Machine dynamics

Derive equations of motion, identify modes and damping, and use them to design isolation, balance rotors and assess finite-amplitude vibration.

Read the subject in depth

Each volume develops its arguments in sequence, with worked problems and links to experiments.

- ### Vibration and identification in machinery

  Connect modes, moving components and measured frequency responses. Investigate how forcing, damping and the observation point change an inferred dynamic model.

Advanced study across the subject

Each sequence connects the course to detailed chapters in neighbouring subjects. The complete course PDF includes their derivations, examples and exercises.

1. ### Coordinates, invariants and integration

   Derive a coordinate description for a loaded mechanism, follow its energy through a cycle and assess numerical drift before assigning a measured decay to damping.

2. ### Dynamic coupling and measured response

   Combine structural elasticity, a transducer and an estimator. Compare a split resonance, a free decay and a periodically loaded shaft.

How the course fits together

1. **Chapters 1–4**

   Choose a coordinate and compute the kinetic and elastic energy it carries. Free and driven vibration follow from inertia, stiffness and damping. Coupled coordinates permit several natural frequencies and mode shapes.

2. **Chapters 5–8**

   Distinguish base motion from a directly applied force and a transient from a steady response. Rigid-body kinematics and impulse balances tell you how an initial motion is created before vibration begins.

3. **Chapters 9–12**

   An unbalanced rotor provides a periodic load. Lagrange’s equations assemble a multibody model, modal forces identify which vibrations a load excites, and nonlinear stiffness can move the resonances with amplitude.

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Chapter 1 · Coordinates, energy and equivalent inertia

Reduce a constrained drive to one coordinate and use a power balance to check its equation of motion.

FROM FIRST PRINCIPLES

Refer moving masses to one shaft by matching kinetic energy

Choose shaft angle θ as the independent coordinate of a rigid mechanism. Suppose the angular speeds of its rotating components and the linear speeds of its translating components are fixed multiples of $\dot{\theta}$. Assume ideal constraints with no slip, so specifying θ and $\dot{\theta}$ fixes the motion. Total kinetic energy is the sum of the individual translational and rotational energies.

$$T = \sum_i \tfrac{1}{2} m_i v_i^2 + \sum_j \tfrac{1}{2} J_j \omega_j^2 = \tfrac{1}{2} J_{\mathrm{eq}} \dot{\theta}^2.$$

Here rotational inertias are about the component centres or fixed rotation axes as appropriate, with no motion counted twice. Dividing by $\dot{\theta}^2/2$ gives the inertia seen at the chosen shaft.

$$J_{\mathrm{eq}} = \sum_i m_i \left(\frac{v_i}{\dot{\theta}} \right)^2 + \sum_j J_j \left(\frac{\omega_j}{\dot{\theta}} \right)^2.$$

A force on a translating part contributes the shaft torque that gives the same instantaneous power,

$$T_i \dot{\theta} = F_i v_i.$$

For constant speed ratios and constant equivalent inertia, energy balance then gives

$$\sum T_i = J_{\mathrm{eq}} \ddot{\theta}.$$

The square on the speed ratio in inertia reflection follows from energy and is essential when changing the reference shaft.

Background · Generalised inertia

$$M(q) = 2T/\dot{q}^2$$

This scalar definition applies to one independent moving coordinate with purely quadratic kinetic energy. Changing the coordinate changes the numerical inertia.

DERIVATION AND APPLICATIONS

Derive the extra drive term for position-dependent inertia

A motor can require a torque at constant angular speed if a mechanism changes the linear speed of its payload during a revolution. Use a single angular coordinate θ, input inertia J_0, and a guided mass m with position $x = f(\theta)$. Assume a rigid lossless linkage and a horizontal guide, so gravity does no work. The kinematic constraint sets the payload velocity and fixes the instantaneous reflected inertia by equating the total kinetic energy to that of one shaft.

$$\dot{x} = f'(\theta)\dot{\theta},$$
$$T = \tfrac{1}{2}[J_0 + m(f')^2]\dot{\theta}^2 = \tfrac{1}{2}J(\theta)\dot{\theta}^2.$$

Differentiate the energy with respect to time. The shaft input supplies the full change in kinetic energy through its power $\tau\dot{\theta}$. Equating the two and cancelling speed gives the required drive torque. At a turning point the same equation follows directly from Lagrange's equation.

$$\tau = J(\theta)\ddot{\theta} + \tfrac{1}{2}J'(\theta)\dot{\theta}^2.$$

For an ideal sinusoidal guide $x = a\sin\theta$, substitution makes the speed-squared contribution explicit. It changes sign within each turn because the translating payload first absorbs and then releases kinetic energy.

$$J = J_0 + ma^2\cos^2\theta,$$
$$\tau = J\ddot{\theta} - ma^2\sin\theta\cos\theta\,\dot{\theta}^2.$$

A constant-inertia approximation omits this alternating torque even if it predicts the total energy averaged over a turn. Its integral over a full constant-speed revolution vanishes as the payload returns to its initial speed.

WORKED EXAMPLE

Predict a constant-speed torque measurement

A 2.00 kg carriage follows $x = 0.0500\sin\theta$ metres with shaft speed 20.0 rad/s. At $\theta = 45^\circ$ its required torque is $-2.00(0.0500)^2(0.5)(20.0)^2 = -1.00$ N m. The motor absorbs 20.0 W at that instant. At 135° the required torque is $+1.00$ N m. Both readings occur at the same shaft speed and the same reflected inertia.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Identify a payload from the torque harmonic

A separate horizontal mechanism has stroke parameter $a = 40.0$ mm and runs at 30.0 rad/s. After removing its empty-linkage trace, the measured second-harmonic torque amplitude is 0.720 N m. Since $|\sin\theta\cos\theta|_{\max} = 1/2$, the fitted mass is

$$2(0.720)/[(0.0400)^2(30.0)^2] = 1.00\,\mathrm{kg}.$$

A run at half speed should show 0.180 N m amplitude if the measured harmonic is inertial.

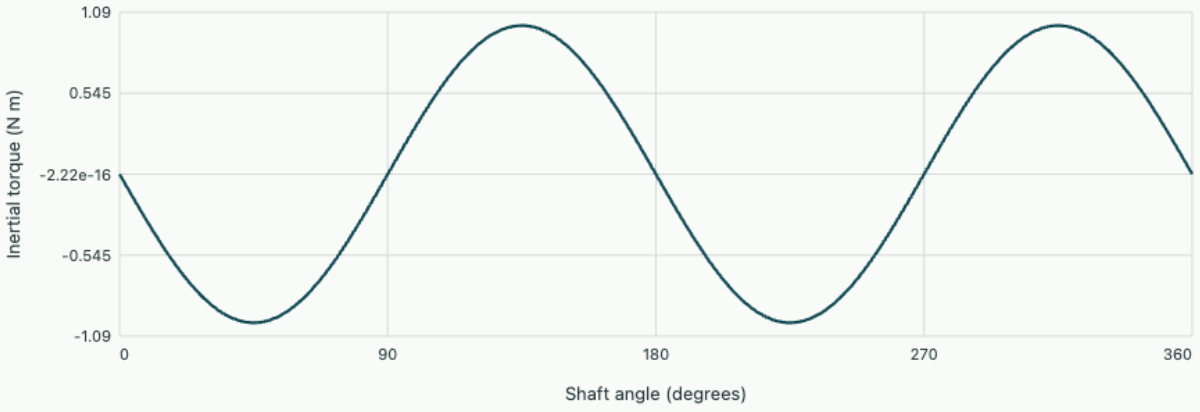
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Carriage mass (kg)	2
Motion amplitude (mm)	50
Constant shaft speed (rad/s)	20



Constant-speed carriage torque

Torque-harmonic amplitude 1.000 N m. Energy is exchanged despite zero shaft acceleration.

