallel and perpendicular to the shaft:

$$F_t = \frac{2T}{d_m},$$

$$|F_a| = F_t \tan\alpha \sin\delta,$$

$$|F_r| = F_t \tan\alpha \cos\delta.$$

For a 20-tooth pinion driving a 40-tooth wheel at 90°, $\delta_1 = 26.57^\circ$. At mean pinion diameter 64 mm and torque 80 N m, $F_t = 2500$ N. With a 20° pressure angle this gives radial and axial magnitudes of 814 N and 407 N. The wheel sees the equal and opposite contact force in space. Its own shaft axes exchange radial and axial roles. Use the direction of rotation and the identified tooth flank to assign signs.

Unroll one turn of the screw

A cylindrical worm with z_s starts advances the wheel z_s teeth each revolution. Consequently $i = z_w/z_s$. Unwrap the mean worm cylinder into a plane. One turn advances axial lead L over distance πd_w, so $\tan\gamma = L/(\pi d_w)$. For sliding contact represented by an effective friction angle $\phi = \arctan\mu_{eff}$, equilibrium along the inclined plane requires input effort proportional to $\tan(\gamma + \phi)$ instead of $\tan\gamma$ in the ideal case. The worm-driving efficiency and heat rejection estimate are

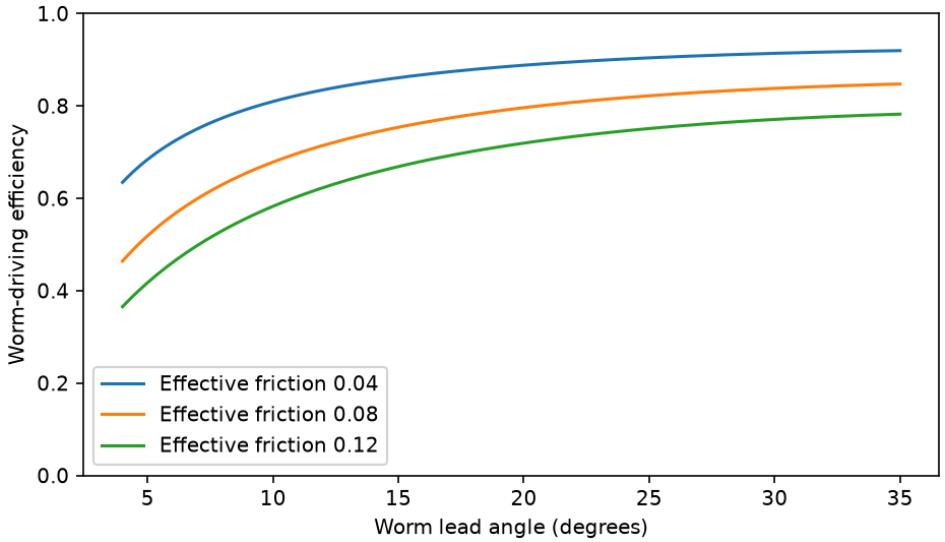
$$\eta = \frac{\tan\gamma}{\tan(\gamma + \phi)},$$

$$\dot{Q}_{loss} \simeq (1 - \eta) P_{in}.$$

For $\gamma = 12^\circ$, $\mu_{eff} = 0.08$ and 2 kW input, efficiency is about 0.714 and heat production about 0.57 kW before bearing and seal losses. A 40-tooth wheel with a two-start worm has ratio 20. Efficiency depends on lubricant and operating temperature. A ratio alone gives no thermal rating.

Can the load backdrive the worm?

Within this ideal Coulomb model, reversed sliding is possible when $\gamma > \phi$. A static friction estimate alone is inadequate as a holding specification when vibration, lubrication, wear or operating direction change.



Efficiency rises with lead angle over this range. Each curve assumes a constant effective friction coefficient.

Interactive calculation · Change lead angle, friction and heat loss

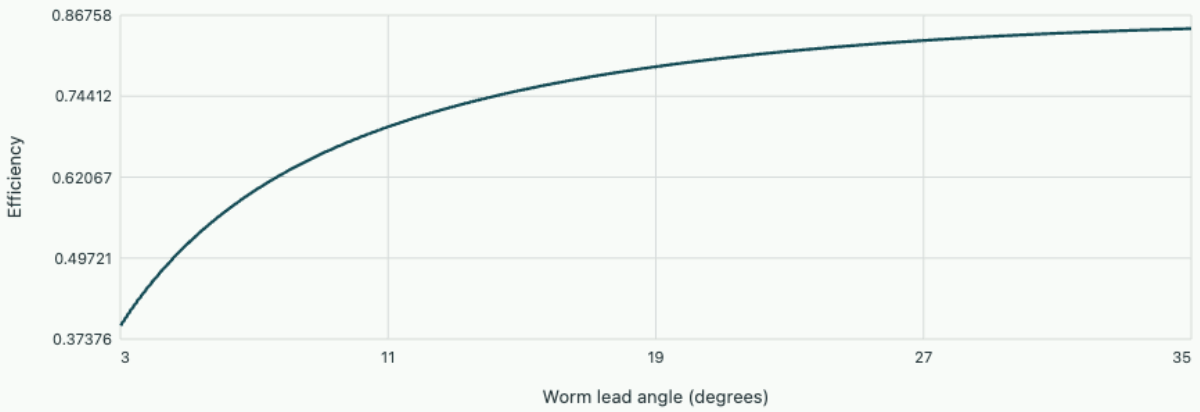
Raise friction while holding transmitted geometry fixed. The input power changes thermal loss but not the dimensionless efficiency in this idealised model.

Open this live calculation

Try the calculation

Effective friction coefficient **0.08**

Input power (W) **1000**



Driving worm

At15° lead angle efficiency=0.754, heat loss=246.4 W. Reverse self-locking in this ideal friction model requires lead below 4.574°.

Exercises on geometry and heat

A bevel pair has equal numbers of teeth. Which separating components on either shaft become equal?

▼ **Worked solution**

Both pitch cones have 45° half-angle, so the axial and radial components are each $F_t \tan\alpha/\sqrt{2}$. The tangential component still carries all transmitted torque.

The worm housing can reject 0.40 kW at the allowable temperature. What continuous input power fits the example model?

▼ **Worked solution**

Divide rejection by the loss fraction. With $\eta \simeq 0.714$, $P_{in} \simeq 1.40$ kW. An operating-temperature efficiency map and other losses are needed before specifying the drive.

Further reading · Budynas and Nisbett, Shigley's Mechanical Engineering Design, 10th ed., Chapter 15.

Design and analysis of machine components · 9. Can a mating pin enter every accepted hole?

A rectangular coordinate tolerance and a circular position tolerance accept different parts. Carry a hole requirement from its datum reference frame through its material condition to an actual inspection decision.

Construct the measurement before assigning error

A primary planar datum fitted by the specified simulator fixes normal translation and two tilts. A secondary side plane arrests one lateral translation and rotation about the primary normal. A tertiary end stop removes the final sliding freedom. Basic coordinates specify the intended hole axis in this frame. Datum order matters when the real surfaces are imperfect. A three-point seating scheme has different sensitivity to dirt or local form than a least-squares plane.

For a hole perpendicular to the primary plane in a thin plate, suppose tilt has been independently shown negligible. If the fitted centre departs from its basic coordinates by $\Delta x, \Delta y$, the smallest circular zone centred on the basic axis that contains it has diameter p. In a deep hole the entire derived axis must remain in the specified cylindrical zone, so a single face reading is insufficient.

$$p = 2\sqrt{\Delta x^2 + \Delta y^2}.$$

Why a larger internal feature can gain position tolerance

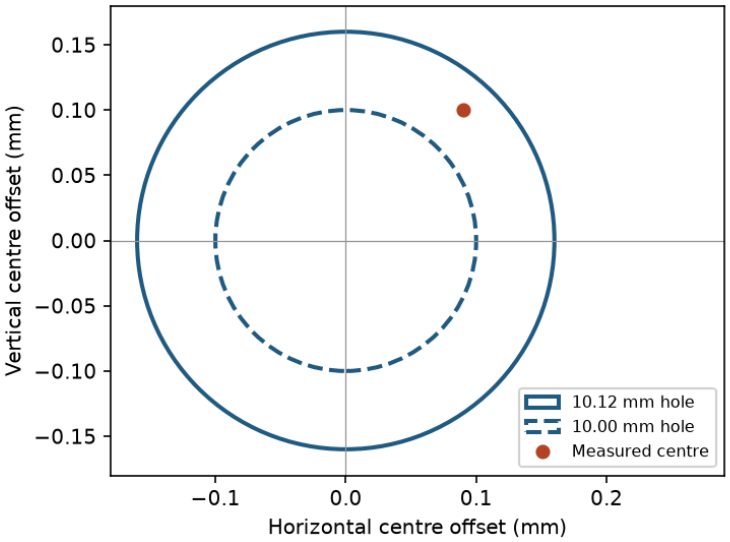
Suppose a mating design uses a hole between 10.00 and 10.20 mm and specifies a 0.20 mm diametral position tolerance at maximum material condition relative to fixed datums. For an internal feature maximum material is the smallest permissible diameter D_{MMC}. Holding a fixed virtual-condition boundary at the basic location gives

$$D_{VC} = D_{MMC} - t_{MMC} = 9.80 \text{ mm},$$
$$t_{allowed} = t_{MMC} + D_{actual} - D_{MMC}.$$

A measured 10.12 mm hole gains 0.12 mm diametral allowance. For offsets (0.09,0.10) mm its measured position is 0.269 mm, within the 0.32 mm allowance. An otherwise identical 10.00 mm hole fails the 0.20 mm position requirement. The smaller hole still satisfies its size limit. Under a rule without the material-condition modifier, the position allowance in this example stays 0.20 mm for both.

What a functional gauge actually checks

A suitably located 9.80 mm pin simulates the worst-case internal boundary only with the specified datum simulators engaged. Passing that pin alone does not check the hole’s upper size limit or all separate requirements on shape. The example assumes ideal form unless a test explicitly evaluates form.



The same measured centre lies inside the enlarged zone of a 10.12 mm hole and outside the zone of a hole at maximum material.

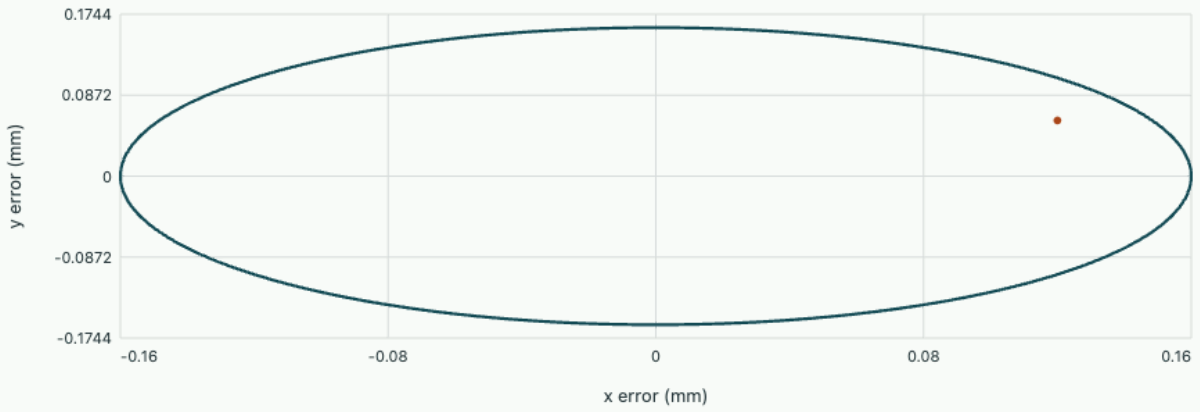
Interactive calculation · Move the measured centre within a material-dependent zone

Increase the measured hole diameter and inspect the allowed centre zone. Move the two measured offsets independently; acceptance uses their vector magnitude rather than separate coordinate limits.

Open this live calculation

Try the calculation

Actual hole diameter (mm)	**10.12**
Measured x error (mm)	**0.12**
Measured y error (mm)	**0.06**



Allowed centre zone Measured centre

Diametral position error 0.268 mm; allowed 0.320 mm: accepted. Virtual condition stays9.80 mm under the stated maximum-material rule.

Exercises on inspection decisions

A 10.08 mm hole has offsets (0.11,0.09) mm. Can the part be accepted on nominal readings?

▼ **Worked solution**

The diameter of its position zone is $2\sqrt{0.11^2 + 0.09^2} = 0.2843$ mm. The allowance is 0.28 mm. It narrowly fails on the stated values. With uncertainty comparable to 0.0043 mm, apply the drawing and inspection decision rule rather than rounding to two decimals before comparison.

A 30 mm long hole with nominal entrance centre is tilted by 0.3°. Where should a second measurement be taken?

▼ **Worked solution**

At the opposite end the centre departs by $30 \tan(0.3^\circ) = 0.157$ mm. The corresponding diametral deviation is about 0.314 mm if the axis is constrained to its basic orientation. Two widely separated centre measurements reveal tilt that an entrance-only check cannot see.

Further reading · Budynas and Nisbett, Shigley’s Mechanical Engineering Design, 10th ed., Chapter 20.

Account for compliance, contact and failure

Determine what the support actually imposes before sizing a shaft, flexure or connection. Check its static deformation, cyclic response and sensitivity to temperature.

1. Elastic curves and virtual work
2. Stress transformation and combined loading
3. Torsional stiffness of shafts and closed sections
4. From a strain rosette to principal stresses
5. Yield fronts and plastic bending capacity
6. Crack driving force and fracture assessment
7. Finding a displacement when no load acts in its direction
8. Why a bonded pair of strips bends under uniform heating
9. Static and couple balance of rotors
10. Virtual work and the forces hidden by a constraint
11. Action density, stress waves and the boundary that reflects them

Elastic curves and virtual work

Engineering mechanics

Predict a specified displacement from the bending-moment field and distinguish a stiffness limit from a stress limit.

FROM FIRST PRINCIPLES

Why cantilever stiffness scales as the inverse cube of span

Take a straight slender beam clamped horizontally, with a downward end force P. The empirical inputs are small-strain linear elasticity and the beam assumption that plane sections remain normal to the deflected centreline. Denote its centroidal bending rigidity by EI. With constant section and modulus the stress moment of the stretched and compressed fibres is proportional to curvature. With downward deflection positive, equilibrium of a cut at x from the clamp supplies

$$M(x) = P(L - x),$$
$$w''(x) = M(x)/(EI).$$

Curvature is the rate at which slope changes. Integrating it once gives the slope, and integrating slope gives deflection. Combine both integrations into a weighted curvature integral: curvature occurring near the clamp rotates a longer remaining segment than curvature near the tip. The clamp fixes initial slope and displacement to zero, hence

$$w(L) = \int_0^L (L - x) w''(x)\, dx = \frac{P}{EI} \int_0^L (L - x)^2 dx = \frac{PL^3}{3EI}.$$

This is a displacement prediction under small slopes, not just a stress limit. Its input $M = EIw''$ is established from individual fibre strains and the neutral-axis condition in the detailed derivation. Short beams can also require shear deflection.

Background · Second moment of area

$$I = \int_A y^2\, dA$$

Measure the transverse coordinate from the stated bending axis. This section property has units of length to the fourth power and enters the flexural rigidity.

DERIVATION AND APPLICATIONS

From the extension of a fibre to cantilever curvature and displacement

Why does section depth enter a bending calculation through a second moment of area? Take a straight prismatic cantilever with small slopes, uniform Young's modulus E, linear elastic response and slender dimensions. Euler–Bernoulli kinematics assumes that cross-sections remain plane and normal to the deflected centreline. Measure x from the clamp and transverse deflection w downward. Measure cross-section coordinate y downward from an initially arbitrary reference line. Sections separated by dx rotate relative to one another by $d\theta$; a fibre at y shortens relative to the reference line by $y\,d\theta$.

$$\epsilon_x(y) = \epsilon_0 - y\,d\theta/dx \simeq \epsilon_0 - y w''(x),$$
$$\sigma_x = E\epsilon_x.$$

With no imposed axial resultant, the integral of this normal stress must vanish. Writing $\bar{y} = A^{-1}\int_A y\,dA$, we obtain the strain of the reference line, and thereby identify the zero-strain fibre:

$$0 = N = \int_A E(\epsilon_0 - y w'')\,dA,$$
$$\epsilon_0 = \bar{y} w'',$$
$$\sigma_x = -E(y - \bar{y})w''.$$

Define bending moment positive for this downward curvature. Each fibre contributes its force times its lever arm about the neutral axis. Integrating introduces the centroidal second moment I:

$$M = -\int_A (y - \bar{y})\sigma_x\,dA = EIw'',$$
$$I = \int_A (y - \bar{y})^2\,dA.$$

A downward tip load P gives $M(x) = P(L - x)$ by equilibrium of the portion beyond the cut. For constant EI, integrate twice and enforce the actual clamp conditions $w(0) = w'(0) = 0$:

$$EIw' = P(Lx - x^2/2),$$
$$EIw = P(Lx^2/2 - x^3/6),$$
$$w(L) = PL^3/(3EI).$$

The slope and displacement constants follow from compatibility at the support. The zero moment at the free tip follows independently from the load on the right-hand portion. Plane sections can also remain plane under pure bending of a short specimen, but shear deformation near loads and supports can matter in short cantilevers.

WORKED EXAMPLE

Calculate a tip reading and maximum fibre strain

A rectangular steel strip has $L = 0.200$ m, $b = 20.0$ mm, $h = 8.00$ mm and $E = 200$ GPa. Depth h lies along the deflection direction, so $I = bh^3/12 = 8.53 \times 10^{-10}$ m⁴. A 10.0 N tip load produces $w(L) = 0.156$ mm. The clamp moment is 2.00 N m; its curvature 0.0117 m⁻¹ gives outer-fibre strain magnitude $(0.00400)(0.0117) = 46.9$ microstrain. The top is tensile and the bottom compressive for the stated load.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Vary the span to distinguish two sources of compliance

On the same strip double the free span to 0.400 m. Bending predicts eight times the displacement, or 1.25 mm; outer-fibre strain doubles to 93.8 microstrain. A nearly span-independent displacement offset would instead suggest motion of the clamp fixture, and a shear-displacement component at fixed load and section scales only linearly with span. The different scalings let two displacement readings test the assumptions behind a bending-only interpretation.

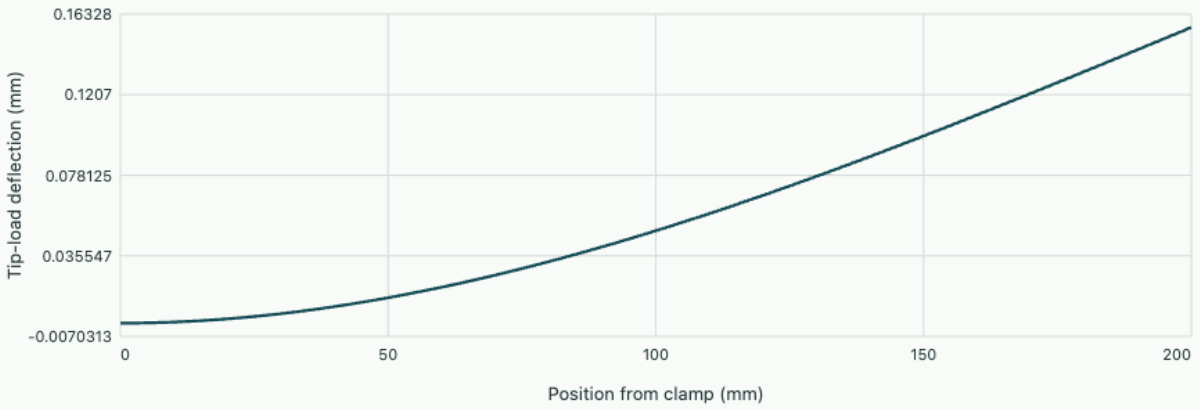
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Length (mm)	**200**
Section depth (mm)	**8**
Tip force (N)	**10**



Integrated curvature

Tip reading 0.1563 mm; clamp outer-fibre strain 46.87 microstrain. Width20 mm and E200 GPa.

Test a beam reduction against a solid in OneSimulation

A 120 mm cantilever of width 18 mm and modulus 70 GPa carries 30 N uniformly distributed over its end face. Make a prediction before using the slider. If the height changes from 12 to 24 mm, which changes more, the section area or its bending stiffness?

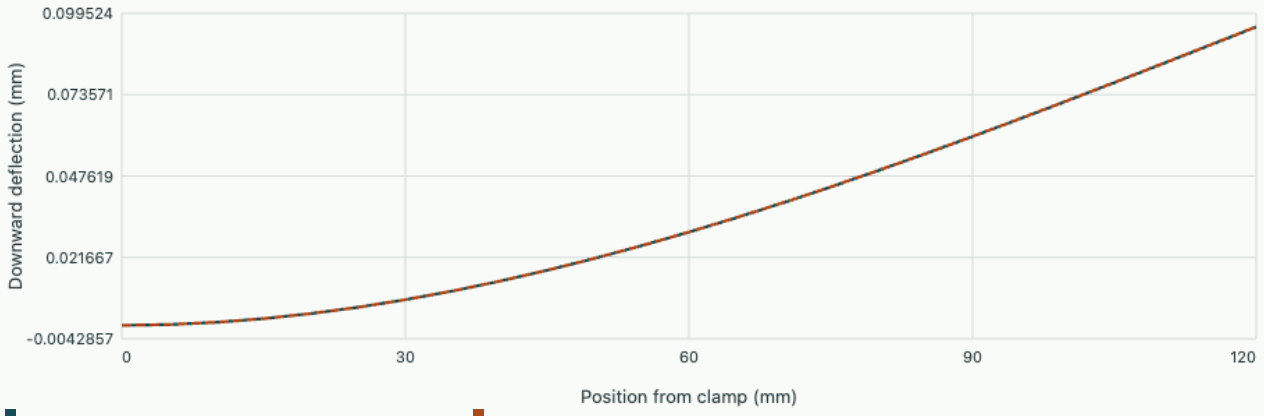
The end moment about a section a distance x from the clamp has magnitude $F(L-x)$. The small-slope beam model first converts it into curvature $w'' = F(L-x)/(EI)$, then integrates twice with $w(0) = w'(0) = 0$. The browser performs both integrations on the selected grid by trapezoidal sums. The first integral is exact for linear curvature. The second integrates a quadratic slope and retains a grid error. Its reference is $w(x) = Fx^2(3L-x)/(6EI)$.

Open this experiment online

Try the calculation

Section height (mm) **12**

Axial cells **24**



Integrated curvature on the grid Euler–Bernoulli solution

Tip deflection 0.095197 mm on this 1D grid, 0.095238 mm analytically. The OneSimulation model solves the same load on a 3D solid.

Download the OneSimulation model

In your OneSimulation environment, run the model with these inputs.

```
python cantilever.py --height 12 --cells 24
```

Download the model for the same dimensions and change the parameters shown with the graph. OneSimulation discretises the three-dimensional solid with quadratic hexahedral elements and applies distributed end traction. Its CSV compares centreline deflection with the analytical curve. First fix height and refine axially. Then increase height at a converged mesh. The first experiment addresses numerical error within a model, the second probes the slender-beam assumptions. Under those assumptions doubling height reduces deflection eightfold because $I = bh^3/12$.

When a beam is strong enough but too flexible

A tool rail may stay below its elastic limit while its cutting head moves outside a position tolerance. The maximum bending stress depends on the largest moment at one section. Deflection accumulates the effect of curvature over a length. A displacement sensor near an unloaded end can therefore move even if the nearby bending moment is zero. To predict it, identify both the point and direction of the required displacement.

Consider a straight, slender Euler–Bernoulli beam. Cross-sections remain plane and perpendicular to the deformed centreline. Take w positive upward and M positive in sagging. For small slopes the

signed curvature is w'', with $\kappa = M/(EI)$. Integrate once to find the slope and once more to find the displacement. Each constant of integration has physical meaning. A clamp fixes displacement and rotation. A simple support fixes its vertical displacement while permitting rotation. At an ordinary join without a hinge, both displacement and slope are continuous.

Work through a tip reading

Clamp a beam at $x = 0$. Apply a downward point load P at $x = a$ on a beam of total length $L > a$. The signed moment in $0 < x < a$ is $-P(a - x)$. For $x > a$ the free segment has no external force and no moment. Taking constant stiffness and using the clamp conditions gives

$$EIw'(x) = -P(ax - x^2/2),$$

$$EIw(x) = -P(ax^2/2 - x^3/6), \quad 0 \le x \le a.$$

At the load point the displacement is $-Pa^3/(3EI)$ and the slope is $-Pa^2/(2EI)$. The free extension beyond it is straight but tilted. Use those values as initial conditions for the unloaded segment:

$$w(L) = w(a) + (L - a)w'(a) = -Pa^2(3L - a)/(6EI).$$

This expression correctly vanishes when the force is applied at the clamp and reduces to the standard end-load expression when $a = L$.

Calculate both position and slope

Let $EI = 8400\,\mathrm{N\,m^2}$, $L = 1.20$ m, $a = 0.80$ m and $P = 100$ N. Under the load the downward displacement is 2.03 mm and the rotation magnitude is

$$100(0.80)^2/[2(8400)] = 0.00381\,\mathrm{rad}.$$

The 0.40 m overhang contributes another 1.52 mm at the tip, for a total downward reading of 3.56 mm. The slope is small compared with a radian, consistent with the linear curvature assumption. A gauge at the load and one at the tip would give different stiffness estimates if their positions were omitted.

Obtain one displacement directly

There is another way to compute a single sensor reading. Introduce a fictitious force Q at the sensor in its desired positive direction. The moment becomes $M_P + Qm$, where m is the moment per unit sensor force. The bending energy is

$$U = \int (M_P + Qm)^2/(2EI)\,dx.$$

Differentiating with respect to Q gives the displacement conjugate to that force. Set $Q = 0$ afterwards to recover the original beam.

$$\delta = \int_0^L \frac{M_P(x) m(x)}{E(x) I(x)}\, dx.$$

For a downward unit force at the free tip, $m = -(L - x)$. Only the portion up to a contributes because $M_P = 0$ further along. The integral gives $Pa^2(3L - a)/(6EI)$, the positive downward reading obtained above. This form is especially useful for stepped sections: split the integral where either stiffness or loading changes. Its positive self-flexibility for a load and a sensor pointing the same way is a useful sign check.

Exchange the load and the gauge

The unit-load formula supplies a test that can be made on the same beam without knowing its exact stiffness distribution. Put the same 100 N force at the tip and move the displacement sensor to $x = 0.80$ m. The new integral has exactly the previous pair of moment fields with their roles exchanged. Since their product is unchanged, the new reading is also 3.56 mm downwards. This Maxwell–Betti reciprocity applies to these work-conjugate forces and displacements in a linear elastic structure with unchanged restraints. A slipping clamp, a support that lifts off or motion large enough to change the load geometry can cause discrepancies between reciprocal tests. Swapping load and sensor positions is therefore also a practical check of the assumed support behaviour.

Quantify the deformation omitted by a bending-only model

The straight-normal assumption of Euler–Bernoulli theory omits transverse shear distortion. For a slender beam this is often small, but for a deep beam or a material with a small effective shear modulus it can dominate a displacement measurement. In a simple Timoshenko description let $\kappa_s G A$ be the effective shear stiffness. The dimensionless shear correction factor κ_s accounts for the nonuniform cross-sectional shear field when a single transverse shear resultant V is used. For the illustration use $\kappa_s = 5/6$ for a homogeneous rectangular section.

For an end-loaded cantilever the bending moment magnitude is $P(L - x)$ and the shear force magnitude is P throughout the span. Add the elastic shear energy to the bending energy. Differentiating with respect to the end force yields the total deflection under that force:

$$U = \int_0^L \left[\frac{P^2 (L - x)^2}{2EI} + \frac{P^2}{2\kappa_s G A} \right] dx,$$

$$\delta = \frac{PL^3}{3EI} + \frac{PL}{\kappa_s G A}.$$

The shear contribution grows linearly with span while the bending contribution grows cubically. Their ratio is $3EI/(\kappa_s G A L^2)$. Writing $I/A = h^2/12$ for a rectangle gives $[E/(4\kappa_s G)](h/L)^2$. This form

identifies both the geometric slenderness and material stiffness ratio required for a sensible model choice. A fixed rule based on span-to-depth alone can be poor for a sandwich section with a compliant core.

Interpret a short cantilever stiffness test

An aluminium fixture projects 120 mm from a rigid clamp. Its cross-section is 20 mm wide and 40 mm deep in the bending direction. Take $E = 70{,}000\,\mathrm{MPa}$, $G = 26{,}000\,\mathrm{MPa}$, and an end load of 500 N. The area is 800 mm² and the bending second moment is

$$20(40)^3/12 = 106667\,\mathrm{mm}^4.$$

In consistent N–mm units the bending deflection is

$$500(120)^3/[3(70000)(106667)] = 0.0386\,\mathrm{mm}.$$

The shear deflection is

$$500(120)/[(5/6)(26000)(800)] = 0.00346\,\mathrm{mm}.$$

A dial indicator would measure about 0.0420 mm under these ideal assumptions.

Fitting a bending-only formula to this reading suggests an apparent modulus of about $70(0.0386/0.0420) = 64.2\,\mathrm{GPa}$. The discrepancy arises from omitted shear deformation despite a perfectly elastic material. Repeat the test at twice the overhang while keeping the force unchanged: bending deflection rises by eight, shear deflection by two. The fractional shear error falls to one quarter of its original value. Measuring several spans can therefore separate an incorrect modulus from a shear correction. Clamp compliance is another candidate in a real experiment. A clamp rotation produces a term with a different span dependence, and a gauge mounted close to the fixture can reveal that motion independently.

For equal applied loads at spans L and $2L$, express measured compliances as $C_1 = B + S$ and $C_2 = 8B + 2S$, where B and S are the original bending and shear contributions per unit load. Then $B = (C_2 - 2C_1)/6$ and $S = C_1 - B$. A negative inferred S signals measurement uncertainty or an omitted boundary deformation, since the assumed elastic shear energy must be positive. This inversion makes the two-span proposal a quantitative model test.

Taking three spans instead of two permits a residual check, because the third measured displacement is then predicted without adjusting either fitted contribution.

Stress transformation and combined loading

Engineering mechanics

Find principal stresses and elastic yield measures at a shaft surface carrying bending and torsion.

FROM FIRST PRINCIPLES

Recover principal normal stresses from traction balance

For a small material tetrahedron, equilibrium of the tractions on its faces gives Cauchy's relation $\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma}\mathbf{n}$. In a classical continuum without body couples, local moment balance makes $\boldsymbol{\sigma}$ symmetric. In plane stress select a section normal $\mathbf{n} = (n_x, n_y)$ in the sheet. A principal section carries traction parallel to its normal, so it solves $\boldsymbol{\sigma}\mathbf{n} = \lambda\mathbf{n}$. A nonzero normal exists only if

$$(\sigma_x - \lambda)(\sigma_y - \lambda) - \tau_{xy}^2 = 0.$$

Solve the quadratic by completing the square about the mean normal stress. This gives both principal values:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}.$$

The associated real eigenvectors are orthogonal when the values differ. For a pure shear, the principal stresses are equal and opposite even though normal stresses on the original coordinate cuts are zero. For a traction-free sheet the third principal value is zero; this third direction must be included when applying a three-dimensional yield criterion.

DERIVATION AND APPLICATIONS

Find the plane carrying the largest shear traction

A tensile test measures a normal stress on one specified cut. In a machine part another cut through the same point can carry shear even when the original axes coincide with the loading. Assume plane stress: the free faces normal to z have zero traction and the nonzero components are $\sigma_x, \sigma_y, \tau_{xy}$. Choose an in-plane unit normal $\mathbf{n} = (\cos\theta, \sin\theta)$ and tangent $\mathbf{s} = (-\sin\theta, \cos\theta)$. Cauchy's traction relation gives the force per actual area on that cut:

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = (\sigma_x \cos\theta + \tau_{xy}\sin\theta,\ \tau_{xy}\cos\theta + \sigma_y \sin\theta).$$

Project onto the normal and tangent. Introduce $m = (\sigma_x + \sigma_y)/2$ and $d = (\sigma_x - \sigma_y)/2$. The double-angle identities rearrange the two projections into a rotation:

$$\sigma_n - m = d\cos 2\theta + \tau_{xy}\sin 2\theta,$$
$$\tau_{ns} = -d\sin 2\theta + \tau_{xy}\cos 2\theta.$$

Squaring and adding cancels the cross terms. This supplies both the in-plane principal stresses and the range of shear traction:

$$(\sigma_n - m)^2 + \tau_{ns}^2 = d^2 + \tau_{xy}^2 = R^2,$$
$$\sigma_{1,2} = m \pm R,$$
$$|\tau_{ns}|_{\max} = R.$$

Planes of largest in-plane shear are rotated 45° from principal planes. The three-dimensional maximum also includes planes involving the out-of-plane principal stress, which equals zero under plane stress. That distinction matters whenever both in-plane principal stresses have the same sign.

WORKED EXAMPLE

Resolve the stress on a particular cut

Take $(\sigma_x, \sigma_y, \tau_{xy}) = (90, 30, 40)$ MPa. The centre and radius are 60 and 50 MPa, giving in-plane principal stresses 110 and 10 MPa. On a cut with normal at 45°, the projections give $\sigma_n = 100$ MPa and $\tau_{ns} = -30$ MPa. Their point lies on the circle because $40^2 + 30^2 = 50^2$. Including the zero out-of-plane stress gives absolute maximum shear 55 MPa.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Test a reported shear measurement

For equal biaxial tensile stresses 70 MPa with $\tau_{xy} = 0$, both d and R vanish. Every in-plane cut has 70 MPa normal stress and zero shear. A gauge claiming 20 MPa shear on a plane normal entirely in the sheet contradicts this state. A plane tilted 45° out of the sheet can carry 35 MPa, since it mixes principal stresses of 70 and zero MPa.

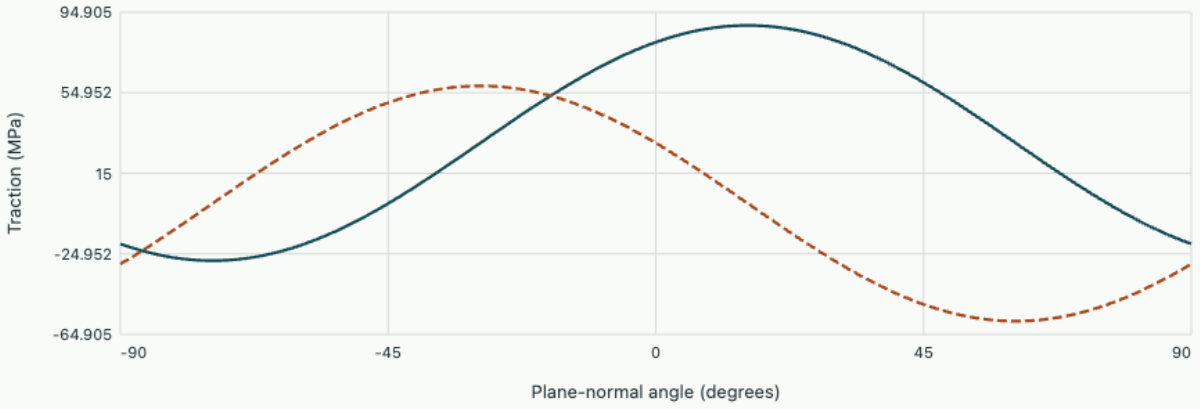
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Stress σx (MPa)	80
Stress σy (MPa)	-20
Shear τxy (MPa)	30



Principal stresses 88.310 and -28.310 MPa; principal direction 15.482°. In-plane maximum shear 58.310 MPa.

A section resultant does not specify a failure plane

The free-body diagram of a shaft gives bending moments and torque. A potential crack, however, can cut the shaft on any plane. To compare the local stress with a material test, first evaluate the stress components at the same material point. At the outer fibre of a circular shaft the bending normal stress may be large. At the neutral axis the bending stress vanishes. Values from different points cannot be combined into one stress state.

For a circular shaft of diameter d bending moment M gives surface stress $32M/(\pi d^3)$, and torque T gives surface shear $16T/(\pi d^3)$. The torsion relation uses $J = \pi d^4/32$. The combination assumes elastic small deformation in a smooth prismatic portion, sufficiently far from shoulders, load

introduction and keyways. At a traction-free outer surface the radial traction is zero. Use a local basis with longitudinal and circumferential axes for the remaining plane-stress components.

Change the orientation of a cut

In two dimensions write the stress tensor with normal components σ_x, σ_y and shear τ_{xy}. The traction vector on a cut with unit normal $\mathbf{n}$ is $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$. Project that vector along and perpendicular to $\mathbf{n} = (\cos\theta, \sin\theta)$. The normal stress becomes a constant average plus a sinusoidal term; the shear varies with twice the same angle:

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta.$$

The largest and smallest in-plane normal stresses differ from the mean by $R = \sqrt{[(\sigma_x - \sigma_y)/2]^2 + \tau_{xy}^2}$. The shear on those principal planes vanishes. Taking the third principal stress as zero allows a three-dimensional maximum-shear check: evaluate half the difference of the largest and smallest of all three principal stresses. In a biaxial tensile state, ignoring the zero third stress can underestimate this difference.

Compare principal stresses on a machine shaft

Let a 40 mm diameter steel shaft carry $M = 300$ N·m and $T = 200$ N·m. At its tensile bending surface, $\sigma_x = 47.75$ MPa, $\sigma_y = 0$ and $\tau_{xy} = 15.92$ MPa. The in-plane mean is 23.87 MPa and the radius is 28.69 MPa. The principal stresses are about 52.57 and −4.82 MPa. The plane of greatest tension is rotated through half of $\operatorname{atan2}(2\tau_{xy}, \sigma_x - \sigma_y)$, here 16.8° from the original plane normal. The maximum shear magnitude is 28.7 MPa.

A useful check is the trace: the principal stresses sum to 47.75 MPa, the same sum as the original normal components. A second check is the determinant: the two principal values have opposite signs since the determinant of this particular matrix is $-\tau_{xy}^2$. A picture with the two directions marked prevents confusing the angle of a plane normal with that of the line drawn for the plane itself.

Interpret an equivalent stress

For isotropic ductile materials a common onset-of-yield model uses von Mises equivalent stress. Under plane stress it is $\sqrt{\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2}$, giving 55.1 MPa here. Compare this scalar with the uniaxial yield strength from a material test under a comparable temperature and loading rate. A given yield strength of 220 MPa corresponds to a nominal static factor of about 4.0. That number answers a defined static question. A shaft rotating under a stationary transverse force cycles the tensile and compressive stress of a given material point once per revolution. Evaluating its fatigue resistance requires the stress range and a separate material model.

Change the mean stress without changing the distortion

Apply a uniform external hydrostatic pressure of 50 MPa to a material point whose original principal stresses are 100, 0 and 0 MPa. The new principal stresses are 50, –50 and –50 MPa, since pressure acts on every plane. Their pairwise differences remain 100, 0 and –100 MPa, so the von Mises stress remains 100 MPa. This illustrates why a distortion-based yield condition must include all three normal directions under confinement. Applying a plane-stress formula after adding pressure only to the in-plane directions would represent a different three-dimensional loading.

Derive the two membrane stresses of a thin cylinder

A hydraulic accumulator behaves differently in its axial and circumferential directions even if it is made of isotropic steel. Consider a long cylindrical shell of mean radius r, wall thickness $t \ll r$, closed ends and internal gauge pressure p. Away from nozzles and end transitions, symmetry permits a membrane model with tensile stresses uniform through the thickness. The radial stress changes from approximately $-p$ at the inner wall to zero outside. Its magnitude is small compared with pr/t if the shell is thin.

Cut the cylinder longitudinally into two halves of length L. The net force from pressure acts on the projected rectangle of area $2rL$, not on the curved wetted area. The two cut edges carry hoop tension. A different cut across the full circular end isolates the cap thrust and the single axial membrane stress. The resulting balances are

$$2\sigma_\theta t L = p(2rL),$$

$$2\pi r t \sigma_z = p\pi r^2,$$

$$\sigma_\theta = pr/t,$$

$$\sigma_z = pr/(2t).$$

The factor of two follows from geometry and end load transmission. If the ends are open and pressure thrust returns through a separate frame, the cylinder may carry little axial membrane force. Always follow the cap force into the structure before assigning axial stress. A cap anchored externally and a welded pressure shell have different load paths even at the same gauge pressure.

Infer shell strain and the effect of axial restraint

For a closed vessel with $r = 0.20\,\mathrm{m}$, $t = 4.0\,\mathrm{mm}$ and $p = 1.2\,\mathrm{MPa}$, the membrane stresses are 60 MPa circumferentially and 30 MPa axially. These directions are principal at an unperturbed surface. Under plane stress the von Mises stress is

$$\sqrt{60^2 + 30^2 - 60(30)} = 52.0\,\mathrm{MPa}.$$

With $E = 200\,\mathrm{GPa}$ and $\nu = 0.30$, an outer hoop gauge reads $(60 - 0.30 \times 30)/200000 = 255$ microstrain and an axial gauge reads only $(30 - 0.30 \times 60)/200000 = 60$ microstrain. The radius increases by about $0.20(255 \times 10^{-6}) = 51\,\mu\mathrm{m}$. The smaller axial signal is particularly sensitive to temperature correction.

Now imagine an axial restraint strong enough to hold the length constant without affecting the radius directly. The condition $\varepsilon_z = 0$ gives

$$\sigma_z = \nu \sigma_\theta = 18\,\mathrm{MPa}.$$

The cap thrust has not disappeared. An external axial reaction must remove the difference of 12 MPa integrated over the shell area $2\pi r t$, about 60.3 kN. A gauge-based interpretation using the unrestrained 30 MPa estimate would miss this load. Conversely, where the restraint attaches, local bending and stress concentration may dominate the smooth membrane solution. Locate comparison gauges several characteristic wall dimensions away from such load introduction.