Start from this chapter. Write translation and rotation as distinct kinetic-energy terms. Adding material near the reference point mostly changes the translational term. Predict which change the frame model makes when only rotary inertia is doubled.

OneSimulation laboratory · Separate the effect of mass from rotary inertia

Follow the motion through a mechanism

A motor accelerating a load feels inertia from components that turn at different speeds. Simply adding their physical moments of inertia would treat all of those speeds as equal. Instead, choose one independent coordinate and express every kinetic energy in terms of it. No-slip belts, rigid gears and inextensible cables give useful constraints when their compliance and backlash are negligible over the motion of interest.

Consider a drum of radius R winding a cable carrying a vertically guided mass m. Choose the drum angle θ positive when the mass rises. If the effective winding radius is constant, displacement, velocity and acceleration satisfy $x = R\theta$, $\dot{x} = R\dot{\theta}$ and $\ddot{x} = R\ddot{\theta}$. These equalities are kinematic. A torque relation requires force balance or work in addition. When the cable builds successive layers on a spool, the effective radius varies with angle and the constant-ratio assumption must be revisited.

Background · What changes when the reflected inertia depends on position?

For a piston linkage let the piston position be $x(q)$, where q is shaft angle. Its kinetic energy contributes $m(dx/dq)^2$ to the shaft inertia. As the linkage moves through a stroke this contribution varies, even for a piston of constant mass. Use the energy derivative to find the generalised inertial torque.

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = J(q)\ddot{q} + \tfrac{1}{2}J'(q)\dot{q}^2,$$

$$T = \tfrac{1}{2}J(q)\dot{q}^2.$$

Uniform shaft speed sets the first term to zero. The second term represents kinetic energy flowing into and out of the accelerating piston. Its shaft work over a complete cycle vanishes when the mechanism returns to the same angle and speed. An average torque meter can consequently miss a substantial alternating torque.

Reflect mass and forces to a common shaft

Let J_d be the drum inertia. Add the rotational energy of the drum to the translational energy of the load, then express both in terms of the drum speed.

$$
\begin{aligned}
K &= \tfrac{1}{2} J_d \dot{\theta}^2 + \tfrac{1}{2} m \dot{x}^2 \\
&= \tfrac{1}{2} \left(J_d + m R^2 \right) \dot{\theta}^2, \\
J_{\mathrm{eq}} &= J_d + m R^2 .
\end{aligned}
$$

A small positive angle raises the mass through $R\,d\theta$, increasing its potential energy by $mgR\,d\theta$. Thus gravity contributes a resisting torque mgR. A positive applied torque T_d supplies power $T_d\dot{\theta}$. Applying the energy balance gives the equation of motion.

$$
\begin{aligned}
\frac{d}{dt}\left(\tfrac{1}{2} J_{\mathrm{eq}} \dot{\theta}^2 + mgR\theta \right) &= T_d \dot{\theta}, \\
J_{\mathrm{eq}} \ddot{\theta} &= T_d - mgR .
\end{aligned}
$$

The second expression also holds at zero speed by force balance. It separates steady effort, needed to hold or raise the load against gravity, from the extra effort needed to accelerate the two inertias. With the motor disconnected, an ideal light cable does not make the mass accelerate quite at g, since the drum must gain rotational kinetic energy too.

Calculate a motor torque

A drum with $R = 0.10$ m and $J_d = 0.15\,\mathrm{kg\,m^2}$ lifts 25 kg. A reduction gear gives

$$
\dot{\theta}_m = 4\dot{\theta}.
$$

The rotor has inertia $J_m = 0.020\,\mathrm{kg\,m^2}$. Refer the train to the motor coordinate. The drum-plus-load inertia becomes $(0.15 + 25(0.10)^2)/4^2 = 0.025\,\mathrm{kg\,m^2}$, so the total at the motor is 0.045 kg·m². Gravity contributes $25(9.81)(0.10)/4 = 6.13$ N·m at that shaft.

Demand upward acceleration 0.80 m/s². The drum must accelerate at 8.0 rad/s² and the motor at 32 rad/s². The motor supplies

$$
6.13 + 0.045(32) = 7.57\,\mathrm{N\,m}.
$$

A separate check at load speed 1.0 m/s gives motor speed 40 rad/s. Its 303 W then divide into about 245 W raising the weight and 57.6 W increasing kinetic energy. Ideal gearing changes torque and angular speed inversely while preserving transmitted power.

Identify which approximation controls the answer

At constant lifting speed the inertial torque vanishes. At constant angular acceleration, doubling the transmission ratio decreases reflected load inertia by four but also doubles the motor acceleration for the same load motion. The rotor’s own inertia remains at the motor and eventually dominates. A gear ratio therefore cannot reduce every part of acceleration torque indefinitely. Bearings, cables and gears

also dissipate energy or store it elastically. Include their effects when a measured transient shows a repeatable lag, oscillation or loss that an ideal rigid constraint cannot explain.

Is the largest reduction always preferable?

For a prescribed drum acceleration α_d and reduction $n = \omega_m/\omega_d$, collect gravity and load-side acceleration into the drum demand $T_d = mgR + (J_d + mR^2)\alpha_d$. The motor torque becomes

$$T_m = T_d/n + J_m n \alpha_d.$$

The first term falls as reduction rises, while rotor acceleration grows. Differentiating with respect to n gives an ideal minimum at $n = \sqrt{T_d/(J_m\alpha_d)}$. For the stated lift $T_d = 27.73$ N m and $\alpha_d = 8$ rad/s², giving $n = 13.2$ approximately. This torque minimum assumes the motor can achieve the associated speed. At a drum speed of 10 rad/s it would turn at 132 rad/s, over three times its speed with the original reduction. Shaft-speed and voltage limits can therefore select a different practical ratio.

Derive speed ripple from a torque diagram

A crank-driven machine can require very different torque at different angles even when the average load is constant. A flywheel stores kinetic energy when driving torque exceeds resisting torque, then releases it when the load peaks. Refer all inertias and torques to the same shaft and suppose the equivalent inertia J is constant. The signed difference $M(\theta) = T_{\text{drive}} - T_{\text{load}}$ gives

$$J\dot{\omega} = M.$$

For continuous positive rotation write $\dot{\omega} = \omega\, d\omega/d\theta$. Integration between two angular positions gives an exact work–energy relation.

$$\tfrac{1}{2}J[\omega_b^2 - \omega_a^2] = \int_{\theta_a}^{\theta_b} M(\theta)\, d\theta.$$

A periodic steady sequence requires zero net area under the torque-difference curve over a whole revolution. Its cumulative area nevertheless has a maximum and a minimum. Their difference ΔE is the energy that must swing into and out of the rotating inertia. Define mean speed in the ripple specification as the arithmetic mean of the two extreme speeds $\bar{\omega} = (\omega_{\max} + \omega_{\min})/2$ and fluctuation coefficient $C_s = (\omega_{\max} - \omega_{\min})/\bar{\omega}$. Factoring the difference of squared speeds gives

$$J = \Delta E/(\bar{\omega}^2 C_s).$$

A time-average tachometer reading approximates this mean for small ripple, but the definitions are distinct for a strongly pulsating shaft.

Size a flywheel for a repeating work demand

A laboratory indexing machine runs at a nominal 300 rpm. A motor supplies approximately constant 30 N m. The resisting torque is 70 N m for the first quarter revolution and a lower constant value for the remaining three quarters. Equate total input and output work over each revolution: $30(2\pi) = 70(\pi/2) + T_l(3\pi/2)$, giving $T_l = 16.67\,\mathrm{N\,m}$. The first quarter produces a 40 N m deficit, so the cumulative energy falls by $40(\pi/2) = 62.83\,\mathrm{J}$. The remaining 13.33 N m surplus restores exactly that energy.

For a maximum peak-to-peak speed variation of 2% and $\bar{\omega} = 31.42\,\mathrm{rad/s}$, total inertia required is

$$62.83/[31.42^2(0.02)] = 3.18\,\mathrm{kg\,m^2}.$$

If the machine already has 1.00 kg m² on this shaft, an extra 2.18 kg m² is required. A thin-ring estimate at mean radius 0.30 m gives mass $2.18/0.30^2 = 24.2\,\mathrm{kg}$. A solid disc of the same radius needs twice the mass because its material lies closer to the shaft. At half speed the same torque-versus-angle work deficit requires four times the total inertia for the same fractional ripple. Startup torque or bearing load may therefore limit an inertia-based solution even when steady motor power suffices.

The torque balance also predicts where a shaft encoder should record the two speed extremes. The energy derivative with respect to angle equals the torque difference, so the maximum occurs just before the high resisting plateau begins and the minimum where that plateau ends. The maximum torque itself persists over a finite angular interval, while minimum speed occurs only at its end. This distinction matters when timing a torque pulse from a speed record: the sign of the speed derivative, rather than the absolute value of speed, identifies when the flywheel is discharging.

Chapter 2 · Free vibration, energy loss and ring-down data

Infer stiffness and damping from a released machine mount and check the implications of a viscous damping model.

FROM FIRST PRINCIPLES

Determine the frequency and decay of a released mount

Displace a mass m from its static equilibrium on a linear spring of stiffness k and release it. A viscous damper exerts force opposite to velocity with coefficient c. Measuring displacement x relative to static equilibrium removes any constant force already balanced by the spring. Newton's second law gives the free-motion equation.

$$m\ddot{x} = -kx - c\dot{x}.$$

An exponential trial e^{st} turns differentiation into multiplication by s, so nonzero motion requires a root of the characteristic polynomial. Its two roots depend on the undamped natural frequency ω_n and damping ratio ζ.

$$ms^2 + cs + k = 0,$$

$$\omega_n = \sqrt{k/m},$$

$$\zeta = \frac{c}{2\sqrt{km}}.$$

For $0 < \zeta < 1$ the roots are a conjugate pair. Their oscillatory contributions combine to give a real damped sinusoid with envelope $e^{-\zeta\omega_n t}$.

$$s = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}.$$

The initial position and velocity determine amplitude and phase, while the parameters set decay and period. Same-sign peaks one period apart therefore measure the exponential envelope without requiring an accurately known release time.

Background · Damping ratio

$$\zeta = c/(2\sqrt{km})$$

This compares a viscous damper with critical damping for a single linear oscillator. It cannot represent Coulomb friction at every amplitude.

DERIVATION AND APPLICATIONS

Infer damping from the loss of mechanical energy per cycle

A lightly damped mount sheds a measurable fraction of stored energy on each oscillation. Consider a single translating mass m with linear stiffness k, viscous damping c, and no forcing after release. Measure displacement from static equilibrium. Multiply the equation of motion by velocity to convert inertia and spring terms into total time derivatives.

$$m\ddot{x} + c\dot{x} + kx = 0,$$

$$\frac{d}{dt}\left(\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2\right) = -c\dot{x}^2.$$

For weak damping, amplitude A changes little during one period and $x \simeq A\cos\omega_n t$, with $\omega_n = \sqrt{k/m}$. Integrating viscous power over the cycle then gives the leading loss. Divide by the stored maximum spring energy evaluated at the same amplitude.

$$\Delta E_{\text{loss}} \simeq cA^2\omega_n^2\frac{\pi}{\omega_n} = \pi c\omega_n A^2, \qquad \frac{\Delta E_{\text{loss}}}{E} \simeq 4\pi\zeta,$$

$$\zeta = \frac{c}{2m\omega_n}.$$

For appreciable damping use two same-sign turning points directly. At each peak the kinetic energy vanishes, so the exact energy ratio equals the square of the amplitude ratio. The measured logarithmic decrement relates to viscous damping for an underdamped oscillator.

$$\frac{E_{j+1}}{E_j} = \left(\frac{A_{j+1}}{A_j}\right)^2 = e^{-2\delta},$$

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}},$$

$$\delta = \ln\frac{A_j}{A_{j+1}}.$$

The energy fraction is independent of starting amplitude for linear viscous damping. Amplitude dependence in repeated runs is therefore a direct reason to examine extra friction or nonlinear damping.

WORKED EXAMPLE

Determine heat release from two peak readings

For $k = 800$ N/m, consecutive positive peaks are 10.0 and 9.00 mm. The spring energies are 40.0 mJ and 32.4 mJ, so 7.60 mJ is dissipated. The decrement is $\ln(10/9) = 0.1054$, giving $\zeta = 0.0168$. If the first amplitude were doubled and the same linear damper retained, the energy lost in that cycle would quadruple.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Test a damper from its measured cycle work

A separate mount has $k = 2000$ N/m, $\omega_n = 50.0$ rad/s and small oscillations of amplitude 2.00 mm. A damper dynamometer measures 0.250 mJ of work per cycle. The leading estimate gives $c = \Delta E/(\pi\omega_n A^2) = 0.398$ N s/m. From $m = k/\omega_n^2 = 0.800$ kg, $\zeta = 0.00497$. The estimated loss fraction $4\pi\zeta = 0.0625$ supports the nearly constant-amplitude approximation for a single cycle.

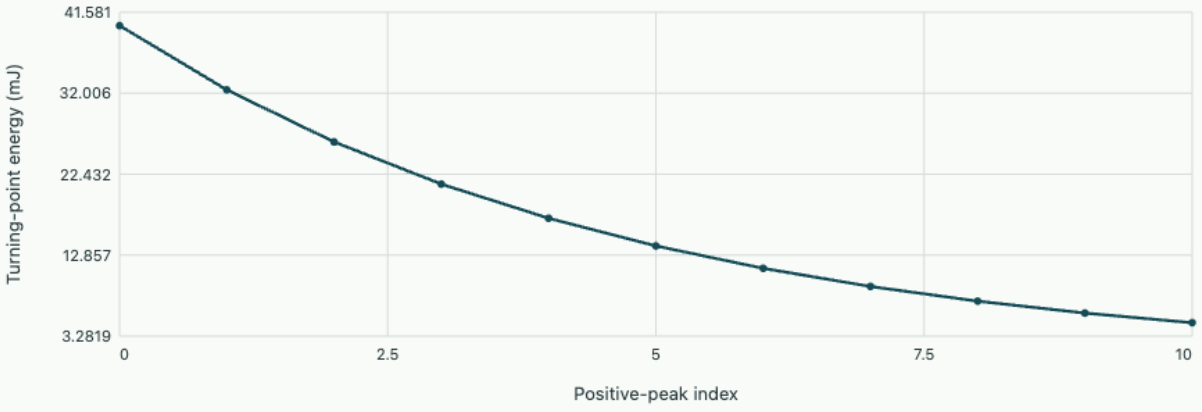
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Peak displacement (mm)	**10**
Next/previous peak	**0.9**
Stiffness (N/m)	**800**



Measured geometric peak decay

First-cycle loss 7.600 mJ; inferred ζ=0.01677.