The force calculation can be audited directly. Pressure thrust on the end cap is $1.2\pi(200)^2 = 151\,\mathrm{kN}$ in MPa and millimetres. An 18 MPa shell stress carries

$$18[2\pi(200)(4)] = 90.5\,\mathrm{kN}.$$

Their difference, 60.3 kN, is precisely the required external load path. A restraint instrumented with a load cell would test the assumed axial boundary condition independently of the strain readings.

Extend the traction balance to three dimensions

In three dimensions a cut normal has three components. Cauchy’s relation remains $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$. Project onto the unit normal to obtain signed normal stress, then remove that projection to obtain the shear traction. Rotating a diagram does not rotate the tensor physically; changing its components does.

$$\sigma_n = \mathbf{n}^T \boldsymbol{\sigma} \mathbf{n},$$

$$\mathbf{t}_s = \boldsymbol{\sigma}\mathbf{n} - \sigma_n \mathbf{n}.$$

The eigenvectors give principal cut directions with zero shear traction. The following notebook evaluates the symmetric eigensystem and independently verifies both the tensor reconstruction and the orthogonality of each shear traction to its cut normal.

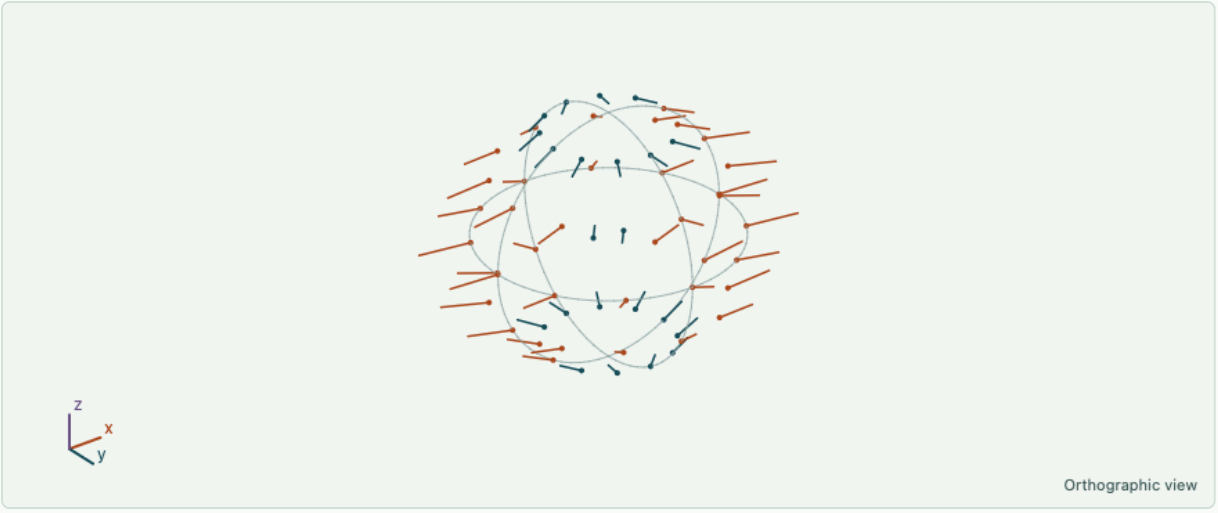
Interactive calculation · Inspect three-dimensional traction on every cut

Change the out-of-plane shear to leave the original plane-stress example. Rotate the sphere of cut normals, then tilt the plotted slice through it. The force segments use a common symbolic scale; they show traction directions, not displacements.

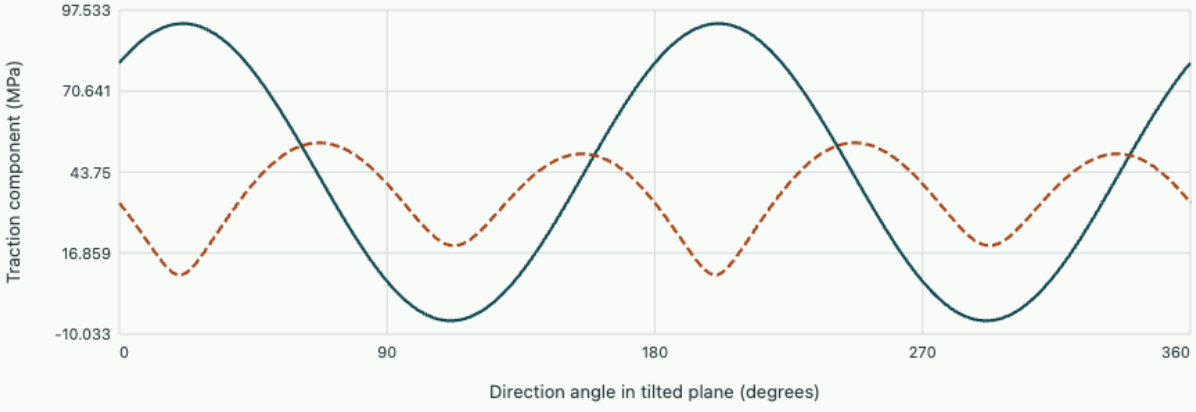
Open this live calculation

Try the calculation

Normal stress σx (MPa)	80
Shear τxy (MPa)	30
Shear τxz (MPa)	15
Slice-plane tilt (degrees)	30



Unit-normal sphere with three-dimensional traction vectors. Each segment starts at a cut normal and ends in the traction direction, using one common symbolic force scale. Orange indicates tensile normal traction, teal compressive. The sphere is a directional diagram, not a deformed body. The OneSimulation notebook also computes and verifies the principal directions. Axes: x = normal x; y = normal y; z = normal z.



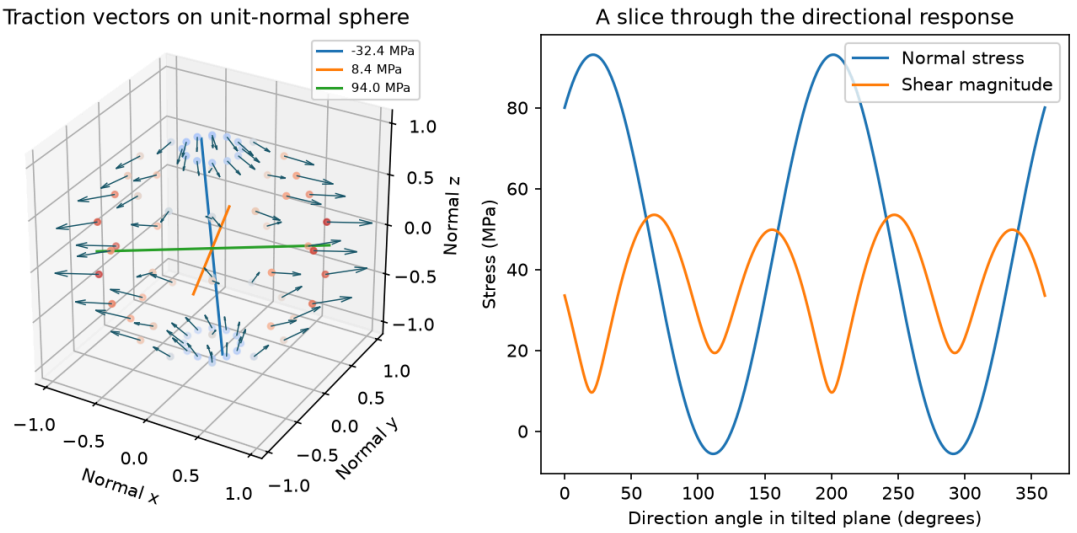
Cauchy traction t=σn is resolved into a normal part and a tangential part for every cut. Fixed σy=20 MPa and σz=−30 MPa; τyz=0. At slice angle45°, normal=77.231 MPa and shear=39.925 MPa.

OneSimulation notebook · solved spatial stresses, code and output

The code solves the illustrated stress model with OneSimulation and checks its equilibrium or eigensystem. The figure is the output of that code.

```
"""Compute principal stresses with OneSimulation and show Cauchy traction in 3D."""
import numpy as np
import matplotlib.pyplot as plt
from onesim.numerics.linear.matrices.dense_linear_algebra import symmetric_eigensystem

# Symmetric Cauchy tensor in MPa; edit the entries to reproduce the browser controls.
sigma=np.array([[80.,30.,15.],[30.,20.,0.],[15.,0.,-30.]])
principal, axes = symmetric_eigensystem(sigma)
assert np.allclose(axes@np.diag(principal)@axes.T,sigma)
assert np.allclose(axes.T@axes,np.eye(3))
print('Principal stresses [MPa]:',np.round(principal,5))
print('Absolute maximum shear [MPa]:',round((principal[-1]-principal[0])/2,5))
phi=np.linspace(0,2*np.pi,12,endpoint=False)
theta=np.linspace(.25,np.pi-.25,6)
n=np.array([[np.sin(t)*np.cos(p),np.sin(t)*np.sin(p),np.cos(t)] for t in theta for p in phi])
traction=n@sigma.T
normal=np.sum(n*traction,axis=1)
shear=traction-normal[:,None]*n
assert np.max(abs(np.sum(n*shear,axis=1)))<1e-12
assert normal.min()>=principal[0]-1e-10 and normal.max()<=principal[-1]+1e-10
fig=plt.figure(figsize=(9,4.2),layout='constrained');ax=fig.add_subplot(121,projection='3d')
ax.quiver(*n.T,*(traction/120).T,length=.5,normalize=False,color='#19596a',linewidth=.7)
ax.scatter(*n.T,c=normal,cmap='coolwarm',s=12,vmin=-100,vmax=100)
for j in range(3):ax.plot(*np.array([-axes[:,j],axes[:,j]]).T,label=f'{principal[j]:.1f} MPa')
ax.set(xlabel='Normal x',ylabel='Normal y',zlabel='Normal z',title='Traction vectors on unit-normal
sphere')
ax.set_box_aspect((1,1,1));ax.legend(fontsize=7)
a=np.linspace(0,2*np.pi,361);incline=np.deg2rad(30)
directions=np.column_stack((np.cos(a),np.sin(a)*np.cos(incline),np.sin(a)*np.sin(incline)))
t=directions@sigma.T;sn=np.sum(t*directions,axis=1);st=np.linalg.norm(t-
sn[:,None]*directions,axis=1)
ax2=fig.add_subplot(122);ax2.plot(np.rad2deg(a),sn,label='Normal
stress');ax2.plot(np.rad2deg(a),st,label='Shear magnitude')
ax2.set(xlabel='Direction angle in tilted plane (degrees)',ylabel='Stress (MPa)',title='A slice
through the directional response');ax2.legend()
fig.savefig('complete-engineering-traction3d.png',dpi=170);plt.show()
```

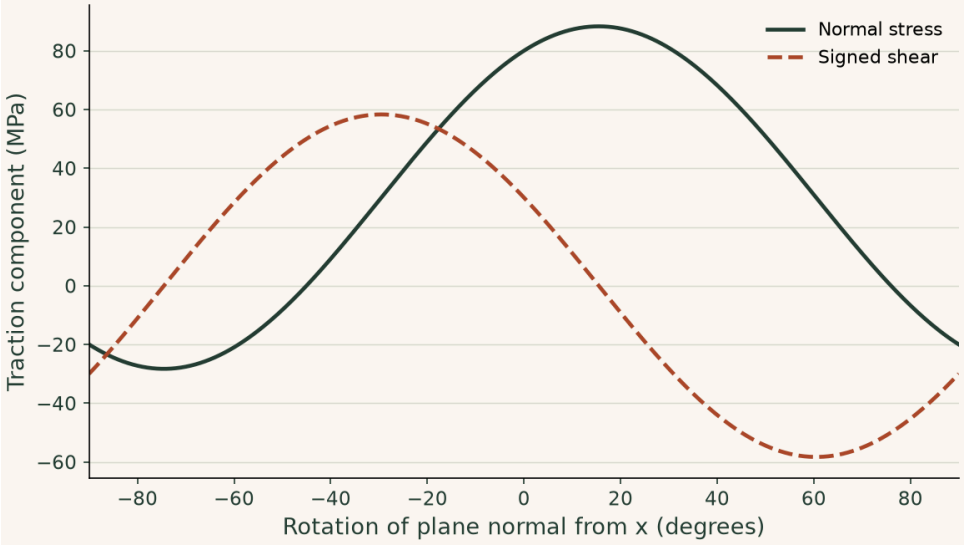


Recorded output from the executed OneSimulation notebook; change its inputs to reproduce the browser study.

Download the executed editable notebook · Download the Python model

Choose a cut plane for a combined stress state

A material point has $\sigma_x = 80,\ \sigma_y = -20,\ \tau_{xy} = 30$ MPa. Resolve the traction on a rotated plane into normal and tangential components. The stress transformation contains twice the rotation angle because both the traction vector and the basis change.



The major principal stress is $30 + \sqrt{3400} \approx 88.3$ MPa at about 15.5°, where the shear vanishes. The maximum absolute in-plane shear is 58.3 MPa and occurs on planes 45° from the principal planes.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

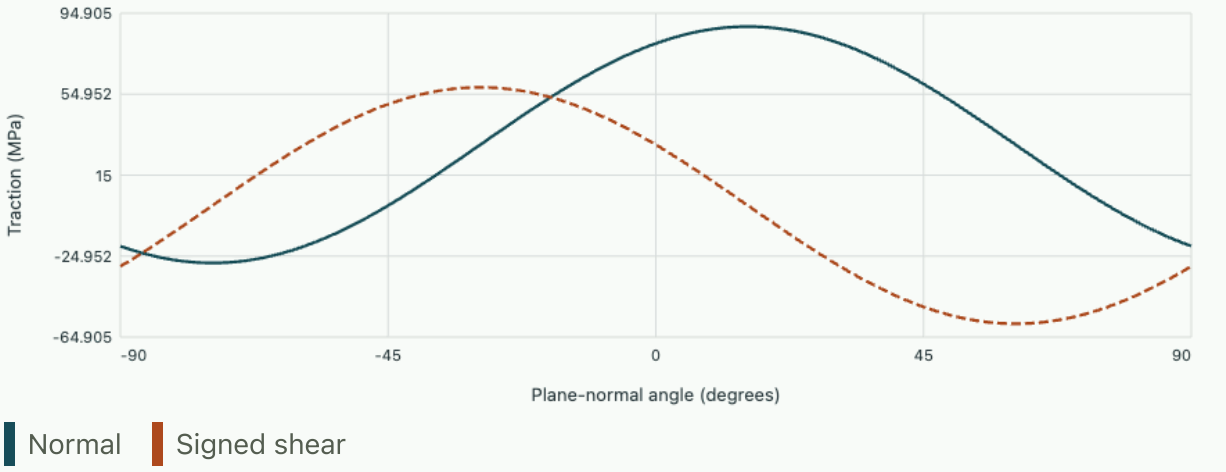
Vary each tensor component and locate the plane on which shear vanishes. Compare that angle with the principal-direction readout.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Stress σx (MPa)	80
Stress σy (MPa)	-20
Shear τxy (MPa)	30



Principal stresses 88.310 and -28.310 MPa; principal direction 15.482°. In-plane maximum shear 58.310 MPa.

Torsional stiffness of shafts and closed sections

Engineering mechanics

Derive torque–twist relations and explain the large difference in stiffness between an intact tube and an open strip.

FROM FIRST PRINCIPLES

Derive the torque–twist relation from local shear deformation

Consider a straight circular shaft in small elastic torsion. In Saint-Venant torsion of a circular section, each cross-section can rotate as a rigid plane relative to the next. Let its rotation be $\phi(x)$ and take two sections separated by dx. At radius r their relative tangential displacement is $r\,d\phi$. Dividing by the axial spacing gives engineering shear strain and, through the measured shear modulus G, shear stress:

$$\gamma_{x\theta} = r\,d\phi/dx,$$
$$\tau_{x\theta} = Gr\,d\phi/dx.$$

Each patch of area dA carries tangential force $\tau\,dA$ with moment arm r. Sum these moments over the section and integrate the resulting twist rate along the shaft:

$$T = \int_A r\tau\,dA = GJ\phi',$$
$$J = \int_A r^2\,dA,$$
$$\Delta\phi = \int \frac{T(x)}{G(x)J(x)}dx.$$

For a homogeneous solid circular shaft $J = \pi R^4/2$, so doubling radius strongly increases torsional stiffness. Noncircular sections generally warp and their torsion constant need not equal this polar area moment. The circular-shaft kinematics should not be used unchanged for an open thin-walled section.

DERIVATION AND APPLICATIONS

How two bonded shaft materials share one applied torque

Two shafts in series carry the same torque; a bonded core and sleeve instead undergo the same twist per length. Consider a circular core of radius a and shear modulus G_c, bonded to a concentric sleeve with outer radius b and modulus G_s. Use a long prismatic region away from load introduction, small elastic deformations, perfect bond and no interfacial slip. Let $\phi(x)$ describe section rotation. Two cross-sections separated by dx acquire a relative tangential motion $r\,d\phi$ at radius r:

$$\gamma_{x\theta} = r\phi',$$
$$\tau_i(r) = G_i r\phi'.$$

Compatibility makes ϕ' common. The shear stress may change at the material boundary because the moduli differ. The torque contribution from a thin annulus is its area $2\pi r\,dr$ times shear stress times radius. Integrating in each material gives

$$T_c = G_c J_c \phi',$$
$$T_s = G_s J_s \phi',$$
$$J_c = \frac{\pi a^4}{2},$$
$$J_s = \frac{\pi(b^4 - a^4)}{2}.$$

The applied torque T is their sum. Solving first for the common twist and then for the load split yields

$$\phi' = \frac{T}{G_c J_c + G_s J_s},$$
$$\frac{T_c}{T} = \frac{G_c J_c}{G_c J_c + G_s J_s}.$$

The denominator has units of N m². Setting both shear moduli equal recovers the homogeneous solid shaft independently of the arbitrary division at a. If the sleeve is unbonded and torque is applied only to the core, equal twist is no longer a valid compatibility condition.

WORKED EXAMPLE

Predict the twist of a reinforced shaft

Take $a = 10.0$ mm, $b = 15.0$ mm, $G_c = 80.0$ GPa and $G_s = 30.0$ GPa. The rigidities are 1257 and 1914 N m². Under $T = 60.0$ N m the twist rate is 0.0189 rad/m; over 0.500 m the end rotates 0.00946 rad. The core carries 23.8 N m and the sleeve 36.2 N m. Their stresses at a are 15.1 and 5.68 MPa respectively despite equal shear strain there.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Infer a sleeve modulus from a rotation test

A second shaft with the same dimensions and core rotates through 0.0100 rad over 0.500 m under 50.0 N m. The measured total rigidity is $TL/\phi = 2500$ N m². Subtracting the known core leaves 1243 N m². Since $J_s = 6.38 \times 10^{-8}$ m⁴, $G_s = 19.5$ GPa. The test would yield an apparent modulus if compliance of the grips were included in the measured rotation.

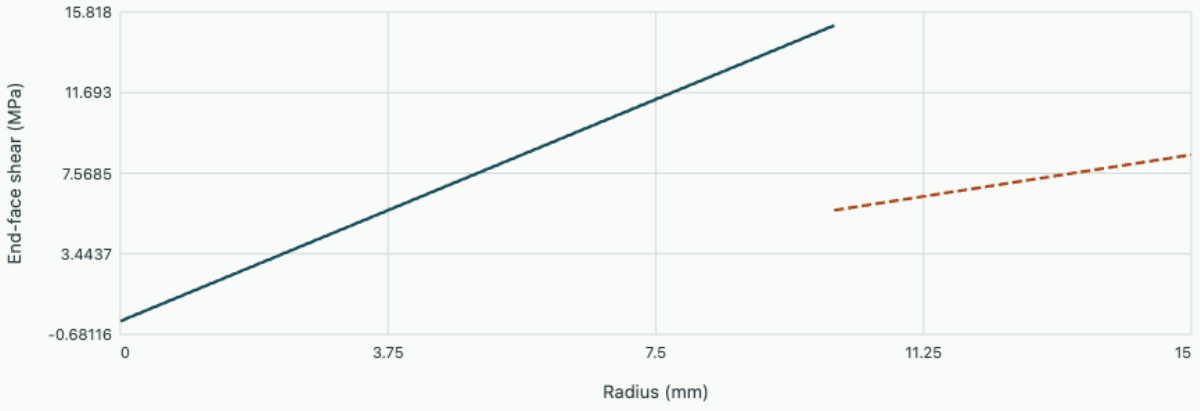
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Core radius (mm)	10
Outer radius (mm)	15
Torque (N m)	60



Twist rate 0.01892 rad/m; core torque 23.777 N m; sleeve 36.223 N m. Compatibility sets equal twist, not equal interface stress.

How much torque does the sleeve actually carry?

A steel core and an aluminium sleeve extend the full length of a circular shaft. They are bonded along their cylindrical interface, and the end fittings transmit the resultant torque to the complete section. A student dividing the torque in proportion to cross-sectional area obtains a plausible answer that misses both lever arm and material stiffness. We can reconstruct the division from the displacement at the interface before writing any torque balance.

Let a denote the core radius and b the outside radius. A cross section rotating by angle $\varphi(x)$ moves a point at radius r tangentially by $r\varphi$. Over a short axial interval the resulting engineering shear strain is $\gamma_{x\theta} = r\,d\varphi/dx$. Bonding requires a common twist per length φ' for the two materials. The shear on a

cross-sectional end face is $Gr\varphi'$, so it can jump at $r = a$ even though the strain there is continuous. It is the traction on a radial-normal interface that must equilibrate on opposite sides of the bond. Confusing it with the traction on the shaft end face would wrongly force the plotted shear stresses to coincide.

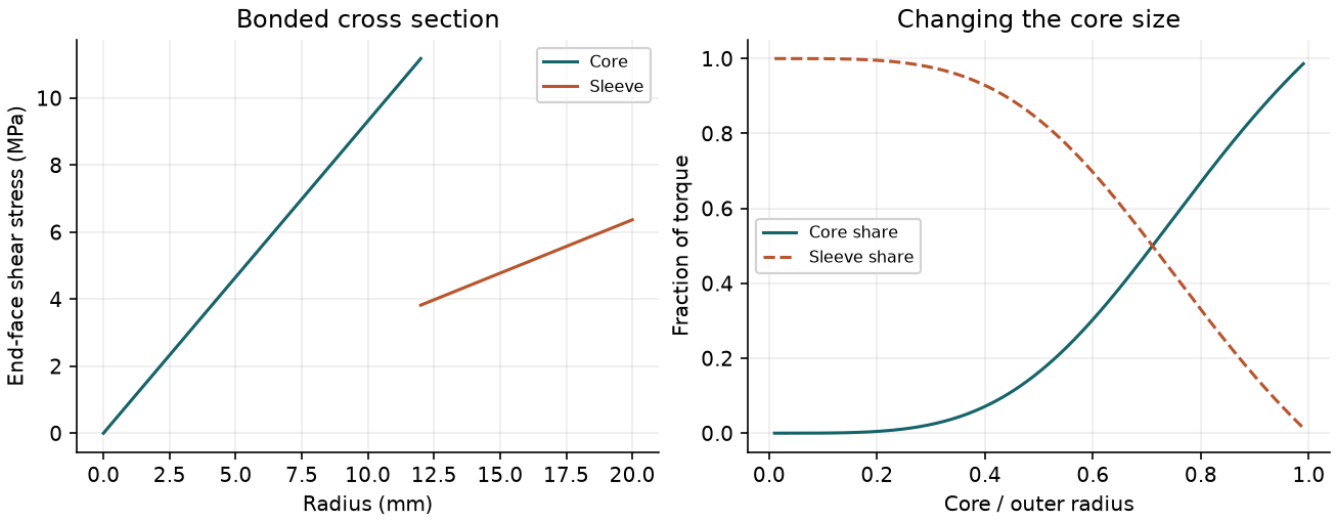
The torque on an annular element is the shear times its area times its lever arm. Integration of $r(Gr\varphi')(2\pi r\,dr)$ gives

$$T = [G_c J_c + G_s J_s]\varphi',$$

$$J_c = \pi a^4/2,$$

$$J_s = \pi(b^4 - a^4)/2.$$

Use $a = 12$ mm, $b = 20$ mm, $G_c = 79$ GPa and $G_s = 27$ GPa. Under 100 N m the core carries 30.35 N m and the sleeve 69.65 N m. A 600 mm length rotates by 0.405 degrees. Immediately inside the interface the end-face shear is 11.18 MPa. On the other side of that same radius the lower modulus reduces it to 3.82 MPa; continuing outward increases it again. An accurate shaft twist cannot by itself identify the local peak stress unless the section and material distribution are known.



The radial stress is local, while each torque share is an integral weighted by the lever arm. A thin outer ring may contribute disproportionately to the total rigidity.

In the notebook, integrate the stress directly over the core and sleeve and compare the total with the imposed torque. Evaluate the two integrals separately so a numerical trapezoid does not smear the material jump. Then make the two moduli equal: the end-face shear curve becomes straight across the entire section, while the fraction of torque in the core equals $(a/b)^4$. This limit offers an independent check against a uniform solid shaft.

A separate way to check the torque split uses stored energy. In a unit length of shaft the strain energy density is $\tau\gamma/2$. Integrating it over each annulus gives $GJ(\varphi')^2/2$ for that material. Because the twist is common, the energy fractions equal the torque fractions. For the example the total strain energy is $T\varphi/2$, about 0.35 J over 600 mm. If the two materials were joined end to end, the applied torque would be common and the angles of twist would add. The continuous cylindrical bond is the feature that selects the parallel arrangement used here.

At what radius fraction will the default materials divide torque exactly in half?

▼ **Work through the solution**

Set $G_c a^4 = G_s(b^4 - a^4)$. Rearranging gives

$$a/b = [G_s/(G_c + G_s)]^{1/4} = 0.7106.$$

At $b = 20$ mm this corresponds to a core radius of about 14.2 mm. Run the notebook with the revised radius and read both shares. This calculation assumes a bonded circular shaft away from the ends; it cannot predict the local loading of a short end fitting.

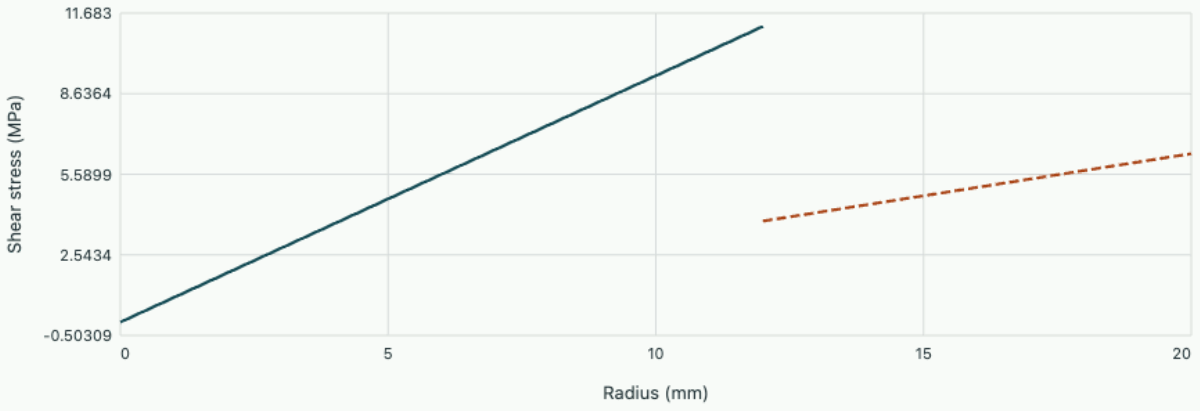
Interactive calculation · Bonded torsion

Vary the notebook inputs and compare both plotted observables. The same equations generate the browser curves and the notebook reference; fixed material, geometry and model assumptions are stated in the readout.

Open this live calculation

Try the calculation

Core radius (mm)	12
Core shear modulus (GPa)	79
Sleeve shear modulus (GPa)	27
Applied torque (N m)	100



At length0.60 m the rotation is 0.405°. Core torque 30.346 N m. Shear strain is continuous while stress jumps when the moduli differ.

Download the editable notebook for this example

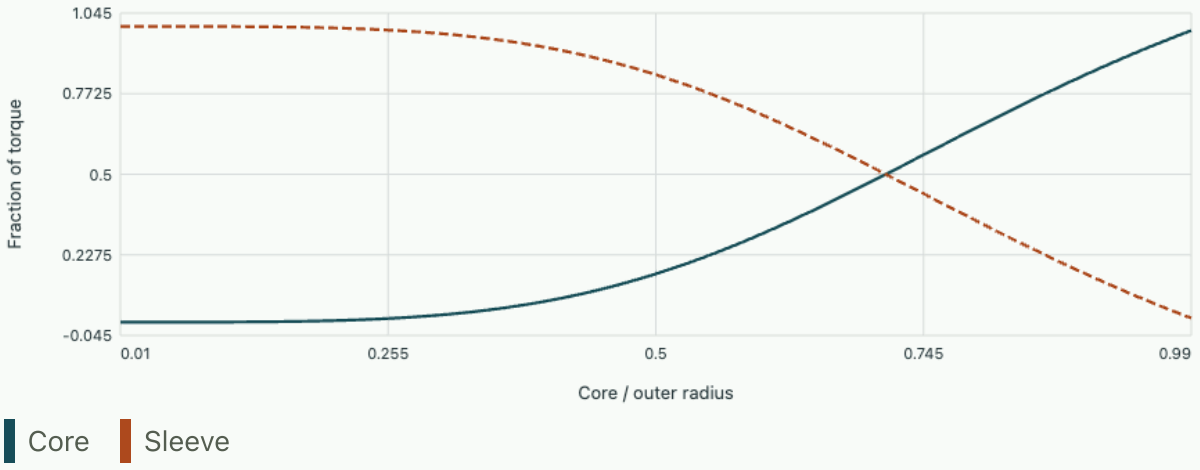
Interactive calculation · Bonded torque share

Vary the notebook inputs and compare both plotted observables. The same equations generate the browser curves and the notebook reference; fixed material, geometry and model assumptions are stated in the readout.

Open this live calculation

Try the calculation

Core radius (mm)	12
Core shear modulus (GPa)	79
Sleeve shear modulus (GPa)	27
Applied torque (N m)	100



At length0.60 m the rotation is 0.405°. Core torque 30.346 N m. Shear strain is continuous while stress jumps when the moduli differ.

Download the editable notebook for this example

Open editable notebook Download .ipynb

Rotate two adjacent sections

A torque meter measures twist between two stations on a shaft. To infer torque, first ask how material between the stations deforms. A straight circular shaft in uniform, small-deformation torsion has cross-sections that rotate as rigid discs. Two discs separated by dx rotate relative to one another by $d\theta$. A fibre at radius r moves sideways by $r\,d\theta$, giving shear strain $\gamma = r\,d\theta/dx$. With elastic shear modulus G, the shear stress increases linearly from the centre to the surface.

Each area element transmits a torque $r\tau\,dA$. Integrating produces the polar second moment of area J, whose dimensions are length to the fourth power. With constant torque and cross-section, integration along the shaft then converts twist per length into total relative angle.

$$T = \int_A r\tau\,dA = G\theta' \int_A r^2 dA,$$

$$J = \frac{\pi}{2}(r_o^4 - r_i^4),$$

$$\theta = \frac{TL}{GJ}.$$

Follow a torque through a stepped shaft

A 0.75 m aluminium tube of outer radius 15 mm and inner radius 12 mm transmits 45 N m. For $G = 26\,\mathrm{GPa}$, $J = 4.695 \times 10^{-8}\,\mathrm{m}^4$. Its twist is

$$45(0.75)/(26\times10^{9}J) = 0.02765\,\mathrm{rad} = 1.58^\circ.$$

Surface shear is

$$45(0.015)/J = 14.4\,\mathrm{MPa}.$$

The inside surface still carries 80% of this stress. Drilling a central hole removes the least highly stressed material first.

Twist can also identify an unknown joint compliance. If an instrument measures 2.00° across this tube plus its end couplings, and all parts carry the same torque, the couplings contribute about 0.42°. Their combined torsional compliance is approximately $0.42\pi/[180(45)] = 1.6\times10^{-4}\,\mathrm{rad/(N\,m)}$. Such a comparison assumes that the gauges do not include a second shaft segment and that the torque is transmitted without slip.

Carry torque around a single closed cell

For a long, thin-walled closed tube with a single cell, use the shear flow $q = \tau t$, the force per unit length of wall. Longitudinal equilibrium makes this flow approximately constant around the perimeter if no intermediate longitudinal force is introduced. A length ds carries tangential force $q\,ds$. Its moment about an arbitrary origin is q times twice the triangular area swept by the radius vector. Around the median wall contour this gives

$$T = 2A_m q.$$

For a rectangular cell with median dimensions 80 by 40 mm, a torque of 320 N m gives

$$q = 320000/[2(80)(40)] = 50\,\mathrm{N/mm}.$$

A 2 mm wall carries 25 MPa while a 4 mm wall carries 12.5 MPa. A slender longitudinal slit interrupts the closed path. The constant circulating flow solution then ceases to apply. A seam that reconnects the two lips must transmit the longitudinal shear flow if the component is to recover its closed-section behaviour.

Distinguish two section properties

For a noncircular closed tube, the polar area moment and torsion constant generally differ. Compatibility around a single thin-walled cell gives

$$\theta' = T \oint ds/(Gt)/(4A_m^2).$$

Making only a short wall segment very thin substantially increases this integral. A wall segment can therefore control angle of twist even when equilibrium fixes the same shear flow everywhere around the cell.

Determine torque sharing between two closed cells

A single closed cell supports one constant circulating shear flow. In an aircraft-style box or a partitioned machine boom, adjacent cells share a web. Torque balance alone cannot tell how much flow each cell carries. The missing equation is twist compatibility: because the cells are bonded along the common web, their twist per unit length must agree. Assume a long, thin-walled prismatic box of one material carrying Saint-Venant torsion with negligible distortion. Use median-line areas and length-to-thickness ratios throughout.

Assign positive clockwise circulations q_1, q_2 to both cells. They oppose each other on the common web, where physical flow becomes $q_1 - q_2$ measured in the first cell's direction. Let S_1, S_2 denote $\int ds/t$ over the exterior walls of each cell and S_w that quantity along their common web. The torque and the two contour compatibility relations become

$$T = 2A_1 q_1 + 2A_2 q_2,$$
$$2A_1 G\theta' = (S_1 + S_w)q_1 - S_w q_2,$$
$$2A_2 G\theta' = (S_2 + S_w)q_2 - S_w q_1.$$

These equations express the same shear-strain compatibility that leads to $J_t = 4A_m^2 / \oint ds/t$ for one cell. If both cells and exterior wall thicknesses are mirror images, symmetry requires equal circulations and zero common-web flow under pure torque. The common web may still carry transverse shear from a separate load case. A missing web flow under pure torque is therefore a property of that loading and symmetry.

Compute the stress in an internal dividing web

Take adjacent median rectangles 40 mm high, 60 and 30 mm wide, with every wall 2 mm thick. Then

$$A_1 = 2400,\ A_2 = 1200\,\mathrm{mm}^2.$$

Their three exterior walls give $S_1 = (60+40+60)/2 = 80$ and $S_2 = (30+40+30)/2 = 50$, while $S_w = 20$. Equal twist requires $(100q_1 - 20q_2)/2400 = (70q_2 - 20q_1)/1200$, giving $140q_1 = 160q_2$. Thus $q_2 = 0.875q_1$, larger than one might expect from the smaller area alone.

Applying $T = 345\,\mathrm{N\,m} = 345000\,\mathrm{N\,mm}$ gives $345000 = 4800q_1 + 2400(0.875)q_1$, so $q_1 = 50.0$ and $q_2 = 43.75\,\mathrm{N/mm}$. The dividing web carries only 6.25 N/mm, or 3.13 MPa. The primary exterior walls carry 25.0 and 21.9 MPa. Using $G = 27000\,\mathrm{MPa}$, either contour gives $\theta' = (100\times 50 - 20\times 43.75)/(2\times 2400\times 27000) = 3.18\times 10^{-5}\,\mathrm{rad/mm}$, about 1.82° per metre. Checking the other contour independently catches a reversed sign on the shared wall. If a weld along that wall fails, equal twist and the closed paths that led to these flows can no longer be assumed.

Individual cell torque contributions are informative when interpreting a section cut. The larger cell contributes $2(2400)(50) = 240\,\mathrm{N\,m}$, and the smaller contributes $2(1200)(43.75) = 105\,\mathrm{N\,m}$. Their 345 N m sum is independent of the coordinate origin. These are bookkeeping circulations rather than two torques applied independently to separate tubes. Cutting through the common web changes the compatibility and available flow paths, so using the original partition after a cut is not justified. Surface strain measurements on the upper exterior walls and the shared web would distinguish the predicted 50, 43.75 and 6.25 N/mm flows more effectively than measuring a single outer-wall strain alone.

A comparison with a thin unpartitioned box is possible without changing outer dimensions. For the outer median rectangle 90 by 40 mm, $A_m = 3600\,\mathrm{mm}^2$ and $\oint ds/t = 2(90+40)/2 = 130$. Its single-cell torsion constant is

$$4A_m^2/130 = 398769\,\mathrm{mm}^4.$$

The partitioned solution implies

$$J_t = T/(G\theta') = 345000/[27000(3.183\times 10^{-5})] \simeq 401500\,\mathrm{mm}^4.$$

Under this specific pure torque the additional web therefore raises torsional stiffness by less than one percent. It experiences little shear because the neighbouring circulations are similar. Its contribution to transverse load sharing, panel buckling resistance or local attachment stiffness can nevertheless be important and needs separate calculations.

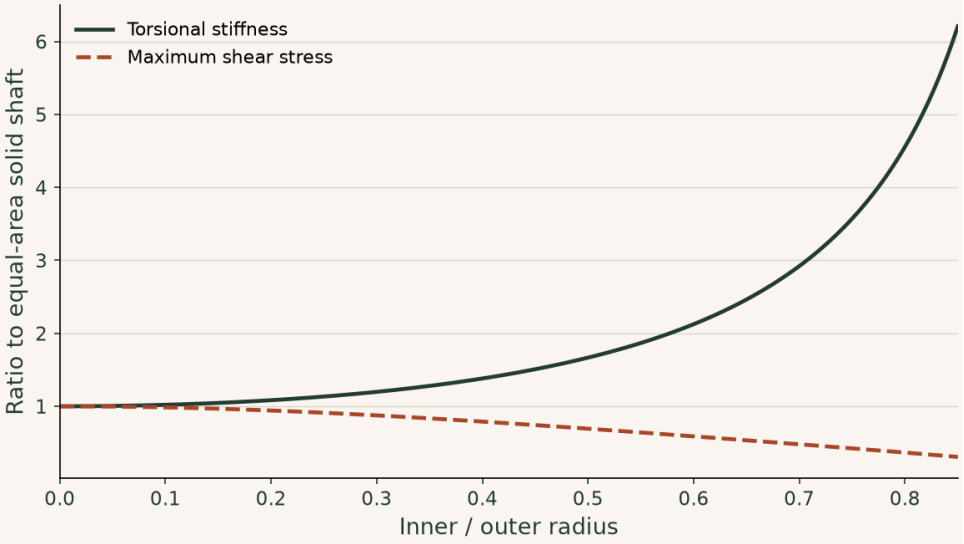
Move the web to the centre, producing identical cells of median width 45 mm. Pure torque then gives equal circulations, zero shear flow on the common web and precisely the single outer-cell twist in this idealisation. This limiting case is a demanding model check. A numerical section calculation that predicts double the torsional stiffness simply because there are now two cells has likely treated the common web as two independent exterior walls.

Compare a tube and a shaft at equal material area

Hold length, material, torque and cross-sectional area fixed while hollowing a circular shaft and increasing its outer radius. For radius ratio r, area conservation gives

$$R/R_s = 1/\sqrt{1 - r^2}.$$

Substitute in $J = \pi R^4(1 - r^4)/2$ and $\tau_{\max} = TR/J$.



At inner/outer radius 0.6 the torsional stiffness is 2.125 times the solid-shaft value, while peak shear is about 0.588 times as large. This comparison requires 25% greater outer radius; under a fixed envelope it would answer a different design question.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

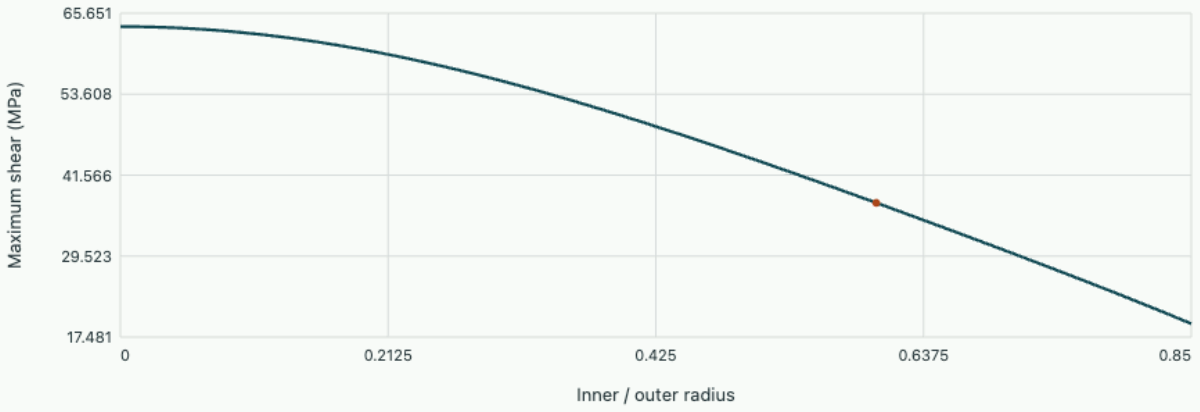
Hollow the shaft while keeping material area fixed. Watch both the required outer envelope and the peak shear change.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Equivalent solid radius (mm)	**10**
Torque (N m)	**100**
Hollow radius ratio	**0.6**



Equal material area Selected tube

Selected outer radius 12.500 mm; stiffness / solid 2.125; peak shear 37.448 MPa. The outer envelope increases.