ONESIMULATION LABORATORY

When does a released mass cross equilibrium?

Release a 1 kg mass from rest 10 mm to the right of equilibrium on a spring of stiffness 4 N/m. A viscous damper exerts a force opposite to the velocity. In a driven machine this free response is the transient left by the initial displacement. Here the external drive is zero, so the transient can be measured on its own.

Predict a crossing. With displacement measured from static equilibrium, the weight has already been balanced. Newton's second law gives one restoring term for displacement and one for velocity. Write the trial motion as an exponential. Dividing the resulting equation by the nonzero exponential leaves a quadratic:

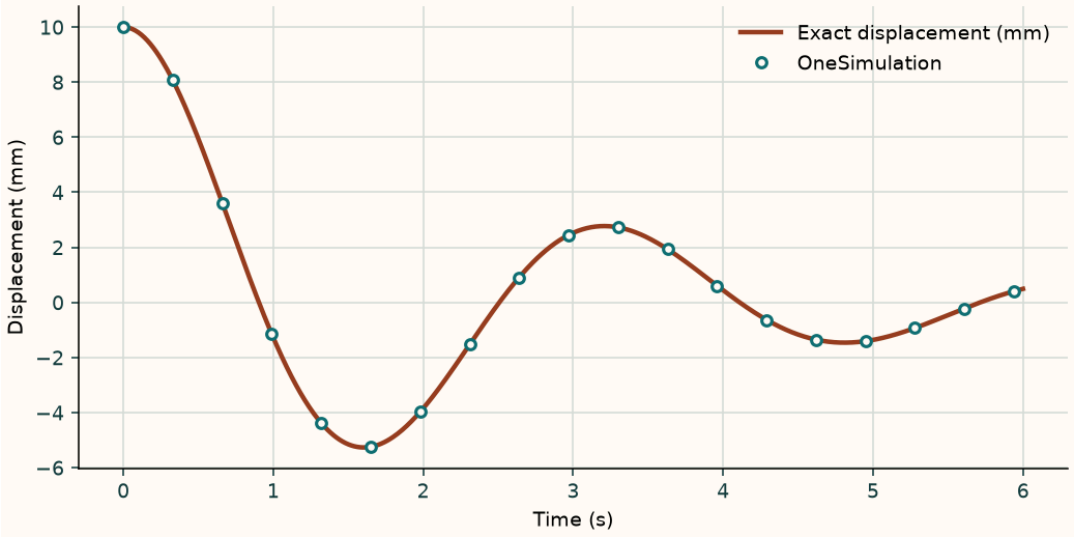
$$m\ddot{x} = -kx - c\dot{x},$$

$$x = Ae^{st} \quad \Longrightarrow \quad s^2 + 2\zeta\omega_n s + \omega_n^2 = 0,$$

$$\omega_n = \sqrt{k/m},$$

$$\zeta = \frac{c}{2\sqrt{mk}}.$$

Here $\omega_n = 2$ rad/s. At $\zeta = 0.2$ the roots are $-0.4 \pm 1.960i$ s⁻¹. The imaginary part permits a zero crossing. At critical damping the double root is -2 s⁻¹. Applying the initial conditions to the two independent solutions gives $x = 10(1 + 2t)e^{-2t}$ mm. At $\zeta = 2$ the slowest root is about -0.536 s⁻¹. Which of the last two releases is closer to equilibrium at three seconds?



Displacement versus time for a mass released from rest, with the exact solution and sampled numerical displacements. Damping ratio = **0.2**.

Download this run (CSV)

Read the comparison. At three seconds the critical solution is about 0.174 mm from equilibrium. Increasing damping again leaves about 2.16 mm. The slower real root explains the long tail. The

underdamped trajectory has already crossed the equilibrium. The points come from integrating the two first-order equations for displacement and velocity. The line comes from their characteristic roots.

Try a fourth case. Change the damping ratio to 0.7. Predict the first zero from the damped frequency and the phase fixed by release from rest, then find the first sign change in the output CSV.

Further reading · MIT 18.03, Natural frequency and damping ratio.

Download the OneSimulation model (Python). Set up the local runtime, then run the downloaded file:

```
python oscillator.py --ratio 0.7
```

> **Start from this chapter.** The one-coordinate frequency varies as the inverse square root of mass at fixed stiffness. Test that expectation against three frame modes when the payload mass changes.
>
> OneSimulation laboratory · Separate the effect of mass from rotary inertia

Displace a machine from its static position

A pump on a spring mount settles under its weight. Pull it slightly farther down and release it. The weight still acts, but the extra motion is driven by the change in spring force. Define x as displacement from the loaded equilibrium, take constant local spring stiffness k, and represent dissipation by a force opposing velocity of magnitude $c|\dot{x}|$. Newton's law becomes

$$m\ddot{x} + c\dot{x} + kx = 0.$$

This model describes one isolated mode over an amplitude range where inertia, stiffness and damping are approximately constant.

Multiply the equation by $\dot{x}$. The mass term differentiates kinetic energy, and the spring term differentiates stored elastic energy. Consequently $d(m\dot{x}^2/2 + kx^2/2)/dt = -c\dot{x}^2$. Energy cannot grow for positive damping. Displacement can increase temporarily as kinetic energy converts back into spring energy, so the instantaneous displacement alone is not an energy meter.

Obtain the decay and oscillation rates

Trying $x = e^{st}$ gives

$$ms^2 + cs + k = 0.$$

Define $\omega_n = \sqrt{k/m}$ and $\zeta = c/(2m\omega_n)$. The roots are

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}.$$

For $0 < \zeta < 1$ their imaginary part gives the damped angular frequency $\omega_d = \omega_n\sqrt{1 - \zeta^2}$. The real part gives a decaying exponential. A general underdamped trace is

$$x(t) = e^{-\zeta\omega_n t}\,[A\cos(\omega_d t) + B\sin(\omega_d t)].$$

At $\zeta = 1$ the repeated root gives $(A + Bt)e^{-\omega_n t}$. Above critical damping both roots are negative and real. A lack of visible oscillation therefore does not by itself determine the stiffness. For an underdamped trace, use two peaks of the same sign: their separation is $T_d = 2\pi/\omega_d$, and their amplitude ratio depends only on the envelope decay.

Background · Which energy is represented by a sequence of displacement peaks?

At a turning point the velocity is zero. If displacement is measured from static equilibrium, the energy of the linear oscillator there is entirely elastic. Two peaks of the same sign one damped period apart give an exact energy ratio even though the instantaneous dissipation varies within that period.

$$\frac{E_{n+1}}{E_n} = \left(\frac{x_{n+1}}{x_n}\right)^2,$$

$$\frac{\Delta E_{\text{lost}}}{E_n} = 1 - e^{-2\delta},$$

$$\delta = \ln\frac{|x_n|}{|x_{n+1}|}.$$

If successive positive peaks fall from 10 to 8 mm, 36% of the initial peak energy has been lost in one cycle. The amplitude has fallen by only 20%. Near critical damping it may be difficult to resolve a full period above sensor noise. The definition then remains valid for a measurable underdamped cycle, but the inferred ratio becomes sensitive to peak errors.