From a strain rosette to principal stresses

Engineering mechanics

Recover a complete plane-strain measurement at a surface from three gauges and infer the corresponding plane-stress state.

FROM FIRST PRINCIPLES

Calculate the extension recorded by an angled strain gauge

A bonded straight gauge records fractional change of length along its own axis. Let adjacent material points be separated by $d\mathbf{x} = \mathbf{n}\, ds$, where $\mathbf{n} = (\cos\theta, \sin\theta)$ is a unit direction. For a smooth small displacement field $\mathbf{u}$, their relative displacement is approximately $(\nabla \mathbf{u}) d\mathbf{x}$. To first order only the projection along the original separation changes its length:

$$\frac{\delta ds}{ds} = \mathbf{n}^{\mathsf{T}} (\nabla \mathbf{u}) \mathbf{n} = \mathbf{n}^{\mathsf{T}} \boldsymbol{\epsilon}\, \mathbf{n},$$

$$\boldsymbol{\epsilon} = (\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathsf{T}})/2.$$

The antisymmetric part represents infinitesimal rotation and has zero quadratic form. Writing engineering shear as $\gamma_{xy} = 2\epsilon_{xy}$ yields the observation equation

$$\epsilon_\theta = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta \cos\theta.$$

Three independently oriented grids provide three scalar observations of a uniform in-plane strain tensor. For directions $0, 45, 90$ degrees, their combination $2\epsilon_{45} - \epsilon_0 - \epsilon_{90}$ recovers engineering shear. Converting this strain to stress additionally requires elastic properties and a local plane-stress or plane-strain model.

DERIVATION AND APPLICATIONS

Remove transverse gauge sensitivity before reconstructing strain

A resistance grid responds primarily to extension along its long axis. Its transverse sensitivity may still matter when a small shear strain is inferred from differences of readings. Model a gauge at angle θ as reporting $g_\theta = \epsilon_\theta + \xi \epsilon_{\theta+90^\circ}$ after longitudinal calibration, with known dimensionless transverse response ξ. Assume uniform infinitesimal in-plane strain over a rectangular $0^\circ, 45^\circ, 90^\circ$ rosette, identical temperature compensation and the same ξ for each grid. Readings and strains below are dimensionless.

At the two axial gauges the transverse direction of one is the longitudinal direction of the other. Sum and subtract their equations to separate mean extension from normal-strain difference:

$$g_0 + g_{90} = (1+\xi)(\epsilon_x + \epsilon_y),$$
$$g_0 - g_{90} = (1-\xi)(\epsilon_x - \epsilon_y).$$

At 45° and 135°, small-strain transformation gives normal strain $(\epsilon_x + \epsilon_y)/2 \pm \gamma_{xy}/2$, where γ_{xy} is engineering shear strain. Inserting both projections into the calibrated reading gives

$$2g_{45} = (1+\xi)(\epsilon_x + \epsilon_y) + (1-\xi)\gamma_{xy}.$$

Remove the trace using the first equation and recover the corrected shear and principal strains:

$$\gamma_{xy} = \frac{2g_{45} - g_0 - g_{90}}{1-\xi},$$
$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \frac{1}{2}\sqrt{(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2}.$$

If $\xi = 0$ this reduces to the ideal rosette. As ξ approaches unity all readings respond to the trace and directional information becomes poorly conditioned. A common temperature offset cancels from this shear combination but changes the inferred mean strain if no thermal correction is made.

WORKED EXAMPLE

Recover a strain tensor from raw readings

For $\xi = 0.0200$ suppose $(g_0, g_{45}, g_{90}) = (298, 200, -94)$ microstrain. The true strain sum is $204/1.02 = 200$ microstrain, while its difference is $392/0.98 = 400$ microstrain. Thus $\epsilon_x = 300$ and $\epsilon_y = -100$ microstrain. The corrected engineering shear is $(400 - 298 + 94)/0.98 = 200$ microstrain. The principal values are 100 ± 223.6, or 323.6 and -123.6 microstrain.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Determine which calibration error matters for a pure shear test

Apply a reference pure engineering shear of 500 microstrain in the same axes. The corrected model predicts $g_0 = g_{90} = 0$ and $g_{45} = (1 - 0.0200)(500)/2 = 245$ microstrain. The ideal-gauge inversion would report 490 microstrain, low by 2.0%. A uniform offset of 12 microstrain on every channel leaves this shear unchanged, separating the transverse response from the common offset in a calibration test.

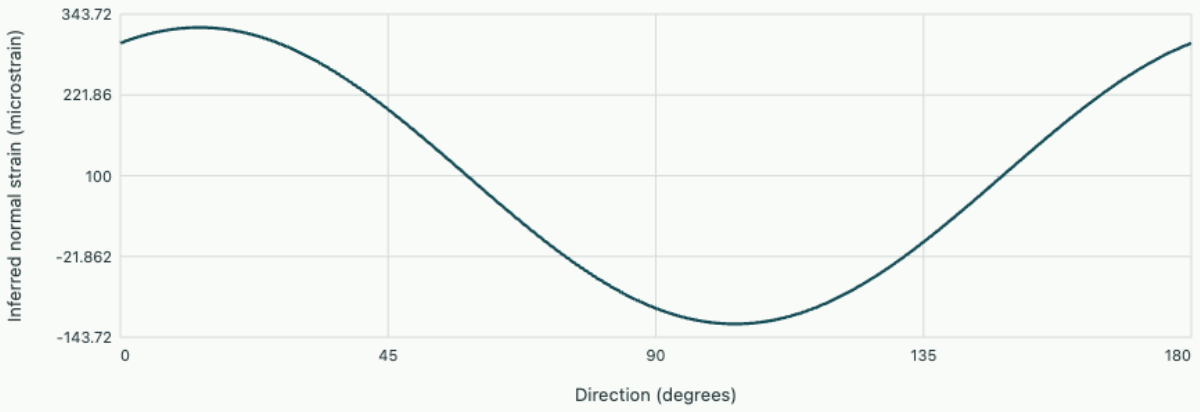
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Transverse sensitivity ξ	**0.02**
Raw0° reading (microstrain)	**298**
Raw45° reading (microstrain)	**200**
Raw90° reading (microstrain)	**-94**



Calibrated strain tensor

εx=300.000, εy=-100.000, γxy=200.000 microstrain. Engineering shear is twice the off-diagonal tensor strain.

What a bonded gauge actually measures

A resistance gauge aligned with a line on a free surface records extension along that line. A single gauge cannot distinguish uniaxial tension from combined stretching and shear arranged to give the same directional extension. Bond three gauges at different angles near enough together that the local strain is approximately uniform. A rectangular rosette uses directions of zero, 45 and 90 degrees, all referred to the same positive surface axis.

Under infinitesimal deformation, a unit direction vector $n = (\cos\theta, \sin\theta)$ experiences normal strain $n^T \boldsymbol{\varepsilon} n$. Engineering shear strain γ_{xy} is twice the off-diagonal strain tensor component. Expanding the

directional expression yields

$$\varepsilon_\theta = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy}\sin\theta\cos\theta.$$

The zero and 90° readings give $\varepsilon_x, \varepsilon_y$ directly. The middle reading gives

$$\gamma_{xy} = 2\varepsilon_{45} - \varepsilon_x - \varepsilon_y.$$

A gauge at minus 45° measures the opposite shear contribution. A sketch of the physical gauge labels avoids reversing the inferred shear sign during data reduction.

Introduce the local stress assumption

At an unloaded outer surface the normal and shear tractions on that surface vanish. With z normal to it, this gives local plane stress. For isotropic linear elasticity and mechanical strains obtained after temperature correction, invert the plane-stress compliance equations and use the shear modulus for the remaining component.

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y),$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x),$$

$$\tau_{xy} = \frac{E\gamma_{xy}}{2(1+\nu)}.$$

A strain rosette measures only the in-plane quantities. A nonzero through-thickness strain is consistent with plane stress because the in-plane stresses cause Poisson contraction. Beneath a contact patch the local traction condition changes, so the same stress recovery would need additional information.

Interpret the readings on a plate

Suppose a steel surface gives $400,\ 250,\ -100$ microstrain at zero, 45 and 90 degrees. The recovered shear strain is $500 - 400 + 100 = 200$ microstrain. With $E = 210\,\mathrm{GPa}$ and $\nu = 0.30$, the stresses are $85.38,\ 4.62,\ 16.15\,\mathrm{MPa}$ for x, y, xy, respectively. The shear is positive under the stated gauge orientation. Converting microstrain to 10^{-6} before multiplying by MPa is essential.

The principal stresses are the eigenvalues of this symmetric matrix. Its mean normal stress is 45.0 MPa, and its Mohr-circle radius is

$$\sqrt{[(85.38-4.62)/2]^2 + 16.15^2} = 43.50\,\mathrm{MPa}.$$

Thus the two in-plane principal stresses are approximately 88.5 and 1.5 MPa. The major direction is $\tfrac{1}{2}\operatorname{atan2}(2\tau, \sigma_x - \sigma_y) = 10.9^\circ$ from the zero gauge. A strain reading of –100 microstrain along y

therefore does not imply compressive normal stress along that axis: Poisson contraction from the dominant x load accounts for much of that reading.

Resolve a uniform thermal signal

Uniform heating of an unconstrained isotropic plate adds the same free thermal strain $\alpha \Delta T$ to each gauge reading, after allowing for the gauge instrumentation itself. It cancels from the inferred engineering shear because $2 - 1 - 1 = 0$, but it persists in both recovered normal components. For a 20 K temperature rise and linear expansion $12 \times 10^{-6}\,\mathrm{K}^{-1}$, subtract 240 microstrain from each reading before applying the mechanical plane-stress relation.

Reconstruct a surface strain from a delta rosette

The zero, 45 and 90 degree rosette is convenient but not the only useful placement. Introduce the strain-circle coordinates $m = (\varepsilon_x + \varepsilon_y)/2,\ d = (\varepsilon_x - \varepsilon_y)/2,\ g = \gamma_{xy}/2$. Then each directional reading is $e_\theta = m + d\cos 2\theta + g \sin 2\theta$. Every installed gauge supplies one linear equation in these three unknowns. Gauge directions that are nearly parallel give nearly dependent equations, so a small error in an individual reading can then cause a large error in a recovered component.

For a delta rosette take directions $0, 60, 120^\circ$. The doubled angles are equally spaced round the strain circle, so their cosine and sine values each sum to zero. Adding all three readings gives

$$m = (e_0 + e_{60} + e_{120})/3.$$

The first reading then gives

$$d = e_0 - m;$$

subtracting the last two cancels both m and their identical cosine terms and gives

$$g = (e_{60} - e_{120})/\sqrt{3}.$$

The in-plane principal strains and their physical orientation follow without a material assumption:

$$e_{1,2} = m \pm \sqrt{d^2 + g^2},$$

$$\theta_1 = \tfrac{1}{2}\,\mathrm{atan}\,2(g, d).$$

The half-angle is necessary because a straight gauge has the same reading after a physical rotation of 180°. For an isotropic elastic solid the principal stress directions coincide with these strain directions. Orthotropic laminates can give stress and strain axes that differ, so converting a strain circle into stresses would require the material compliance in the specimen’s axes.

OneSimulation · How does a small gauge error become a large transverse strain?

A bonded gauge observes the normal strain along its axis. Reconstruct the two normal components and the engineering shear from three readings. Give only the middle reading a positive bias of 5 με. The factor multiplying the shear in the observation matrix follows from rotating a symmetric tensor with $\epsilon_{xy} = \gamma_{xy}/2$.

$$\epsilon_\theta = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta \cos\theta.$$

OneSimulation solves the same three-unknown system for two different layouts. At 0°, 45° and 90°, the last gauge directly measures transverse strain. At 0°, 10° and 20°, all gauges see mostly the longitudinal direction. The reconstructed transverse strain now misses by −155.8 με. Fitting three readings exactly supplies no independent check on accuracy.

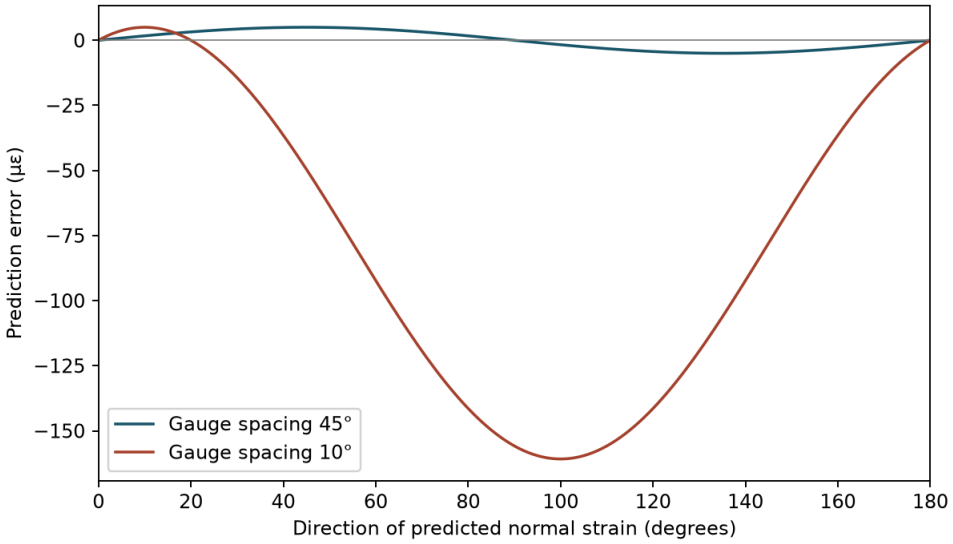
```
"""Compare strain-rosette geometries with the same middle-gauge error."""
import matplotlib.pyplot as plt
import numpy as np
from onesim.numerics.linear.matrices.dense_linear_algebra import solve_dense

true_strain = np.array([100., -60., 80.])  # ex, ey, engineering shear in με
bias = 5.0  # με added to the middle gauge
rotation = np.arange(0, 181)

def gauge_rows(degrees):
    angle = np.deg2rad(degrees)
    c, s = np.cos(angle), np.sin(angle)
    return np.column_stack((c**2, s**2, c*s))

fig, ax = plt.subplots(figsize=(7, 4), constrained_layout=True)
for spacing, colour in [(45, "#19596a"), (10, "#a5442e")]:
    layout = gauge_rows([0, spacing, 2*spacing])
    readings = layout @ true_strain
    assert np.allclose(solve_dense(layout, readings), true_strain)
    readings[1] += bias
    inferred = solve_dense(layout, readings)
    error = gauge_rows(rotation) @ (inferred - true_strain)
    ax.plot(rotation, error, color=colour, label=f"Gauge spacing {spacing}°")
    print(f"{spacing}°: error at 90° = {error[90]:.2f} με")

ax.axhline(0, color="0.5", linewidth=0.7)
ax.set(xlabel="Direction of predicted normal strain (degrees)",
       ylabel="Prediction error (με)", xlim=(0, 180))
ax.legend()
fig.savefig("engineering-mechanics.png", dpi=170)
```



Change the middle-gauge bias and rerun. Its error remains 5 με at that gauge; the much larger error occurs in directions far outside the narrow set of measured angles.

Download the Python example

Vary the parameters of this exact example in the chapter's interactive calculation.

Check an installed rosette using an independent gauge

On a loaded aluminium plate a delta rosette reads $e_0 = 350,\ e_{60} = 200,\ e_{120} = -100$ microstrain. The mean is 150 microstrain; $d = 200$ and $g = 300/\sqrt{3} = 173.2$ microstrain. Thus $\varepsilon_x = 350, \varepsilon_y = -50, \gamma_{xy} = 346.4$ microstrain. The strain-circle radius is $\sqrt{200^2 + 173.2^2} = 264.6$ microstrain. Principal strains are 414.6 and –114.6 microstrain, with largest extension along $\theta_1 = 20.45^\circ$ from the zero gauge.

Before converting these figures to stress, predict an independent reading at 90°. The directional equation gives $m - d = -50$ microstrain, which can be checked with a fourth gauge. If it instead reads +50 microstrain, possible explanations include angular mislabelling, an adhesive defect or a spatial strain gradient across the footprint. In fact the maximum local slope of the directional reading with respect to angle is $2\sqrt{d^2 + g^2}$ per radian. A two-degree placement error can therefore contribute at most about $529(2\pi/180) = 18.5$ microstrain for this field to first order. An observed 100 microstrain discrepancy would be too large to explain by that tolerance alone. This cross-check assesses the measurement geometry before uncertainty in elastic moduli enters the interpretation.

Close to an equibiaxial strain state both d and g are small, so angle estimates from $\operatorname{atan}2(g, d)$ can jump substantially between repeated readings. That sign of numerical sensitivity has a physical interpretation: when the strain circle collapses, every direction is principal. The nearly equal principal strain values may still be well determined even though a reported principal orientation is almost arbitrary. Quoting the circle radius together with the angle gives a reader enough information to decide whether the orientation is useful.

Interactive calculation · Change the inputs of this example

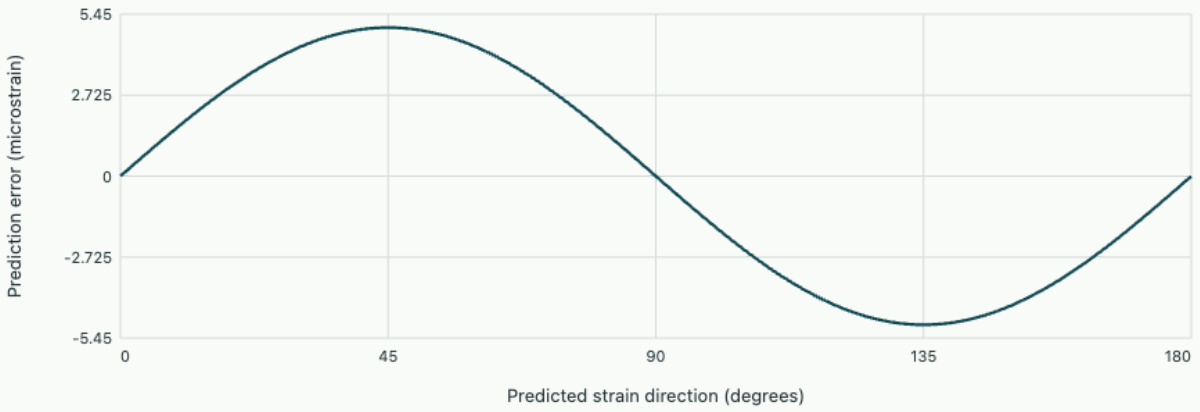
Change the parameters of this reference plot and compare its limiting cases. The fixed illustration remains beside the derivation; the adjustable calculation exposes the same dependence on geometry, material properties or loading.

Open this live calculation

Try the calculation

Gauge angular spacing (degrees) **45**

Middle-gauge bias (microstrain) **5**



Reconstructed tensor error

The0° gauge is exact; all error comes from the middle gauge. At90° prediction error=-0.000 microstrain. Crowding gauge angles amplifies the same measurement error.

Yield fronts and plastic bending capacity

Engineering mechanics

Track the spread of yielding through a rectangular section and distinguish the first-yield moment from the fully plastic moment.

FROM FIRST PRINCIPLES

Separate first yield from the plastic moment of a section

Take a homogeneous symmetric beam section in pure bending, with plane sections remaining plane and elastic-perfectly plastic material of yield stress σ_y. Let c be the largest distance from the neutral axis. For small curvature κ longitudinal strain varies as κy, so the elastic stress distribution and bending moment are

$$\sigma_x = E\kappa y,$$

$$M = \int_A \sigma_x y \, dA = EI\kappa.$$

First yield occurs when the outermost stress magnitude reaches σ_y. Continuing to bend can still raise the moment as deeper fibres reach yield. In the ideal fully plastic limit, the tensile and compressive blocks have stresses $\pm\sigma_y$. Symmetry makes their axial forces cancel. The two distinct moment scales are therefore

$$M_y = \sigma_y I/c,$$

$$M_p = \sigma_y \int_A |y| \, dA.$$

For a rectangle of breadth b and total depth $2c$, the moments become $2b\sigma_y c^2/3$ and $b\sigma_y c^2$. Their ratio $3/2$ measures the stress redistribution available in the section. A complete frame mechanism generally needs several suitable plastic hinges, so section yield alone does not determine the collapse load.

DERIVATION AND APPLICATIONS

Predict springback after partial plastic bending

A bent coupon can retain curvature after the applied moment is removed. Consider a homogeneous rectangular section of breadth b, half-depth c, Young's modulus E and yield stress σ_y. Assume an elastic-perfectly plastic material in monotone pure bending, plane sections and symmetric tension/compression response. Take curvature $\kappa > 0$ and magnitudes on the upper half. After yielding begins the elastic core extends from the centre to $y_e = \sigma_y/(E\kappa)$; beyond it stress has saturated.

Integrate stress times lever arm separately in the core and yielded zones, then double by symmetry:

$$M = 2b\left[\int_0^{y_e} E\kappa y^2\,dy + \int_{y_e}^{c} \sigma_y y\,dy\right] = b\sigma_y(c^2 - y_e^2/3).$$

At first yield

$$\kappa_y = \sigma_y/(Ec);$$

the fully plastic limiting moment is

$$M_p = b\sigma_y c^2.$$

Eliminating y_e relates peak moment to peak curvature:

$$M/M_p = 1 - \tfrac{1}{3}(\kappa_y/\kappa)^2, \qquad \kappa \geq \kappa_y.$$

Upon unloading, initially all fibres respond elastically. The removed moment corresponds to a change $M/(EI)$ in curvature, with $I = 2bc^3/3$. Since $M_p/(EI) = 3\kappa_y/2$, the residual curvature is

$$\kappa_r = \kappa - \frac{M}{EI} = \kappa - \tfrac{3}{2}\kappa_y\left[1 - \tfrac{1}{3}(\kappa_y/\kappa)^2\right].$$

At first yield the predicted residual is zero. The derivation assumes the unloading increment does not drive any fibre into reverse yield; that can be checked by its surface stress change. Real strain hardening or different moduli during unloading change the inferred springback.

WORKED EXAMPLE

Calculate the unloaded radius of a bent strip

Take $E = 200$ GPa, $\sigma_y = 250$ MPa and thickness $2c = 5.00$ mm. First-yield curvature is 0.500 m⁻¹. Bend to $\kappa = 1.00$ m⁻¹ so only the middle half of the depth remains elastic. The peak moment is $(11/12)M_p$, unloading removes 0.6875 m⁻¹, and $\kappa_r = 0.3125$ m⁻¹. The strip springs back from a 1.00 m radius to a radius of 3.20 m.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Check for reverse yielding and residual surface stress

At that same peak, the tensile surface was at 250 MPa. Elastic unloading changes its stress by $Ec(0.6875) = 343.75$ MPa in the opposite direction, leaving -93.8 MPa. This compressive residual remains below the 250 MPa reverse yield magnitude, consistently with elastic unloading. The unloaded

part carries no applied moment even though its surface stress is nonzero; residual stresses elsewhere across the section balance its moment.

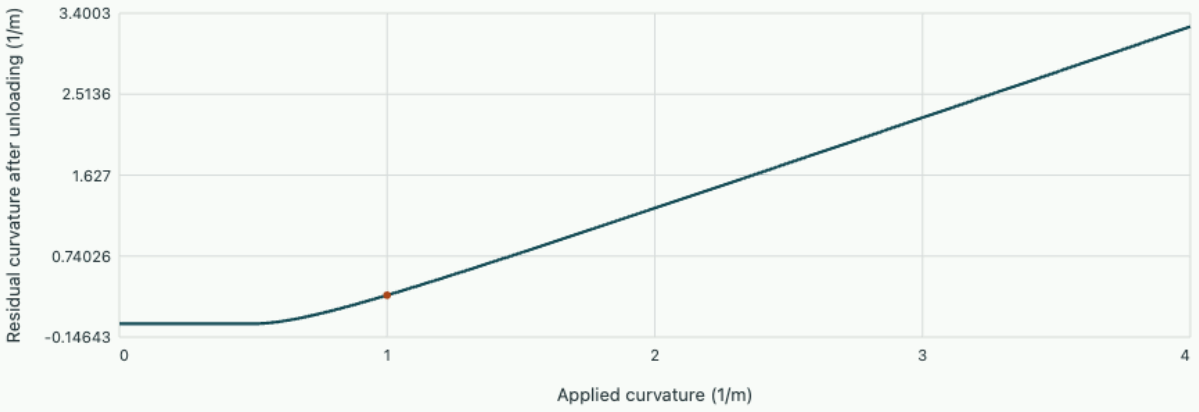
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Thickness (mm)	5
Yield stress (MPa)	250
Maximum curvature (1/m)	1



First-yield curvature 0.500 /m; retained curvature 0.313 /m. Elastic unloading is assumed; reverse yielding must be checked separately.

First yield occurs at a surface

A beam can retain moment capacity after its first surface fibres yield. Consider a straight rectangular bar of width b, total depth $2c$, equal tensile and compressive yield stresses σ_Y, and an elastic, perfectly plastic law. Assume pure bending about its centroid, small strains and a compact section that avoids local buckling. Plane sections remain plane, so longitudinal strain varies as κy. Symmetry keeps the neutral axis at mid-depth for this material and section.

While the whole section is elastic, $\sigma = E\kappa y$. As curvature increases, the first yield occurs when $E\kappa_Y c = \sigma_Y$. Integrating stress times moment arm gives the elastic first-yield moment $M_Y = EI\kappa_Y = \sigma_Y I/c$. A section with a narrow outer flange or unequal tensile and compressive material behaviour requires a fresh stress distribution and force equilibrium.

Integrate the elastic core and yielded layers

At a greater curvature let $a = \sigma_Y/(E\kappa)$ be the half-depth of the remaining elastic core. Stress in the core grows linearly as $\sigma_Y y/a$. Outside the core it remains at the yield magnitude. Each half of the section contributes the same bending moment. Splitting the integral at the moving yield front gives

$$M = 2b\left[\int_0^a \frac{\sigma_Y y^2}{a}\,dy + \int_a^c \sigma_Y y\,dy\right] = b\sigma_Y(c^2 - a^2/3).$$

At $a = c$ this reproduces $M_Y = (2/3)b\sigma_Y c^2$. As a approaches zero, M tends to $M_p = b\sigma_Y c^2$, which is 1.5 times first yield for this solid rectangle. The yield fronts advance towards the centre while progressively larger curvature increments produce smaller changes of moment. The bending stiffness at first loading is therefore history dependent after yielding begins.

Quantify the onset and extent of plastic deformation

A 20 mm wide bar is 40 mm deep, with $E = 200\,\mathrm{GPa}$ and $\sigma_Y = 250\,\mathrm{MPa}$. Its first-yield curvature is $250/(200000 \times 20) = 6.25 \times 10^{-5}\,\mathrm{mm^{-1}}$, or 0.0625 m⁻¹. First yield occurs at 1.333 kN m and the limiting plastic moment is 2.000 kN m. At twice the yield curvature the elastic core half-depth has fallen to 10 mm. Substitution gives

$$M = 20(250)[400 - 100/3] = 1.833\,\mathrm{kN\,m}.$$

Upon unloading, assume the stress changes elastically without reverse yield. The immediate moment–curvature slope is then EI, so the recovered curvature is $M/(EI)$, where $I = 20(40)^3/12 = 106667\,\mathrm{mm^4}$. It is $8.59 \times 10^{-5}\,\mathrm{mm^{-1}}$. The peak curvature was $1.25 \times 10^{-4}\,\mathrm{mm^{-1}}$, leaving about 0.0391 m⁻¹ residual curvature. A permanent bend can therefore coexist with zero applied bending moment. The residual internal stresses balance their own force and moment across the section.

Compare equal areas in different locations

Redistributing material towards the outer fibres changes the reserve between first yield and full plasticity. In the limiting idealisation of two very thin equal flanges of area A_f at $y = \pm c$, essentially all material reaches yield together. The first-yield and plastic moments both approach $2A_f\sigma_Y c$. A shape factor of 1.5 is specific to the solid rectangle and cannot be carried over to this ideal flange pair.

Build a collapse mechanism from plastic hinges

A section that reaches its plastic moment can undergo substantial further rotation in an ideal perfectly plastic model. A structure may still support additional external load because other sections can take up incremental bending. Collapse in a rigid–plastic beam idealisation occurs when enough hinges form to allow a compatible mechanism. Use virtual work at collapse: the external forces do work through compatible virtual displacements, and each plastic hinge dissipates M_p times its relative rotation. Elastic strain energy changes are neglected in this limit calculation.

Take a uniform beam fixed at both ends under a concentrated downward force P at midspan. Assume equal plastic moments of magnitude M_p in sagging and hogging, compact sections and adequate rotation capacity. Consider hinges at the two clamps and at midspan, joined by two rigid half-spans. A small downward virtual centre movement δ rotates each half through $\theta = 2\delta/L$. The clamp hinges each rotate by θ, while the centre hinge rotates through 2θ because its adjacent halves slope in opposite directions. Work equivalence gives

$$P\delta = M_p(\theta + 2\theta + \theta) = 4M_p(2\delta/L),$$

$$P_c = 8M_p/L.$$

This kinematically admissible mechanism gives an upper bound on true plastic collapse load under these assumptions. Here a matching statically admissible diagram can be constructed: end moments are $-M_p$, the centre moment is $+M_p$, and constant shear on each half gives

$$P/2 = 4M_p/L.$$

The matching bounds identify the exact ideal collapse load for this geometry and loading.

Separate first yield, section capacity and structural collapse

A mild-steel rectangular strip 30 mm wide and 60 mm deep spans 1.20 m between fixed clamps. Take yield stress 240 MPa. The first-yield section moment is $M_Y = \sigma_Y b h^2/6 = 4.32\,\mathrm{kN\,m}$, and the fully plastic moment is

$$M_p = \sigma_Y b h^2/4 = 6.48\,\mathrm{kN\,m}.$$

For a central point force, elastic symmetry and compatibility give clamp and centre moment magnitudes $PL/8$. First yield thus occurs at $8(4.32)/1.20 = 28.8\,\mathrm{kN}$ and the ideal mechanism load is

$$8(6.48)/1.20 = 43.2\,\mathrm{kN}.$$

The factor of 1.5 here comes from the rectangular section’s reserve as yielding spreads. The three critical sections have equal absolute moment in this symmetric loading.

Release the right clamp rotation and the mechanism changes. For the simpler limiting case with both ends pinned, only one interior plastic hinge is needed and the moment maximum is $PL/4$. The same

section would collapse at $4(6.48)/1.20 = 21.6\,\mathrm{kN}$, half the fixed-ended result. The comparison explains why a connection described as fixed must actually develop bending moment and plastic rotation before its contribution is credited. A brittle welded detail, lateral instability or a shear failure can prevent the assumed redistribution. The calculation supplies a limit for the stated deformation mode, not a prediction of every possible failure mechanism.

The location of the mechanism can be checked in an overload test by measuring rotations near both supports and on either side of midspan. A large rotation at a single connection before substantial rotation elsewhere indicates that the assumed distribution of hinge resistance is not being realised.

Crack driving force and fracture assessment

Engineering mechanics

Use an energy balance and a stress-intensity model to distinguish crack-controlled fracture from yielding of the gross section.

FROM FIRST PRINCIPLES

Relate crack extension to the energy released by elastic compliance

Consider a slowly loaded linear elastic specimen of thickness B, applied force P, measured displacement δ and crack coordinate a. Write $C(a) = \delta/P$. The projected area convention is $dA = B\,da$. On a fixed crack the linear loading curve stores strain energy equal to the triangle below its force–displacement graph:

$$U = \tfrac{1}{2} P\delta = \tfrac{1}{2} P^2 C(a).$$

For a dead force held constant while the crack extends, the external force performs additional work $P\,d\delta$. Subtract the load potential $P\delta$ from the stored energy and differentiate with respect to area to obtain the driving force for extension:

$$\Pi = U - P\delta = -\tfrac{1}{2} P^2 C,$$

$$G = -\left.\frac{d\Pi}{dA}\right|_P = \frac{P^2}{2B}\frac{dC}{da}.$$

The units are J/m². An increase of compliance can free energy to create new surfaces and to dissipate energy near the crack. A material resistance measured under matching conditions is needed to predict onset. In linear elastic mode I, the local intensity relation $G = K_I^2/E'$ connects this global energy measure with the near-tip field.

Background · Stress intensity

$$K_I = Y\sigma\sqrt{\pi a}$$

Opening-mode intensity has units of stress times the square root of length. Geometry and the definition of crack length are part of the calibration.

DERIVATION AND APPLICATIONS

How machine compliance changes the energy available to a crack

A tensile fixture can release stored elastic energy as a crack grows even when the grips do not move. To quantify this, treat a specimen of thickness B with crack coordinate a and measured opening compliance $C(a) = \delta_s / P$. Use a single projected crack-area increment $B\,da$ consistent with the stated convention. Let an elastic testing machine in series have constant compliance C_m, and assume quasistatic linear elastic loading with negligible dissipative processes outside the crack tip. At fixed total grip opening Δ the force must change as the specimen compliance changes:

$$\Delta = P(C + C_m),$$

$$U = \tfrac{1}{2} P^2 (C + C_m) = \frac{\Delta^2}{2(C + C_m)}.$$

No further external work enters at fixed Δ. Differentiate the stored energy with respect to projected area. Since C_m is independent of the crack coordinate, only the specimen derivative appears:

$$G = -\frac{1}{B}\left.\frac{dU}{da}\right|_{\Delta} = \frac{\Delta^2}{2B(C + C_m)^2}\frac{dC}{da} = \frac{P^2}{2B}C'(a).$$

At a given instantaneous force this has the familiar force-based value, but its variation during crack extension depends on compliance control. Take the idealized measured relation $C = ca^3$ over a limited range. Logarithmic differentiation at fixed imposed opening gives

$$\frac{1}{G}\frac{dG}{da} = \frac{2}{a} - \frac{6C}{a(C + C_m)}.$$

For constant fracture resistance, a decreasing driving force after slight growth favours arrest. In this compliance model it occurs when $C_m < 2C$. This local comparison assumes a small crack extension within the calibrated range and a material whose fracture resistance is not changing appreciably.

WORKED EXAMPLE

Estimate driving force from a coupon measurement

At $a = 40.0$ mm take $C = 2.00$ micrometres/N, $C_m = 1.00$ micrometre/N, $B = 10.0$ mm and $\Delta = 0.600$ mm. The force is 200 N, and the local cubic slope is $C' = 3C/a = 1.50 \times 10^{-4}$ N⁻¹. Substitution yields $G = 300$ J/m². The calculated logarithmic slope is -50.0 m⁻¹, predicting about a 5% decrease for an initial additional millimetre of extension.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Compare a compliant test frame at the same force

Increase machine compliance to 6.00 micrometres/N and impose $\Delta = 1.60$ mm so the starting force stays 200 N. The initial G remains 300 J/m², but the slope is now $+12.5$ m⁻¹. A millimetre of additional crack length initially raises the driving force by about 1.25%. The specimen itself has not changed; the frame now delivers a different load history as the crack extends.

Interactive calculation · Change the inputs of this example

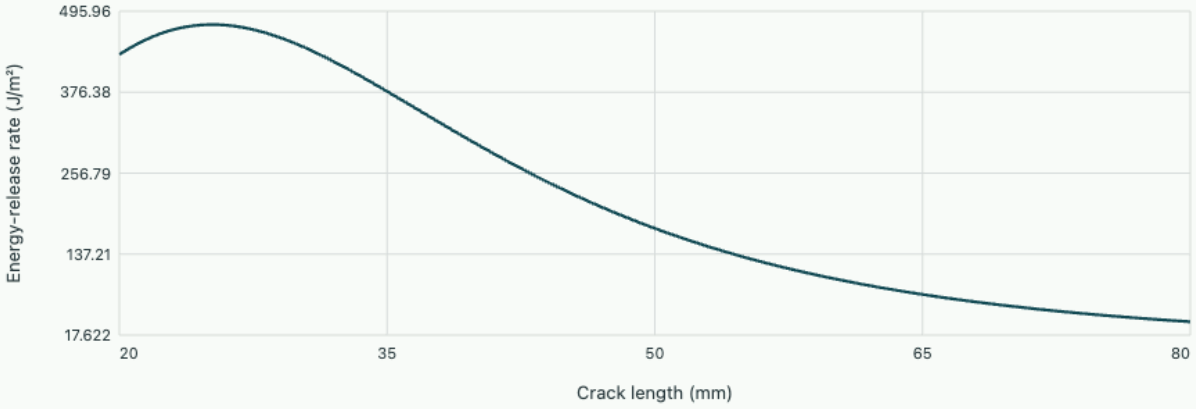
Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Machine compliance (µm/N) **1**

Crosshead opening (mm) **0.6**



Fixed crosshead displacement

At40 mm crack length, G=300.000 J/m². Specimen compliance is2 µm/N at40 mm and varies cubically; width10 mm.

A sharp crack changes the question

An uncracked tensile coupon can often be assessed by comparing nominal stress with a yield stress. A sharp pre-existing crack concentrates the stress into a small region near its tip. Within linear elastic

fracture mechanics the leading opening-mode field has magnitude proportional to $K_I/\sqrt{2\pi r}$ at distance r ahead of the tip. The unbounded mathematical stress signals the limits of the elastic tip description. A material toughness measured under comparable conditions instead sets a threshold for crack extension.

A common representation is $K_I = Y\sigma\sqrt{\pi a}$, where a is the specified crack dimension, σ is a stated remote stress, and Y describes the geometry and loading. For a central crack in a sufficiently wide plate a is the half-length and Y approaches one. For a short edge crack in a wide plate a is its depth and a common approximation is $Y \approx 1.12$. Finite width, bending or a different crack shape changes the factor.

Relate compliance to an energy release rate

In a linear test specimen let load P, opening displacement u, and compliance $C(a) = u/P$ refer to the same load point. At fixed load the stored elastic energy is $P^2C/2$. The external load does work $P\,du = P^2\,dC$ as the crack extends. The released potential energy is the difference, $P^2dC/2$. For a through crack with one advancing front and specimen thickness B, its projected area increases by $B\,da$, giving

$$G = \frac{P^2}{2B}\frac{dC}{da} = \frac{K_I^2}{E'},$$

$$E' = E \ (\text{plane stress}),$$

$$E' = E/(1-\nu^2) \ (\text{plane strain}).$$

The compliance argument explains how a distant load–displacement measurement can identify energy available at a crack. Define the crack area consistently: a symmetric specimen in which two tips advance together changes projected area twice as quickly for the same increment of half-length.

Infer a critical flaw size

Take a wide plate with an edge crack of depth 3.0 mm, tensile remote stress 160 MPa and illustrative applicable toughness $K_c = 35\,\mathrm{MPa}\sqrt{\mathrm{m}}$. The estimate $K_I = 1.12(160)\sqrt{\pi(0.003)} = 17.4\,\mathrm{MPa}\sqrt{\mathrm{m}}$ uses metres for crack depth. Holding geometry factor and stress fixed, the critical depth is $a_c = [35/(1.12\times 160)]^2/\pi = 0.0121\,\mathrm{m}$, about 12.1 mm.

At twice the stress this depth falls by a factor of four, to approximately 3.0 mm. Before using that extrapolation, check remaining net section, applicability of the wide-plate geometry and whether plasticity near the crack is small compared with crack and ligament dimensions. Toughness depends on temperature, constraint and loading rate. An inspection interval would additionally require a crack-growth law, a load history and a detection limit. The calculated critical depth describes one load state and cannot by itself predict the number of cycles remaining.

Interpret a crack-length uncertainty

At fixed remote stress and geometry factor a small relative error in crack depth gives approximately half that relative error in stress intensity, because $dK/K = da/(2a)$. A 0.3 mm uncertainty on a 3 mm measurement therefore changes this estimated intensity by about five percent.

Integrate stable fatigue growth up to inspection

Fracture toughness characterises an instability at a given load. Damage tolerance also asks how long a detected crack takes to reach an inspection or retirement size under repeated loading. For illustration, suppose tests for the service material and stress ratio support a Paris relation $da/dN = C(\Delta K)^m$ in its intermediate stable growth range. Use crack depth a in metres and intensities in MPa$\sqrt{\text{m}}$. The coefficient C must correspond to those units, stress ratio and environment. This empirical law generally changes near threshold and near final instability.

For constant stress range $\Delta\sigma$, fixed geometry factor Y and exponent $m = 3$, substitute $\Delta K = Y\Delta\sigma\sqrt{\pi a}$. Crack advance per cycle then scales as $a^{3/2}$; the same stress history grows a larger crack faster. Separate variables between measured depth a_i and removal depth a_f. Integrating yields

$$N = \frac{2(a_i^{-1/2} - a_f^{-1/2})}{C(Y\Delta\sigma)^3\pi^{3/2}}.$$

This prediction decreases to zero as the two depths approach each other. It scales with the inverse cube of stress range at fixed bounds. If the geometry factor changes appreciably with crack depth, retain $Y(a)$ inside the integral. The maximum stress in the cycle is needed separately to compare $K_{\max}$ with applicable toughness, because it is the range that drives the adopted fatigue law.

Compare two inspection thresholds on the same plate

Consider a wide plate with $Y = 1.12$, $\Delta\sigma = 80\,\text{MPa}$, $\sigma_{\max} = 140\,\text{MPa}$, $m = 3$, and illustrative $C = 2.0 \times 10^{-12}\,(\text{m/cycle})/(\text{MPa}\sqrt{\text{m}})^3$. Suppose the measured growth-law range includes depths between 2 and 8 mm for this loading. At 2 mm,

$$\Delta K = 1.12(80)\sqrt{\pi(0.002)} = 7.10\,\text{MPa}\sqrt{\text{m}};$$

at 8 mm it doubles to 14.2. The growth rate at the upper limit is consequently eight times the lower-limit rate.

The integral denominator is $2.0 \times 10^{-12}(89.6)^3\pi^{3/2} = 8.01 \times 10^{-6}$ in the stated units, and the numerator is

$$2(22.36 - 11.18) = 22.36\,\text{m}^{-1/2}.$$

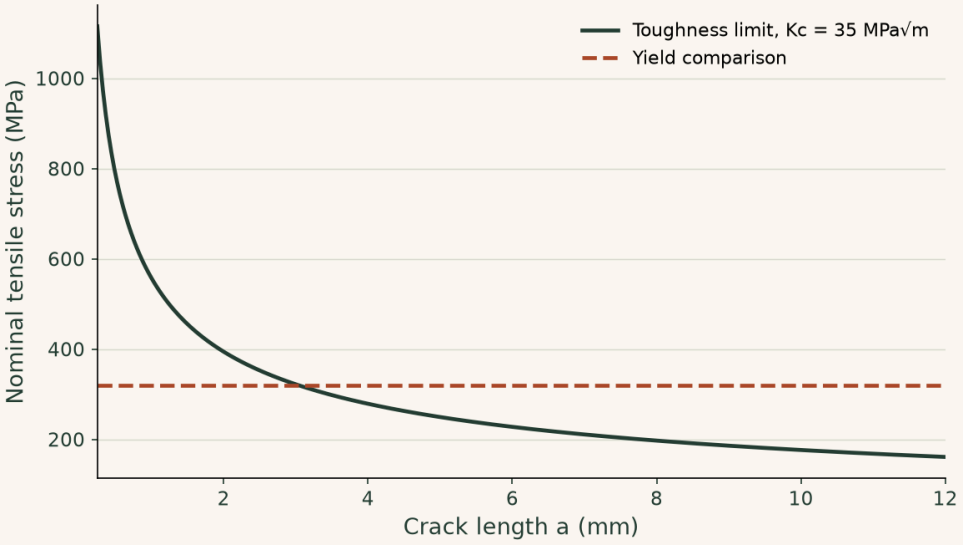
The estimated interval is 2.79×10^{6} cycles. If the smallest reliably detected crack were 4 mm, the available interval to 8 mm would fall to $2(15.81 - 11.18)/(8.01 \times 10^{-6}) = 1.16 \times 10^{6}$ cycles. Improving early detection has a measurable value in this model. At retirement depth $K_{\max} = 24.9\,\mathrm{MPa}\sqrt{\mathrm{m}}$; compare that independently with the applicable toughness. Uncertainty in detection, loading, environment and growth coefficients would enter an actual inspection decision rather than treating this calculated mean growth interval as a guaranteed service interval.

A useful sensitivity check is to increase the entire stress range by ten percent while keeping the inspection depths and coefficient fixed. The Paris estimate falls by $1/1.1^{3} = 0.751$, reducing the 2.79 million-cycle interval to about 2.10 million cycles. If the maximum stress also rises by ten percent, retirement intensity increases by the same ten percent to 27.4 MPa√m. Those two changes affect different checks and cannot be combined into a single linear safety multiplier. A record of both stress range and stress maximum is therefore more informative than a peak-only reading for planning inspections.

Both quantities can be recovered from the two measured stress extremes if the counting convention and gauge reference are retained.

Which flaw sizes make fracture the controlling limit?

Consider a wide plate with a small edge crack, geometry factor $Y = 1.12$ and conditional toughness $K_c = 35\,\mathrm{MPa}\sqrt{\mathrm{m}}$. Rearrange $K = Y\sigma\sqrt{\pi a}$ and compare with a 320 MPa yield scale. Convert the crack length to metres.



The two estimates intersect near a = 3.04 mm. As cracks grow, the estimated fracture stress falls as the inverse square root. Applying this simple comparison to a real component requires geometry-specific factors, valid toughness data and a check on plastic-zone size.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

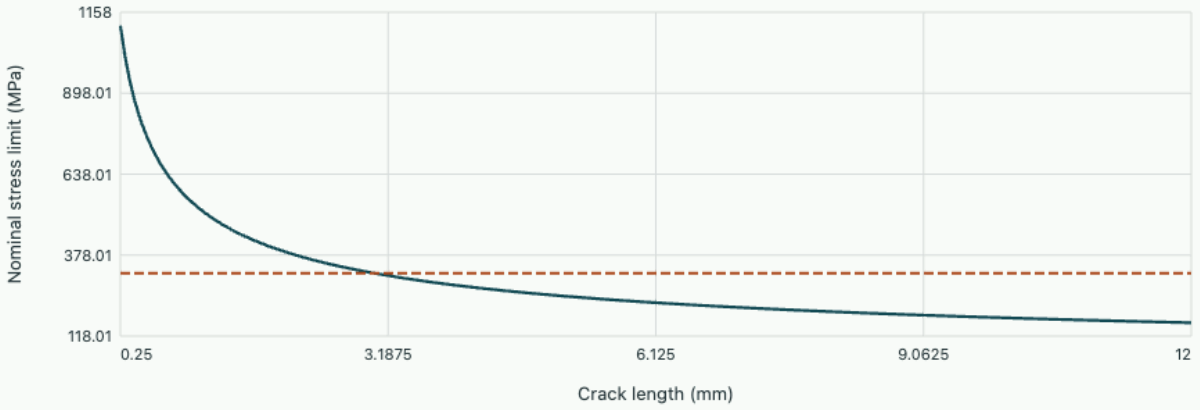
Vary toughness and yield scale independently. Locate the crack length where the controlling estimate changes.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Conditional toughness (MPa√m)	**35**
Geometry factor Y	**1.12**
Yield scale (MPa)	**320**



Conditional fracture Yield comparison

The two estimates cross at 3.036 mm. This comparison does not replace geometry and plastic-zone validity checks.

Finding a displacement when no load acts in its direction

Engineering mechanics

A displacement gauge measures the lateral travel of a truss joint under a vertical load. A force balance determines the bar forces, but cannot directly give the sideways motion. Introduce a dummy load, differentiate the stored energy and recover the gauge reading with its sign.

Link an external force coordinate to stored strain energy

For a linear elastic pin-jointed truss, neglect support movement and temperature changes and assume small displacements. Bar i has constant axial force N_i, length L_i, modulus E_i and area A_i. Integrating its force against its extension gives

$$U_i = N_i^2 L_i/(2E_i A_i).$$

Superposition makes each force linear in the external loads. A small increment of load changes the equilibrium configuration. For a linear elastic conservative system the complementary energy equals U and its derivative with respect to an independent applied force gives displacement along that force.

$$U = \sum_i \frac{N_i^2 L_i}{2E_i A_i},$$

$$\delta_Q = \frac{\partial U}{\partial Q} = \sum_i \frac{N_i n_i L_i}{E_i A_i},$$

$$n_i = \frac{\partial N_i}{\partial Q}.$$

The derivative n_i is the member force produced per unit trial load in the requested direction, with all other independent loads held fixed. Solve this trial equilibrium even if $Q = 0$ physically. Setting $Q = 0$ before differentiating erases its entire contribution. As a dimensional check, n_i is dimensionless when Q and N_i both use newtons, and the sum has dimensions of length.

Compute the two-bar flexibility matrix

A joint hangs from two overhead pins by straight bars slanting up-left and up-right, each at 45° to horizontal. Positive joint coordinates are rightward and downward. Let V be the applied downward force and H the rightward force; bars carry tension positive. Equilibrium gives

$$N_L = (V + H)/\sqrt{2},\ N_R = (V - H)/\sqrt{2}.$$

Define bar axial compliances $c_i = L_i/(E_i A_i)$. Insert the forces into the energy and differentiate with respect to H and V independently:

$$\begin{bmatrix} \delta_H \\ \delta_V \end{bmatrix} = \frac{1}{2} \begin{bmatrix} c_L + c_R & c_L - c_R \\ c_L - c_R & c_L + c_R \end{bmatrix} \begin{bmatrix} H \\ V \end{bmatrix}.$$

The cross coefficients are equal because the mixed derivatives of the stored energy commute. This is Maxwell–Betti reciprocity. A cross displacement can be negative even though both direct compliances and stored energy are positive. The eigenvalues of this particular flexibility matrix are c_L, c_R, corresponding to the two independent bar-extension directions.

Predict a sideways gauge reading and reverse the test

Take $c_L = 0.0020,\ c_R = 0.0040$ mm/N. At $V = 100$ N and $H = 0$, both members carry 70.71 N. The vertical displacement is 0.300 mm, and the horizontal displacement is −0.100 mm. The more compliant right bar stretches farther; the joint moves left and down to accommodate both bar extensions. Half the vertical load times its displacement gives 15 N mm of strain energy. Summing the two bar energies independently gives the same result.

Apply instead $H = 100$ N and $V = 0$, assuming the slender right bar remains stable under compression. The vertical reading is −0.100 mm and horizontal motion is +0.300 mm. Thus the lateral response to the vertical test predicts the vertical response to a lateral test of equal magnitude. Equalizing bar areas would remove the cross response if the lengths and moduli matched. In practice a pin with clearance or a cable that goes slack can break the assumed linear bilateral response.

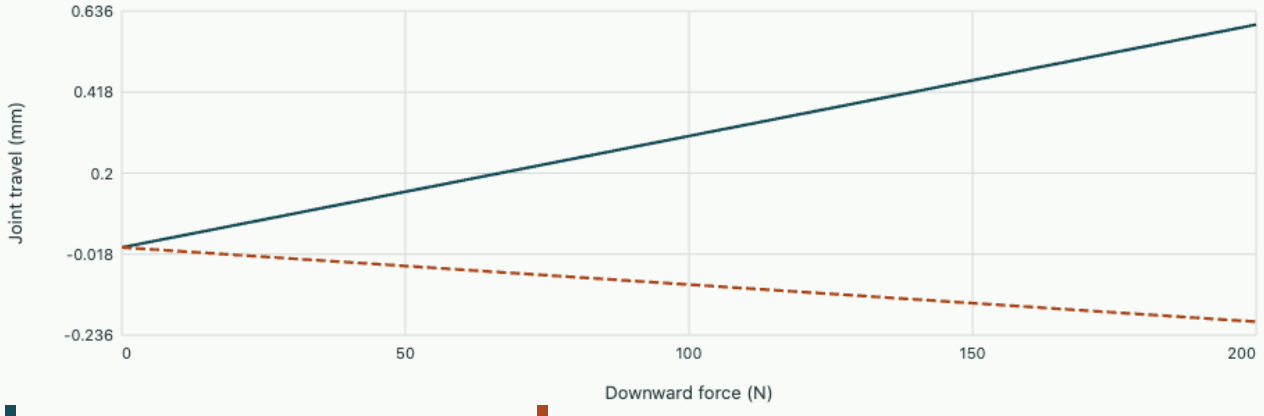
Exchange the left and right compliances. Before looking at the new graph, mark which readout should reverse sign and which should remain equal. Then set the two values alike and identify the symmetry that removes cross-coupling.

Open this experiment online

Try the calculation

Left bar compliance (µm/N) **2**

Right bar compliance (µm/N) **4**



At 100 N down the joint travels 0.300 mm down and -0.100 mm right. By reciprocity the latter also gives downward travel under 100 N right.

A related OneSimulation calculation

Use the course truss to test the same flexibility scaling on a larger assembly. Double every member area at fixed loads and geometry, then compare the joint displacement with the separately calculated chord force.

Run the OneSimulation truss experiment

Tutorial · identify the two member compliances from one load case

A 200 N vertical test gives +0.50 mm down and +0.10 mm right. Estimate each compliance and predict horizontal travel under simultaneous 60 N rightward and 100 N downward loading. How can one additional gauge reading test reciprocity?

▼ **Show the identification and loading prediction**

Direct compliance is $0.50/200 = 0.0025$ mm/N and cross compliance is $0.10/200 = 0.0005$ mm/N. Adding and subtracting gives $c_L = 0.0030,\ c_R = 0.0020$ mm/N. The predicted horizontal motion is

$$0.0025(60) + 0.0005(100) = 0.200\,\mathrm{mm}.$$

A 200 N rightward test should give +0.10 mm down. A disagreement after changing load direction suggests checking seating and the linear-elastic domain before assigning a single compliance to each bar.

Further reading · David Roylance, MIT 3.11 · truss displacements, strain energy and Castigliano's theorem

Why a bonded pair of strips bends under uniform heating

Engineering mechanics

Two perfectly bonded strips would expand by different amounts if separated. After bonding they must share a continuous longitudinal strain field. Solve simultaneously for its mean and its curvature and recover stresses that can be present even when the entire strip is free.

Separate total strain into free and constrained parts

Consider a narrow long strip of width b and total thickness $2t$, with layers of equal thickness on either side of $z = 0$. The upper layer has expansion coefficient α_+, the lower α_-, and both have the same E here. Start stress-free, then impose a uniform temperature increase ΔT. With plane sections remaining plane, define signed curvature by writing longitudinal strain $\epsilon = \epsilon_0 + \kappa z$. Stress is proportional to the difference from the free thermal strain:

$$\sigma_\pm(z) = E[\epsilon_0 + \kappa z - \alpha_\pm \Delta T].$$

Perfect adhesion enforces continuity of longitudinal displacement at the interface. It permits longitudinal stress to jump there because the stress-free thermal strain changes discontinuously. Traction continuity across the horizontal interface concerns its normal and shear traction components. This beam model describes the middle portion away from free ends where interfacial shear transfers load between layers.

Use both a zero axial force and a zero moment

For an unconstrained strip no external force or moment acts on its section. The thickness integral of stress must vanish, and its first moment must also vanish. Because z is measured from the symmetric section midpoint, the odd integrals of $\epsilon_0 z$ and κz cancel. Separate the thermal integrals over the two layers:

$$2t\epsilon_0 = t(\alpha_+ + \alpha_-)\Delta T,$$

$$\frac{2t^3}{3}\kappa = \frac{t^2}{2}(\alpha_+ - \alpha_-)\Delta T.$$

Hence $\epsilon_0 = (\alpha_+ + \alpha_-)\Delta T/2$ and $\kappa = 3(\alpha_+ - \alpha_-)\Delta T/(4t)$. The higher-expansion side takes the greater total strain at large positive z. For unequal moduli, the two equilibrium equations still work but must retain E inside the integrals; symmetry no longer removes every strain–curvature coupling term.

A tiny curvature can coexist with measurable stress

Let $t = 0.50$ mm, $E = 100$ GPa, $\alpha_+ = 20$ and $\alpha_- = 10$ micrometres per metre per kelvin, with $\Delta T = 40$ K. The mean strain is 600 microstrain, curvature is 0.600 m⁻¹, and the radius of curvature is 1.67 m. Across a free 100 mm length with one end setting the reference tangent, the small-slope tip offset magnitude is

$$\kappa L^2/2 = 3.0\,\mathrm{mm}.$$

A free single layer with either coefficient would expand without this curvature.

At the interface the upper layer has stress $100\,\mathrm{GPa}(600-800)10^{-6} = -20$ MPa and the lower +20 MPa. At the outer surfaces the bending term contributes ±300 microstrain; the upper stress becomes +10 MPa and the lower –10 MPa. If a fixture holds the same strip flat while allowing average extension, the mean strain is still 600 microstrain but the layer stresses are uniform ±20 MPa. The fixture must now supply a restraining moment. Compare the stress fields at fixed temperature before interpreting a strain-gauge reading as stress.

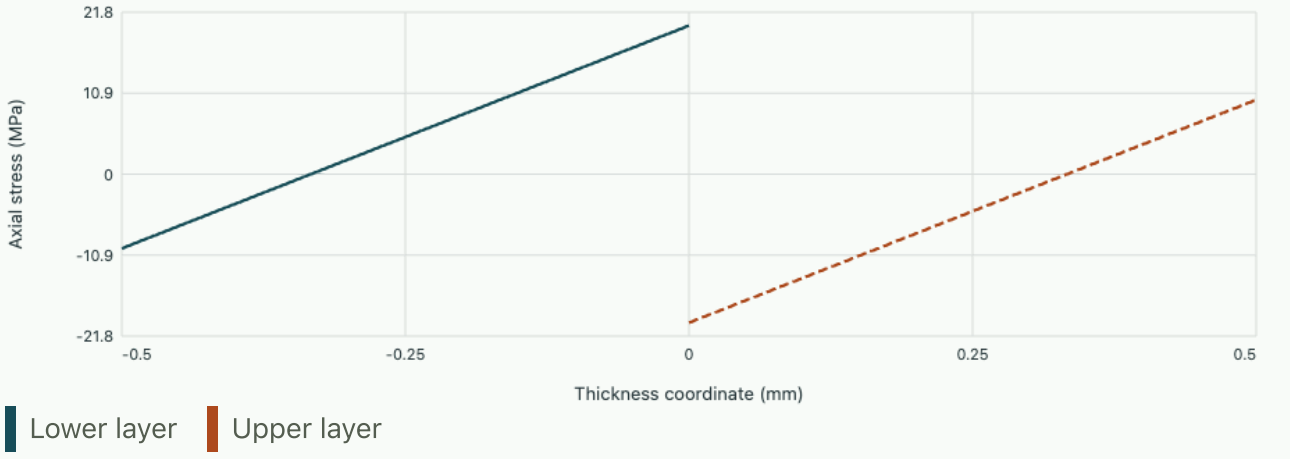
First set both expansion coefficients to 10 and heat the strip. Next restore the mismatch and cool by the same temperature magnitude. Locate stress reversals at the interface and within each layer. Which of these reversals would a gauge on an external face detect?

Open this experiment online

Try the calculation

Temperature change (K) **40**

Upper expansion coefficient (µm/m/K) **20**



Free curvature is 0.600 m⁻¹. With a 100 mm reference tangent, the small-slope tip offset is 3.00 mm. The two interface stresses are 20.0 and -20.0 MPa.

Tutorial · infer thermal mismatch from shape

For $t = 0.40$ mm a free 80 mm strip bends 1.20 mm from its initial tangent after a 30 K rise. Estimate the expansion-coefficient difference and the upper-interface stress magnitude for $E = 100$ GPa. State what independent measurement identifies the average expansion coefficient.

▼ **Recover mismatch and mean expansion separately**

The curvature magnitude is $2(0.00120)/(0.080)^2 = 0.375$ m⁻¹. Inversion gives $|\Delta\alpha| = 4t\kappa/(3\Delta T) = 6.67 \times 10^{-6}$ K⁻¹. The interface stress magnitude is

$$E|\Delta\alpha|\Delta T/2 = 10.0\,\mathrm{MPa}.$$

Curvature alone determines only the difference. Measuring strain on both external surfaces and averaging removes the bending term and gives the common mid-plane strain from which the mean expansion coefficient follows.

Further reading · David Roylance, MIT 3.11 · laminate force and moment resultants and thermal strains

Static and couple balance of rotors

Machine dynamics

Represent rotating imbalance as a vector and determine the corrections needed to cancel both net force and rocking moment.

FROM FIRST PRINCIPLES

Represent a rotating offset by a force and moment phasor

A rigid rotor carries a small mass m at eccentricity e from its shaft axis. At constant angular speed ω, its radial acceleration has magnitude $e\omega^2$. The shaft supplies the inward force and receives an equal outward reaction whose direction rotates with the shaft. Encode the offset orientation by phase ϕ and the unbalance by $U = me\,e^{i\phi}$. Complex notation retains both perpendicular force components.

$$F_x + iF_y = \omega^2 U e^{i\omega t}.$$

For several masses spinning together, force superposition makes the net phasor proportional to $\sum U_j$. Axially separated forces can also produce a rotating couple even when the net force vanishes. Taking moments about a reference plane weights each unbalance by its axial distance z_j.

$$\sum_j U_j = 0,$$

$$\sum_j z_j U_j = 0.$$

These are the two rigid-rotor balance conditions. Changing the reference plane adds a multiple of the first sum to the second, so once net force vanishes the couple balance is independent of that choice. Flexible rotor modes at operating speed require additional information.

DERIVATION AND APPLICATIONS

Separate the imbalance force and couple at the bearings

A rotor can have little net rotating force yet cause large opposed bearing reactions. Treat the shaft as rigid with bearings at axial positions $z = 0$ and $z = L$. Assume speed ω is constant and sufficiently below the first bending resonance for a rigid load distribution. Represent each unbalance by the complex vector $U_j = m_j e_j e^{i\phi_j}$. The physical rotating force on the support is the real part of the phasor times $e^{i\omega t}$. Summing forces and taking moments about the first support gives the transmitted load phasors F_A, F_B.

$$F_A + F_B = \omega^2 \sum_j U_j,$$

$$L F_B = \omega^2 \sum_j z_j U_j.$$

The moment equation weights each unbalance by axial position. Solving for the individual support loads identifies the force distribution even for a resultant whose direction rotates with the shaft. The weights can be negative for overhung masses.

$$F_B = \frac{\omega^2}{L} \sum_j z_j U_j,$$

$$F_A = \omega^2 \sum_j (1 - z_j/L) U_j.$$

Conversely, two synchronised bearing measurements supply both the summed unbalance and its first axial moment. Their relative phases must be retained, since adding magnitude readings would erase the signature of a rotating couple.

$$U_{\text{sum}} = \frac{F_A + F_B}{\omega^2},$$

$$M_U = \sum_j z_j U_j = \frac{L F_B}{\omega^2}.$$

With $U_{\text{sum}} = 0$, equal and opposite support phasors can remain. Their mount loads grow with squared speed even though the instantaneous vector sum vanishes.

WORKED EXAMPLE

Predict bearing reactions from a pure couple

Take $L = 0.600$ m, $\omega = 100$ rad/s, and equal opposite unbalances of magnitude 200 g mm in planes at 0.100 and 0.500 m. Taking the first unbalance as positive, $M_U = 0.000200(0.100 - 0.500) = -8.00 \times 10^{-5}$ kg m². The bearing load $F_B = -1.33$ N and $F_A = +1.33$ N. A single accelerometer measuring common support translation could miss the rotating moment.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Identify the plane of a single residual mass

In a different machine, in-phase bearing load amplitudes are 3.00 N and 1.00 N at A, B respectively, $\omega = 200$ rad/s and $L = 0.800$ m. The inferred net unbalance is $4.00/200^2 = 1.00 \times 10^{-4}$ kg m, or 100 g mm. For a single residual plane its equivalent position is

$$LF_B/(F_A + F_B) = 0.200\,\mathrm{m}.$$

A pair of distributed errors may produce the same two rigid-body measurements.

Interactive calculation · Change the inputs of this example

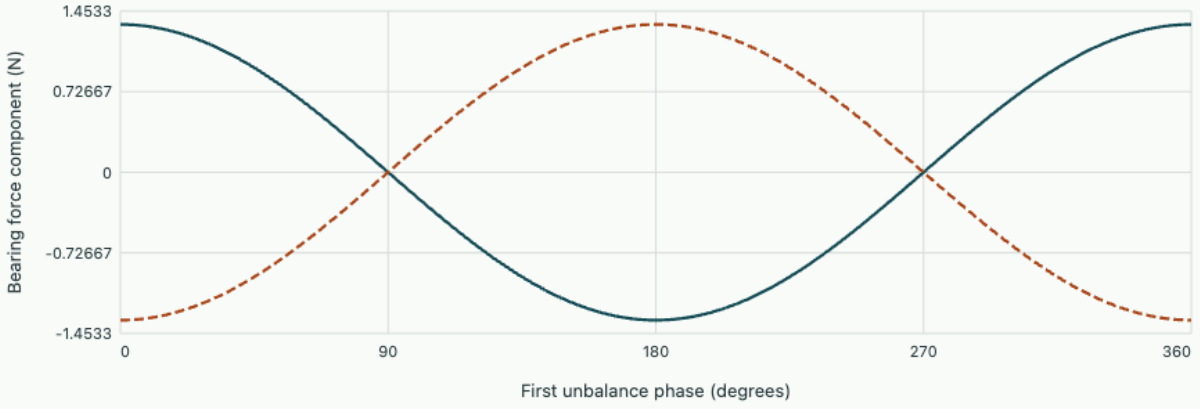
Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Unbalance magnitude (g mm) **200**

Spin speed (rad/s) **100**



Equal opposite unbalances at0.10 and0.50 m on a0.60 m span give opposite bearing-force amplitudes 1.333 N. Net force is zero while the rotating couple persists.

Translate a mass offset into a periodic bearing load

A rotor can have its total mass centre on the shaft axis and still shake its housing. The reason is spatial: two equal and opposite centrifugal force vectors in separate transverse planes form a rotating couple. First describe one small excess mass m at radial offset e and clock angle ϕ. At constant angular speed ω, its required radial acceleration has magnitude $e\omega^2$. The equal and opposite reaction transmitted to the rotating assembly has magnitude $me\omega^2$.

Represent its radial unbalance by a complex number $U = me(\cos\phi + i\sin\phi)$. This is shorthand for two transverse components at a reference shaft orientation, not a complex physical force. Multiplication by $e^{i\omega t}$ rotates all vectors together. Since both ω^2 and this rotation are common to all masses, cancellation can be carried out using U alone.

$$\sum U_j = 0,$$

$$\sum z_j U_j = 0.$$

Here z is axial location from a chosen reference plane. The second condition cancels the rotating transverse moment; its vector direction includes a common quarter-turn from the cross product that drops out of the zero condition. If the first condition holds, shifting the axial origin leaves the second condition unchanged.

Split one known unbalance between two planes

A rigid rotor has an 18 g excess mass at radius 40 mm, angle 0°, and axial location 120 mm between two correction planes at $z_A = 0$ and $z_B = 300\,\mathrm{mm}$. Its unbalance is $720\,\mathrm{g\,mm}$ along the reference direction. Moment balance gives $300U_B = -120(720)$, hence $U_B = -288\,\mathrm{g\,mm}$. Force balance then gives $U_A = -432\,\mathrm{g\,mm}$. Both corrections point at 180° because the original mass lies between the planes.

At correction radius 60 mm these vectors require added masses 4.80 g in plane B and 7.20 g in plane A. If material is removed instead, the equivalent removed masses lie at the original 0° clock position. At 1800 rpm the original force amplitude was $0.000720(188.5)^2 = 25.6\,\mathrm{N}$. Doubling speed quadruples that force, yet the same mass corrections still balance the assumed rigid rotor.

Identify what a single-plane test can miss

Consider instead two unbalances $+400$ and $-400\,\mathrm{g\,mm}$ in planes 200 mm apart. Their resultant force is zero at every speed. On ideal low-friction supports there is no preferred heavy side in a static test. Their couple magnitude at 1800 rpm is $0.000400(0.200)(188.5)^2 = 2.84\,\mathrm{N\,m}$, sufficient to generate equal opposing reactions at bearings. Measurements in two support planes can distinguish this pattern from a net shaking force. Near a flexible rotor’s bending critical speed, shaft deflection changes the mass distribution and response phase; the rigid-body correction formulas then require a modal or measured influence-coefficient model.

Interpret correction accuracy

An angular placement error also leaves a residual vector. A correction of the correct magnitude set one degree away leaves a residual approximately $U\pi/180$ for small angle, since the tip of the vector follows a circle. At large operating speed even this residual produces a force scaling with the square of speed.

Derive a balance correction from a trial mass

An assembled rotor rarely reveals the angular location of its excess mass directly. A once-per-revolution vibration measurement supplies an amplitude and phase referred to a shaft mark. At fixed operating speed and linear support conditions, represent the measured complex vibration V by $V = \alpha U$, where U is the complex unbalance in one plane and α is the response per unit unbalance. This coefficient includes bearing compliance, dynamic amplification and the sensor's phase lag. The phase of largest displacement is therefore not necessarily the angle where correction mass should be added.

Record the original vibration V_0, add a known trial unbalance U_t and measure V_t at the same speed with the same phase reference. Subtraction cancels the original signal and gives

$$\alpha = (V_t - V_0)/U_t,$$

$$U_c = -V_0/\alpha.$$

Remove the trial before installing the calculated total correction. If the trial remains attached, install only $U_c - U_t$ as further correction. Subtract complex vectors, not their magnitudes. A trial too small to resolve above measurement noise makes the inferred phase and correction unreliable; an unnecessarily large trial can take the rotor outside the measured linear range. A repeat baseline after removing the mass can expose a changing thermal or mounting condition.

Convert measured shaft phase into a physical correction

At fixed speed a laser reports an original synchronous displacement $V_0 = (30 + 40i)\ \mu\mathrm{m}$. Add a 5.0 g trial mass at a 40 mm radius at the reference angle, so $U_t = 200\ \mathrm{g\,mm}$. The new reading is $V_t = (50 + 60i)\ \mu\mathrm{m}$. The influence coefficient is $(20 + 20i)/200 = 0.10 + 0.10i$ micrometres per g mm. The required correction is

$$U_c = -\frac{30 + 40i}{0.10 + 0.10i} = -(350 + 50i)\ \mathrm{g\,mm}.$$

Its magnitude is $353.6\ \mathrm{g\,mm}$ and angle is 188.1° from the trial reference in the same positive rotational convention. With a correction radius of 50 mm, add about 7.07 g after removing the trial. Multiplying the correction vector by the measured coefficient gives $-30 - 40i$ micrometres, cancelling the baseline in the linear model. If the trial must stay, the extra vector is $-550 - 50i$ g mm. Installing the full correction in addition to the trial would leave its trial response as a predictable residual.

Single-plane correction minimises one measured response at one speed. For a long rotor with a separate couple unbalance, a response measured at one support may vanish while the opposite support continues to vibrate. Measuring both supports under trial masses in two planes produces a two-by-two complex influence matrix. Nearly proportional measured columns mean that the two trial planes cannot be distinguished reliably at those sensor locations, a conditioning issue that no amount of arithmetic precision can cure.

The initial amplitude in the example is 50 micrometres; the trial amplitude is 78.1 micrometres. Their scalar difference, 28.1 micrometres, happens to be close to the actual complex response increment’s magnitude of 28.3 micrometres, but the scalar comparison contains no correction angle. Even apparently favourable numbers cannot replace the shaft reference.

A balanced rotor can load its bearings during a turn

After a rotor is balanced, its spin angular momentum still has to change direction whenever its supporting structure turns. Consider an axisymmetric flywheel rotating rapidly at ω_s about unit vector e_s, with polar inertia I_s. If its axis is steered at angular rate Ω, then $\dot{e}_s = \Omega \times e_s$. Isolate the spin part of angular momentum and differentiate in an inertial frame:

$$H_s = I_s \omega_s e_s,$$

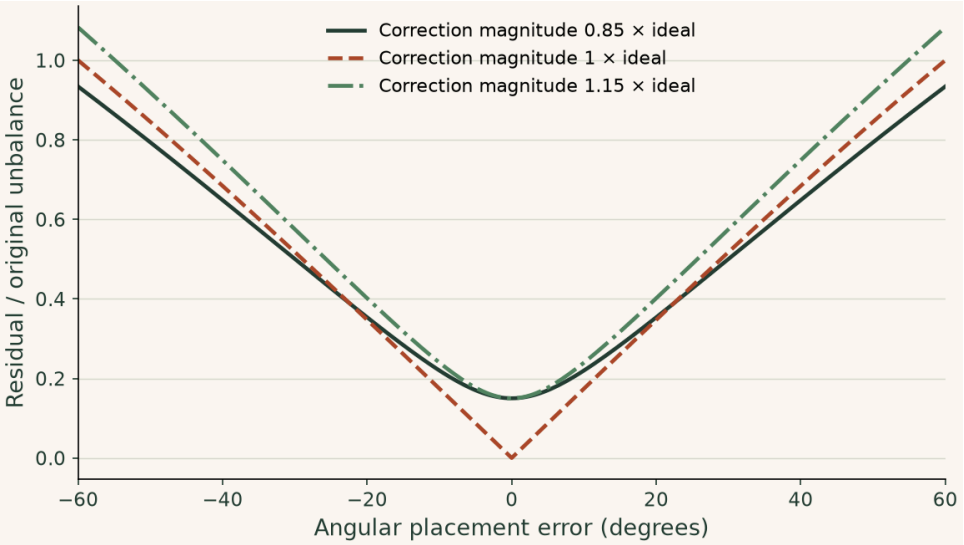
$$M_s = I_s \dot{\omega}_s e_s + \Omega \times H_s.$$

The first term spins the rotor up or down. The second turns its axis without changing its spin energy to leading order. For steady orthogonal steering the moment magnitude is $I_s |\omega_s \Omega|$. Components of Ω parallel to the spin axis do not turn the angular momentum. For slow regular precession of a fast rotor this spin-dominated relation also gives $\Omega \simeq M / H_s$ for a transverse applied moment. Rapid steering acceleration, strong nutation or significant transverse angular momentum requires the full tensor equations.

A horizontally mounted flywheel on a turntable has $I_s = 0.12\,\mathrm{kg\,m^2}$, spins in the positive x direction at 3000 rpm and is turned about positive z at 0.40 rad/s. Angular momentum is $0.12(3000)(2\pi/60) = 37.7\,\mathrm{kg\,m^2/s}$ along x. Since $e_z \times e_x = e_y$, the bearings must exert 15.1 N m along positive y on the rotor. A 0.30 m support separation along x requires a pair of equal opposite vertical bearing forces of 50.3 N to deliver this moment, in addition to weight and ordinary radial loads. Reversing the steering reverses the couple immediately at a given speed. A rigid imbalance force at fixed shaft speed does not reverse just because steering changes direction. That distinction is a practical way to diagnose the origin of a load measured on a moving platform.

How accurately must a balance mass be placed?

A single-plane correction should cancel the original unbalance vector. If its magnitude ratio is k and its angle differs from perfect opposition by δ, vector addition leaves the normalised magnitude $\sqrt{1 + k^2 - 2k\cos\delta}$.



Even an otherwise exact correction leaves 0.174 of the original unbalance for a 10° angular error. A magnitude tolerance creates a nonzero floor at zero angle. The plotted ratio also scales the residual centrifugal force at any chosen speed.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

Change magnitude error and spin speed. Identify the residual force that remains even with perfect angular placement.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Correction / ideal magnitude	**1**
Original unbalance (g mm)	**100**
Spin speed (rpm)	**3000**



Selected magnitude error Ideal magnitude

With zero placement error, residual force is 0.000 N. Doubling speed multiplies every force by four.

Virtual work and the forces hidden by a constraint

Analytical mechanics from constraints to invariants

A bead follows a wire, a pendulum remains the same length, and a rolling wheel obeys a relation between translation and rotation. Their reduced equations describe the motion but can conceal the largest forces in the apparatus. Determine which forces disappear under a virtual displacement, recover their values, and decide when the assumed motion ceases to be physically admissible.

Hold the clock fixed when defining a virtual displacement

A system of particles has Cartesian positions $\mathbf{r}_i$ and applied forces $\mathbf{F}_i$. A holonomic constraint is an equation $g_\alpha(\mathbf{r}_1, \dots, \mathbf{r}_N, t) = 0$. At a fixed instant an admissible infinitesimal change of configuration obeys $\sum_i \nabla_i g_\alpha \cdot \delta\mathbf{r}_i = 0$. The physical velocities obey a different relation, $\sum_i \nabla_i g_\alpha \cdot \dot{\mathbf{r}}_i + \partial_t g_\alpha = 0$. A virtual displacement compares geometrically allowed configurations at the same time. It has no prescribed duration and makes no claim that the system's momentum actually points along it.

The distinction becomes measurable for a bead on a moving horizontal table. Every allowed virtual displacement has zero vertical component, so the table's upward normal force gives zero virtual work. An upward-moving table can nevertheless raise the bead and perform positive ordinary work. Ordinary power involves the actual velocity, which includes the table motion. A careless statement that constraint forces do no work confuses two different operations. For a fixed ideal frictionless surface the real velocity is tangent as well, and vanishing virtual work then does imply vanishing ordinary power.

Let $\mathbf{R}_i$ be the reaction forces and use $m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i + \mathbf{R}_i$. Multiply by an admissible $\delta\mathbf{r}_i$ and sum over particles. We call the constraints ideal when $\sum_i \mathbf{R}_i \cdot \delta\mathbf{r}_i = 0$ for every allowed virtual displacement. Eliminating those reactions gives d'Alembert's principle,

$$\sum_i (\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i) \cdot \delta\mathbf{r}_i = 0.$$

This is a dynamical equation expressed through a static-looking virtual-work test. It applies instantaneously along the motion. It does not assert that the total force vanishes or that the acceleration is an independent virtual displacement. One advantage is that the detailed construction of a rigid support need not be known to determine the admissible tangential acceleration; its force can be calculated afterwards when designing the support.

Pass from virtual work to equations for coordinates

Choose independent coordinates q_a with $\mathbf{r}_i = \mathbf{r}_i(q, t)$ and independent virtual increments δq_a. The velocity is $\sum_a \partial_{q_a}\mathbf{r}_i\,\dot{q}_a + \partial_t\mathbf{r}_i$. With kinetic energy $T = \sum_i m_i\dot{\mathbf{r}}_i^2/2$, differentiation at fixed q, t gives

$$\partial_{\dot{q}_a} T = \sum_i m_i\dot{\mathbf{r}}_i \cdot \partial_{q_a}\mathbf{r}_i.$$

Differentiate in time. The product rule produces an acceleration term and a second term involving $d(\partial_{q_a}\mathbf{r}_i)/dt$. The latter equals $\partial_{q_a}\dot{\mathbf{r}}_i$, with the generalised velocities held fixed, and supplies $\partial_{q_a}T$. Hence

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_a} - \frac{\partial T}{\partial q_a} = \sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_a}.$$

The projected applied force is

$$Q_a = \sum_i \mathbf{F}_i \cdot \partial_{q_a}\mathbf{r}_i.$$

Independence of the coordinate variations in d'Alembert's principle gives one equation $d(\partial_{\dot{q}_a}T)/dt - \partial_{q_a}T = Q_a$ per coordinate. When $\mathbf{F}_i$ comes from a potential $V(q,t)$, the conservative contribution is $-\partial_{q_a}V$. Other contributions such as a measured resisting torque remain explicit. A coordinate expressed as an angle has a generalised force in joules per radian; a coordinate expressed as a length has one in newtons. Checking $Q_a\,dq_a$ as work avoids mistaking a generalised force for a Cartesian component.

For a simple pendulum of length l and mass m, choose θ measured from downward vertical. Then $T = ml^2\dot{\theta}^2/2$ and $V = mgl(1-\cos\theta)$. The equation $ml^2\ddot{\theta} = -mgl\sin\theta$ follows at once. The string tension vanished under the angular displacement. It has not been shown to vanish physically. Projecting Newton's equation onto the direction from the bead toward the pivot gives

$$T_{\text{string}} = mg\cos\theta + ml\dot{\theta}^2.$$

This second observable is sensitive to speed even when the reduced angular acceleration depends only on angle.

Recover multipliers before choosing coordinates

Cartesian coordinates keep reaction directions explicit. Put all coordinates into a vector x, let M be the mass matrix and let G contain the rows ∇g_α. The tangent virtual displacements are precisely those satisfying $G\delta x = 0$. A reaction doing zero virtual work must be a linear combination of the normal covectors, $R = G^T\lambda$, when the constraints are independent. Newton's equation becomes

$$M\ddot{x} = F + G^T\lambda.$$

Differentiate the constraints twice along the trajectory: $G\ddot{x} + \gamma = 0$, where γ includes the known velocity, curvature and explicit time terms. Substituting for $\ddot{x}$ determines reactions from a much smaller system,

$$(GM^{-1}G^T)\lambda = -\gamma - GM^{-1}F.$$

For any vector c, $c^T G M^{-1} G^T c = (G^T c)^T M^{-1} (G^T c)$ is positive if G has independent rows and particle masses are positive. Dependent constraint equations make the individual multipliers ambiguous. For instance writing both $g = 0$ and $2g = 0$ declares the same geometrical restriction twice. Only the combined reaction is then determined. The magnitude of a raw multiplier also depends on the units and normalisation of g. Reporting the force $G^T \lambda$ makes comparisons between alternative constraint descriptions meaningful.

Use $g = (x^2 + y^2 - l^2)/2 = 0$ for a planar pendulum, with y upward, so the gradient is the radius vector $\mathbf{r}$. Newton's equation reads $m\ddot{\mathbf{r}} = -mg\hat{y} + \lambda \mathbf{r}$, where the symbol g in the gravitational term denotes the acceleration magnitude. The acceleration constraint $\mathbf{r} \cdot \ddot{\mathbf{r}} = -v^2$ gives

$$\lambda = (mgy - mv^2)/l^2.$$

The inward tension is $-\lambda l = mv^2/l - mgy/l$. Since $y = -l\cos\theta$, this is the radial-force result already obtained. The multiplier in this convention has units of force per length. Using the radial-distance constraint instead would make its magnitude a force.

Background · What if the support can push but cannot pull?

Replacing a surface by an equality constraint assumes the object stays in contact. A particle sliding along the inside of a smooth vertical circular track illustrates the additional physical test. At the top, gravity and the inward normal reaction together must supply the centripetal acceleration.

$$N = \frac{mv^2}{R} - mg, \qquad N \geq 0.$$

At a top speed below $\sqrt{gR}$ the equality-constrained solution requires negative N. An ordinary track cannot exert that pull. Its normal force then falls to zero and the particle must follow free flight until it meets the track again. A bead threaded on a wire could receive the oppositely directed force and would follow a different trajectory. Both models have the same geometrical path during contact.

Recognise an inequality at a contact

A bead threaded on a stiff loop can experience reaction in either direction, whereas a string can pull only. A surface without adhesion can push only. Derive the constrained motion, calculate its reaction, and compare the required sign with the physical mechanism. When a calculated tension turns negative, replace the constrained trajectory by free flight until a new contact event. Continuing to integrate the pendulum equation would retain a mathematically valid circle and a physically unavailable pushing string.

A mass starts at the bottom of a frictionless vertical loop with speed v_b. Energy conservation gives $v_{\text{top}}^2 = v_b^2 - 4gl$, since the top is $2l$ higher. A string just remaining taut at the top requires

$T_{\text{top}} = m v_{\text{top}}^2/l - mg \geq 0$, giving $v_b^2 \geq 5gl$. The energy-only criterion for reaching the elevation is $v_b^2 \geq 4gl$; the interval between these two values illustrates why a trajectory test must include reactions. With $l = 0.80\,\text{m}$, the threshold bottom speed is $6.26\,\text{m/s}$. At this threshold the bottom tension is $6mg$, six times the weight, even though the tension is about to vanish at the top.