Infer parameters from four cycles

A 20.0 kg mounted component rings down from a positive peak of 8.0 mm to a positive peak of 2.0 mm four cycles later, after 1.60 s. The period is $T_d = 0.400$ s and $\omega_d = 15.71$ rad/s. The log decrement per cycle is

$$\delta = \ln(8/2)/4 = 0.3466.$$

Since $x_j/x_{j+1} = \exp(\zeta\omega_n T_d)$, the envelope decay rate is

$$\alpha = \delta/T_d = 0.8664\,\text{s}^{-1}.$$

Now $\omega_n = \sqrt{\omega_d^2 + \alpha^2} = 15.73$ rad/s and $\zeta = \alpha/\omega_n = 0.0551$. The inferred stiffness is $m\omega_n^2 = 4.95$ kN/m, while $c = 2m\alpha = 34.7$ N·s/m. After eight cycles the prediction is 0.50 mm. Using peaks separated by several cycles reduces sensitivity to noise in any one amplitude or timestamp.

Challenge the damping assumption

Subtract the equilibrium sensor reading before taking amplitude ratios. A fixed offset changes the ratio of small late peaks more strongly than large early peaks. Plot $\ln|x_j|$ against time for peaks whose signal clearly exceeds the noise floor. A straight trend supports the viscous envelope. Approximately constant loss of amplitude per cycle suggests a different mechanism such as dry friction. Two nearby natural frequencies may produce a beat envelope that grows and shrinks even though total energy decays.

A second experiment can separate stiffness from added inertia. If a small known mass is bolted to the same moving assembly and the support stiffness is unchanged, $1/\omega_n^2$ should increase in proportion to effective mass. This check assumes that the added mass participates fully in the chosen coordinate. A flexible appendage that moves out of phase changes the system being identified.

Correct a biased displacement signal

Suppose a displacement sensor reads 0.50 mm high relative to the true equilibrium. The first and fifth positive peaks then appear as 8.50 and 2.50 mm. Their raw log ratio over four cycles gives $0.25\ln(8.5/2.5) = 0.306$, underestimating the true 0.347 decrement. The error grows as the decaying signal approaches the offset. One remedy is to measure equilibrium independently when excitation has stopped. Another is to use the difference between consecutive positive and negative extrema and compare such differences one whole period apart. The constant offset cancels while, for a single underdamped mode, successive peak-to-trough ranges retain the same exponential ratio. Late peaks below the sensor resolution still carry little reliable damping information.

Derive the repeated-root response and its crossing condition

The oscillatory ring-down described above assumes $\zeta < 1$. For a return mechanism holding a fragile component, the question can instead be whether it reaches equilibrium without crossing it. At critical damping, $c = 2\sqrt{mk}$, the characteristic equation is

$$(s + \omega_n)^2 = 0.$$

The repeated root gives two independent free solutions $e^{-\omega_n t}$ and $te^{-\omega_n t}$. Imposing $x(0) = x_0, \dot{x}(0) = v_0$ yields

$$x(t) = [x_0 + (v_0 + \omega_n x_0)t]e^{-\omega_n t}.$$

For release from rest at positive displacement, the bracket stays positive and $\dot{x} = -\omega_n^2 x_0 t e^{-\omega_n t}$ stays negative for $t > 0$. If a mechanism reaches the same x_0 with negative incoming velocity, the bracket can cross zero once even at critical damping. In particular $v_0 < -\omega_n x_0$ gives crossing time $-x_0/(v_0 + \omega_n x_0)$. Critical damping rules out repeated free oscillations, but the initial velocity still determines whether the target is overshot.

For overdamping the distinct real decay rates are $\omega_n[\zeta \pm \sqrt{\zeta^2 - 1}]$. Their product is ω_n^2. Increasing viscous resistance makes the fast root decay more rapidly but the slow root approach zero. Simply adding damping can consequently prolong the final approach to a sensing threshold.

Select a damper for a spring-return slide

A slide of mass 3.0 kg has a return spring of 300 N/m, so $\omega_n = 10\,\mathrm{rad/s}$ and $c_{\mathrm{cr}} = 60\,\mathrm{N\,s/m}$. Released from rest at $x_0 = 50$ mm with critical damping, its normalised position is $(1 + 10t)e^{-10t}$. To enter a 5% band, solve $(1 + s)e^{-s} = 0.05$ with $s = 10t$. Testing 4.7 and 4.8 gives 0.0518 and 0.0477; the crossing is about 4.74, or $t = 0.474$ s.

The same slide at the same 50 mm location might arrive with $v_0 = -0.80\,\mathrm{m/s}$ after an actuator disconnects. The bracket is now $0.050 - 0.30t$, which crosses at 0.167 s. Differentiating gives its negative peak at $t = 0.267\,\mathrm{s}$, with an overshoot magnitude about 2.08 mm. An optical stop positioned only 1 mm beyond equilibrium would register a crossing even though the damper is critical. Compare that with a slow overdamped return from rest: for $\zeta = 2$ its dominant decay rate falls to $10(2 - \sqrt{3}) = 2.68\,\mathrm{s^{-1}}$. An assessment based simply on whether the motion oscillates would overlook both the clearance requirement and this much slower return.

The return time depends on which event the instrumentation calls completion. Some switches trigger when the position first enters a band and remain latched even if it later leaves. For the critical release with negative entering speed, first crossing the equilibrium plainly does not mean motion has stopped. A position-plus-velocity criterion is stricter. At the first zero crossing, the bracket in x vanishes, so the velocity is its constant derivative multiplied by the exponential, $-0.30e^{-10(0.167)} = -0.0567\,\mathrm{m/s}$. A slide with a maximum docking speed of 0.020 m/s would fail that condition despite its position being exactly correct at that instant. The appropriate response test must state both allowed path excursion and the speed at any mechanical contact.

Checking only the first displacement peak cannot identify this velocity-limited docking event.

Distinguish sliding-friction decay from a viscous ring-down

Suppose a slide moves on a dry horizontal guide under a spring force $-kx$. While the slide moves left, kinetic friction F_c acts right, and $m\ddot{x} + kx = +F_c$. The solution is harmonic about the shifted equilibrium $+F_c/k$. During rightward sliding the equilibrium switches to $-F_c/k$. A complete undisturbed sliding half-cycle still lasts $\pi\sqrt{m/k}$, but friction dissipates energy in proportion to distance travelled. At zero speed, static friction can take any value up to F_s, so checking the spring force before starting the next half-cycle is essential.

Let a leftward excursion start at $x = A > 0$ and reach a turning point at $x = -B$, with $A, B > 0$. The spring energy drop equals friction work,

$$k(A^2 - B^2)/2 = F_c(A + B).$$

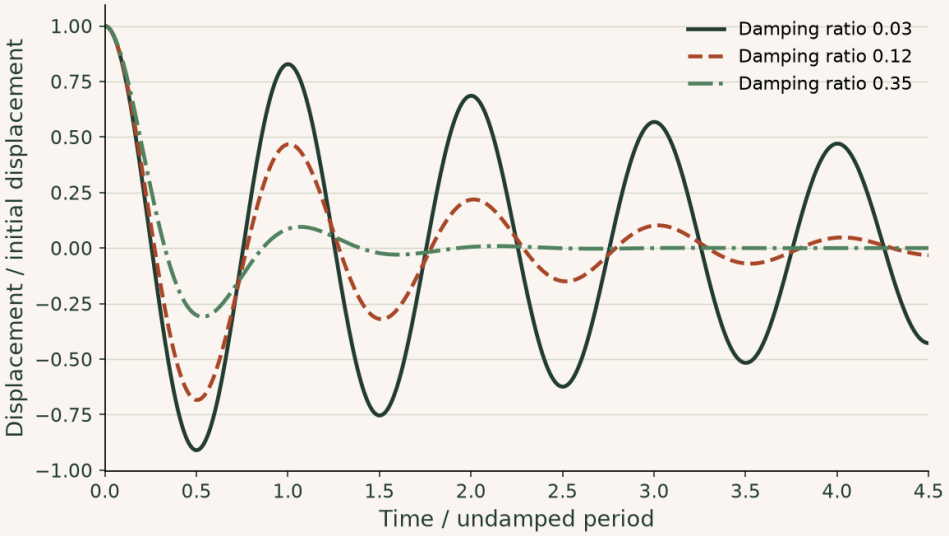
Cancellation leaves $B = A - 2F_c/k$. Same-side peak magnitudes fall by $4F_c/k$ each complete sliding cycle. This is a constant absolute decrement. A viscous exponential gives a constant ratio instead. If a late half-cycle does not cross zero, use its signed turning-point coordinate $x_{\text{next}} = 2F_c/k - x_{\text{start}}$ for leftward motion and apply the static criterion there.