Recover the forces on a wedge that is free to slide

Place a wedge of mass M on a smooth horizontal floor and a block of mass m on its smooth upper face. Let the slope descend to the right at angle α. Choose X as the wedge's horizontal position and s as distance down the face. The block's coordinates apart from fixed offsets are $x = X + s\cos\alpha,\ y = -s\sin\alpha$. Its relative velocity points down the slope but its absolute acceleration need not. The kinetic energy contains a mixed term $m\dot{X}\dot{s}\cos\alpha$. Dropping that term is equivalent to holding the wedge stationary while allowing it to recoil, two incompatible mechanical models.

The reduced equations are $(M+m)\ddot{X} + m\ddot{s}\cos\alpha = 0$ and $\ddot{s} + \ddot{X}\cos\alpha = g\sin\alpha$. The first is conservation of total horizontal momentum, since the block–wedge normal force is internal and the floor is frictionless. Elimination gives

$$\ddot{s} = (M+m)g\sin\alpha/(M + m\sin^2\alpha),$$

$$\ddot{X} = -mg\sin\alpha\cos\alpha/(M + m\sin^2\alpha).$$

The acceleration down the moving face exceeds $g\sin\alpha$ because the wedge moves left as the block slides right. Nevertheless the absolute vertical acceleration of the block remains less than or equal to free fall because the upper face pushes upward.

To find the block's normal force, project its absolute acceleration onto the outward unit normal $(\sin\alpha, \cos\alpha)$. The part proportional to $\ddot{s}$ cancels, leaving $\ddot{X}\sin\alpha$. Newton's equation along that direction gives

$$N = m(g\cos\alpha + \ddot{X}\sin\alpha) = mMg\cos\alpha/(M + m\sin^2\alpha).$$

With $M = 3.0\,\text{kg}$, $m = 1.0\,\text{kg}$, $\alpha = 30^\circ$, one finds $\ddot{s} = 6.04\,\text{m/s}^2$,

$$\ddot{X} = -1.31\,\text{m/s}^2,$$

$$N = 7.84\,\text{N}.$$

The normal based on a stationary wedge would be $8.50\,\text{N}$. A force sensor under the moving face could distinguish the two models even when a short record of relative slope displacement cannot resolve the difference.

Which reaction does no virtual work here? The normal force on the block has virtual work $N\hat{n}\cdot(\delta X\hat{x} + \delta s\hat{t}) = N\sin\alpha\,\delta X$. The opposite normal force on the wedge gives $-N\sin\alpha\,\delta X$, so the sum cancels. Considering only the block and asserting zero work from the normal for arbitrary δX

would be incorrect. At the real velocities the same reaction transfers energy between the bodies even though its net internal power on their pair is zero. The floor's normal reaction balances the vertical momentum change of the whole system. During the slide it is $(M+m)g - m\ddot{s}\sin\alpha$, less than total weight while the block accelerates downwards.

Take the limits as an additional test. At fixed m, α, increasing M to infinity returns the familiar fixed-wedge results. At a nearly horizontal face, downslope acceleration and wedge recoil vanish, restoring the full block weight as normal force. At a vertical face the normal vanishes and the block falls. These limits hold because the contact has no friction and can be maintained with a nonnegative normal. A moving wedge of independently prescribed acceleration would replace the horizontal momentum equation by a control input, changing both the relative acceleration and the support force.

Distinguish rolling constraints from purely geometric restrictions

A cylinder of radius R, mass m and axial inertia I rolls without slip down a fixed rough incline. Choose coordinate s down the incline and spin angle ϕ in the corresponding sense. The constraint $\dot{s} = R\dot{\phi}$ is integrable in this one-dimensional motion, so it can be inserted as $s - R\phi = \text{constant}$. The resulting kinetic energy is $(m + I/R^2)\dot{s}^2/2$. The reduced equation gives

$$\ddot{s} = g\sin\alpha/[1 + I/(mR^2)].$$

Reconstruct the static friction from translation, $f = gm\sin\alpha - m\ddot{s}$, directed up the slope, and compare it with the available maximum $\mu_s mg\cos\alpha$.

For a homogeneous solid cylinder $I = mR^2/2$, the speed-up is $2g\sin\alpha/3$ and friction magnitude $mg\sin\alpha/3$. A ring of the same mass and radius has $I = mR^2$, half the gravitational downslope acceleration and half the downslope weight component as friction. At 30° the minimum static friction coefficients are respectively $\tan 30^\circ/3 = 0.192$ and $\tan 30^\circ/2 = 0.289$. Two objects on the same surface can therefore diverge qualitatively because the ring slips while the cylinder still rolls. Treating an assumed no-slip relation as exact without testing that inequality can produce an unphysical result despite apparently consistent energy conservation.

The friction on a rigid body rolling without slip on a fixed surface has zero power because its instantaneous point of contact is stationary. The force still changes both linear and angular momentum. On a moving conveyor the contact velocity is no longer zero in the laboratory; friction can supply mechanical energy even while the relative contact velocity vanishes. For a wheel steering on a plane the restrictions $\dot{x} = R\dot{\phi}\cos\psi,\ \dot{y} = R\dot{\phi}\sin\psi$ depend on heading ψ. In general they cannot be integrated into two configuration equations relating x, y, ϕ, ψ globally. d'Alembert's fixed-time allowed variations still project out ideal rolling reactions, but replacing those velocity constraints by invented holonomic coordinates gives the wrong dynamics. The distinction can be seen geometrically by rolling and steering through a closed sequence of steering operations that leaves a net displacement.

Find loads on two supports of a suspended bar

A rigid horizontal bar of mass m and length $2a$ is supported at its ends by two vertical inextensible light cords. At one instant both cords are vertical and the bar is horizontal. An extra downward load of magnitude F is applied at horizontal coordinate s relative to the centre. Ask first for a static experiment. The two vertical reaction forces T_L, T_R obey force balance $T_L + T_R = mg + F$ and moment balance $a(T_R - T_L) = sF$. This gives

$$T_R = (mg + F + sF/a)/2,$$
$$T_L = (mg + F - sF/a)/2.$$

The lever arms encode geometry that the vertical balance alone cannot provide.

The same result can be obtained from virtual variations if the reactions are kept as multipliers until the end. For small vertical translation y of the centre and small tilt θ, the endpoint vertical displacements are $\delta y - a\delta\theta$ on the left and $\delta y + a\delta\theta$ on the right. The load point displacement is $\delta y + s\delta\theta$. Before eliminating the endpoint constraints, project all applied and reaction forces against arbitrary independent $\delta y, \delta\theta$; vanishing of both coefficients gives the preceding balances. After imposing both endpoint constraints directly, these two variations vanish and the reduced model for vertical motion contains no equation from which the support loads can be read. Removing a degree of freedom can remove precisely the observable of interest unless reactions are reconstructed.

For $m = 2.0\,\mathrm{kg}$, $a = 0.40\,\mathrm{m}$, $F = 12\,\mathrm{N}$ and $s = 0.25\,\mathrm{m}$, the total load is $31.62\,\mathrm{N}$ and the load difference $7.50\,\mathrm{N}$. Hence $T_R = 19.56\,\mathrm{N}$, $T_L = 12.06\,\mathrm{N}$. Exchanging the load from right to left reverses the difference while preserving the sum. This symmetry is a calibrating manoeuvre for two load cells with different offsets: subtract readings from mirrored configurations to isolate the lever-arm response, and compare their sum to the known total weight and applied force.

Now imagine three vertical supports at $-a, 0, +a$. Their three tensions enter only two independent balances. Perfectly rigid constraints determine no unique allocation to the centre support. The ambiguity is physical; the stiffnesses, preloads and small height differences of real supports decide the result. With three identical unstrained linear springs and a centred load F, each spring extends equally if the bar stays level, so each carries one third of the total weight. If the middle spring is twice as stiff and all have the same unloaded height, it carries one half and each outer spring one quarter. Both answers satisfy exactly the same force and moment balances. Reporting a unique value from a singular ideal multiplier system would manufacture an unmeasured constitutive assumption.

Test an acceleration constraint against drift of the position itself

Solving $G\ddot{x} + \gamma = 0$ guarantees the correct instantaneous normal acceleration at a state lying on the constraint manifold with tangent velocity. It does not magically repair earlier position or velocity errors in approximate time integration. For a particle constrained to a circle, monitor $g = (r^2 - l^2)/2$, $\dot{g} = \mathbf{r} \cdot \mathbf{v}$, and the computed force. A small tangential displacement can give tiny instantaneous force error while the radius gradually drifts. Integrating an acceleration-level constraint twice without a position or velocity correction can cause substantial mismatch over many revolutions.

A simple projection illustrates what each correction does. Normalise an approximate position vector to length l, then remove the radial component of velocity by $\mathbf{v} \leftarrow \mathbf{v} - \hat{r}(\hat{r}\cdot\mathbf{v})$. The corrected point and tangent satisfy the constraints, but the velocity correction removes kinetic energy $m(\hat{r}\cdot\mathbf{v})^2/2$, and the position change can alter potential energy. Projection is therefore a numerical operation whose physical invariant errors must be measured. A structure-preserving constrained integrator builds the position and momentum conditions into the step and often gives better long-time behaviour. Compare convergence of reaction, constraint residual and energy separately instead of treating a rounded radius as proof of a correct force.

A pendulum on a free cart tests the complete mass matrix

Mount a pendulum pivot on a cart of mass M moving horizontally on a smooth track. The bob has mass m, length l and downward angle θ. Choose the cart position X and use bob coordinates $X + l\sin\theta$, $-l\cos\theta$. The kinetic energy is $(M+m)\dot{X}^2/2 + ml\dot{X}\dot{\theta}\cos\theta + ml^2\dot{\theta}^2/2$. Two different off-diagonal mass terms would contradict its origin as a quadratic form; the same factor $ml\cos\theta$ multiplies the companion acceleration in each equation. This is a useful check on a manually assembled multibody model.

The equations are

$$(M+m)\ddot{X} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = 0,$$

$$l\ddot{\theta} + \ddot{X}\cos\theta + g\sin\theta = 0.$$

In the small-angle regime the horizontal centre of mass moves uniformly. Choosing zero total momentum gives

$$\dot{X} \simeq -ml\dot{\theta}/(M+m).$$

Substitute its derivative into the angular equation to obtain

$$\ddot{\theta} + [(M+m)g/(Ml)]\theta = 0.$$

Releasing the support raises the oscillation frequency because the effective angular inertia falls while the gravitational stiffness stays the same. The result is a statement about the relative pendulum angle, measured from an inertial vertical, with an unforced horizontally moving support.

For $M = 0.60\,\mathrm{kg}$, $m = 0.20\,\mathrm{kg}$ and $l = 0.50\,\mathrm{m}$, the small-angle period is

$$2\pi\sqrt{Ml/[(M+m)g]} = 1.229\,\mathrm{s}.$$

Fixing the cart with a clamp gives $1.419\,\mathrm{s}$ instead. At zero total horizontal momentum the cart displacement from its equilibrium is approximately $-0.125\,\theta$ metres, so a $0.10\,\mathrm{rad}$ angle corresponds to $12.5\,\mathrm{mm}$ of recoil. Simultaneous position measurements on cart and bob should reconstruct a

horizontal centre of mass that is stationary to within tracking uncertainty. A missing cart inertia or a wrongly filled cross term can alter both period and recoil; checking just one number may not identify which error occurred.

The reaction on the moving bob can again be recovered. Its radial acceleration along the outward string direction is $\ddot{X}\sin\theta - l\dot{\theta}^2$. Thus the tension is $T = m[g\cos\theta - \ddot{X}\sin\theta + l\dot{\theta}^2]$. The acceleration of the support appears explicitly. Reusing the fixed-pivot formula at all angles would omit a term that vanishes accidentally at the bottom. To test the sign, observe a rightward tilted pendulum whose pivot accelerates right: the support's acceleration has an outward projection along the tilted string, reducing the inward tension needed for the same relative motion. Substituting a directly measured cart acceleration provides an independent load-cell prediction.

Follow a force eliminated by a constraint

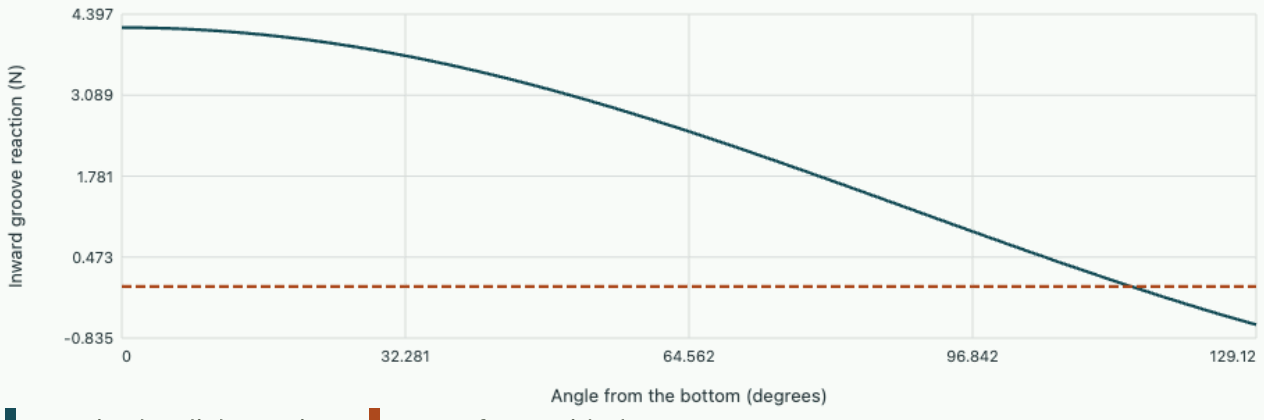
Predict when a two-sided circular groove must reverse its normal force. Check the prediction against the exact energy and radial balances.

Open this experiment online

Try the calculation

Bottom speed (m/s) **4**

Loop radius (m) **0.5**



Reachable angle 129.12°. Inward groove reaction there -0.619 N. One-sided loop threshold v_0 = 4.9523 m/s.

Decide which force a sensor actually reports

Reaction forces can be reported as tension along a member, normal traction at a contact, total support load on a floor, or a single Cartesian component read by a load cell. A rigid-body idealisation alone

does not specify a sensor's orientation or the weight of its housing. For a fixed pendulum, for example, the string carries tension T, while a vertical force channel at its pivot reads the vertical string contribution $T\cos\theta$. A horizontal channel reads $T\sin\theta$. The force on the pivot is opposite the force exerted on the bob. Selecting the wrong sign or leaving out the fixture's own weight can create a discrepancy even when the dynamical calculation is correct.

Suppose an ideal load cell with its sensitive axis vertical is tared with the fixture installed but the $0.40\,\mathrm{kg}$ bob removed. When the bob is held at 60° by a separate purely tangential actuator and is otherwise stationary, the radial balance gives

$$T = mg\cos 60^\circ = 1.96\,\mathrm{N}.$$

The pivot's vertical reading attributable to the string is $0.981\,\mathrm{N}$. The tangential actuator supplies the remaining vertical support as well as a horizontal component. Once the actuator releases, instantaneously the same tension and reading hold because radial speed is still zero. At the bottom of the subsequent swing the load jumps to the dynamical value obtained earlier, even though no upper fixture mass has changed.

A student hangs the bob at the same 60° angle using a separate horizontal cord instead of a tangential actuator. Should the pivot cell read the preceding $0.981\,\mathrm{N}$?

▼ Work through the solution

The horizontal cord contributes no vertical support. Static balance now requires $T\cos 60^\circ = mg$, giving $T = 7.85\,\mathrm{N}$ and vertical pivot contribution $3.92\,\mathrm{N}$. The horizontal holding force has a nonzero radial projection and changes the radial-force balance. After the horizontal cord is removed the tension immediately changes to $mg\cos 60^\circ$ because no momentum impulse is required for a finite jump in force. A schematic that shows only the bob angle is insufficient to infer the support load without saying how the angle is maintained.

In the free cart experiment bearing drag is weak but measurable. Can an observed frequency shift distinguish it from a wrongly assumed cart mass?

▼ Work through the solution

A viscous drag changes the horizontal momentum balance and allows the formerly stationary centre of mass to move as momentum leaks to the track. A mass error in a drag-free model changes the relative period and recoil ratio without supplying an actual external impulse. Track the complete centre of mass using independently weighed masses, compare its momentum change with the integrated bearing force, and measure the decay envelope. The frequency alone is a single scalar and can often be fitted by several combinations of missing mass, spring compliance and drag.

Locate contact loss on a smooth convex surface

A small block starts sliding from near rest just to one side of the top of a smooth fixed sphere of radius R. Measure angle θ from upward vertical. Energy gives $v^2 = 2gR(1-\cos\theta)$ during contact, while the inward radial equation reads $mg\cos\theta - N = mv^2/R$. Thus $N = mg(3\cos\theta - 2)$. Contact first vanishes at $\cos\theta = 2/3$, or 48.2°. For a sphere of radius $0.50\,\mathrm{m}$, this occurs after a vertical drop $R/3 = 0.167\,\mathrm{m}$ and at speed $1.81\,\mathrm{m/s}$. A block balanced exactly at the top at exact rest is an unstable equilibrium in the ideal model; a small initial displacement or speed initiates the measured trajectory.

Once the block leaves, the same spherical coordinates can still be used to describe position, but $r = R$ is no longer an equation of motion. Continue with the measured tangential velocity and free gravitational acceleration. Enforcing the sphere condition beyond release would make the computed normal negative and predict a surface that pulls on the block. The zero normal, the geometric release angle and the subsequent ballistic arc provide three independent observables for the same force reconstruction.

Tutorial on measuring reactions

A $0.40\,\mathrm{kg}$ pendulum of length $0.75\,\mathrm{m}$ is released from rest at 60° from downward vertical. Predict the load-cell reading at the pivot when it passes the bottom. What part does the reduced equation alone omit?

▼ **Work through the solution**

The height change is

$$l(1-\cos 60^\circ) = 0.375\,\mathrm{m}.$$

Energy conservation gives

$$v^2 = 2g(0.375) = 7.36\,\mathrm{m^2/s^2}.$$

At the bottom the tension is $mg + mv^2/l = 7.85\,\mathrm{N}$, twice the weight. The tangential acceleration there vanishes because gravity points along the string, but the radial acceleration remains nonzero. A sensor measuring the string force sees this centripetal contribution directly.

A bead lies on a smooth moving plane with upwards acceleration a_z. Which acceleration first makes a passive contact impossible if the bead initially shares the plane's motion?

▼ **Work through the solution**

Vertical Newton balance requires $N - mg = ma_z$, or $N = m(g + a_z)$. At $a_z = -g$ the reaction vanishes. A more negative plane acceleration would require the plane to pull the bead down, so contact

is lost and the bead follows free fall until the separation closes. For upward acceleration the normal exceeds weight. In all these cases a virtual displacement tangent to the plane produces zero virtual normal work.

Two code implementations use $(x^2 + y^2 - l^2)/2$ and $x^2 + y^2 - l^2$ as constraints for the same state. Their multipliers differ by a factor two. How can the physical result be tested?

▼ Work through the solution

The second gradient is twice the first at the same point. Its multiplier must be half the first multiplier for $G^T \lambda$ to coincide. Compare the reaction vector in newtons, its dot product with an independently chosen tangent direction, and the normal acceleration required by the geometry. A comparison of raw multiplier values alone would flag a difference that is entirely due to normalisation.

Compare a constrained trajectory with the motion and force calculations above.

David Tong develops virtual work and constrained coordinates in his classical dynamics course.

Action density, stress waves and the boundary that reflects them

Analytical mechanics from constraints to invariants

A chain of coupled masses becomes an elastic field when the wavelength is long compared with the spacing. Derive the field equation and its boundary traction from the same variation, follow energy and stress across a junction, and decide whether a measured pulse is limited by material waves or by numerical dispersion.

Pass from a discrete rod to an action density

Let equal masses of spacing a move longitudinally by $u_j(t)$, with nearest-neighbour spring constant k_s and mass m_s per site. Their Lagrangian is $\sum_j [m_s \dot{u}_j^2/2 - k_s(u_{j+1}-u_j)^2/2]$. For variations smooth on the spacing, $(u_{j+1}-u_j)/a \to u_x$ and $\sum_j a\, f_j \to \int f(x)\,dx$. Hold the line density $\rho A = m_s/a$ and axial rigidity $EA = k_s a$ fixed. The continuum Lagrangian density per unit reference length is

$$\mathcal{L} = \rho A\, u_t^2/2 - EA\, u_x^2/2,$$

with A area, ρ volume density and E Young modulus. Both terms have units of joules per metre. This is a slender-rod model with small strain, primarily uniaxial stress and wavelengths long compared with the cross-sectional dimensions.

Vary the field in the action over $[0,L]\times[t_1,t_2]$. Integration of $\rho A\, u_t \delta u_t$ by parts in time, with fixed endpoint fields, produces $-\rho A\, u_{tt}\delta u$. Integration of $-EAu_x\delta u_x$ by parts in space produces $+\partial_x(EAu_x)\delta u$ and a remaining term $-[EAu_x\delta u]_0^L$. Thus the interior equation is $\rho A u_{tt} = \partial_x(EAu_x)$, and the end force conjugate to displacement is axial force $N = EAu_x$. Adding external work $F_L(t)\delta u(L)$ at the right face yields

$$N(L) = F_L.$$

A prescribed displacement instead imposes $\delta u(L)=0$, leaving its reaction as an output. Keeping this boundary term prevents prescribing displacement and independent traction simultaneously at the same end of a second-order rod.

Determine the energy flux and characteristic impedance

Assume time-independent coefficients. Multiply the field equation by u_t and rearrange to obtain

$$\partial_t[\rho A u_t^2/2 + EAu_x^2/2] + \partial_x[-EAu_x u_t] = 0.$$

The energy flux is $-Nu_t$, with positive sign denoting energy moving towards increasing x. The minus sign follows the convention of positive tensile N: positive mechanical power entering through a right face is Nu_t, which is inward with respect to the rod on its left. This identity controls energy transport across both smooth material changes and sharp joints.

For a homogeneous right-moving wave $u = F(x - ct)$, $c = \sqrt{E/\rho}$, particle velocity $v = u_t = -cF'$ and axial force $N = EAF' = -Zv$, where $Z = \rho A c = EA/c$. A left-moving wave has $N = +Zv$. The impedance is force divided by particle velocity and has units $\mathrm{kg/s}$. If one instead uses stress, impedance is ρc per unit area. Doubling the area of the same material keeps wave speed unchanged but doubles the force–velocity impedance. A transit-time measurement and a reflection measurement therefore provide independent material or geometry information.

For a rod with $E = 70\,\mathrm{GPa}$, $\rho = 2700\,\mathrm{kg/m^3}$ and $A = 1.0 \times 10^{-4}\,\mathrm{m^2}$, $c \approx 5092\,\mathrm{m/s}$ and $Z \approx 1375\,\mathrm{kg/s}$. A right-going compressive pulse with particle speed $0.020\,\mathrm{m/s}$ carries axial force $-27.5\,\mathrm{N}$, compressive stress $0.275\,\mathrm{MPa}$, and instantaneous power $Zv^2 = 0.550\,\mathrm{W}$ across a section. A rectangular approximation lasting $100\,\mu\mathrm{s}$ carries about $55\,\mu\mathrm{J}$ and occupies $0.509\,\mathrm{m}$. The pulse speed is kilometres per second; the material particles move only millimetres per second. Treating these as the same velocity would spoil force, impulse and energy estimates.

Match a bonded interface and distinguish amplitude from power

At a perfect massless bond displacement and axial force are continuous. For an incident right-going velocity amplitude v_i in rod 1, let v_r be reflected left-going velocity and v_t transmitted right-going velocity. Velocity continuity gives

$$v_i + v_r = v_t.$$

Force continuity gives

$$-Z_1 v_i + Z_1 v_r = -Z_2 v_t.$$

Thus

$$r_v = \frac{Z_1 - Z_2}{Z_1 + Z_2},$$

$$t_v = \frac{2Z_1}{Z_1 + Z_2},$$

$$\mathcal{R} = r_v^2,$$

$$\mathcal{T} = \frac{Z_2}{Z_1} t_v^2 = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2}.$$

The power fractions add to one for real positive impedances. The reflected force coefficient has opposite sign to the reflected velocity coefficient because the wave direction reverses. At a free end $Z_2 \to 0$, reflected velocity equals incident velocity, so end velocity doubles while total force vanishes. At a rigid end $Z_2 \to \infty$, velocity reverses and force doubles. The sign a strain gauge reports can therefore differ from that of a velocity-sensitive optical measurement at the same reflection.

A step from the worked rod into one of the same material with twice the cross-sectional area gives $Z_2 = 2Z_1$, $r_v = -1/3$, $t_v = 2/3$, and power fractions $1/9$ and $8/9$. A reduced transmitted velocity

amplitude does not imply that most energy is reflected. If the interface includes a concentrated mass M_b, force jumps instead obey $N_2 - N_1 = M_b \dot{v}_b$ with signs determined by forces exerted on that mass from either side. At high frequency the reaction of that mass can mimic a much stiffer termination. Frequency-dependent reflection is then an expected boundary effect, rather than evidence that the bulk modulus varies with frequency.

Connect continuum modes to their lattice predecessor

For a periodic chain the exact discrete equation is

$$m_s \ddot{u}_j = k_s (u_{j+1} - 2u_j + u_{j-1}).$$

Substitute $u_j = U e^{i(kja - \omega t)}$ to obtain

$$\omega = 2\sqrt{k_s/m_s}\,|\sin(ka/2)|.$$

In the first positive Brillouin zone, $\omega/(ck) = \sin(ka/2)/(ka/2)$ and group velocity $d\omega/dk = c\cos(ka/2)$, where $c = a\sqrt{k_s/m_s}$. The continuum rod retains only the leading low-ka term. A wave with eight sites per wavelength has phase speed about $0.9745c$ and group speed $0.9239c$. A narrowband packet's peak can arrive later than the continuum prediction even when a single crest is timed fairly accurately.

A finite-difference code approximating the continuum rod can produce the same spatial dispersion with its mesh spacing in place of a physical lattice spacing. Distinguish them experimentally or numerically by changing the computational mesh while holding the physical structure fixed. Lattice effects persist as the numerical grid is refined; artificial spatial dispersion diminishes. A separate time-step study is needed if a time integrator also disperses the wave. Comparing both peak arrival and phase at two frequencies tests more of the dynamics than fitting a single transit time.

Derive transverse beam waves from curvature energy

A slender beam bending laterally is governed by a different elastic energy from the axial rod. Let $w(x,t)$ denote small transverse deflection of its neutral axis. Under the Euler–Bernoulli assumption that cross sections remain plane and normal to that axis, the axial strain a distance z from the neutral axis is approximately $-z w_{xx}$. Integrating the local elastic energy $E\epsilon_{xx}^2/2$ across the section gives $EI_A w_{xx}^2/2$ per unit length, where $I_A = \int z^2\,dA$ is a geometrical second moment of area, with units m^4. It is distinct from a mass moment of inertia. The kinetic contribution uses line density ρA as before.

Varying $\mathcal{L} = \rho A w_t^2/2 - EI_A w_{xx}^2/2$ requires integrating twice by parts in space. With constant coefficients the interior variation gives

$$\rho A w_{tt} + EI_A w_{xxxx} = 0.$$

The boundary terms are $-[EI_A w_{xx}\delta w_x] + [(EI_A w_{xx})_x \delta w]$. Thus slope variations pair with bending moments and displacement variations with shear forces, with signs set by the chosen face orientation. A clamped end prescribes both deflection and slope; a free end sets moment and shear to zero. A simply supported end prescribes deflection and zero bending moment. Using only one boundary condition per end, as for the axial rod, leaves the fourth-order beam problem underdetermined.

A sinusoid $e^{i(kx-\omega t)}$ gives

$$\omega = \sqrt{EI_A/(\rho A)}\, k^2.$$

Phase speed grows proportionally to wavenumber and group speed is twice phase speed in this model. A narrow transverse pulse in a long beam consequently spreads even in a perfectly homogeneous non-dissipative continuum. By contrast the axial rod was nondispersive in its long-wavelength continuum approximation. At high enough frequency the simple bending model omits rotary inertia, shear deformation and eventually three-dimensional elastic behaviour. Extrapolating its unbounded speed to arbitrary k would use the slender, low-frequency approximation far beyond its derivation.

For a simply supported beam with length L, the sine modes satisfy both end conditions and give

$$\omega_n = (n\pi/L)^2 \sqrt{EI_A/(\rho A)}.$$

For a rectangular section of width w_b and thickness d in the bending direction, $I_A = w_b d^3/12$ and $A = w_b d$. Doubling thickness doubles all the natural angular frequencies at fixed L, E, ρ, whereas doubling width alone leaves the ideal frequencies unchanged. The stiffness increases and so does the inertia. A beam with $L = 0.50\,\mathrm{m}$, $d = 5.0\,\mathrm{mm}$ and the earlier aluminium properties has $\sqrt{EI_A/(\rho A)} \approx 7.35\,\mathrm{m^2/s}$, giving fundamental $46.2\,\mathrm{Hz}$. The second bending mode is near four times that frequency, a clear difference from the approximately harmonic axial mode sequence.

Match a force pulse to a support spring and dashpot

A right termination exerts force on the rod through a dashpot with mechanical impedance Z_L. For a real positive dashpot impedance the reflected velocity fraction is $r = (Z - Z_L)/(Z + Z_L)$. Taking $Z_L = Z$ gives zero reflection at the termination even though energy leaves a finite rod: the dashpot dissipates the arriving power. With $Z_L = 3Z$ one obtains a reflected velocity of $-1/2$ of the incident value, a reflected energy fraction one quarter and absorbed power fraction three quarters. A free or rigid end also reflects energy, but it absorbs no net power in the ideal limit. A measured weak reflection alone is therefore not proof of weak damping inside the rod.

For a pure end spring of stiffness k_L attached to a rigid support, the velocity impedance is $Z_L = k_L/(i\omega)$ under $e^{i\omega t}$. It is imaginary, so the reflection coefficient has unit magnitude at every frequency and a frequency-dependent phase. At low frequency the spring impedance is large and the end looks nearly fixed. At high frequency its impedance is small and the end looks nearly free within the lumped approximation. An impulsive signal contains many frequencies and can return with a changed

waveform even without any transmitted or dissipated energy. A support combining a resistor-like dashpot and a spring gives both amplitude loss and phase variation.

A measurement with strain gauges at two positions can separate forward and backward waves if their path delays are resolved. In a narrow frequency band the two records are linear combinations of advancing and returning phase factors. If the gauge spacing corresponds to an integer number of half wavelengths, these combinations become linearly dependent and the separation is ill-conditioned. Change the spacing, use a broadband pulse, or combine a velocity measurement with a strain measurement at one station so that the two characteristic directions enter with opposite signs. The relation $v = (v_+ + v_-)$, $N = Z(-v_+ + v_-)$ then gives a direct local decomposition.

Derive sound propagation from mass and momentum conservation

The displacement-action argument has a fluid analogue. Take an initially uniform fluid of density ρ_0 and pressure p_0, moving little from equilibrium in one spatial dimension. Write small perturbations ρ', p', v. Linearised mass conservation is

$$\partial_t \rho' + \rho_0 \partial_x v = 0.$$

Momentum balance for a short fluid parcel is $\rho_0 \partial_t v = -\partial_x p'$. If compression over an acoustic cycle is approximately isentropic, $p' = c_s^2 \rho'$ with $c_s^2 = (\partial p/\partial \rho)_s$. Differentiate the first equation in time, eliminate the velocity derivative with momentum balance, and obtain $\partial_t^2 p' = c_s^2 \partial_x^2 p'$.

The thermodynamic derivative matters because pressure can respond differently when compression is slow enough for heat exchange. For a perfect gas with ratio of specific heats γ, isentropic perturbations give

$$c_s^2 = \gamma p_0 / \rho_0.$$

Taking an isothermal derivative would give p_0/ρ_0 instead. For $p_0 = 101\,\mathrm{kPa}$, $\rho_0 = 1.20\,\mathrm{kg/m^3}$, and $\gamma = 1.40$, the isentropic speed is about $343\,\mathrm{m/s}$, while the isothermal calculation gives $290\,\mathrm{m/s}$. A measured transit time can discriminate between these constitutive conditions. Wall heat exchange in a very narrow tube can alter which approximation is appropriate over a chosen frequency band.

For a right-moving sound wave $p' = \rho_0 c_s v$, with the pressure positive in compression. This differs in sign from the tensile-positive axial force used for the rod, so match physical force conventions before borrowing formulas across the two examples. The acoustic power per area is $p' v$. For sinusoidal pressure with rms amplitude $1.0\,\mathrm{Pa}$ in the stated gas, the mean intensity is

$$p_{\mathrm{rms}}^2 / (\rho_0 c_s) = 2.43 \times 10^{-3}\,\mathrm{W/m^2}.$$

A peak amplitude of $1.0\,\mathrm{Pa}$ would yield half this number. Writing rms or peak explicitly prevents a routine factor-of-two error when comparing a microphone measurement with an energy flux.

Determine whether a measured transit reveals a modulus or a geometry

Suppose two axially loaded slender rods have measured wave speeds agreeing within their uncertainties. That alone determines only the ratio E/ρ under the slender-rod assumptions. Measuring force–velocity impedance $Z = \rho A c$ and cross section as well gives $\rho = Z/(Ac)$ and $E = Zc/A$. With $Z = 1375\,\mathrm{kg/s}$, $c = 5092\,\mathrm{m/s}$ and $A = 1.0 \times 10^{-4}\,\mathrm{m^2}$, the inversion gives

$$\rho = 2700\,\mathrm{kg/m^3},$$

$$E = 70.0\,\mathrm{GPa}.$$

If area were overestimated by five percent, both inferred quantities would be underestimated by almost five percent while their ratio remained correct.

Now examine a thick elastic solid under a narrow compressional disturbance with wavelengths short compared with its transverse dimensions. Lateral deformation can be constrained by surrounding material. In an isotropic three-dimensional continuum the longitudinal plane-wave modulus is $\lambda + 2\mu_s$, where λ, μ_s are Lamé coefficients, rather than the uniaxial-stress Young modulus of a slender rod. The shear-wave speed satisfies $c_T^2 = \mu_s/\rho$ and longitudinal speed $c_L^2 = (\lambda + 2\mu_s)/\rho$. Measuring both, together with density, can recover the two independent elastic moduli. Using only one longitudinal speed to infer Young modulus without specifying the deformation geometry can give a physically incorrect value.

For $\rho = 2500\,\mathrm{kg/m^3}$, $c_L = 5000\,\mathrm{m/s}$ and $c_T = 3000\,\mathrm{m/s}$, these relations give

$$\mu_s = 22.5\,\mathrm{GPa},$$

$$\lambda = 17.5\,\mathrm{GPa}.$$

The implied Young modulus $E = \mu_s(3\lambda + 2\mu_s)/(\lambda + \mu_s) = 54.8\,\mathrm{GPa}$, while $\rho c_L^2 = 62.5\,\mathrm{GPa}$. Both numbers represent material stiffness, but they enter different mechanical boundary conditions. A long thin specimen of the same material should have an axial speed closer to $\sqrt{E/\rho} = 4680\,\mathrm{m/s}$. Comparing the bulk and rod experiments tests the constitutive and geometrical assumptions behind the measured speed.

Derive orthogonality when a rod changes material along its length

For a rod with positive varying density $\rho(x)$, area $A(x)$ and modulus $E(x)$, a separated free oscillation $u(x,t) = \phi(x)\cos\omega t$ satisfies

$$-[EA\phi']' = \omega^2 \rho A \phi.$$

Suppose two modes ϕ_n, ϕ_m satisfy the same homogeneous fixed or free boundary conditions and the correct continuity of displacement and axial force across every perfectly bonded interface. Multiply the

first equation by ϕ_m, the second by ϕ_n, subtract and integrate. Interior terms produced by integration by parts cancel pairwise, including the interface flux terms, and the endpoint terms vanish. Thus

$$(\omega_n^2 - \omega_m^2) \int \rho A \phi_n \phi_m \, dx = 0.$$

Mode shapes with distinct frequencies are orthogonal under the mass-weighted inner product, not generally under a simple average of displacements measured at uniformly spaced stations. An initial velocity distribution $v_0(x)$ projects onto mode n with coefficient $\int \rho A v_0 \phi_n \, dx / \int \rho A \phi_n^2 \, dx$. In a rod with a heavy section, the same imposed velocity there stores more energy and carries more modal weight. An experiment interpreting raw camera amplitudes without line-density weighting can misassign energy among modes. Normalising each measured mode shape to unit maximum does not fix the mass weighting needed for energies.

Multiplying one eigenvalue equation by its own shape gives

$$\omega_n^2 = \int EA(\phi_n')^2 dx / \int \rho A \phi_n^2 dx.$$

For an admissible trial function satisfying prescribed displacement conditions, the same quotient bounds the fundamental squared frequency from above when rigid translations are excluded. Consider a uniform fixed–free rod and trial $\phi = x/L$. The numerator is EA/L and denominator $\rho AL/3$, giving $\omega_{\mathrm{trial}} = \sqrt{3}\, c/L$. The exact fundamental is $\pi c/(2L)$, so this one-parameter shape overestimates frequency by about 10.3%. It need not satisfy the free-end traction condition itself because that is a natural condition of the variational minimiser; it does need to satisfy the essential fixed-end displacement.

For a free–free rod the constant displacement mode has zero frequency. Using an arbitrary positive trial function without removing its mass-weighted mean can drive a quotient towards zero by adding a large constant. To estimate the first deforming mode impose orthogonality to the rigid translation, $\int \rho A \phi \, dx = 0$. This has an immediate mechanical interpretation: the deformation mode should not translate the centre of mass of an isolated rod. The null mode and its mass-weighted constraint reappear in numerical eigenproblems built from discretised stiffness and mass matrices.

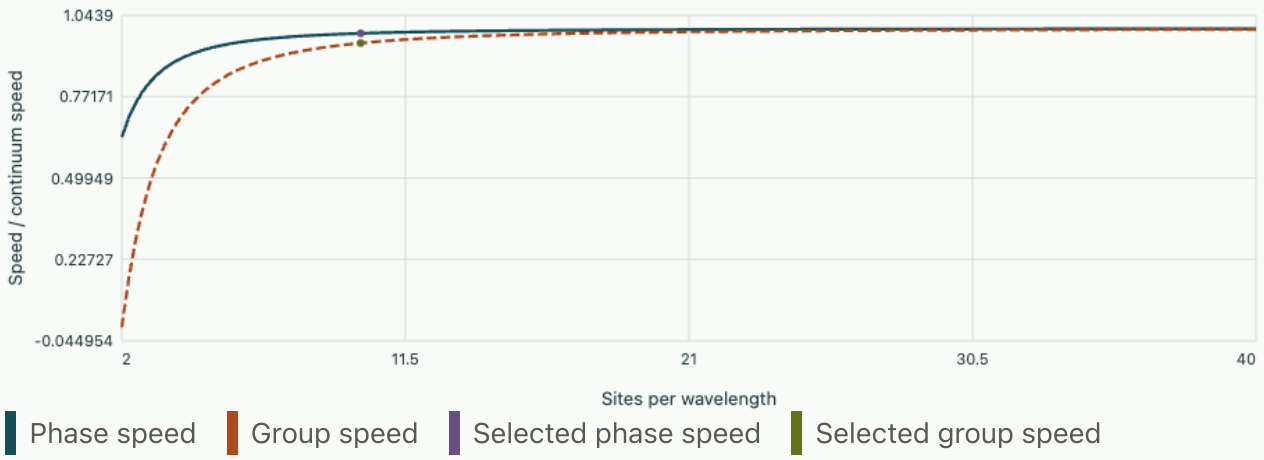
Find where a discrete elastic chain ceases to look continuous

Shorten the wavelength relative to the lattice spacing. Compare the phase and group velocities with those inferred from a continuum action.

Open this experiment online

Try the calculation

Sites per wavelength 10



Frequency and phase speed are 98.36% of the continuum prediction. Group speed is 95.11%.

Track work and impulse in a one-dimensional pulse experiment

A finite displacement pulse provides a useful warning about integrating strain and velocity records. A right-going strain pulse of one sign can carry net momentum $\int \rho A v \, dx$. Because $v = -c u_x$, its integrated material momentum is $-\rho A c[u(+\infty) - u(-\infty)]$ at a fixed time. A localised displacement bump that returns to the same displacement on both sides has zero net ordinary material momentum, even though its energy is positive and travels to the right. Its strain and particle-velocity records contain positive and negative parts. A unipolar velocity pulse leaves behind a displacement step, produced by a source that has supplied net impulse.

For a right-moving rectangular velocity pulse of amplitude $0.020\,\mathrm{m/s}$, duration $100\,\mu\mathrm{s}$ in the worked rod, the local displacement jump measured in time is $2.0\,\mu\mathrm{m}$. Its impulse magnitude across a section is $Z v \Delta t = 2.75 \times 10^{-3}\,\mathrm{N\,s}$, while its transmitted energy is

$$Z v^2 \Delta t = 55.0\,\mu\mathrm{J}.$$

A second equal pulse of opposite velocity sign launched later cancels the net displacement and material momentum but doubles the transmitted energy. Thus integrating a signed velocity signal and integrating its square distinguish different properties of the same experimental record.

A displacement sensor has a small linear drift during a burst. Which inferred observables are particularly sensitive?

▼ **Work through the solution**

Differentiating adds a constant offset to particle velocity. Integrating that apparent velocity over a long window produces a growing spurious displacement jump and momentum estimate. Squaring it adds both a cross term with the physical velocity and a positive baseline energy. Measure a pre-pulse baseline, choose an integration interval covering the physical pulse, and test that the result is stable when the window is moderately extended. An independent strain gauge gives another characteristic combination; comparison with $N = -Zv$ can expose drift or an overlapping reflection.

Can a finite-element rod model with a plausible first eigenfrequency still fail an impulse test?

▼ **Work through the solution**

The first frequency mostly probes a long-wavelength weighted stiffness-to-mass ratio. A brief pulse contains higher modes and is sensitive to temporal resolution, dispersion, boundary tractions and the way discrete source work enters. Verify modal frequencies at several wavelengths, check the pulse's positive energy and signed impulse separately, and compare the measured incoming and reflected characteristic fluxes. A single standing-wave eigenvalue cannot validate all of those physically independent balances.

Recover a jump condition at an interface with a point mass

For a concentrated mass M_b joining two rods include $M_b u_t(x_b)^2/2$ as a separate contribution to the Lagrangian. Its time variation contributes $-M_b u_{tt}\delta u$ at the joint. The left and right rod boundary terms contribute respectively $-N_1\delta u$ and $+N_2\delta u$. Requiring stationarity for an independently varying shared displacement gives exactly $N_2 - N_1 = M_b u_{tt}$. This derivation identifies both the sign and the constitutive assumption of a perfectly bonded, nonslipping joint.

For identical rods and a sinusoidal drive at angular frequency ω, the reflected velocity coefficient becomes $r = -i\omega M_b/(2Z + i\omega M_b)$ under the stated phasor convention. At $\omega M_b = 2Z$ it reflects half the power and transmits half. The mass itself absorbs no cycle-averaged energy, though it stores kinetic energy during a cycle. For $Z = 100\,\mathrm{kg/s}$ and $M_b = 0.10\,\mathrm{kg}$, the equal-power frequency is about $318\,\mathrm{Hz}$. A real joint that loses appreciable power at that frequency needs a damping or radiation term in addition to its measured mass.

Tutorial on stress waves

A gauge $0.80\,\mathrm{m}$ from a free end records the echo of a sharp disturbance in the worked rod. What echo delay and strain sign should it expect?

▼ **Work through the solution**

The extra path from gauge to end and back is $1.60\,\mathrm{m}$, giving $1.60/5092 = 0.314\,\mathrm{ms}$. A free end has zero net axial force, so the reflected force and strain have opposite signs to the incident ones. The

velocity reflection keeps its sign. Finite pulse width and any change of material between gauge and end can obscure that simple result.

A bonded step has measured reflected velocity coefficient -0.20 and no detectable dissipation. Infer its impedance ratio and transmitted energy fraction.

▼ **Work through the solution**

Rearranging $r_v = (1 - r)/(1 + r)$ gives

$$r = Z_2/Z_1 = (1 - r_v)/(1 + r_v) = 1.50.$$

The energy reflection is 0.04 and transmission 0.96. The velocity transmission is 0.80, whose square alone is only 0.64. Multiplying by the transmitted-to-incident impedance ratio recovers the correct energy fraction.

A rod model with a prescribed displacement at its right end reports nonzero work there, even though the variational boundary term vanished in deriving its equation. Is there a contradiction?

▼ **Work through the solution**

Vanishing of $N\delta u$ follows because the allowed virtual displacement at a prescribed boundary is zero. Actual boundary velocity can be nonzero when the prescribed displacement varies in time. The actuator supplies power Nu_t at the right boundary. Compare the time-integrated input with the change of stored energy plus any losses or outgoing energy rather than identifying an admissible variation with actual displacement.

Compare a finite chain with its long-wavelength continuum.

Fließbach includes string vibrations, beam bending and sound in his continuum mechanics treatment.

Actuation, bearings and measurement

Follow the coupled force, motion and dissipation in a lubricating film, a compliant positioning system and a motion sensor.

1. Viscous stresses, duct resistance and wall-driven flow
2. A viscous squeeze film as a rate-dependent support
3. Coordinates, energy and equivalent inertia
4. Why a radial slide presses sideways on a rotating guide
5. When a suspended capacitor loses its stable equilibrium
6. Designing a displacement bridge at its noise limit
7. Eddy-current braking, magnetic penetration and the heat left behind
8. Measure inertia and gravity with two suspension axes

Viscous stresses, duct resistance and wall-driven flow

Fluid mechanics

Derive a developed velocity profile from momentum balance and distinguish a moving-wall contribution from a pressure-driven contribution.

FROM FIRST PRINCIPLES

Derive the parabolic velocity profile between stationary plates

Consider steady fully developed flow between wide parallel plates at $y = 0$ and $y = H$. Assume incompressible Newtonian fluid, constant viscosity μ, no slip, no variation across the channel width and no streamwise body force. Let $G = -dp/dx > 0$ drive flow in the positive x direction. Since the axial velocity depends only on y, the convective acceleration is zero. A thin fluid layer balances pressure force against the change of shear stress across its thickness.

$$0 = -\frac{dp}{dx} + \frac{d\tau_{xy}}{dy},$$

$$\tau_{xy} = \mu \frac{du}{dy},$$

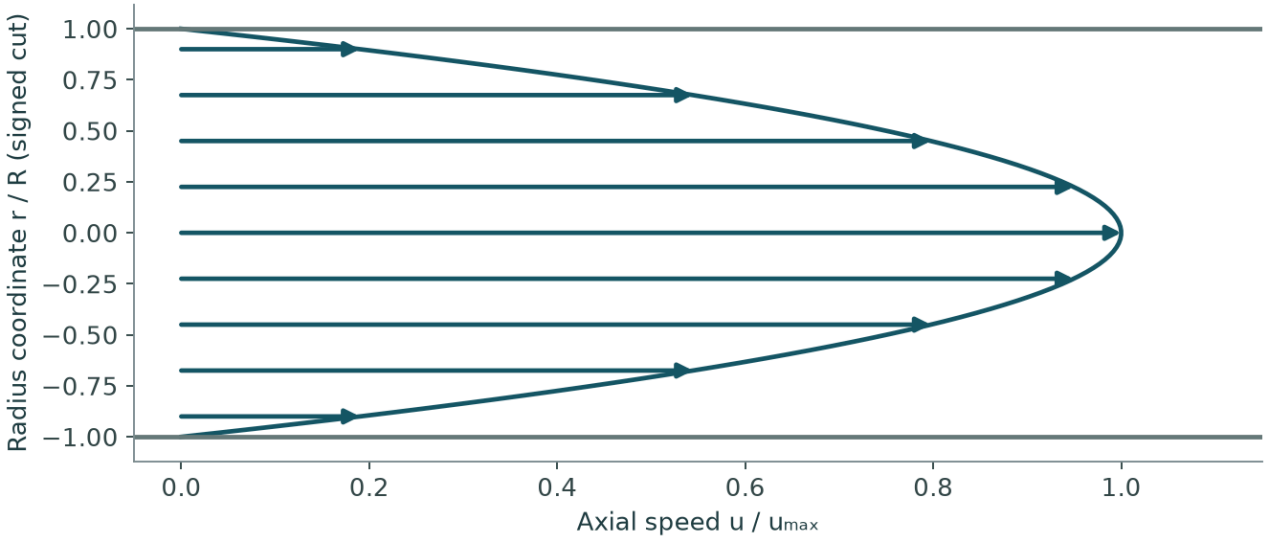
$$\mu u'' = -G.$$

Integrate twice and apply zero velocity at both walls. The resulting velocity is largest halfway between the walls, where the shear stress vanishes. Integrating the profile gives discharge per unit width.

$$u(y) = \frac{G}{2\mu} y(H - y),$$

$$\frac{Q}{b} = \int_0^H u(y)\, dy = \frac{G H^3}{12\mu}.$$

The cube of the gap arises because widening the channel increases both conducting area and the speed at a given pressure gradient. A moving wall adds a linear shear-driven profile through the boundary condition because the governing equation is linear.



Why the mean velocity is half the centreline velocity

Consider steady, fully developed laminar flow of a Newtonian liquid through a straight circular pipe with no slip. The pressure force balances viscous shear on each cylindrical fluid core. Its area grows as r² and its shear surface grows as r, so the needed wall-parallel shear grows linearly in r. Integrating once gives a parabola. The cross-sectional mean uses the annular weight 2πr dr and equals half the centreline value for this parabolic profile.

DERIVATION AND APPLICATIONS

Quantify the effect of wall slip on laminar pipe resistance

A pressure-flow calibration of a small tube can reveal a change in the assumed wall condition. Consider steady, fully developed, axisymmetric flow of a Newtonian liquid of viscosity μ in a round tube of radius R. Take the pressure gradient magnitude $G = -dp/dz > 0$ and neglect body forces along the tube. Balance pressure force against viscous drag on a coaxial fluid cylinder of radius r. Positive axial speed decreases approaching the wall.

$$G\pi r^2 = -\mu \frac{du}{dr}\, 2\pi r,$$

$$\frac{du}{dr} = -\frac{Gr}{2\mu}.$$

Allow a Navier slip length $b \geq 0$ at the stationary wall: the wall speed extrapolates to zero a distance b beyond it in a local linear approximation. This gives

$$u(R) = -b\, u'(R).$$

Integrating the shear rate from the wall into the flow adds a uniform velocity offset to the usual parabola.

$$u(r) = \frac{G}{4\mu}(R^2 - r^2 + 2bR).$$

Integrate the velocity over annuli to turn a local change of boundary condition into a measurable discharge. The contribution from the offset is proportional to the full area because the velocity addition is constant over the cross-section.

$$Q = 2\pi \int_0^R u(r) r \, dr = \frac{\pi G R^4}{8\mu}\left(1 + \frac{4b}{R}\right).$$

The shear stress needed to balance a fixed pressure gradient is unchanged. Effective slip lowers the required pressure gradient for a fixed flow. When b/R is small the fractional change is proportional to b/R, making small tubes more sensitive at a given slip length.

WORKED EXAMPLE

Calculate the extra discharge of a coated tube

Take $R = 100$ µm, $b = 5.00$ µm, $\mu = 1.00$ mPa s and $G = 10.0$ kPa/m. The no-slip flow is $\pi(10^4)(10^{-4})^4/(8 \times 10^{-3}) = 0.393$ µL/s. Slip increases it by $4b/R = 0.200$, yielding 0.471 µL/s. The predicted wall speed $GbR/(2\mu) = 2.50$ mm/s could be examined by a near-wall tracer.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Infer a slip length from pressure reduction

A different tube of radius 40.0 µm requires only 80.0 percent of the reference no-slip pressure drop to sustain the same flow. Equal flow implies $1 + 4b/R = 1/0.800 = 1.25$, giving $b = 2.50$ µm. A radius calibration error can mimic this signal because the no-slip flow varies as R^4. Independent bore measurements are therefore especially informative in this inverse experiment.

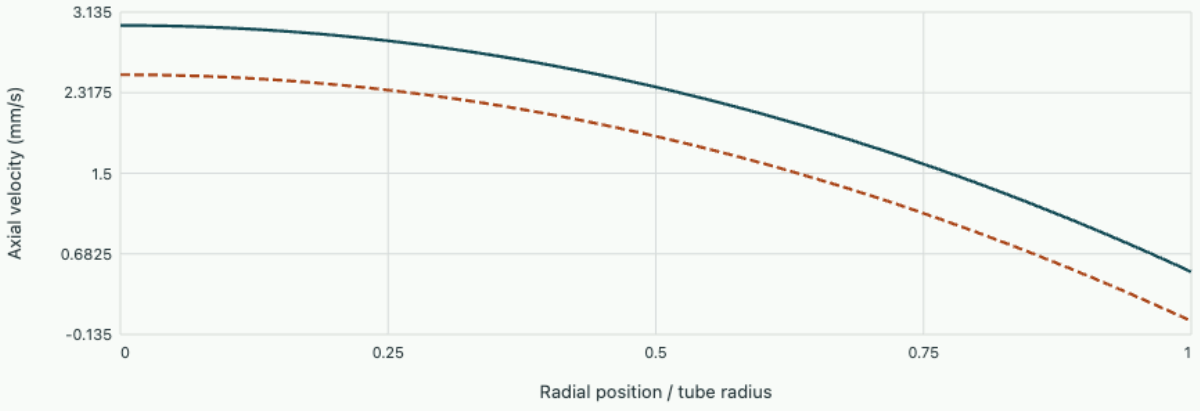
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Tube radius (µm)	**100**
Slip length / radius	**0.1**
Pressure gradient (Pa/m)	**1000**



Flow=0.0550 mm³/s, 1.400 times the no-slip flow. Dynamic viscosity1 mPa s; slip condition changes the wall value and the integrated conductance.

Transport momentum between neighbouring layers

Push two parallel plates past one another with liquid in the gap. Even at constant velocity the moving plate requires force. In a Newtonian fluid the tangential shear stress on a plane parallel to the motion is $\tau_{xy} = \mu\, du/dy$. Dynamic viscosity μ has units Pa·s. Divide it by density to obtain kinematic viscosity ν, an indicator of how rapidly momentum spreads through the fluid. Viscous resistance depends on a velocity gradient; uniform translation of the entire fluid would not create such stress.

Take two long horizontal plates separated by H. Far from inlet and side walls assume constant density, constant viscosity and steady fully developed laminar motion $u = u(y)$. The lower plate is fixed and the upper moves at speed U. Represent the constant driving pressure gradient by $G = -dp/dx$,

positive for pressure falling downstream. A differential slab balances the downstream pressure force with the change in shear, giving $0 = G + d\tau_{xy}/dy = G + \mu u''$.

Impose the two wall velocities

The first integration gives

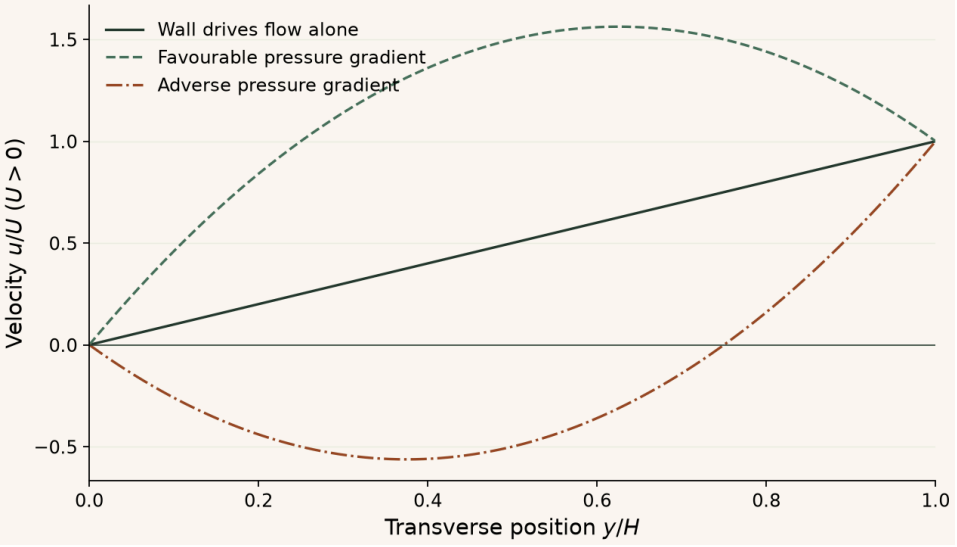
$$\mu u' = -Gy + C_1.$$

Integrating again and imposing no slip $u(0) = 0,\ u(H) = U$ determines the constants. The result is the sum of a straight wall-driven profile and a parabolic pressure-driven one:

$$u(y) = U\frac{y}{H} + \frac{G}{2\mu}y(H - y),$$

$$\frac{Q}{b} = \int_0^H u\,dy = \frac{UH}{2} + \frac{GH^3}{12\mu}.$$

Superposition here follows because the reduced equation and boundary conditions are linear in these inputs. It is not a universal property of the full nonlinear fluid equations. A negative G represents an adverse pressure rise and can produce a reverse-flow layer close to the stationary wall even when the moving wall drags fluid forward elsewhere.



The moving upper plate has the same speed in all three cases. The dimensionless pressure-gradient term multiplying (y/H)(1−y/H) is +4, 0 or −4.

Predict the pressure a moving surface can hold

An oil film has $\mu = 0.050$ Pa·s, $H = 1.0$ mm and $U = 0.20$ m/s. The wall-driven flux per unit width is $UH/2 = 1.0 \times 10^{-4}$ m²/s. An adverse pressure gradient sufficient to reduce net flux to zero must

satisfy $G = -6\mu U/H^2 = -60\,000$ Pa/m. Over 50 mm, pressure rises by 3.0 kPa. Zero net transport does not make the oil stationary.

Set $s = y/H$ and insert the zero-flux gradient. The velocity is $u/U = 3s^2 - 2s$, which changes sign at $s = 2/3$ inside the film. The velocity gradient at the upper moving wall is $4U/H = 800\,\mathrm{s}^{-1}$, so the wall supplies shear of magnitude 40 Pa. Even though the pressure-driven and wall-driven volume rates cancel, the moving surface still performs work, dissipated through the internal shear field.

Connect a local solution to a network

In a horizontal circular tube with fixed wall, axisymmetry changes the area that carries shear. Balancing forces on a cylinder of radius r gives

$$|\tau| = Gr/2.$$

Integrating the velocity gradient with no slip yields

$$u = G(R^2 - r^2)/(4\mu),$$

$$Q = \pi R^4 \Delta p/(8\mu L).$$

Thus a straight developed pipe behaves as a hydraulic resistance $8\mu L/(\pi R^4)$ in Pa·s/m³. Series resistances carry the same Q and their pressure drops add. Parallel branches share pressure drop and their flows add. The analogy requires consistent units and a laminar regime; short entrance sections and strongly accelerating junctions may add losses outside the fully developed solution.

Express pipe resistance as a friction factor

For a circular pipe of diameter $D = 2R$, the developed laminar solution gives

$$\Delta p = 32\mu L\bar{u}/D^2.$$

Define the Darcy friction factor by $\Delta p = f_D (L/D)\rho\bar{u}^2/2$. Equating these expressions produces $f_D = 64/\mathrm{Re}_D$. For 2 m of 10 mm pipe with mean speed 0.10 m/s, water viscosity 0.001 Pa s and density 1000 kg/m³, the Reynolds number is 1000 and the pressure drop is 64 Pa. A friction factor listed in the Fanning convention is one quarter of the Darcy value, so identify the convention before substituting a tabulated number into a loss calculation.

Derive the profile of a shear-thinning liquid in a pipe

A Newtonian viscosity inferred in one duct may give poor predictions for a suspension or polymer solution at another rate. In a simple power-law model the shear stress magnitude is $\tau = K|\dot{\gamma}|^n$, with consistency K in Pa sⁿ and index n. A liquid with $n < 1$ has apparent viscosity decreasing with shear

rate. Assume this law is supported over the shear range being sampled, flow is steady, axisymmetric, laminar and fully developed, and the circular pipe has no wall slip. The axial force balance is independent of rheology: at radius r, $\tau = Gr/2$ for driving gradient $G = -dp/dx > 0$.

Since the positive downstream velocity decreases towards the wall, write $-du/dr = (Gr/2K)^{1/n}$. Integrate from radius r to the stationary wall at R, then integrate the result over the circular cross-section to obtain

$$u(r) = \frac{n}{n+1}(G/2K)^{1/n}(R^{1+1/n} - r^{1+1/n}),$$

$$Q = \frac{\pi n}{3n+1}(G/2K)^{1/n}R^{3+1/n}.$$

Setting $n = 1,\ K = \mu$ gives the parabolic profile and $Q = \pi G R^4/(8\mu)$. For $n < 1$ the velocity is flatter towards the centre and the relationship between pressure gradient and flow rate is nonlinear. The ideal power law can imply unrealistic apparent viscosity near zero shear at the axis; a fitted model with finite limiting viscosities is preferable when measurements resolve those regimes.

Infer a pressure demand from wall shear

A model resin follows $n = 0.50,\ K = 0.80\,\mathrm{Pa\,s^{0.5}}$ in the operating range. A 2.0 m pipe of radius 5.0 mm experiences a pressure drop of 8.0 kPa, giving $G = 4000\,\mathrm{Pa/m}$ and wall shear $\tau_w = GR/2 = 10.0\,\mathrm{Pa}$. The wall shear rate is

$$(10/0.80)^2 = 156.25\,\mathrm{s^{-1}}.$$

Writing the mean speed in terms of this wall value gives

$$\bar{u} = nR|\dot{\gamma}_w|/(3n+1) = 0.15625\,\mathrm{m/s}.$$

The corresponding flow is $\pi(0.005)^2(0.15625) = 1.227 \times 10^{-5}\,\mathrm{m^3/s}$, about 0.736 L/min.

To double flow in this same pipe, the power-law relation requires the gradient to rise by $2^n = \sqrt{2}$. The new pressure drop is 11.3 kPa, and wall shear rate doubles to 312.5 s⁻¹. A linear resistance model extrapolated from the original measurement would predict 16 kPa. Conversely at fixed gradient a 10% radius increase multiplies rate by $1.1^{3+1/n} = 1.61$, a stronger dependence than the Newtonian fourth power. Verify the resulting shear rate still falls in the tested rheological range, and check entrance length and temperature. Temperature changes produced by viscous dissipation can otherwise mimic a shear-dependent viscosity.

Two well-spaced steady pressure tests can independently infer the unknown exponent. At fixed pipe geometry,

$$\ln Q = (1/n)\ln \Delta p + \text{constant}.$$

If pressure drop rises from 8.0 to 12.0 kPa while flow rises from 0.736 to 1.66 L/min, the log slope is $\ln(1.66/0.736)/\ln(12/8) \simeq 2.00$, giving $n \simeq 0.50$. Repeating the higher-pressure point after cooling checks whether heating caused hysteresis between runs. Using an inlet pressure transducer alone would also include a changing downstream pressure; both end pressures or a differential gauge are needed to identify the flow law from this experiment.

The pressure taps should bracket the same developed length at every setting because entrance losses may change differently with flow.

How does a local pressure gradient accelerate a fluid particle?

A duct solution generally begins by cancelling terms in the momentum balance. To decide which terms can be cancelled, follow a small volume of fluid. Its acceleration includes the change in the velocity at its present location and the change as it moves into a different part of the velocity field. Applying the chain rule along its trajectory gives

$$D\mathbf{u}/Dt = \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}.$$

Consider the two faces normal to the first coordinate axis. Their net surface force per unit volume in direction i approaches $\partial_x \sigma_{ix}$ as the face separation tends to zero. Summing the three face pairs and adding weight gives

$$\rho Du_i/Dt = \sum_j \partial_j \sigma_{ij} + \rho g_i.$$

For an incompressible Newtonian liquid of constant viscosity μ, insert $\sigma_{ij} = -p\delta_{ij} + \mu(\partial_j u_i + \partial_i u_j)$. The cross derivative sums to $\partial_i(\nabla\cdot\mathbf{u}) = 0$. The remaining terms are the local equations

$$\rho[\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}] = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{g},$$
$$\nabla\cdot\mathbf{u} = 0.$$

Take a horizontal slice through an ideal planar stagnation region, away from confining walls. The steady field $u = ax,\ v = -ay$ satisfies continuity. Every velocity component is linear, so its Laplacian vanishes. Nevertheless the material acceleration is (a^2x, a^2y). Integrating the two horizontal gradients gives

$$p(x,y) = p_0 - \rho a^2(x^2+y^2)/2,$$

where p_0 is the stagnation-point pressure at this elevation. For water at $1000\,\mathrm{kg\,m^{-3}}$, $a = 4\,\mathrm{s^{-1}}$ and $y = 0.10\,\mathrm{m},\ x = 0$, pressure is 80 Pa below p_0. Fluid entering along positive y moves towards the origin but accelerates in positive y: it slows as pressure rises. At the stagnation point itself both velocity and acceleration vanish. Specifying p_0 fixes the pressure datum. The velocity gradients determine the spatial change.

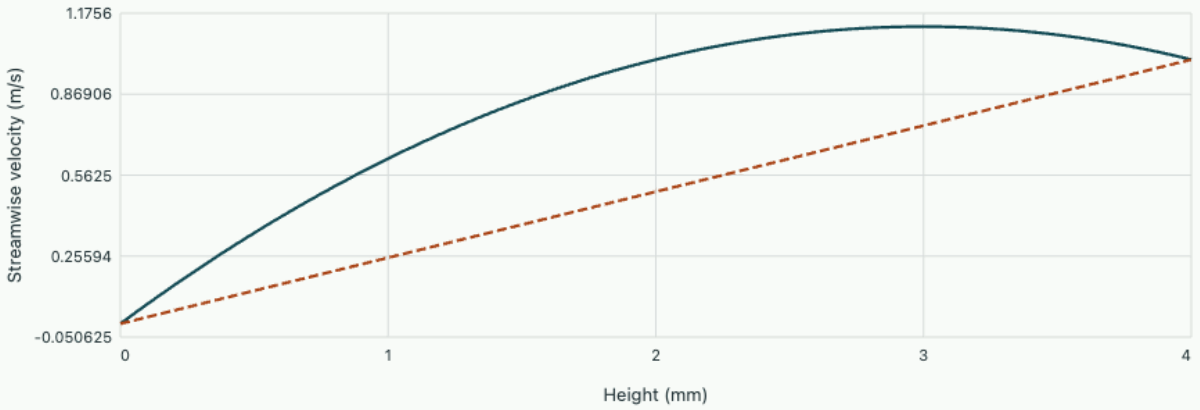
Interactive calculation · Change the inputs of this example

Change the parameters of this reference plot and compare its limiting cases. The fixed illustration remains beside the derivation; the adjustable calculation exposes the same dependence on geometry, material properties or loading.

Open this live calculation

Try the calculation

Moving-wall speed (m/s)	1
Pressure forcing / wall forcing	2
Gap (mm)	4



Couette–Poiseuille Wall driven alone

Mean speed 0.833 m/s. Pressure parameter equals $Gh^2/(2\mu U)$; adverse pressure can reverse flow close to the stationary wall.

A viscous squeeze film as a rate-dependent support

Fluid mechanics

Bring two smooth circular disks slowly together with a thin layer of liquid between them. The force needed depends strongly on gap size even when neither surface is elastic.

Decide which motion the gap permits

Take rigid parallel disks of radius R separated by $h(t) \ll R$, filled with an incompressible Newtonian liquid of viscosity μ. The lower disk is fixed and the upper disk approaches at speed $V = -\dot{h} > 0$. Assume an axisymmetric film discharging to ambient pressure at the edge, no slip on both disks and negligible inertia. The latter requires viscous stress to dominate transient and convective inertia, including the scale $\rho V h/\mu \ll 1$. Fluid displaced from the centre must travel sideways through the thin film. Across-gap derivatives dominate radial derivatives of velocity, and the leading pressure is uniform through the thickness.

$$\mu\, \partial_z^2 u_r = \frac{dp}{dr},$$

$$u_r(0) = u_r(h) = 0.$$

Two integrations give

$$u_r = (dp/dr) z(z-h)/(2\mu).$$

The pressure gradient must be negative for outward flow. Integrating velocity over the gap gives radial volume flow per circumferential length $q_r = -h^3 (dp/dr)/(12\mu)$. The cubic power comes from a velocity scale proportional to h^2 multiplied by the available gap h.

Balance the displaced volume inside every circle

The film inside radius r loses volume at rate $\pi r^2 V$. With no compressibility or leakage through the disks this must equal $2\pi r q_r$. Hence $q_r = rV/2$. Match to the viscous flow relation, integrate inward from $p(R) = p_a$, and sum the excess pressure over the wetted disk:

$$\frac{dp}{dr} = -\frac{6\mu V r}{h^3},$$

$$p(r) - p_a = \frac{3\mu V}{h^3}(R^2 - r^2),$$

$$F = \int_0^R [p(r) - p_a]\, 2\pi r\, dr = \frac{3\pi\mu V R^4}{2h^3}.$$

The pressure distribution is parabolic; its area average is half its centre value. A fixed-speed actuator sees a rapidly increasing load as the film narrows. Conversely a constant load makes the approach

slower. An experiment must state which of speed or force is controlled before two records can be compared.

Worked example · halve a film thickness

Let $R = 20$ mm, $h = 0.20$ mm, $\mu = 0.10$ Pa s and $V = 0.10$ mm/s. The centre excess is

$$3(0.10)(10^{-4})(0.020)^2/(0.00020)^3 = 1500\,\mathrm{Pa}.$$

The net force is 0.942 N. Reducing the gap to 0.10 mm at unchanged speed raises pressure and force by eight, giving 12.0 kPa and 7.54 N. Reducing speed by eight at the thinner gap would recover the original force. The model consequently distinguishes a viscous support from a spring by repeating the same gap at different rates.

Worked example · integrate the approach at constant load

Rearrange the force law to $-dh/dt = 2Fh^3/(3\pi\mu R^4)$. Separation of variables gives

$$t = [3\pi\mu R^4/(4F)](h_f^{-2} - h_i^{-2}).$$

Under a constant 1.00 N load the example disk takes 2.83 s to go from 0.20 to 0.10 mm. The ideal time to reach zero mathematical thickness diverges. Real surfaces eventually develop solid contact at roughness peaks, and departures from parallel alignment can redirect much of the pressure distribution.

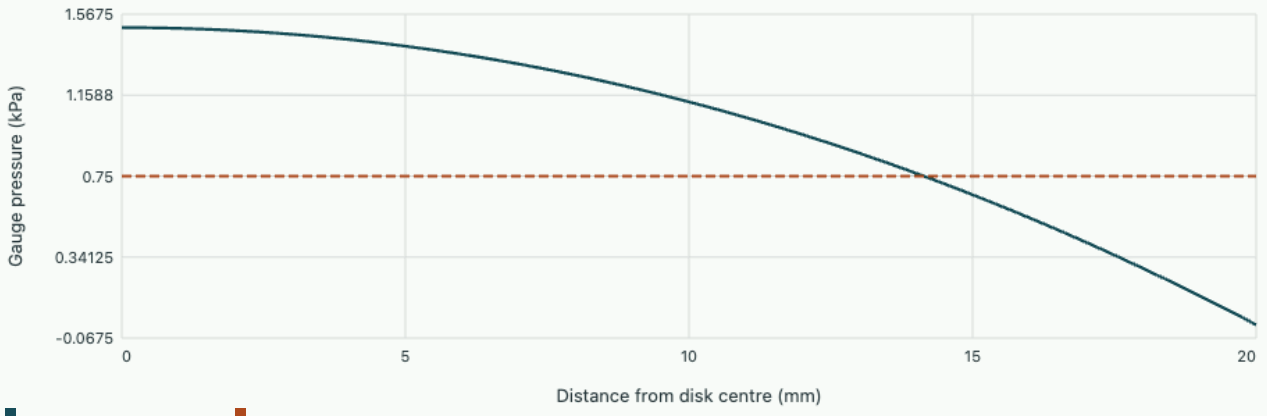
First halve the gap and then independently halve the approach speed. Predict how the centre pressure changes in each trial. Check the area-average line against the radial curve and explain why an unweighted average over radius gives a different value.

Open this experiment online

Try the calculation

Disk gap (mm) **0.2**

Approach speed (mm/s) **0.1**



The central excess pressure is 1.500 kPa and the integrated disk load is 0.942 N.

Tutorial · identify a viscosity and test the boundary assumption

At gap 0.15 mm, disks of radius 15 mm give 2.00 N at 0.25 mm/s. Infer the viscosity. Predict the load if a vented outer annulus reduces the effective wetted radius to 12 mm while the gap and velocity stay fixed.

▼ **Integrate over the wetted radius**

The inverse is $\mu = 2Fh^3/(3\pi VR^4) = 0.113$ Pa s. With the edge now open to ambient at 12 mm, force scales by $(12/15)^4 = 0.4096$, giving 0.819 N. Simply multiplying the original force by the area ratio would miss the change in peak pressure caused by the shorter flow path.

Further reading · MIT 2.25, §6.4, locally developed viscous flows and lubrication

Coordinates, energy and equivalent inertia

Machine dynamics

Reduce a constrained drive to one coordinate and use a power balance to check its equation of motion.

FROM FIRST PRINCIPLES

Refer moving masses to one shaft by matching kinetic energy

Choose shaft angle θ as the independent coordinate of a rigid mechanism. Suppose the angular speeds of its rotating components and the linear speeds of its translating components are fixed multiples of $\dot{\theta}$. Assume ideal constraints with no slip, so specifying θ and $\dot{\theta}$ fixes the motion. Total kinetic energy is the sum of the individual translational and rotational energies.

$$T = \sum_i \tfrac{1}{2} m_i v_i^2 + \sum_j \tfrac{1}{2} J_j \omega_j^2 = \tfrac{1}{2} J_{\mathrm{eq}} \dot{\theta}^2 .$$

Here rotational inertias are about the component centres or fixed rotation axes as appropriate, with no motion counted twice. Dividing by $\dot{\theta}^2/2$ gives the inertia seen at the chosen shaft.

$$J_{\mathrm{eq}} = \sum_i m_i \left(\frac{v_i}{\dot{\theta}} \right)^2 + \sum_j J_j \left(\frac{\omega_j}{\dot{\theta}} \right)^2 .$$

A force on a translating part contributes the shaft torque that gives the same instantaneous power,

$$T_i \dot{\theta} = F_i v_i .$$

For constant speed ratios and constant equivalent inertia, energy balance then gives

$$\sum T_i = J_{\mathrm{eq}} \ddot{\theta} .$$

The square on the speed ratio in inertia reflection follows from energy and is essential when changing the reference shaft.

Background · Generalised inertia

$$M(q) = 2T/\dot{q}^2$$

This scalar definition applies to one independent moving coordinate with purely quadratic kinetic energy. Changing the coordinate changes the numerical inertia.

DERIVATION AND APPLICATIONS

Derive the extra drive term for position-dependent inertia

A motor can require a torque at constant angular speed if a mechanism changes the linear speed of its payload during a revolution. Use a single angular coordinate θ, input inertia J_0, and a guided mass m with position $x = f(\theta)$. Assume a rigid lossless linkage and a horizontal guide, so gravity does no work. The kinematic constraint sets the payload velocity and fixes the instantaneous reflected inertia by equating the total kinetic energy to that of one shaft.

$$\dot{x} = f'(\theta)\dot{\theta},$$

$$T = \tfrac{1}{2}[J_0 + m(f')^2]\dot{\theta}^2 = \tfrac{1}{2}J(\theta)\dot{\theta}^2.$$

Differentiate the energy with respect to time. The shaft input supplies the full change in kinetic energy through its power $\tau\dot{\theta}$. Equating the two and cancelling speed gives the required drive torque. At a turning point the same equation follows directly from Lagrange's equation.

$$\tau = J(\theta)\ddot{\theta} + \tfrac{1}{2}J'(\theta)\dot{\theta}^2.$$

For an ideal sinusoidal guide $x = a\sin\theta$, substitution makes the speed-squared contribution explicit. It changes sign within each turn because the translating payload first absorbs and then releases kinetic energy.

$$J = J_0 + ma^2\cos^2\theta,$$

$$\tau = J\ddot{\theta} - ma^2\sin\theta\cos\theta\,\dot{\theta}^2.$$

A constant-inertia approximation omits this alternating torque even if it predicts the total energy averaged over a turn. Its integral over a full constant-speed revolution vanishes as the payload returns to its initial speed.

WORKED EXAMPLE

Predict a constant-speed torque measurement

A 2.00 kg carriage follows $x = 0.0500\sin\theta$ metres with shaft speed 20.0 rad/s. At $\theta = 45^\circ$ its required torque is $-2.00(0.0500)^2(0.5)(20.0)^2 = -1.00$ N m. The motor absorbs 20.0 W at that instant. At 135° the required torque is $+1.00$ N m. Both readings occur at the same shaft speed and the same reflected inertia.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Identify a payload from the torque harmonic

A separate horizontal mechanism has stroke parameter $a = 40.0$ mm and runs at 30.0 rad/s. After removing its empty-linkage trace, the measured second-harmonic torque amplitude is 0.720 N m. Since $|\sin\theta\cos\theta|_{\max} = 1/2$, the fitted mass is

$$2(0.720)/[(0.0400)^2(30.0)^2] = 1.00\,\mathrm{kg}.$$

A run at half speed should show 0.180 N m amplitude if the measured harmonic is inertial.

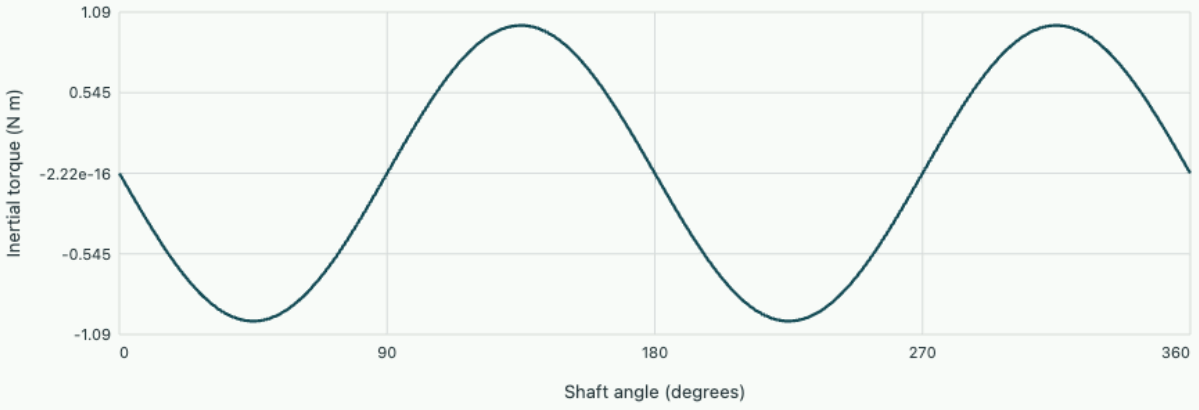
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Carriage mass (kg)	2
Motion amplitude (mm)	50
Constant shaft speed (rad/s)	20



Constant-speed carriage torque

Torque-harmonic amplitude 1.000 N m. Energy is exchanged despite zero shaft acceleration.

Follow the motion through a mechanism

A motor accelerating a load feels inertia from components that turn at different speeds. Simply adding their physical moments of inertia would treat all of those speeds as equal. Instead, choose one independent coordinate and express every kinetic energy in terms of it. No-slip belts, rigid gears and inextensible cables give useful constraints when their compliance and backlash are negligible over the motion of interest.

Consider a drum of radius R winding a cable carrying a vertically guided mass m. Choose the drum angle θ positive when the mass rises. If the effective winding radius is constant, displacement, velocity and acceleration satisfy $x = R\theta$, $\dot{x} = R\dot{\theta}$ and $\ddot{x} = R\ddot{\theta}$. These equalities are kinematic. A torque relation requires force balance or work in addition. When the cable builds successive layers on a spool, the effective radius varies with angle and the constant-ratio assumption must be revisited.

Background · What changes when the reflected inertia depends on position?

For a piston linkage let the piston position be $x(q)$, where q is shaft angle. Its kinetic energy contributes $m(dx/dq)^2$ to the shaft inertia. As the linkage moves through a stroke this contribution varies, even for a piston of constant mass. Use the energy derivative to find the generalised inertial torque.

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = J(q)\ddot{q} + \tfrac{1}{2}J'(q)\dot{q}^2,$$

$$T = \tfrac{1}{2}J(q)\dot{q}^2.$$

Uniform shaft speed sets the first term to zero. The second term represents kinetic energy flowing into and out of the accelerating piston. Its shaft work over a complete cycle vanishes when the mechanism returns to the same angle and speed. An average torque meter can consequently miss a substantial alternating torque.

Reflect mass and forces to a common shaft

Let J_d be the drum inertia. Add the rotational energy of the drum to the translational energy of the load, then express both in terms of the drum speed.

$$\begin{aligned} K &= \tfrac{1}{2}J_d\dot{\theta}^2 + \tfrac{1}{2}m\dot{x}^2 \\ &= \tfrac{1}{2}\left(J_d + mR^2\right)\dot{\theta}^2, \\ J_{\mathrm{eq}} &= J_d + mR^2. \end{aligned}$$

A small positive angle raises the mass through $R\,d\theta$, increasing its potential energy by $mgR\,d\theta$. Thus gravity contributes a resisting torque mgR. A positive applied torque T_d supplies power $T_d\dot{\theta}$. Applying the energy balance gives the equation of motion.

$$\frac{d}{dt}\left(\tfrac{1}{2}J_{\mathrm{eq}}\dot{\theta}^2 + mgR\theta\right) = T_d\dot{\theta},$$
$$J_{\mathrm{eq}}\ddot{\theta} = T_d - mgR.$$

The second expression also holds at zero speed by force balance. It separates steady effort, needed to hold or raise the load against gravity, from the extra effort needed to accelerate the two inertias. With the motor disconnected, an ideal light cable does not make the mass accelerate quite at g, since the drum must gain rotational kinetic energy too.

Calculate a motor torque

A drum with $R = 0.10$ m and $J_d = 0.15\,\mathrm{kg\,m^2}$ lifts 25 kg. A reduction gear gives

$$\dot{\theta}_m = 4\dot{\theta}.$$

The rotor has inertia $J_m = 0.020\,\mathrm{kg\,m^2}$. Refer the train to the motor coordinate. The drum-plus-load inertia becomes $(0.15 + 25(0.10)^2)/4^2 = 0.025\,\mathrm{kg\,m^2}$, so the total at the motor is 0.045 kg·m². Gravity contributes $25(9.81)(0.10)/4 = 6.13$ N·m at that shaft.

Demand upward acceleration 0.80 m/s². The drum must accelerate at 8.0 rad/s² and the motor at 32 rad/s². The motor supplies

$$6.13 + 0.045(32) = 7.57\,\mathrm{N\,m}.$$

A separate check at load speed 1.0 m/s gives motor speed 40 rad/s. Its 303 W then divide into about 245 W raising the weight and 57.6 W increasing kinetic energy. Ideal gearing changes torque and angular speed inversely while preserving transmitted power.

Identify which approximation controls the answer

At constant lifting speed the inertial torque vanishes. At constant angular acceleration, doubling the transmission ratio decreases reflected load inertia by four but also doubles the motor acceleration for the same load motion. The rotor’s own inertia remains at the motor and eventually dominates. A gear ratio therefore cannot reduce every part of acceleration torque indefinitely. Bearings, cables and gears also dissipate energy or store it elastically. Include their effects when a measured transient shows a repeatable lag, oscillation or loss that an ideal rigid constraint cannot explain.