A 2 kg slide has $k = 800\,\text{N/m}$, $F_c = 2.0$ N and $F_s = 2.4$ N. Release it from 18 mm. Its initial spring force is 14.4 N, so it moves. Each half-cycle reduces the alternating peak magnitude by 5.0 mm, giving signed positions $+18, -13, +8, -3$ mm. At −3 mm the spring pull is 2.4 N, equal to the static threshold. Under an ideal static-friction law it can remain there. It took $3\pi\sqrt{2/800} = 0.471$ s to reach that position. The final displacement need not be zero. Interpreting its first two positive peaks with a logarithmic decrement would give a damping estimate that fails to predict the finite stop.

Practise on sheet 1 · Identify a dry guide from three same-side peaks. Further reading · IIT Kharagpur · Aircraft Structural Dynamics · Module 4, Lecture 3, Free Vibration of Damped System (Contd.), “Coulomb Damping”.

Estimate damping from repeated peaks

Release an initially displaced mass with zero initial velocity. The underdamped waveform contains both cosine and sine terms; the latter enforces zero initial velocity. Compare successive same-sign peaks using a logarithmic decrement.



At damping ratio 0.12 each positive peak retains $\exp(-2\pi\zeta/\sqrt{(1-\zeta^2)}) \approx 0.468$ of the previous one. Taking consecutive positive and negative extrema instead halves the interval and changes the inferred decrement.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

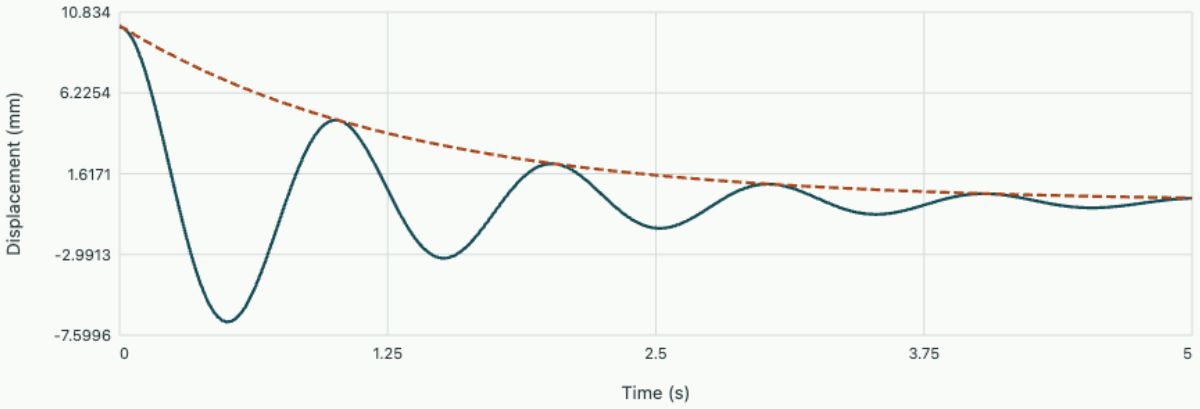
Change damping before changing natural frequency. Separate the decay per cycle from the number of cycles per second.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Damping ratio	0.12
Natural frequency (Hz)	1
Initial displacement (mm)	10



Release from rest Exponential envelope

Successive same-sign peak ratio 0.468. Damped period 1.007 s.

Study the full derivation

Parametric resonance under periodic stiffness modulation · Vibration and identification in machinery

A carriage supported by a spring whose stiffness changes periodically can gain oscillation energy with no additive force at its natural frequency. Compare this mechanism with an ordinary forced resonance, estimate the first instability band and learn how to measure growth before nonlinear saturation invalidates the prediction.

Problem sheet 1 · Equivalent inertia and ring-down

Refer all energy terms to one coordinate, then use measured peaks to identify a vibrating support.

1. Accelerate a conveyor

A roller of inertia 0.080 kg·m² and radius 0.12 m drives an effective translating belt mass of 30 kg without slip. A constant resisting belt force of 40 N acts. (a) Find the roller-referred inertia. (b) Find the torque for belt acceleration 0.60 m/s². (c) At belt speed 1.20 m/s, check the input power against friction and the kinetic-energy increase.

▼ **Reduce and check the drive**

The inertia is $0.080 + 30(0.12)^2 = 0.512$ kg·m². Angular acceleration is $0.60/0.12 = 5.0$ rad/s² and the resistance torque is $40(0.12) = 4.80$ N·m. The total torque is

$$4.80 + 0.512(5) = 7.36\,\mathrm{N\,m}.$$

At 10 rad/s this supplies 73.6 W. Resistance consumes 48.0 W and kinetic energy increases at $0.512(10)(5) = 25.6$ W, giving the same total.

---

2. Identify an unknown damper

A 12 kg instrument rings down from 6 mm to 3 mm over exactly three full cycles taking 0.75 s. (a) Infer ω_d and envelope rate α. (b) Obtain k, c, ζ. (c) Predict when the envelope will reach 1.5 mm and state one measurement that could falsify the viscous model.

▼ **Fit the measured peaks**

The period is 0.25 s, so $\omega_d = 25.13$ rad/s. The envelope satisfies $\alpha = \ln(6/3)/0.75 = 0.924$ s⁻¹. Thus $c = 2m\alpha = 22.2$ N·s/m, $k = m(\omega_d^2 + \alpha^2) = 7.59$ kN/m and $\zeta = 0.0367$. Another 0.75 s halves the envelope again, so 1.5 mm is expected 1.50 s after the first peak. If successive same-sign peak ratios change substantially above the noise floor, a constant viscous damper does not explain the trace by itself.

---

3. Identify an effective mass with an added mass

A nearly undamped mount oscillates at 5.00 Hz. A compact added mass of 4.50 kg lowers the frequency to 4.00 Hz. (a) Infer the original effective mass. (b) Find the spring stiffness. (c) Estimate the static additional sag and identify the assumption needed to compare with a ruler.

▼ **Solve the two frequency equations**

If the stiffness does not change, $(5/4)^2 = (m + 4.50)/m$, giving $m = 8.00$ kg. Therefore $k = 8(2\pi 5)^2 = 7.90$ kN/m. The extra weight gives sag $4.5(9.81)/7896 = 5.59$ mm for a vertical mount. This comparison requires the measured displacement and added mass to participate in the same coordinate and the spring to remain linear over the extra static travel.

Identify a dry guide from three same-side peaks

A slide with stiffness 1.20 kN/m has successive positive maxima at 21, 15 and 9 mm before sticking occurs. Find the kinetic friction and predict its following negative turning point. Would it restart from there if static friction can supply 8 N?

▼ **Show the worked solution**

The constant full-cycle decrement is 6 mm, hence

$$F_c = k(0.006)/4 = 1.80\,\mathrm{N}.$$

The half-cycle decrement is 3 mm, so a release from +9 mm reaches –6 mm. There the spring force is $1200(0.006) = 7.2$ N, within the 8 N static limit. It remains at –6 mm. The 9 mm positive peak can start this last half-cycle because its 10.8 N spring force exceeds the same static bound.