Is the largest reduction always preferable?

For a prescribed drum acceleration α_d and reduction $n = \omega_m/\omega_d$, collect gravity and load-side acceleration into the drum demand $T_d = mgR + (J_d + mR^2)\alpha_d$. The motor torque becomes

$$T_m = T_d/n + J_m n \alpha_d.$$

The first term falls as reduction rises, while rotor acceleration grows. Differentiating with respect to n gives an ideal minimum at $n = \sqrt{T_d/(J_m \alpha_d)}$. For the stated lift $T_d = 27.73$ N m and $\alpha_d = 8$ rad/s², giving $n = 13.2$ approximately. This torque minimum assumes the motor can achieve the associated speed. At a drum speed of 10 rad/s it would turn at 132 rad/s, over three times its speed with the original reduction. Shaft-speed and voltage limits can therefore select a different practical ratio.

Derive speed ripple from a torque diagram

A crank-driven machine can require very different torque at different angles even when the average load is constant. A flywheel stores kinetic energy when driving torque exceeds resisting torque, then releases it when the load peaks. Refer all inertias and torques to the same shaft and suppose the equivalent inertia J is constant. The signed difference $M(\theta) = T_{\mathrm{drive}} - T_{\mathrm{load}}$ gives

$$J\dot{\omega} = M.$$

For continuous positive rotation write $\dot{\omega} = \omega\, d\omega/d\theta$. Integration between two angular positions gives an exact work–energy relation.

$$\tfrac{1}{2} J[\omega_b^2 - \omega_a^2] = \int_{\theta_a}^{\theta_b} M(\theta)\, d\theta.$$

A periodic steady sequence requires zero net area under the torque-difference curve over a whole revolution. Its cumulative area nevertheless has a maximum and a minimum. Their difference ΔE is the energy that must swing into and out of the rotating inertia. Define mean speed in the ripple specification as the arithmetic mean of the two extreme speeds $\bar{\omega} = (\omega_{\max} + \omega_{\min})/2$ and fluctuation coefficient $C_s = (\omega_{\max} - \omega_{\min})/\bar{\omega}$. Factoring the difference of squared speeds gives

$$J = \Delta E/(\bar{\omega}^2 C_s).$$

A time-average tachometer reading approximates this mean for small ripple, but the definitions are distinct for a strongly pulsating shaft.

Size a flywheel for a repeating work demand

A laboratory indexing machine runs at a nominal 300 rpm. A motor supplies approximately constant 30 N m. The resisting torque is 70 N m for the first quarter revolution and a lower constant value for the remaining three quarters. Equate total input and output work over each revolution: $30(2\pi) = 70(\pi/2) + T_l(3\pi/2)$, giving $T_l = 16.67\,\mathrm{N\,m}$. The first quarter produces a 40 N m deficit, so the cumulative energy falls by $40(\pi/2) = 62.83\,\mathrm{J}$. The remaining 13.33 N m surplus restores exactly that energy.

For a maximum peak-to-peak speed variation of 2% and $\bar{\omega} = 31.42\,\mathrm{rad/s}$, total inertia required is

$$62.83/[31.42^2(0.02)] = 3.18\,\mathrm{kg\,m^2}.$$

If the machine already has 1.00 kg m² on this shaft, an extra 2.18 kg m² is required. A thin-ring estimate at mean radius 0.30 m gives mass $2.18/0.30^2 = 24.2\,\mathrm{kg}$. A solid disc of the same radius needs twice the mass because its material lies closer to the shaft. At half speed the same torque-versus-angle work deficit requires four times the total inertia for the same fractional ripple. Startup torque or bearing load may therefore limit an inertia-based solution even when steady motor power suffices.

The torque balance also predicts where a shaft encoder should record the two speed extremes. The energy derivative with respect to angle equals the torque difference, so the maximum occurs just before the high resisting plateau begins and the minimum where that plateau ends. The maximum torque itself persists over a finite angular interval, while minimum speed occurs only at its end. This distinction matters when timing a torque pulse from a speed record: the sign of the speed derivative, rather than the absolute value of speed, identifies when the flywheel is discharging.

Why a radial slide presses sideways on a rotating guide

Machine dynamics

A low-friction carriage moves along a radial slot in a horizontal turntable. It can move at constant speed relative to the guide and still require a changing force. Keeping track of the moving basis exposes both acceleration components.

Differentiate the direction vectors

Let $r(t)$ be displacement from the turntable centre along the guide and $\theta(t)$ the angle of the guide. In fixed Cartesian components the radial unit vector is $(\cos\theta, \sin\theta)$ and the tangential vector is $(-\sin\theta, \cos\theta)$. Differentiating shows $\dot{\mathbf{e}}_r = \dot\theta \mathbf{e}_\theta$ and $\dot{\mathbf{e}}_\theta = -\dot\theta \mathbf{e}_r$. Apply the product rule to the position $r\mathbf{e}_r$ to obtain the velocity and then differentiate both terms.

$$\mathbf{v} = \dot r \mathbf{e}_r + r\dot\theta \mathbf{e}_\theta,$$

$$\mathbf{a} = (\ddot r - r\dot\theta^2)\mathbf{e}_r + (r\ddot\theta + 2\dot r\dot\theta)\mathbf{e}_\theta.$$

One contribution to $2\dot r\dot\theta$ comes from differentiating the radially directed velocity as its basis turns. The other comes from differentiating the magnitude of the tangential velocity $r\dot\theta$. For positive rotation and outward travel, both point in the positive tangential direction. The expression already gives the acceleration in an inertial frame. Multiplying by mass gives the net real force required on the carriage.

Check sideways force with angular momentum and energy

Hold the angular speed at Ω, and drive the radial position as $r = r_0 + ut$ with constant u. Gravity and support cancel vertically. The net radial drive force is $-mr\Omega^2$ and the guide must exert a side force $2mu\Omega$. The resulting torque about the axis is $2mru\Omega$. This equals the derivative of the angular momentum $mr^2\Omega$, providing an independent sign and factor-of-two check.

$$P_r + P_\theta = (-mr\Omega^2)u + (2mu\Omega)(r\Omega) = mru\Omega^2 = \frac{d}{dt}\left[\frac{m}{2}(u^2 + r^2\Omega^2)\right].$$

The side force does work in the laboratory since its contact point moves tangentially. Describing a smooth slot as doing zero work without specifying the frame would lose this energy transfer. The radial actuator absorbs power during outward constant-speed travel.

Forces and power on two passes

A 0.40 kg carriage moves from $r = 0.10$ m outwards at 0.20 m/s, with $\Omega = 5$ rad/s. At $t = 0.50$ s it reaches 0.20 m. Its radial force is −2.0 N, tangential force is +0.80 N and net mechanical power is

$$-2(0.20) + 0.80(1) = 0.40\,\mathrm{W}.$$

Setting u to zero removes the side force, while the guide still requires a radial restraint to maintain circular travel.

On an inward pass at −0.20 m/s, evaluated at the same radius, the radial force remains −2.0 N. The side force reverses and the turntable now receives angular momentum from the carriage. The magnitude of each force depends on a different combination of radius, radial speed and angular speed. Use the trace to distinguish the constant side load from the changing inward force.

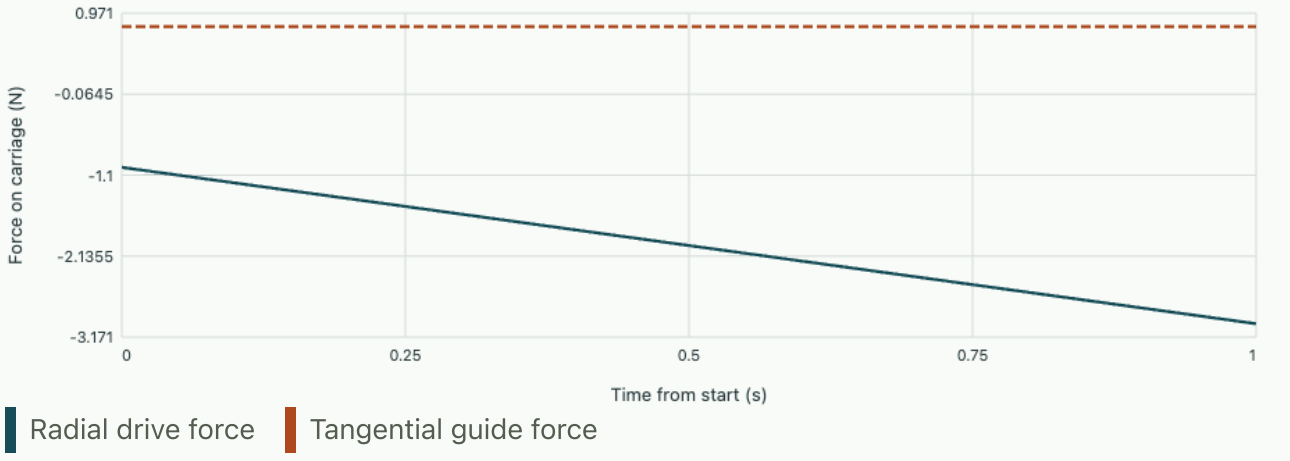
Set radial speed to zero before changing angular speed. Restore outward motion and predict how doubling angular speed changes the two force components at the same radius. Use mechanical power as a separate sign check.

Open this experiment online

Try the calculation

Angular speed (rad/s) **5**

Outward speed (m/s) **0.2**



At 0.50 s the radial and tangential forces are -2.00 N and 0.80 N. The net mechanical power is 0.40 W.

Tutorial · reconstruct angular acceleration from a side load

At one instant the carriage is at 0.20 m, with mass 0.40 kg, $\dot{r} = 0.10$ m/s and $\dot{\theta} = 4$ rad/s. A tangential force transducer measures +0.48 N. Estimate $\ddot{\theta}$ and explain what a rigidly fixed carriage would read at the same instant.

▼ **Show the reconstruction**

The tangential acceleration is

$$0.48/0.40 = 1.20\,\mathrm{m/s^2}.$$

Subtract $2(0.10)(4) = 0.80$ m/s² to isolate $r\ddot{\theta} = 0.40$ m/s². Thus $\ddot{\theta} = 2.0$ rad/s². With no radial sliding the second term disappears and the side load falls to $0.40(0.20)(2.0) = 0.16$ N. The radial centripetal component points along the guide and does not enter this reading.

Further reading · MIT 2.003SC · velocity, acceleration and rotational motion

When a suspended capacitor loses its stable equilibrium

Electromechanics, optics and measurement

An electrode suspended on a spring can remain motionless at a bias that makes the calculated electrostatic force grow as it moves. Static force balance alone misses the voltage at which that equilibrium disappears. Deriving the small-signal frequency gives an independent measurement of the same instability.

Start with the electrical and mechanical control variables

A movable conducting plate of area A faces a fixed electrode. Its unstressed gap is g. Define positive displacement x toward the fixed plate, with spring force $-kx$, mass m and viscous damping coefficient b. For $0 \le x < g$ assume ideal parallel plates, negligible tilt and fringing, constant ϵ, and a linear elastic spring. The capacitance is $C(x) = \epsilon A/(g - x)$. The model ends when the plates approach contact or when the suspending structure departs from its specified elastic law. It cannot by itself calculate adhesion, contact resistance or post-contact motion.

Before differentiating electrical energy specify whether the source holds voltage or charge. At fixed charge Q, electrical energy is $Q^2/[2C(x)]$, and its negative displacement derivative gives the force toward increasing x. At fixed voltage V, a source supplies $V dQ$ as the plates move. The appropriate potential for mechanical equilibrium is the spring energy plus the field energy minus QV, which here is $\mathcal{U}_V = kx^2/2 - C(x)V^2/2$. Differentiating gives the outward spring reaction and the attractive electrical force $F_e = V^2 C'(x)/2 = \epsilon A V^2/[2(g - x)^2]$. An independent derivation multiplies the conductor pressure $\epsilon E^2/2$ by area with $E = V/(g - x)$. Both give a positive force. Replacing the source potential with the stored energy at fixed voltage would reverse its sign.

Locate all equilibria before choosing a branch

Define $s = x/g$ and $\Lambda = \epsilon A V^2/(2kg^3)$. The static force equation reduces to

$$\Lambda = s(1 - s)^2.$$

At a specified voltage one can find s by scanning the curve $s \mapsto s(1 - s)^2$ over the physically admissible interval $0 \le s < 1$. Its derivative is $(1 - s)(1 - 3s)$, giving an interior maximum $4/27$ at $s = 1/3$. For $0 < \Lambda < 4/27$ there are two physical roots. The smaller is a local minimum of $\mathcal{U}_V$ while the larger is a local maximum. Their merger is a fold of the static equilibrium branch:

$$x_{\mathrm{pi}} = g/3,$$

$$V_{\mathrm{pi}} = \sqrt{\frac{8kg^3}{27\epsilon A}}.$$

The often-quoted one-third displacement is the result for a linear spring, ideal voltage drive and this capacitance geometry. It is not a universal travel limit determined by the gap alone. Additional mechanical restoring forces or a different electrical boundary condition change the fold. Neither positive net electrical force nor the mere existence of a real solution is evidence of stable operation. A

root finder started near the almost closed gap can converge to the unstable large-displacement root with an extremely small static residual.

The two branches can be distinguished experimentally without deliberately reaching contact. At static equilibrium $kx = F_e$. Linearise the total force about x_0 while maintaining voltage at its prescribed value. The incremental restoring stiffness and corresponding weak-damping natural frequency are

$$k_{\mathrm{eff}} = k - \frac{\epsilon A V^2}{(g - x_0)^3} = k\,\frac{1 - 3s_0}{1 - s_0},$$

$$f_{\mathrm{small}} = \frac{1}{2\pi}\sqrt{\frac{k_{\mathrm{eff}}}{m}}.$$

A small push toward the fixed electrode is restored only when $k_{\mathrm{eff}} > 0$. The familiar electrostatic softening is the reason the smaller equilibrium is stable. The small-signal frequency tends to zero as the fold is approached. With appreciable damping the observed amplitude resonance and ringdown oscillation have slightly different frequencies; fit the full mechanical response rather than identifying an arbitrary peak with the undamped f_{small}.

Background · Can a sudden step pull in a statically stable actuator?

An ideal voltage step applied to a plate initially at rest can push it past the unstable equilibrium. During the motion at fixed voltage the effective potential per kg^2 is $u(s) = s^2/2 - b/[2(1 - s)]$, where $s = x/g$ and $b = \epsilon A V^2/(kg^3)$. Neglect damping and begin at $s = 0$. At the dynamical threshold the initial energy equals the saddle energy and the slope vanishes at that saddle.

$$u(s) = u(0) \implies b = s(1 - s),$$

$$u'(s) = 0 \implies b = 2s(1 - s)^2.$$

The two equations meet at $s = 1/2$, $b = 1/4$. The quasistatic fold gives $b = 8/27$. Consequently the undamped step threshold is only $\sqrt{27/32} \simeq 0.919$ times the slow-ramp pull-in voltage. A device operated between the two voltages can survive a ramp but collapse after an abrupt step. Dissipation and the actual rise time change this prediction.

Reconstruct the prepared state from a frequency measurement

Take a 100 µm square electrode in vacuum, $g = 2.00$ µm, $k = 2.00$ N/m and $m = 5.00 \times 10^{-10}$ kg. The predicted pull-in voltage is 7.32 V, the zero-bias capacitance 44.3 fF and the zero-bias natural frequency 10.07 kHz. On the low branch $s_0 = 0.20$ gives $x_0 = 0.400$ µm and voltage 6.80 V. Then $k_{\mathrm{eff}} = 1.00$ N/m and the small-signal frequency is 7.12 kHz. A measured frequency ratio f/f_0 can be inverted via $s = (1 - r)/(3 - r)$, where $r = (f/f_0)^2$. The result is independent of absolute optical displacement calibration if the mass and underlying spring stiffness have not changed during the bias sweep.

At the same voltage the other physical root is $s = 0.48769$. A numerical force balance at this root would give the same supply reading but a negative tangent stiffness. The ideal voltage-controlled potential in units kg^2 is $u(s) = s^2/2 - \Lambda/(1 - s)$. The barrier height $kg^2[u(0.48769) - u(0.20)]$ for this example is 7.26×10^{-14} J. A sufficiently energetic disturbance can carry the plate out of the stable basin even below static pull-in. This is different from loss of the static equilibrium itself. A quasistatic model predicts the existence and local stiffness of the equilibrium; a transient simulation must additionally specify initial displacement, velocity, drive ramp and damping.

Change the preparation to fixed charge

Disconnect a perfect isolated capacitor after placing a known charge Q on it, neglecting leakage and parasitic capacitance. Then $F_Q = Q^2/(2\epsilon A)$ is independent of the remaining gap, because $D = Q/A$ is fixed. Static equilibrium is $x_0 = Q^2/(2\epsilon A k)$, with incremental stiffness exactly k. At the same initial voltage and gap the instantaneous force agrees with the voltage-controlled value, since $Q = C(x_0)V$. The derivatives governing subsequent motion differ. At $x_0 = 0.400$ µm the required charge in the numerical example is 0.376 pC. After ideal electrical isolation the small-signal frequency returns to the underlying elastic value even though the loaded static position stays the same.

Ideal charge isolation is delicate in a small device. A fixed parasitic capacitance C_p from the isolated node to a reference means the conserved charge belongs to $C(x) + C_p$. Motion changes the fraction of total charge residing on the actuator. The electrical energy becomes $Q_{\text{total}}^2/[2(C + C_p)]$, so the force is $Q_{\text{total}}^2 C'/[2(C + C_p)^2]$, generally displacement dependent. If $C_p \gg C$, the node voltage changes little with a small displacement and behavior is closer to voltage control. Interpreting an observed softening after disconnecting the supply as evidence against electromagnetic force balance would overlook this boundary change.

A different practical preparation puts an intentional series capacitor C_s between a fixed voltage supply and the actuator. Its series combination is $C_{\text{ser}} = \epsilon A/[g - x + \epsilon A/C_s]$. For static mechanical work at fixed supply voltage use $F = V_s^2 C'_{\text{ser}}/2$. Defining an effective electrical spacing $g_e = g + \epsilon A/C_s$, the static fold occurs at $x = g_e/3$ if this lies within the open physical gap. If $\epsilon A/C_s \geq 2g$, contact would occur before any such static fold in the ideal model. This calculation shows a direct connection between passive circuit design and allowed mechanical travel, while making clear that the maximum actuator voltage can differ from the voltage printed on the supply.

Why a sudden voltage change can reach contact below static pull-in

For a rapid electrical step the plate has a position and velocity inherited from the previous preparation. Suppose it starts at rest at its zero-bias position, the supply steps to a constant V much faster than a mechanical period, and mechanical damping is initially negligible. Immediately after the step, energy in the mechanical potential $u(s) = s^2/2 - \Lambda/(1 - s)$ is $u(0) = -\Lambda$ in units of kg^2. Motion under this constant voltage can reach the force barrier if the initial value exceeds the potential at the unstable

stationary point. This question can be solved by comparing two energies without integrating an entire time history.

Let s_b be the unstable root of $\Lambda = s_b(1 - s_b)^2$. At threshold the initial state is on the separatrix, the trajectory separating oscillations in the well from excursions to the closing gap. Set $u(0) = u(s_b)$, subtract the two expressions and cancel s_b from the nontrivial solution. The two conditions become

$$\Lambda = \frac{s_b(1 - s_b)}{2},$$

$$\Lambda = s_b(1 - s_b)^2.$$

Thus $s_b = 1/2$, $\Lambda = 1/8$, and the ideal undamped step threshold is

$$V_{\mathrm{step}}/V_{\mathrm{pi}} = \sqrt{27/32} = 0.919.$$

In our example this is 6.72 V. A slow increase to 6.80 V leads to the observed stable 0.40 µm deflection, whereas a sudden ideal voltage step to the same value from rest at the origin contains enough energy to escape the well. Starting on the biased equilibrium before applying a small perturbation is a different experiment. The direction of the threshold shift makes physical sense: damping removes mechanical energy during the first excursion, and a slow ramp allows the equilibrium to move with the plate. The undamped value specifies a particular ideal initial-value experiment, rather than a voltage rating for an unspecified drive waveform.

One can check a simulated step by monitoring $m\dot{x}^2/2 + \mathcal{U}_V$ after the supply settles. With viscous damping its derivative is $-b\dot{x}^2$, so any sustained increase during a constant-voltage segment points to a numerical or modelling error. Before the step ends there is an explicit drive term $-CV\dot{V}$ in this energy balance. The mechanical potential already includes the source work. Adding $V\dot{Q}$ to the same balance a second time gives an artificial energy imbalance. A transient solver should stop at a physically specified minimum gap instead of marching across the singularity; a statement about contact time depends on what counts as contact.

A finite source resistance changes what a ringdown measures

A dc supply with series resistance R can maintain the static voltage accurately while failing to hold the actuator voltage fixed over a mechanical cycle. Let its steady node voltage be V, with equilibrium capacitance $C = C(x_0)$. A small sinusoidal displacement δx causes current $i\omega(C\,\delta v + VC'\delta x)$ into the actuator. The small-signal supply voltage is zero, so $\delta v = -Ri$. Eliminating current gives

$$\delta v = -\frac{VC'}{C}\,\frac{i\omega\tau}{1 + i\omega\tau}\delta x,$$

$$\tau = RC.$$

Linearise $F_e = V^2C'/2$ allowing both voltage and position to vary. The direct derivative is $K_{e0} = V^2C''/2$, while the change from node voltage is $VC'\,\delta v$. For an ideal parallel plate

$C''/2 = (C')^2/C$, so the two terms combine to the simple complex relation $\delta F_e = K_{e0}\delta x/(1 + i\omega\tau)$. At low frequency the electrical force follows displacement with its fixed-voltage slope. At high frequency the slope vanishes, as it must for an isolated ideal plate carrying fixed charge over one oscillation. At intermediate frequency a motion-induced current dissipates heat in the resistor.

Bring δF_e to the left side of the mechanical equation. Its real part subtracts $K_{e0}/[1 + (\omega\tau)^2]$ from the mechanical stiffness; its imaginary part adds $K_{e0}\tau/[1 + (\omega\tau)^2]$ to the viscous damping coefficient. This damping is positive for a passive resistor. At $s = 0.20$ the static electrical stiffness is 1.00 N/m, the operating capacitance is 55.3 fF and $R = 400$ MΩ gives

$$\tau = 22.1\,\mu\text{s}.$$

A 7.2 kHz interrogation has $\omega\tau \simeq 1$. Only about half the static electrical slope survives and the added damping is roughly $11\ \mu\text{N s/m}$. Treat these as frequency-dependent coefficients in the driven susceptibility. Substituting the low-frequency stiffness into a ringdown fit at this frequency would give the wrong displacement when the fitted frequency is inverted. Fitting the complex response at several frequencies avoids that ambiguity.

An experiment can separate the circuit effect from mechanical nonlinearity. Keep the same static bias and displacement, insert two known series resistances, then measure both in-phase and quadrature response to a small sinusoidal force. The displaced equilibrium should remain essentially unchanged at dc if leakage stays small. A stiffness dispersion and dissipative peak that shift in proportion to $1/R$ support an electrical cause. Then repeat at a smaller oscillation amplitude. Amplitude-dependent frequency shift that survives the change in resistance can arise from nonlinear spring or force curvature, which the first-order susceptibility leaves out. A simple change to the external circuit has turned the slope of an energy curve into a prediction for a measured phase lag.

Compare force intersections for voltage and charge control

The initial gap is 2.00 µm, the area is a 100 µm square and the mass is 0.50 µg. Use the voltage ratio to move the force curve through the fold. At 93% of pull-in the low-displacement fixed-voltage intersection has a positive restoring slope. The high-displacement intersection has a negative slope. The charge-controlled line is prepared at the selected voltage at the original open gap. It gives the same initial force but a different subsequent curve. The readout reports both open-gap roots even if the upper root lies beyond the plotted interval. At the baseline stiffness, static pull-in is 7.32 V.

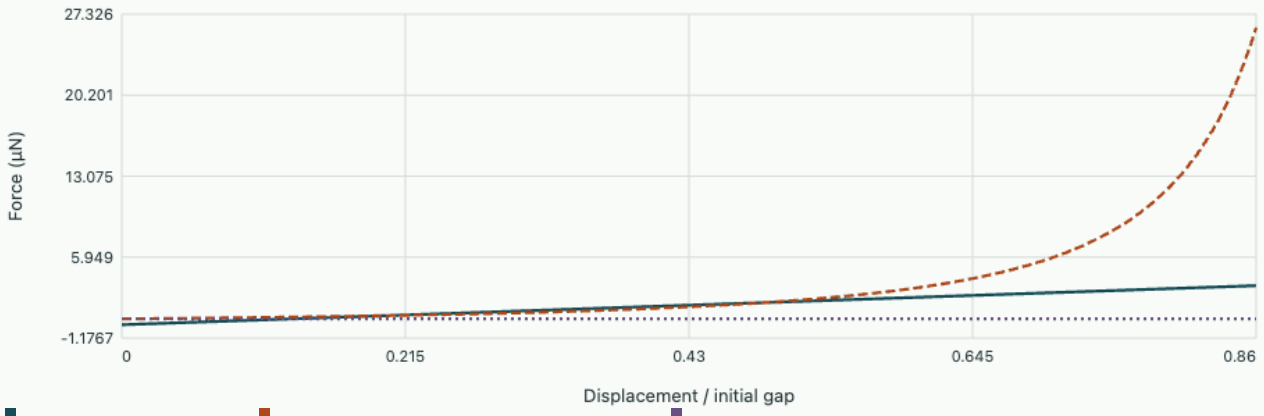
Change the spring stiffness while holding the voltage ratio fixed. The dimensionless locations of the roots stay in place, but their dimensional force and small-signal frequency change. This checks the scaling that eliminated the physical dimensions in the fold equation. Then set the ratio between the dynamic step threshold and unity. Use the force intersections to locate the potential maximum and sketch the area between attraction and spring reaction from the initial position. Its signed integral supplies the work which distinguishes a bounded excursion from crossing the maximum. The displayed curves predict initial-value behaviour only when combined with a choice of initial kinetic energy and damping.

Open this experiment online

Try the calculation

Bias as fraction of static pull-in **0.93**

Spring stiffness (N/m) **2**



Bias 6.805 V; static pull-in 7.317 V. Stable x/g = 0.200, unstable x/g = 0.487, small-signal frequency 7.11 kHz.

Tutorial · distinguish stability from force balance

Infer a gap fraction and a bias from a resonance

An otherwise identical actuator shows $f/f_0 = 0.80$ at a stable fixed voltage. Find the deflection as a fraction of gap, the voltage relative to pull-in, and the independent capacitance change. What could spoil this comparison?

▼ **Work through the solution**

Here $r = 0.64$ and $s = (1 - 0.64)/(3 - 0.64) = 0.1525$. The bias ratio is $\sqrt{(27/4)s(1-s)^2} = 0.860$ approximately. The parallel-plate capacitance should rise by $1/(1-s) - 1 = 18.0\%$. Measure the change relative to the active capacitance after removing any constant wiring capacitance. Disagreement can come from changing mode shape, a nonlinear spring, a tilted plate, a measurement frequency above the electrical equilibration bandwidth, or an incorrect extraction of mechanical stiffness from a strongly damped resonance. With a measured gap and mode shape these explanations yield different quantitative corrections.

Choose a series capacitor to move the static fold

The original 44.3 fF zero-bias actuator should remain on its stable slope as far as $x = 0.60g$, ignoring contact and other limits. Find the series capacitance whose fold falls exactly at this displacement. How

much supply voltage appears across the actuator at that point?

▼ **Work through the solution**

Set $g_e/3 = 0.60g$, giving $\epsilon A/C_s = 0.80g$ and $C_s = C(0)/0.80 = 55.4$ fF. At $0.60g$ the actuator capacitance is $C(0)/0.40 = 110.7$ fF. Series charge is identical in both capacitors, so

$$V_{\text{act}}/V_{\text{supply}} = C_s/(C_s + C_{\text{act}}) = 1/3.$$

Choosing exactly this value places operation at vanishing ideal stiffness, so a strictly positive measured restoring slope at that displacement requires a smaller series capacitance or a different margin. The result is sensitive to additional capacitance connected between the series node and ground.

Further reading

- Derive capacitor pressure from field stress and work

Designing a displacement bridge at its noise limit

Electromechanics, optics and measurement

A moving electrode changes two capacitances in opposite directions. Derive the displacement signal, reconstruct its noise in metres and select a carrier voltage consistent with measured mechanical stiffness and electrical losses.

Find the voltage of the moving electrode before calculating sensitivity

Consider three plane electrodes with common active area. At the centred position the middle conductor is separated by g from each outer electrode and each gap has capacitance C_0. Displacement x toward the positive drive electrode reduces one gap and increases the other. Use $s = x/g$, where $|s| < 1$. Drive the outer electrodes with coherent sinusoidal voltages $+V_d$ and $-V_d$, with V_d the rms amplitude of either electrode relative to ground. The middle conductor leads to a high-impedance amplifier. Its cable and input provide a fixed total capacitance C_p to ground. Leakage is initially negligible over one carrier period; the input draws negligible signal current. All formulas for carrier voltages in this project use rms amplitudes.

$$C_+ = \frac{C_0}{1 - s},$$

$$C_- = \frac{C_0}{1 + s}.$$

The middle node has zero net charge at the carrier frequency. Its instantaneous ac charge is $C_+(V_n - V_d) + C_-(V_n + V_d) + C_p V_n$. Set the sum to zero and solve the resulting linear equation for V_n. After combining the gap capacitances the result simplifies considerably:

$$\frac{V_n}{V_d} = \frac{C_+ - C_-}{C_+ + C_- + C_p} = \frac{2C_0 s}{2C_0 + C_p(1 - s^2)}.$$

For zero parasitic capacitance the response is exactly $V_d x/g$ throughout the ideal geometry. Each individual capacitance is strongly curved near an electrode. Charge conservation divides one nonlinear combination by another and removes that curvature from their voltage ratio. Additional capacitance reduces the small-signal gain and introduces an odd cubic term. Let $\eta = 2C_0/(2C_0 + C_p)$. Expanding about the centre gives

$$V_n/V_d = \eta s[1 + (1 - \eta)s^2 + \cdots].$$

The derivative at zero is $K = \eta V_d/g$ in volts per metre. A small-signal calibration taken only at the origin implicitly accepts this cubic distortion at larger displacements.

For $C_0 = 2.00$ pF, $C_p = 4.00$ pF, $g = 100\,\mu\text{m}$ and $V_d = 1.00$ V, $K = 5000$ V/m. A displacement of 100 nm produces 0.5000 mV. At 20 µm, the exact signal is 0.10204 V against the linear prediction 0.10000 V. The 2.04% curvature would be apparent in a calibration over the wider range. A single proportional calibration cannot remove it. A fit to the ratio itself has a physical parameter, $C_p/(2C_0)$,

which can also be checked by measuring total input capacitance with the mechanical element held at centre.

Translate each readout fluctuation into equivalent displacement

Represent the amplifier near the carrier by an input series voltage noise density e_n and a shunt current noise density i_n. With the drives treated as small-signal grounds for a noise perturbation, the central node presents $C_T = 2C_0 + C_p$ at the centre. The impedance magnitude for current fluctuations is $1/(\omega C_T)$. For mutually independent noise sources and a narrow demodulation band the corresponding input-referred displacement amplitude spectral density is

$$S_{x,e}^{1/2} = \frac{1}{K}\sqrt{e_n^2 + \left(\frac{i_n}{\omega C_T}\right)^2}.$$

Use noise densities and gain referred to the same detection channel. An rms sine amplitude, a peak-amplitude display and a lock-in reading with an unreported normalisation can differ by factors involving $\sqrt{2}$. Here noise figures denote the effective one-sided densities referred to the recovered rms carrier channel. For a white spectrum and an equivalent noise bandwidth B_e the displacement standard deviation equals the density times $\sqrt{B_e}$. The displayed cut-off frequency of a multi-pole lock-in filter need not equal its equivalent noise bandwidth. A calibration using a known injected sideband measures both the gain convention and the actual band response.

At 100 kHz, $e_n = 10\,\mathrm{nV}/\sqrt{\mathrm{Hz}}$, $i_n = 10\,\mathrm{fA}/\sqrt{\mathrm{Hz}}$ and $C_T = 8.00$ pF, current noise produces $1.99\,\mathrm{nV}/\sqrt{\mathrm{Hz}}$. Adding variances gives $10.20\,\mathrm{nV}/\sqrt{\mathrm{Hz}}$, or $2.04\,\mathrm{pm}/\sqrt{\mathrm{Hz}}$ at the stated gain. A 100 Hz equivalent noise bandwidth gives 20.4 pm rms of electrical readout noise. Cable capacitance reduces the voltage made by current noise but also reduces signal gain. In fact $C_T K = 2C_0 V_d/g$, so their ratio for current noise is independent of C_p in this model. The voltage-noise contribution grows with cable capacitance. Higher carrier frequency initially reduces the current contribution but reaches a plateau set by voltage noise.

Derive backaction for a floating sensing plate

Electrical drive can change the mechanical stiffness of the sensor. For each gap take half the squared rms gap voltage times its capacitance derivative. This follows either by averaging the instantaneous plate pressure over a carrier cycle or by differentiating electrical coenergy with the middle node at its charge-balanced voltage. Mechanical motion must be slow compared with the carrier. Since C'_- is negative, the two signed forces cancel at the centre. For a small positive s, write $V_n \simeq \eta V_d s$,

$$C'_+ \simeq (C_0/g)(1 + 2s),$$

$$C'_- \simeq -(C_0/g)(1 - 2s).$$

The leading imbalance is

$$F = \tfrac{1}{2}C'_{+}(V_d - V_n)^2 + \tfrac{1}{2}C'_{-}(-V_d - V_n)^2 \simeq \frac{2C_0V_d^2}{g^2}(1-\eta)x.$$

The electromagnetic stiffness $K_e = 2C_0V_d^2(1-\eta)/g^2$ subtracts from the mechanical restoring stiffness k. If $C_p = 0$, the equal field magnitudes on both sides cancel exactly for every displacement in ideal plane geometry. To check directly, the positive gap field is

$$[V_d(1-s)]/[g(1-s)] = V_d/g.$$

The negative gap field has the same magnitude. A charge amplifier holding the central electrode at virtual ground would change this condition and have a different backaction; its calibration must use its own circuit. For a finite C_p, the measured change of resonance with carrier voltage can identify electrical coupling independently of the signal voltage.

With $k = 0.020$ N/m the nominal configuration has $K_e = 0.00020\,\mathrm{N/m}$ at 1 V and $0.00080\,\mathrm{N/m}$ at 2 V. At 2 V the resonance ratio for the same effective mass is

$$\sqrt{1 - 0.0008/0.020} = 0.980.$$

A 2% frequency decrease measures roughly a 4% reduction of stiffness. If the displacement is imposed by a very stiff external positioner this effect changes the force needed to hold position. If the sample is itself a force sensor, the same effect changes the mechanical conversion between applied force and displacement. Quoting an electronic noise floor without specifying which mechanical situation is intended can hide the larger uncertainty.

Choose a feasible drive and include Brownian motion

Raising the carrier at fixed frequency increases displacement signal. It may also heat the dielectrics, exceed amplifier headroom or significantly soften the spring. Suppose each gap at centre has an effective loss tangent $\tan\delta$, with a real conductance $\omega C_0 \tan\delta$ at the carrier. Its dissipated power is conductance times the squared rms voltage. Add both gaps to find

$$P = 2\omega C_0 \tan\delta V_d^2.$$

Fixed parasitic capacitance at the stationary middle node has negligible carrier voltage at exact balance in this model. An actual shield or cable dielectric connected directly to one of the drive nodes will have its own loss. A calorimetric or current phase measurement should include that extra load.

With effective $\tan\delta = 0.010$, the two active gaps dissipate 25.13 nW per squared volt at 100 kHz. A 100 nW heat budget gives $V_d \le 1.995$ V. A driver that remains linear to 2.50 V and a softening allowance $K_e \le 0.1k$ permit 2.50 V and 3.16 V respectively. The thermal bound sets the largest practical drive under these assumptions. For the electrical readout alone, the best feasible setting is

thus 1.995 V and the electrical floor is about $1.02\,\mathrm{pm}/\sqrt{\mathrm{Hz}}$. A separate limit on the electric field or measurement amplitude could be introduced as another experimentally justified constraint.

Now let the mechanical element have viscous damping $b = 1.00 \times 10^{-6}\,\mathrm{N\,s/m}$ at $T = 300$ K. In thermal equilibrium the one-sided Brownian force density is

$$\sqrt{4k_B T b} = 1.29 \times 10^{-13}\,\mathrm{N}/\sqrt{\mathrm{Hz}}.$$

Far below mechanical resonance its displacement density is this force divided by $k - K_e$. At 1 V it is $6.50\,\mathrm{pm}/\sqrt{\mathrm{Hz}}$, already greater than the electrical term. At 1.995 V it is about $6.70\,\mathrm{pm}/\sqrt{\mathrm{Hz}}$. The combined density adds in quadrature, giving about 6.81 at 1 V and 6.78 at the heat limit. A minimum of the total can occur between them. For an imposed position of a rigid stage the Brownian motion of this compliant probe may be outside the measurement model; for a freely suspended sensing mass it is part of the signal uncertainty. The optimisation criterion follows the physical quantity the instrument measures.

Use the phase and polarity of a calibration to track false signals

Middle-node leakage to ground with conductance G changes the circuit equation to $[G + i\omega C_T]V_n = i\omega(C_+ - C_-)V_d$ at small displacement. The ideal gain acquires a factor $i\omega C_T/(G + i\omega C_T)$. For a 10 MΩ leakage path and 8 pF, the corner is 1.99 kHz. At 100 kHz the magnitude reduction is about 0.02% and the phase lead about 1.14°. At 2 kHz both quadratures matter. A drift in humidity that lowers the resistance can look like a gain change if only the in-phase channel is recorded. Measure a known displacement at two carrier frequencies and fit both quadratures before interpreting it as a change in geometry.

Suppose the two actual drives are $V_d(1 + a)$ and $-V_d(1 - a)$. At centre the output becomes $\eta V_d a$, equivalent in the small-signal calibration to displacement ag. An amplitude imbalance of one part in ten thousand gives a 10 nm offset at $g = 100\,\mu\mathrm{m}$, many orders above a short-band noise floor expressed in picometres. Reverse the mechanical calibration displacement to isolate odd signal, measure the voltages at the electrode terminals and compare the nulled bridge under exchange of the drive channels. Reversing both carrier phases simply reverses both desired signal and this coherently driven offset; by itself it cannot distinguish them. An offset stable over a short scan can be estimated from two symmetric calibration displacements, whereas a drifting imbalance requires a reference or simultaneous drive measurement.

Find the minimum for a compliant probe instead of guessing at it

It is possible to optimise the free probe analytically up to a scalar root. Write $y = V_d^2$, $K_e = hy$, and a for the readout displacement density at unit drive multiplied by one volt. Let $d_F = \sqrt{4k_B T b}$ be the Brownian force density. At displacement frequencies well below the resonance, the total variance density has an electrical term falling as $1/y$ and a mechanical term that grows as the spring softens:

$$S_x(y) = \frac{a^2}{y} + \frac{d_F^2}{(k - hy)^2},$$

$$\frac{dS_x}{dy} = -\frac{a^2}{y^2} + \frac{2hd_F^2}{(k - hy)^3}.$$

On the stable interval the first derivative strictly increases when $h > 0$. The first term starts at negative infinity and the second diverges at the stiffness boundary. Hence there is one unconstrained minimum. Solve a bracketed scalar equation for y, then compare with each experimental upper bound in volts squared. At zero parasitic capacitance the ideal floating plate has $h = 0$; total noise then falls monotonically toward an asymptotic Brownian floor. A numerical routine that divides by h would fail precisely at this informative physical limit. In either case, a force-noise specification referred to the external sample can produce a different optimum because force conversion multiplies displacement noise by stiffness.

With the baseline values $a = 2.039 \times 10^{-12}\ \mathrm{m\,V/\sqrt{Hz}}$, $h = 2.00 \times 10^{-4}\ \mathrm{N/(m\,V^2)}$, $d_F \simeq 1.287 \times 10^{-13}\ \mathrm{N/\sqrt{Hz}}$. Substitution brackets the stationary point between 1.45 and 1.50 V. At roughly 1.47 V the density is about $6.72\ \mathrm{pm/\sqrt{Hz}}$, a shallow minimum compared with the heat-limited result. A tiny uncertainty in amplifier noise or in the support damping could shift the exact optimum appreciably while having very little effect on achievable resolution. Reporting a numerically optimised voltage to six decimals would imply a knowledge of the physical system absent from these inputs.

A direct bench experiment can test whether the excess really comes from compliance. Compare the demodulated spectrum with the centre plate tightly held and with it free. The electrical floor should be similar if the clamping keeps cable capacitance and carrier coupling fixed. The free probe should acquire a mechanical resonance which moves with V_d^2, and the low-frequency component should track the inferred static stiffness. A large peak locked to carrier-drive amplitude modulation in both experiments suggests coherent feedthrough. Also check that the noise estimate comes from a frequency band above drift yet below resonance, and that the inferred amplitude density times the square root of the bandwidth predicts repeated stationary readings. These practical checks distinguish a readout-limited displacement estimate from a calibrated measurement of actual thermal motion.