Revisit chapter 2 · Distinguish sliding-friction decay from a viscous ring-down.

Explore the governing equations after solving the sheet: Coordinates, energy and equivalent inertia; Free vibration, energy loss and ring-down data. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Chapter 3 · Harmonic forcing, phase and dissipated power

Predict the steady vibration under a rotating force and relate the phase lag to the power absorbed by damping.

FROM FIRST PRINCIPLES

Obtain forced vibration amplitude from dynamic equilibrium

A machine on a linear spring and viscous damper is driven by force $F_0 \cos \omega t$ on its mass m. Newton's second law sums inertia, damping and stiffness forces. Assume a stable mount and wait until the transient part has decayed. Seek a sinusoidal response at the forcing frequency, written $x(t) = \mathrm{Re}[X e^{i\omega t}]$ to retain both magnitude and phase.

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t, \qquad (k - m\omega^2 + ic\omega)X = F_0.$$

The real part of the dynamic stiffness combines spring and inertia, while the damping contribution is in quadrature. Taking its modulus gives displacement amplitude; taking its argument gives the lag of displacement behind the applied force.

$$|X| = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}},$$

$$\phi = \operatorname{atan2}(c\omega, k - m\omega^2).$$

At low frequency the displacement approaches static deflection F_0/k. Near $\sqrt{k/m}$ the spring and inertia terms cancel, leaving damping to limit the steady response. High-frequency response falls approximately as $F_0/(m\omega^2)$ for a fixed force amplitude.

Background · Damped frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

This free-decay oscillation frequency exists in the underdamped regime. A resonance peak in a driven response uses a different frequency condition.

DERIVATION AND APPLICATIONS

Separate displacement resonance from maximum absorbed power

The frequency of largest displacement can differ from the frequency of largest mean heating in a damper. Drive a single mass m on a linear spring k and viscous damper c with a fixed peak force F at angular frequency ω. Assume sinusoidal steady state, a rigid base and coefficients independent of

amplitude. With complex displacement amplitude X, mean energy dissipated per unit time is c times mean squared velocity. Force equilibrium gives

$$X = \frac{F}{k - m\omega^2 + ic\omega},$$
$$\bar{P} = \frac{1}{2} c\omega^2 |X|^2.$$

Rewrite the power by dividing numerator and denominator by ω^2. The resulting mechanical impedance has a real dissipative part c and a reactive part $m\omega - k/\omega$. The power peak occurs where this latter term vanishes.

$$\bar{P} = \frac{cF^2/2}{c^2 + (m\omega - k/\omega)^2},$$
$$\omega_{\text{peak}} = \sqrt{k/m},$$
$$\bar{P}_{\max} = \frac{F^2}{2c}.$$

At half maximum the magnitude of the reactive term equals c. The positive solutions of the two resulting quadratic equations give lower and upper half-power frequencies. Their difference is exact for this force-driven, velocity-power measurement.

$$\omega_{\pm} = \frac{\sqrt{c^2 + 4mk} \pm c}{2m},$$
$$\omega_+ - \omega_- = \frac{c}{m},$$
$$\omega_+ \omega_- = \frac{k}{m}.$$

By comparison, the displacement peak contains the additional $1/\omega$ in converting velocity to displacement, so applying these exact bandwidth relations to a displacement trace requires care.

WORKED EXAMPLE

Predict the heating bandwidth

Use $m = 2.00$ kg, $k = 800$ N/m, $c = 8.00$ N s/m and $F = 4.00$ N. The peak is at 20.0 rad/s and maximum mean dissipation is

$$16/16 = 1.00\,\text{W}.$$

The half-power frequencies are 18.10 and 22.10 rad/s. At the peak, velocity is in phase with force and has peak amplitude $F/c = 0.500$ m/s. The corresponding displacement peak amplitude at this frequency is

25.0 mm.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Identify spring and damper from two half-power points

Another oscillator of mass 1.50 kg has measured half-power frequencies of 12.0 and 18.0 rad/s. Their difference gives $c = 1.50(6.00) = 9.00$ N s/m and their product gives

$$k = 1.50(12)(18) = 324\,\mathrm{N/m}.$$

The predicted power peak is at the geometric mean, $\sqrt{216} = 14.7$ rad/s. A direct power sweep can test this prediction without a calibrated displacement amplitude.

Interactive calculation · Change the inputs of this example

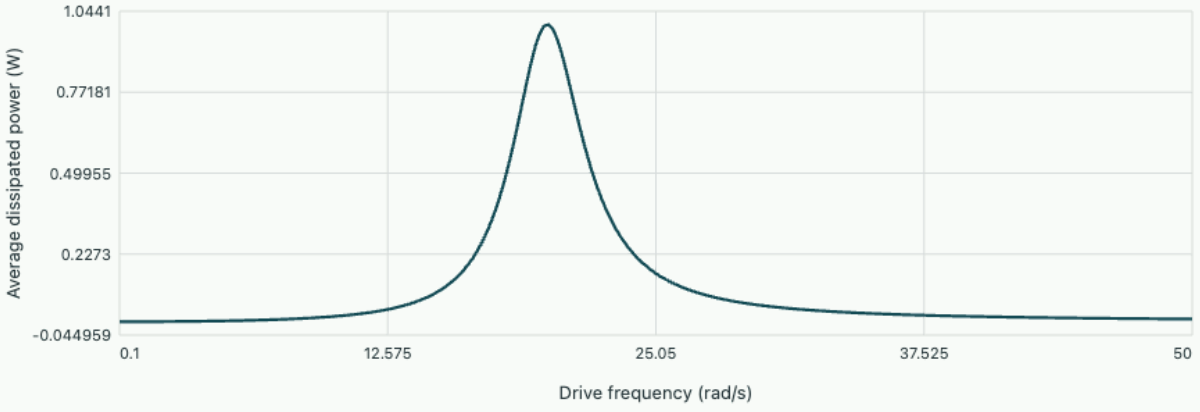
Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Damper (N s/m) **8**

Force amplitude (N) **4**



Damping heat

At natural frequency20 rad/s, maximum power=1.000 W. Half-power bandwidth c/m=4.000 rad/s for mass2 kg.

Extract one sinusoidal component

A slightly eccentric rotating part exerts a periodic force on its housing. Measure the housing displacement from static equilibrium and first represent a single transverse force component by $F_0 \cos \omega t$. Linear response then satisfies

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t.$$

Its total solution contains the free transient plus a steady periodic part. Wait several decay times before comparing measured peak amplitudes with a harmonic prediction.

Write the steady part as $\mathrm{Re}\{X e^{i\omega t}\}$, taking the real force phasor F_0 as phase reference. Each differentiation contributes $i\omega$. The displacement phasor therefore satisfies

$$(k - m\omega^2 + ic\omega)X = F_0.$$

The factor in parentheses is dynamic stiffness in N/m. At low speed the phasor is close to F_0/k. At high speed the inertia term dominates and displacement is almost opposite the applied force. A phase measurement can distinguish those regimes even if their displacement amplitudes happen to be equal.

Read magnitude and phase separately

Introduce $r = \omega/\omega_n$, $\omega_n = \sqrt{k/m}$ and $\zeta = c/(2m\omega_n)$. Take the magnitude and argument of the dynamic stiffness to obtain

$$|X|/(F_0/k) = [(1 - r^2)^2 + (2\zeta r)^2]^{-1/2}.$$

The displacement lags by $\phi = \operatorname{atan2}(2\zeta r, 1 - r^2)$. At $r = 1$ elastic and inertia forces cancel in phase and the remaining quadrature force is viscous. The displacement lags by 90 degrees and has amplitude $F_0/(c\omega_n)$.