Explore the usable voltage range of the bridge

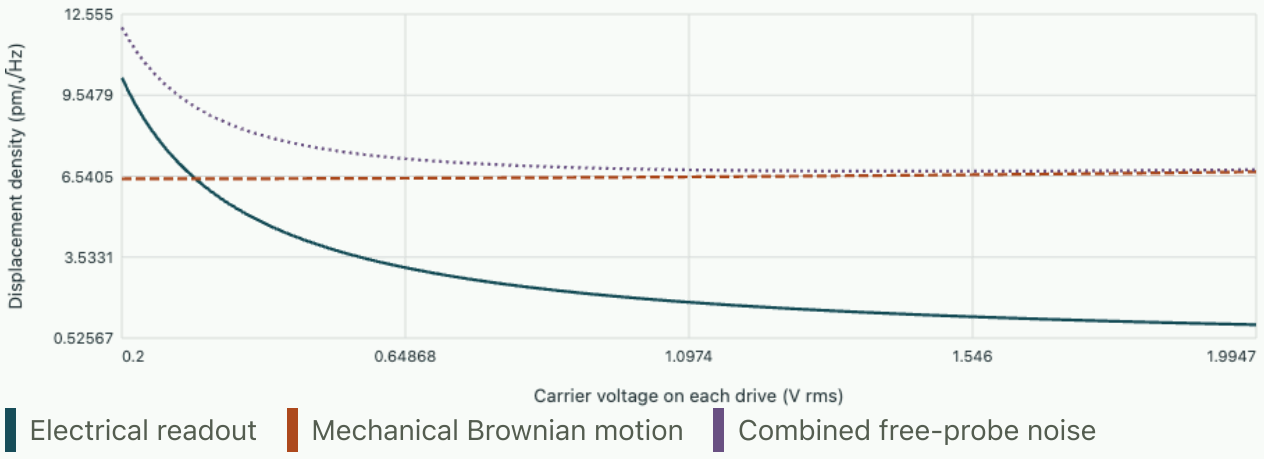
The horizontal axis ends at the first of the heat, driver and mechanical limits. With 4 pF fixed parasitic capacitance at 100 kHz the heat ceiling is 1.995 V rms, while a freely suspended probe reaches its minimum combined noise near 1.47 V rms. Reduce the added capacitance to zero and explain why a finite optimum arising from mechanical softening disappears. Then lower carrier frequency. The current-noise term grows and dielectric heating at a given voltage falls. Observe which effect changes the allowed voltage range and which changes the height of the electrical noise curve.

Open this experiment online

Try the calculation

Input and stray capacitance (pF) **4**

Carrier frequency (kHz) **100**



Maximum allowed drive 1.995 V rms. Minimum combined noise 6.72 pm/√Hz at 1.472 V rms.

Tutorial · separate electronic resolution from mechanical response

Find the frequency at which current and voltage noise contribute equally

Retain the baseline input capacitances and noise sources. Calculate the crossover frequency and the effect of doubling cable capacitance from 4 pF to 8 pF. What happens to input-referred current-noise displacement at fixed frequency?

▼ **Work through the solution**

Set $i_n/(2\pi f C_T) = e_n$. With $C_T = 8$ pF the crossover is 19.9 kHz. The longer cable gives $C_T = 12$ pF and a crossover at 13.3 kHz. At 100 kHz, the current noise at the input falls to $1.33\,\mathrm{nV}/\sqrt{\mathrm{Hz}}$ but gain falls from 5000 to 3333 V/m. The current contribution remains $0.398\,\mathrm{pm}/\sqrt{\mathrm{Hz}}$. The voltage contribution increases from 2.00 to 3.00 pm/√Hz. Both results follow independently by substituting $K = 2C_0V_d/(gC_T)$ into the noise formula and cancelling C_T in only one of the two terms.

Diagnose a frequency shift and a noise claim

At 2.00 V the resonance changes from 500.0 Hz to 490.0 Hz. Determine the implied stiffness reduction. A second experiment quotes 10 pm rms in a 100 Hz noise bandwidth for the freely

suspended probe at 300 K. Can thermal equilibrium and the measured spring and damping account for this number?

▼ **Work through the solution**

At unchanged moving mass $k_{\mathrm{eff}}/k = (490/500)^2 = 0.9604$. The measured electrical stiffness is 0.000792 N/m, consistent with the 0.00080 N/m prediction. At frequencies well below resonance the Brownian density is approximately 6.70 pm/√Hz, and a flat 100 Hz equivalent band would contain approximately 67 pm rms of thermal motion alone. Ten picometres rms would correspond to about 2.2 Hz equivalent noise bandwidth at this density, or require a changed mechanical condition, frequency band or calibration. To discriminate experimentally, measure the mechanical resonance spectrum with a known input-force calibration and integrate the noise with the actual filter transfer. Include mechanical susceptibility over the band if the filter extends close to resonance.

Further reading

- Maxwell stress and electrical coenergy in mechanical force balances
- MIT · Woodson and Melcher, lumped-parameter electromechanics, chapter 3

Eddy-current braking, magnetic penetration and the heat left behind

Electromechanics, optics and measurement

A conducting loop moves above a row of magnetic poles. The drag first increases with speed, but a rapid crossing need not generate proportionally greater braking. Trace the work done against the Lorentz force into circuit energy, heat and a measurable temperature change. Then decide which features of a solid plate the single-loop model can capture.

Generate a voltage from a spatially periodic field

Let a rectangular loop of height h along y and width w along x translate to the right above a stationary magnetic pattern. At the loop let the normal induction be $B_z(x) = B_0 \cos(kx)$, with $k = 2\pi/\lambda$. Its two active sides are centred symmetrically about the coordinate x_c. Assume the field varies negligibly along y over this height. Integrating through the loop gives

$$\Phi = h \int_{x_c - w/2}^{x_c + w/2} B_0 \cos(kx)\, dx = [2hB_0 \sin(kw/2)/k] \cos(kx_c).$$

A uniform translation $x_c = ut$ therefore produces an external emf with peak amplitude $E_{\mathrm{pk}} = Ku$, where $K = 2hB_0 |\sin(kw/2)|$.

This spatial averaging explains an initially surprising design feature. A loop spanning precisely one full spatial period always links zero net flux from the sinusoidal field and therefore generates no emf in this ideal geometry. A loop of half a period has maximum coupling at a specified height. If w is much smaller than the period,

$$\sin(kw/2) \simeq kw/2,$$

$$K \simeq hwB_0 k.$$

This is exactly the small-loop result from using its area times the local gradient of induction. Reversing velocity reverses the emf at a fixed position. At a fixed positive speed the temporal frequency is $f = u/\lambda$, so the same conductor experiences different electrical frequencies at different mechanical speeds.

Include finite inductance and derive the drag curve

Model the current path by a constant resistance R and self-inductance L. Its steady periodic current solves $L\dot{i} + Ri = \mathcal{E}(t)$. In phasor notation its magnitude is $E_{\mathrm{pk}}/\sqrt{R^2 + (\omega L)^2}$, lagging the emf by $\arctan(\omega L/R)$, where $\omega = ku$. The current develops field energy $Li^2/2$ and dissipates Ri^2. Multiplying the circuit equation by i and averaging over a period eliminates the derivative of stored energy. The average mechanical power delivered against the drag is therefore the mean Joule heating $P = E_{\mathrm{pk}}^2 R/[2(R^2 + \omega^2 L^2)]$. Dividing by the speed gives the drag magnitude

$$F(u) = \frac{K^2 u}{2R[1 + (u/u_c)^2]},$$

$$u_c = \frac{R}{kL}.$$

At low speed the drag is proportional to velocity. At the crossover $u = u_c$ it reaches $K^2 u_c/(4R)$, and at much larger speeds it falls approximately as $1/u$ within the assumed lumped model. The incoming mechanical power then tends to a constant rather than growing without bound. Why is there a maximum? Increasing speed raises the emf, but it also raises the reversal frequency. An inductive current increasingly lags the driving emf, and its component in phase with the emf does the net dissipative work. A measured turnover therefore tests both an electrical time scale and the spatial scale of the magnet pattern.

Use $h = 50\,\mathrm{mm}$, $w = 25\,\mathrm{mm}$, $\lambda = 0.100\,\mathrm{m}$, $B_0 = 0.20\,\mathrm{T}$, $R = 0.050\,\Omega$ and $L = 0.150\,\mathrm{mH}$. Then $K = 0.01414\,\mathrm{T\,m}$, $u_c = 5.31\,\mathrm{m/s}$, and maximum mean drag is $5.31\,\mathrm{mN}$. At $u = 2.0\,\mathrm{m/s}$ the field alternates at $20\,\mathrm{Hz}$, giving an emf of $28.3\,\mathrm{mV}$ peak. The current is $0.529\,\mathrm{A}$ peak and its phase lag 20.7°. The resistor receives approximately $7.00\,\mathrm{mW}$, and the motor maintaining translation supplies a mean force of $3.50\,\mathrm{mN}$. Using peak current in I^2R without its factor one half would overstate the mechanical force by a factor two.

Determine whether the current penetrates a real plate

A continuous metal plate contains many possible current paths, unlike a preformed loop. Their resistances, inductances and mutual couplings depend on the plate boundary, magnet geometry and frequency. For a good homogeneous conductor the magnetic diffusion equation gives a skin depth $\delta = \sqrt{2/(\mu\sigma\omega)}$ under sinusoidal excitation. At a depth δ a field entering a locally planar half-space has lost a factor e in amplitude and acquired a phase delay of one radian. The frequency to use is the frequency experienced by the moving material; the fact that the magnets are stationary in the laboratory does not make this frequency zero.

For copper with $\sigma = 5.8 \times 10^7\,\mathrm{S/m}$ and permeability close to μ_0, the skin depth is about $14.8\,\mathrm{mm}$ at $20\,\mathrm{Hz}$ and $1.48\,\mathrm{mm}$ at $2.0\,\mathrm{kHz}$. A $2.0\,\mathrm{mm}$ plate therefore sees little through-thickness attenuation in the first experiment but may need a diffusion calculation in the second. The plate's lateral current pattern can still matter when the through-thickness current density is nearly uniform. Slots interrupt wide current loops and usually reduce drag even when the mass and the bulk conductivity change little. Comparing a slotted and an unslotted plate directly tests the importance of current closure.

If a plate experiences sharply separated poles, its applied field in the moving frame contains a fundamental and higher harmonics. Harmonic n has a skin depth smaller by $\sqrt{n}$, so checking only the fundamental can miss surface concentration associated with narrow features. A Hall scan along the track supplies the actual spatial field spectrum. Multiplying each spatial wavenumber by the speed converts it into a temporal angular frequency for a constant-speed pass. Once the speed changes significantly over one pole spacing, a time-domain calculation using the measured position trace is preferable to this fixed-frequency conversion.

Close a mechanical and thermal energy budget

Take a rotating brake as a second experiment. A flywheel of known inertia $J_{\mathrm{rot}} = 0.020\,\mathrm{kg\,m^2}$ slows from 120 to $80\,\mathrm{rad/s}$ while no drive supplies power. The released rotational kinetic energy is

$$J_{\mathrm{rot}}(120^2 - 80^2)/2 = 80.0\,\mathrm{J}.$$

A separate run with the magnet removed measures bearing and air losses. Suppose the corresponding speed-dependent losses integrated over the magnetic braking trajectory total $8.0\,\mathrm{J}$. Then the electromagnetic channel receives $72.0\,\mathrm{J}$. Using the reference run's total energy over an arbitrary equal time could give a different answer, since the two speed histories affect the parasitic losses.

A calorimeter enclosing the conducting rotor has effective heat capacity $C = 60\,\mathrm{J/K}$ and a thermal link of conductance $G = 0.12\,\mathrm{W/K}$ to an environment at constant temperature T_a. Its balance is

$$C\,dT/dt = P_{\mathrm{deposited}} - G(T - T_a).$$

During a $20\,\mathrm{s}$ run, suppose the time-averaged rise above ambient is $0.50\,\mathrm{K}$. Then $1.20\,\mathrm{J}$ flowed through the thermal link. If all electromagnetic heat entered the calorimeter, its final rise from an ambient start is

$$(72.0 - 1.20)/60 = 1.18\,\mathrm{K}.$$

A measured rise near $0.80\,\mathrm{K}$ would leave an energy discrepancy of approximately $23\,\mathrm{J}$. Check conduction into a shaft, heat stored outside the instrumented volume, and the estimate of parasitic torque before adjusting the electromagnetic theory.

Find the spatial features that dominate a measured braking signal

A single Fourier component of field already produces an informative turnover. In a tracked experiment the field often has a measurable third harmonic. Let $B(x) = B_1 \cos kx + B_3 \cos(3kx + \varphi)$, with constant translation speed. Flux integration across the finite loop multiplies each harmonic by its own width factor. The corresponding emf peak is $2h|B_n|u|\sin(nkw/2)|$, and its temporal angular frequency is nku. The linear RL equation treats the two drives independently. Orthogonality over a complete fundamental period makes the mean of their product vanish, so total mean heat can be found by adding the individual resistive powers. Within one period the interference terms do affect instantaneous torque and power.

Suppose $B_3/B_1 = 0.20$ and keep the preceding $w = \lambda/4$. Here $\sin(kw/2)$ and $\sin(3kw/2)$ both have magnitude $1/\sqrt{2}$. At low speed the third harmonic therefore adds about four percent to mean drag. At $u = u_c$, however, its angular frequency sees inductive impedance $3R$ while the fundamental sees R. The third-harmonic share relative to the fundamental becomes $0.20^2(1+1)/(1+9) = 0.008$, only 0.8%. A peak-to-peak field reading can change visibly without a comparable change in mean drag. Conversely, if the loop width happens to null one spatial harmonic,

another smaller harmonic can supply most of the signal. A field map and the loop geometry together determine what source drives the circuit.

In a solid plate, predicting force from field and speed needs a field solution that enforces current closure. In the slowly moving conductor, $\mathbf{J} = \sigma[-\nabla\phi + \mathbf{u} \times \mathbf{B}]$ when the imposed magnetic pattern is effectively time independent in the laboratory and the induced magnetic field can be neglected. Charge conservation then gives

$$\nabla \cdot [\sigma(-\nabla\phi + \mathbf{u} \times \mathbf{B})] = 0.$$

On an electrically insulated plate edge, enforce $\mathbf{n} \cdot \mathbf{J} = 0$. Solving this potential problem accounts for charge accumulation that redirects currents at a boundary. Simply setting $\mathbf{J} = \sigma\mathbf{u} \times \mathbf{B}$ can predict current flowing straight out of an insulating edge. The force follows from integrating $\mathbf{J} \times \mathbf{B}$ through the conducting volume, and Joule heat follows from $\int J^2/\sigma \, dV$.

A useful check on such a field calculation is a finite plate translating wholly inside an exactly uniform static magnetic field. Charges can rearrange until a uniform electrostatic field cancels $\mathbf{u} \times \mathbf{B}$. An isolated plate in this steady limit then has zero transport current and zero drag. Motion alone does not guarantee braking. At an entry edge, the moving conductor spans a spatial change of induction, and one scalar potential generally cannot cancel the motional drive around every closed path. Slits, thickness variations, a conducting contact at a boundary, and the induced field at higher speeds each change this test problem in an identifiable way.

Interpret a thermal transient without assuming it is instantaneous

If the brake runs at a stable speed and equilibrium electrical conditions, a local single-temperature model sees a nearly constant power. Its step response from ambient is $\Delta T = (P/G)[1 - \exp(-Gt/C)]$. Removing the drive gives an exponential decay with the same time constant C/G. Measuring that cooling tail fixes the ratio of heat capacity to conductance but does not set either one separately. A known heater power or an independent heat-capacity measurement supplies the missing absolute scale. A slope evaluated over a long interval without correcting leakage will underestimate the input power as the sample warms.

Take a small bench calorimeter enclosing the single test loop with $C = 1.20\,\mathrm{J/K}$ and $G = 0.020\,\mathrm{W/K}$. The $7.00\,\mathrm{mW}$ low-speed run gives a $0.350\,\mathrm{K}$ eventual rise and a $60\,\mathrm{s}$ time constant. After $30\,\mathrm{s}$ the rise is about $0.138\,\mathrm{K}$, whereas a thermally isolated calculation would give $0.175\,\mathrm{K}$. Its stored heat $0.166\,\mathrm{J}$ is less than the $0.210\,\mathrm{J}$ mechanical work over that interval; the thermal link accounts for the remainder. Loading the conductor thermally with an additional copper holder increases the response time and changes the initial slope without changing the final rise if conductance to the bath remains the same.

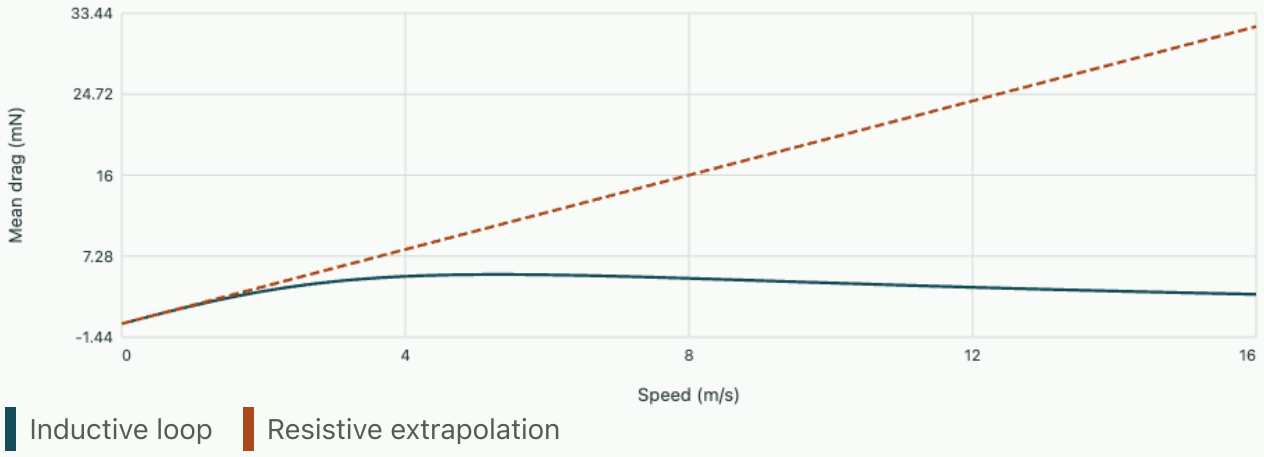
Compare predictions at a chosen travel speed

The live graph displays the force predicted by the finite-inductance loop alongside the resistive low-frequency extrapolation. Choose a speed, calculate the apparent excitation frequency, and compare the returned skin depth to the actual conductor thickness before interpreting the prediction as a plate measurement. Try choosing an inductance twice as large. The location of maximum force halves in speed, but the low-speed slope at fixed geometry and resistance is unchanged. A real plate whose high-speed drag does not follow the single-loop curve can still agree with the model on its low-speed slope. Plotting the measured residuals against speed can reveal the point where extra current paths or through-thickness diffusion become important.

Open the live experiment

Try the calculation

Travel speed (m/s)	**2**
Loop inductance (mH)	**0.15**
Peak field (T)	**0.2**



At 2.00 m/s the drag is 3.50 mN and the mean heat is 7.00 mW. The copper skin depth at 20.0 Hz is 14.78 mm. Peak drag occurs at 5.31 m/s.

A plate and holder cool from $0.240\,\mathrm{K}$ above ambient to $0.120\,\mathrm{K}$ in $42\,\mathrm{s}$ after a run. Can these two readings establish the braking power?

▼ **Work through the solution**

The half-life gives

$$C/G = 42/\ln 2 = 60.6\,\mathrm{s},$$

which agrees approximately with the small calorimeter example. It determines only a ratio. If an independent steady $10.0\,\mathrm{mW}$ heater then produces a $0.500\,\mathrm{K}$ temperature rise, $G = 0.0200\,\mathrm{W/K}$

and $C = 1.21\,\mathrm{J/K}$. A magnetic run that settles at $0.350\,\mathrm{K}$ now corresponds to $7.00\,\mathrm{mW}$, provided the thermal boundary and deposition pattern are sufficiently similar to the calibration.

Check the sign of instantaneous electromagnetic work

At a phase where the applied emf has just crossed zero, a finite-inductance loop may still carry considerable current. During part of a cycle the emf and current can have opposite signs. The product $\mathcal{E} i$ is then negative even though $R i^2$ remains positive. The difference is a negative derivative of the magnetic store $L i^2/2$. The loop briefly returns energy mechanically while its stored field decreases, and some of that release also becomes heat. A force transducer resolving separate pole passages can therefore see instantaneous variations that a mean braking-force estimate suppresses.

For the numerical $2.0\,\mathrm{m/s}$ case the peak field energy is $L I_{\mathrm{pk}}^2/2 \approx 21.0\,\mu\mathrm{J}$. The dissipated energy per full pole period is

$$7.00\,\mathrm{mW} / 20\,\mathrm{Hz} = 350\,\mu\mathrm{J}.$$

These quantities serve different purposes. The first is a reversible instantaneous store; the second is net irreversible energy delivered each time the conductor moves one spatial period. Dividing the second by the spatial period of $0.100\,\mathrm{m}$ recovers the mean drag of $3.50\,\mathrm{mN}$ without using the translational speed explicitly.

Tutorial on braking measurements

At $u = 2.0\,\mathrm{m/s}$ the loop above experiences a mean drag of $3.50\,\mathrm{mN}$. Predict what halving the induction and doubling the speed does, keeping its geometry and electrical properties fixed.

▼ Work through the solution

Halving induction quarters K^2. The speed factor contributes $2[1 + (2/5.31)^2]/[1 + (4/5.31)^2] \approx 1.46$. The combined change gives a drag about 0.364 times the original, or $1.27\,\mathrm{mN}$. The mechanical power decreases less because speed has doubled. A linear-speed extrapolation would miss the inductive phase lag, already significant at $4\,\mathrm{m/s}$.

In a repeated flywheel test the brake becomes warmer with every run. Which electrical parameter change can explain a change of drag even with the magnets held fixed?

▼ Work through the solution

The resistance of an ordinary metal generally increases with temperature. At a fixed speed F is proportional to $R/[R^2 + (\omega L)^2]$. Raising resistance reduces drag on the low-frequency resistive side and can increase it on the high-frequency inductive side. The maximum occurs at $R = \omega L$ if electrical

geometry and inductance stay fixed. Measuring the temperature and resistance before each run helps distinguish this effect from a weakening field or a changed gap.

Compare a step-driven diffusion profile with sinusoidal skin penetration.

Measure inertia and gravity with two suspension axes

Mechanics in the laboratory

A rigid pendulum can oscillate from either of two knife edges. If the periods agree at distinct distances from the centre of mass, the spacing of the edges determines gravity. Derive the cancellation, identify a second way equal periods can arise and find why a poorly chosen geometry magnifies timing errors.

Torque about an offset pivot

Take a rigid body of mass m with parallel suspension axes perpendicular to its plane of oscillation. Its moment of inertia about the centre of mass is $I_C = m\rho^2$; ρ is the radius of gyration. For a pivot a distance $a > 0$ from the centre, the parallel-axis theorem gives

$$I_a = m(\rho^2 + a^2).$$

Gravity exerts torque $-mga\sin\theta$. In the small-angle limit,

$$T_a^2 = \frac{4\pi^2}{g}\,\frac{\rho^2 + a^2}{a},$$

$$L_a = a + \frac{\rho^2}{a}.$$

Here L_a is the length of a point pendulum with the same period. It can lie beyond the physical centre of mass. A tap applied at this distance from a pivot gives the centre of percussion when the ideal instantaneous pivot reaction is to vanish. The same combination of mass distribution and geometry appears in an impulsive and in an oscillatory measurement because both depend on angular momentum about that pivot.

Match the periods and keep both algebraic branches

Put the second knife edge on the opposite side of the centre at distance b. Equal periods require

$$a + \rho^2/a = b + \rho^2/b.$$

Rearrange before dividing by $a - b$:

$$(a - b)\left(1 - \frac{\rho^2}{ab}\right) = 0.$$

Two equal distances always give equal periods. That branch provides no information about ρ. If the distances are distinct, equality implies

$$\rho^2 = ab.$$

Both equivalent lengths then equal $a + b$, the directly measurable edge separation. For $a = 0.200\,\mathrm{m},\ b = 0.450\,\mathrm{m},\ \rho = 0.300\,\mathrm{m}$, the equivalent length is 0.650 m and both periods at standard laboratory gravity are about 1.62 s. A small value of $a - b$ is an experimental warning sign, even if the observed equality is exceptionally clean.

Background · Can unequal release amplitudes imitate a mismatch between pivots?

A rigid pendulum about either knife edge obeys the same sine nonlinearity as a simple pendulum. To leading order in its release angle θ in radians, the measured period exceeds the small-amplitude period by the fraction $\theta^2/16$. At matched small-amplitude periods, measurements with amplitudes θ_a, θ_b therefore give

$$\frac{T_a - T_b}{T_0} \simeq \frac{\theta_a^2 - \theta_b^2}{16}.$$

Releasing at 6° about the first edge and 2° about the other creates a fractional mismatch of about 6.1×10^{-4}. For a two-second period this is 1.2 ms. Record the two amplitudes along with the timing data and reduce both periods to the same limit before moving the trim mass. Otherwise the trim can compensate a preparation error.

Extract gravity when the two measured periods differ

Multiply the first period equation by a and the second by b. Subtract to eliminate ρ^2. The exact small-angle result for unequal distances is

$$g = 4\pi^2 \frac{a^2 - b^2}{aT_a^2 - bT_b^2}.$$

At matched periods this reduces to the simple expression in terms of the edge spacing. The general formula involves a difference in its denominator. If a and b nearly coincide, the denominator is small and two independent timing errors subtract with poor relative precision. This is a property of the inverse measurement, even if computing each individual period is perfectly well conditioned.

Once g is known, use $\rho^2 = gaT_a^2/(4\pi^2) - a^2$ to recover the body inertia per unit mass. A negative value may indicate a wrong distance measured to the centre or a timing observable that is actually a half period. Verify the result using the second pivot and recover a separately predicted period after shifting a suspension point. A single successful cancellation is weaker than reproducing an additional measurement.

Move one knife edge in the experiment

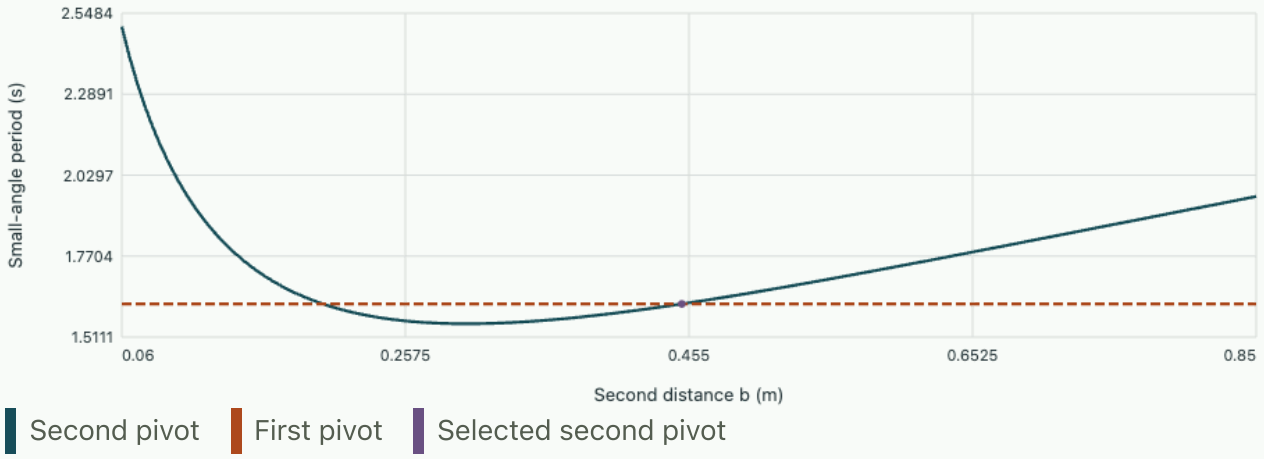
Hold $a = 0.200\,\mathrm{m}$ fixed and vary both the radius of gyration and the second distance. The graph plots the period against the possible second distance. Before inspecting intersections, predict where the second match should lie as ρ grows. The dependence on distance has a minimum at $b = \rho$ because effective length comprises a growing and a decreasing term.

Open this experiment online

Try the calculation

Radius of gyration (m) **0.3**

Second pivot distance (m) **0.45**



First period 1.6173 s; second 1.6173 s. The other matching distance is 0.45 m.

With a finite oscillation amplitude, the physical pendulum has the same elliptic correction as a point pendulum of length L_a, provided the body is rigid and the frictionless pivot stays fixed. Matching different amplitudes can bias the cancellation. Measuring both at the same small angle removes the leading difference, and extrapolating both to zero amplitude provides a check. Rotational compliance of the knife edge and a shifting centre of mass require a different model.

Tutorial

For $a = 0.240$ m and $\rho = 0.300$ m, where should the second distinct match occur and what length should be used in $g = 4\pi^2 L/T^2$?

▼ **Work through the solution**

The second branch gives

$$b = 0.0900/0.240 = 0.375\,\mathrm{m}.$$

Because the pivots straddle the centre of mass the equivalent length is 0.615 m.

Both pivots lie on the same side of the centre, at the same distances a, b as in the previous question. What changes in the cancellation?

▼ Work through the solution

The periods depend on the magnitudes of the offsets, so equal periods still imply $\rho^2 = ab$ on the distinct branch. The sum $a + b$ still gives the equivalent length, but the physical ruler distance between pivots is $|a - b|$. The direct measurement shortcut relies on the chosen geometry.

Why does setting both axes exactly at the same distance make even noise-free measurements insufficient to determine both g and ρ?

▼ Work through the solution

The two equations are then identical. For every positive choice of inertia radius the first period determines a corresponding value of gravity. Improving timing precision cannot provide the missing independent equation.

Connect the offset-pivot calculation to the rigid-body inertia chapter.

Applied study

Determine how load and curvature set the contact area, pressure and stresses between components.

How much area really carries the load?

A ball presses against a bearing race. The unloaded bodies touch at a point, which would imply infinite pressure for any finite force. Both bodies deform elastically under load, opening a contact patch whose size is part of the solution. A pin in a clevis carries a transverse load through a different combination of contact and shear. The relevant area depends on which failure mechanism is being examined.

Start with a sphere of radius R on a flat body. Assume a smooth nonadhesive interface, negligible tangential traction, isotropic linear elasticity and a small contact patch relative to curvature and body dimensions. Hertz's local solution then treats each solid as an elastic half-space. Their normal compliances add through the effective modulus $1/E^* = (1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2$. A grooved race with appreciable curvature can require different principal radii and an elliptical patch.

From contact pressure to the normal force

In a circular contact of radius a, the elastic solution gives

$$p(r) = p_0 \sqrt{1 - r^2/a^2},$$

with the greatest pressure at the centre. Equilibrium of a loaded body connects this local field to a measurable force. Integrating over rings of area $2\pi r\,dr$ gives $F = 2\pi p_0 a^2/3$, so the peak pressure is 1.5 times the patch mean.

Matching the curved gap to the elastic deflections gives

$$a^2 = R\delta,$$

with δ the mutual approach of the remote bodies, and $F = (4/3)E^*\sqrt{R}\,\delta^{3/2}$. Eliminating the approach yields

$$a = \left(\frac{3FR}{4E^*}\right)^{1/3},$$

$$p_0 = \frac{3F}{2\pi a^2},$$

$$k_n = \frac{dF}{d\delta} = 2E^* a.$$

The tangent stiffness k_n increases as load enlarges the patch. A single constant contact stiffness can describe small oscillations around a chosen preload. Following the surface during rolling subjects material below the track to a changing stress state. For Poisson ratio 0.30 the elastic solution puts the largest shear stress on the central axis near $z = 0.48a$, with magnitude about $0.31p_0$.

Worked example · Steel ball on a steel seat

Let $R = 10.0$ mm, $F = 120$ N and $E = 210$ GPa, $\nu = 0.30$ for both bodies. Their combined modulus is

$$210/[2(1-0.3^2)] = 115.4\,\mathrm{GPa}.$$

The patch radius is 0.198 mm and the approach is

$$a^2/R = 3.93\,\mu\mathrm{m}.$$

The pressure averaged over the patch is 971 MPa, and $p_0 = 1.46$ GPa. The maximum shear is about 452 MPa near a depth of 0.095 mm.

The 120 N force alone is a poor guide to stress at the contact. These are elastic predictions. Judging service life also requires material condition, lubrication, repeated loading and the rest of the bearing geometry. Compare how force and geometry determine nominal stress in a beam.

Transfer problem · A pin with two shear planes

A 12 kN load passes through a central 6 mm thick lug between two 4 mm thick clevis plates. Initially the pin diameter is 8 mm. Assume symmetric load sharing and negligible pin bending for this preliminary check. A cut through both shear planes gives mean pin stress $\tau = F/[2(\pi d^2/4)]$. Nominal bearing pressure uses the projected pin area td in each plate, with the central lug carrying F and the outer plates $F/2$ each.

For the 8 mm pin the mean shear is 119 MPa. Nominal bearing pressure is $12000/(6 \times 8) = 250$ MPa in the central lug, with dimensions here in millimetres, and $6000/(4 \times 8) = 188$ MPa in each outer plate. Suppose the exercise specifies limits of 125 MPa in mean shear and 200 MPa in nominal bearing. The central lug controls the diameter. Increasing d to 10 mm gives 76.4 MPa shear, 200 MPa in the central lug and 150 MPa outside. This meets the two stated nominal checks at the limiting diameter. Edge tear-out, net-section failure and pin bending would require additional dimensions and material data.

Exercises

1. Increase the contact preload

Increase the sphere load eightfold, maintaining geometry and elastic properties. Predict the factors of change in contact radius, peak pressure, approach and incremental stiffness. Calculate the new approach.

▼ **Show the load scalings**

From $a \propto F^{1/3}$, the radius doubles. The patch area quadruples, so $p_0 \propto F/a^2$ also doubles. The approach rises fourfold to about 15.7 µm, and the tangent stiffness doubles. A load path based on a geometrically fixed patch would overestimate the pressure increase.

---

2. Widen the contact or change the material?

At the original 120 N, compare doubling the sphere radius and halving the reduced modulus. Which choice reduces the peak pressure more, and which produces less movement?

▼ **Show the design comparison**

Both changes double R/E^*, increasing a by $2^{1/3} = 1.260$ and reducing p_0 to $2^{-2/3} = 0.630$ times its former value. Since $\delta = a^2/R$, doubling R gives $2^{-1/3}\delta = 3.12$ µm, whereas halving E^* gives

$$2^{2/3}\delta = 6.24\,\mu\mathrm{m}.$$

The larger sphere gives the same pressure reduction with a stiffer response at fixed load.

ONESIMULATION LABORATORY

Test the contact law under imposed approach

Compare recorded OneSimulation runs on this page. To test your own inputs, download the notebook and model files and run the model in an activated OneSimulation environment.

The example brings a 10 mm sphere towards a finite block. It prescribes the approach and measures the resulting force. Use the first mesh labelled 7 cells per reference radius for the comparisons below. Its baseline has `youngs_modulus=210e9`, `poisson_ratio=0.30` and a final approach of 25 µm. The computed contact radius comes from a pressure-weighted moment of the contact patch. A force integral over the quarter model is multiplied by four in the supplied script.

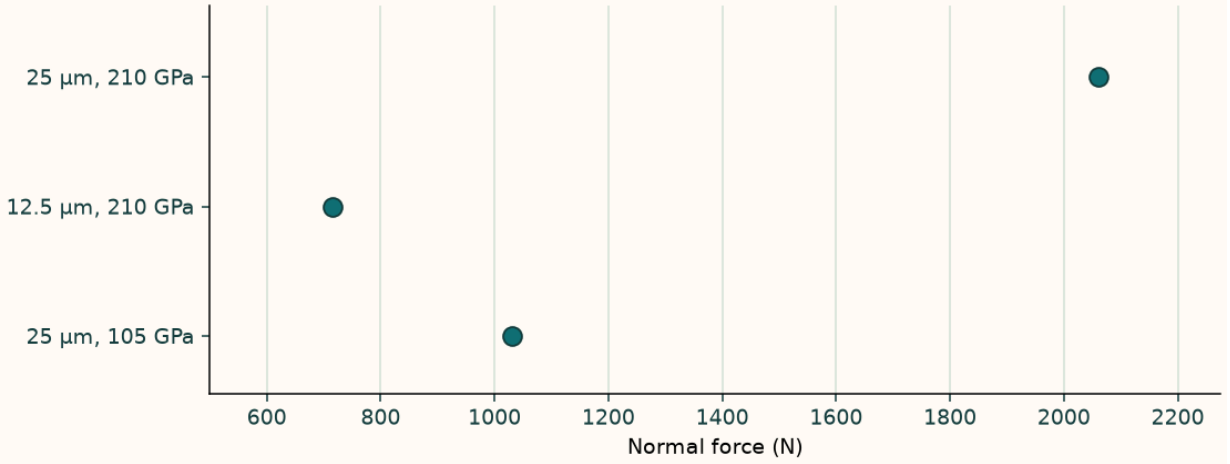
1. Multiply the array `approaches` by `0.5`, keeping material properties and mesh unchanged. The final approach is now 12.5 µm. Relative to baseline the half-space prediction gives a radius factor $\sqrt{0.5} = 0.707$ and a force factor $0.5^{3/2} = 0.354$. Reuse the same mesh to see how closely this finite-body calculation follows these ratios.
2. Restore the original approach and set `youngs_modulus=105e9` Pa for both bodies. Under displacement control the elastic displacement pattern and patch should stay nearly fixed while the reaction and stress halve. At fixed force instead, the Hertz radius varies as $(E^*)^{-1/3}$ at fixed geometry. Which control condition describes the numerical run?

To read the first mesh directly, run the cells through “Contact patch and pressure” and evaluate the following line. For a shorter run you can set `levels=(7,)` in the mesh setup and stop at that cell.

```
print(forces[0][-1], 1000 * contact_radius[0])  # newtons and millimetres
```

Contact results on the first mesh, with seven cells per reference radius

Approach and modulus	Normal force (N)	Patch radius (mm)
25 µm, 210 GPa	2060.2	0.5033
12.5 µm, 210 GPa	714.5	0.3481
25 µm, 105 GPa	1030.1	0.5033



At the same imposed approach, halving the modulus halves the normal force. Halving the approach instead reduces the force to about 35% of baseline.

View the OneSimulation sphere and block contact calculation.

Download experiment files (ZIP)

Extract the files together and open `hertz_sphere_on_flat_3d.ipynb` in a local OneSimulation environment.

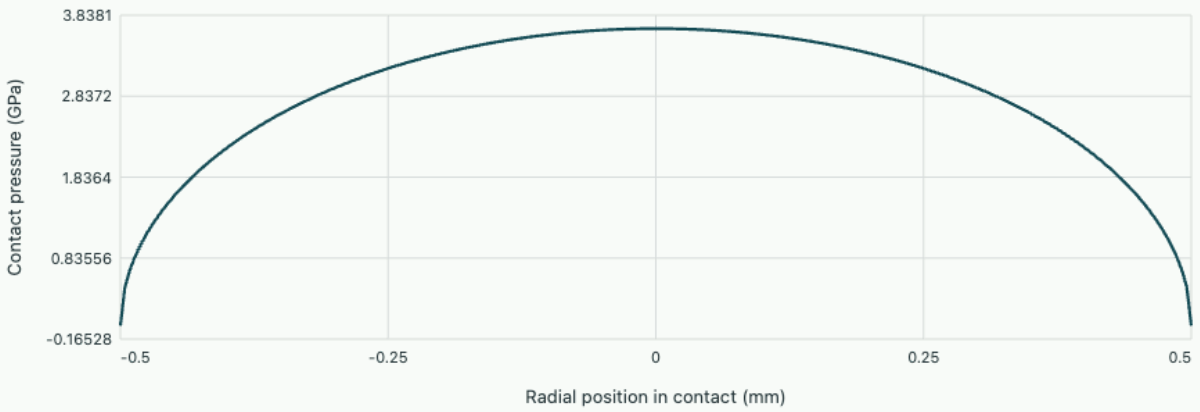
Interactive reference for the OneSimulation laboratory

Vary approach, elastic modulus and sphere radius. The Hertz half-space pressure profile is an independent reference for the linked finite-block3D contact calculation.

Open the live laboratory reference

Try the calculation

Imposed approach (µm)	25
Young modulus of both bodies (GPa)	210
Sphere radius (mm)	10



Elastic half-space pressure

Hertz reference: force 1923.1 N; patch radius 0.5000 mm; peak 3.673 GPa. The linked3D finite-block model can depart from this half-space limit.

Problem sheet 2 · Measure contact-law exponents from two solutions

Notation and assumptions

Read each defining equation with its assumptions and units, then follow the link to the argument that gives the equation its meaning.

Contact ratio

$$\epsilon_\alpha = L_{\text{contact}}/p_b$$

For a transverse involute pair compare the active path of contact with the base pitch. This determines how many pairs of teeth can share the load.

Derivation · How an involute gear pair keeps contact from one tooth to the next

Gear ratio

$$i = \omega_{\text{in}}/\omega_{\text{out}}$$

Specify the chosen convention and rotation signs. Reflecting an inertia to the input of an ideal reduction divides it by the squared speed ratio.

Derivation · Matching a gear ratio to load inertia and move time

Joint stiffness fraction

$$C = k_b/(k_b + k_m)$$

Before separation, bolt and clamped-member stiffnesses decide what fraction of a separating external force increases the bolt load.

Derivation · Preload, joint stiffness and separation

Lead angle

$$\tan\lambda = l/(2\pi r_m)$$

Lead is axial advance per revolution and can differ from the thread pitch. Friction and this angle together determine backdrivability.

Derivation · Deriving screw-drive torque, efficiency and backdrivability

Slip power

$$P_{\text{slip}} = \tau |\omega_1 - \omega_2|$$

Friction power during engagement depends on relative speed. Integrating it over engagement gives energy entering the thermal model.

Derivation · Friction clutches, braking torque and heat

Torsional stiffness

$$k_\theta = GJ/L$$

For a uniform elastic circular shaft use its polar second moment. A noncircular section generally needs a distinct torsion constant and warping analysis.

Derivation · Shaft reactions, twist and rotating bending

Further reading

- Read the design assignments alongside the shaft and bearing chapters. Translate a motion requirement into load paths, component stiffness and a test of the assembled mechanism. MIT · Elements of Mechanical Design

Compare specific topics with the university sources