For constant force amplitude, maximum displacement occurs at $r = \sqrt{1 - 2\zeta^2}$ if $\zeta < 1/\sqrt{2}$. This follows by differentiating the squared denominator with respect to r. Thus the frequency of greatest displacement need not coincide exactly with the undamped natural frequency. A rotating imbalance also changes force amplitude with speed, so its peak cannot be inferred from a constant-force plot alone.

Check a resonant measurement by energy

Let $m = 10$ kg, $k = 4000$ N/m, $c = 40$ N·s/m and $F_0 = 8.0$ N. Then $\omega_n = 20$ rad/s and $\zeta = 0.10$. At resonance the predicted displacement amplitude is

$$8/[40(20)] = 10.0\,\mathrm{mm}.$$

The velocity amplitude is 0.200 m/s. The time-average input power of a sinusoidal force and velocity is half the product of their peak magnitudes times the cosine of their phase difference. Since velocity is in

phase with force at this frequency, the average is 0.800 W.

Independently, the damper dissipates $\overline{P} = c\,\overline{\dot{x}^2} = c\omega^2|X|^2/2 = 0.800$ W. The spring returns the energy it stores each cycle and the mass has no net change in kinetic energy. If force and displacement peaks are simply multiplied and reported as watts, the dimensions and the missing phase both reveal the error.

Apply the result to a rotor

A rotating eccentric mass m_e displaced a radius e generates force amplitude $m_e e\omega^2$ when its circular motion is prescribed. With $m_e e = 0.020\,\mathrm{kg\,m}$ and $\omega = 20$ rad/s, this is the 8.0 N used above. At 40 rad/s the applied force grows to 32 N. The dynamic stiffness magnitude becomes $\sqrt{(4000 - 10(40)^2)^2 + (40(40))^2} = 12.1$ kN/m, giving about 2.64 mm displacement. The force grew fourfold while the response fell because the system moved away from resonance. A speed sweep therefore needs both the changing excitation and the changing dynamic stiffness.

Estimate damping from a frequency sweep

A useful second identification uses the vibration velocity at constant force amplitude. At resonance the velocity amplitude is F_0/c. At the two frequencies where mean dissipated power has fallen to half its maximum, the velocity amplitude is $F_0/(\sqrt{2}c)$. Inverting the velocity response gives

$$|k/\omega - m\omega| = c.$$

Solving the two sign cases and subtracting gives the bandwidth $\omega_2 - \omega_1 = c/m$ for this viscous single-mode system. In the numerical example the difference is 4 rad/s; the predicted limits are about 18.10 and 22.10 rad/s. This result applies to the power or velocity response at constant force amplitude. A sweep driven by a growing rotor imbalance would also require correcting the measured amplitudes for their changing excitation.

Decompose a nonsinusoidal periodic machine force

A cam, pump or reciprocating actuator can generate a nearly rectangular force over each cycle. For a linear system in periodic steady operation, expand this force into Fourier components and apply the dynamic stiffness separately at each harmonic. The construction assumes linear coefficients over the motion range and that the free transient has decayed. A periodic input does not imply that the fundamental component dominates the measured displacement; a higher harmonic near a natural frequency can be amplified strongly.

For a symmetric square force $F(t) = F_s\ \mathrm{sgn}(\sin \Omega t)$, the mean vanishes and half-wave symmetry cancels all even harmonics. Its sine coefficient over one period is $b_n = (2/\pi)\int_0^{\pi} F_s \sin(n\theta)d\theta$ for the half-wave-symmetric terms, giving $4F_s/(n\pi)$ for odd n. Each odd harmonic has its own displacement amplitude and phase:

$$X_n = \frac{4F_s/(n\pi)}{\sqrt{[k - m(n\Omega)^2]^2 + (cn\Omega)^2}},$$

$$\phi_n = \operatorname{atan2}(cn\Omega, k - m(n\Omega)^2).$$

Reconstruct the time history by adding $X_n \sin(n\Omega t - \phi_n)$. Adding the magnitudes gives a peak upper bound, not generally the actual peak because phase shifts differ. Orthogonality gives an exact mean-square contribution $X_n^2/2$ for each distinct harmonic, providing a convenient comparison of vibration levels without finding coincident maxima.

Identify the dominant vibration in an indexing machine

Take $m = 5.0\,\mathrm{kg}$, $k = 4500\,\mathrm{N/m}$, and $c = 9.0\,\mathrm{N\,s/m}$, so $\omega_n = 30\,\mathrm{rad/s}$ and $\zeta = 0.03$. The force alternates between ±12 N at fundamental $\Omega = 10\,\mathrm{rad/s}$. The first-harmonic force is

$$48/\pi = 15.28\,\mathrm{N}.$$

Its dynamic stiffness magnitude is $\sqrt{4000^2 + 90^2} = 4001\,\mathrm{N/m}$, giving $X_1 = 3.82\,\mathrm{mm}$ with a small lag. The third harmonic has force 5.09 N but falls exactly at the natural frequency, giving $X_3 = 5.09/(9 \times 30) = 18.9\,\mathrm{mm}$ at 90° lag.

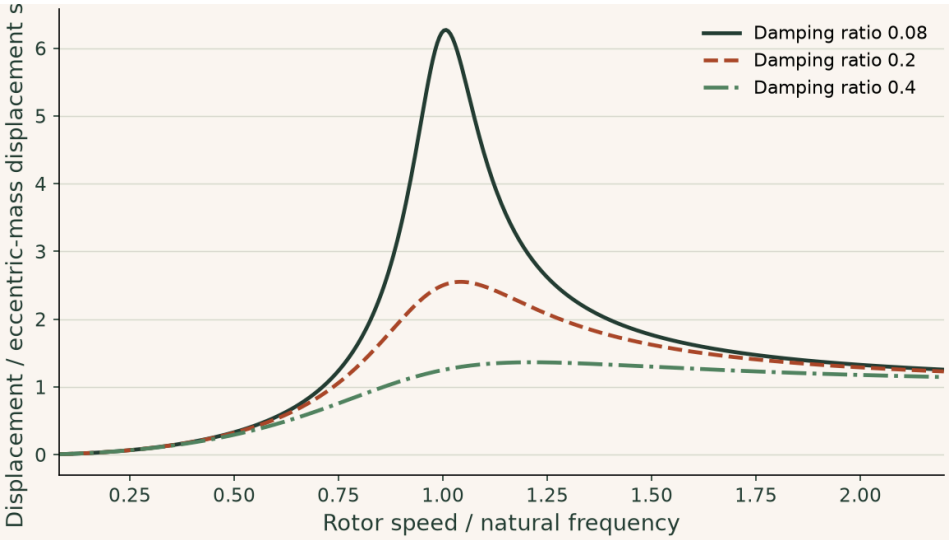
At 50 rad/s the fifth component has force 3.06 N and stiffness magnitude $\sqrt{8000^2 + 450^2}$, giving only 0.381 mm. The first three nonzero harmonics give RMS displacement $\sqrt{(3.82^2 + 18.9^2 + 0.381^2)/2} = 13.6\,\mathrm{mm}$. The largest measured spectral peak would appear at three times the operating frequency. It could arise from the drive's waveform rather than a separate three-times-speed rotating fault. Altering the rise time of the periodic force can reduce high-order Fourier coefficients, while adding damping around this resonance reduces the mechanical amplification. Which intervention addresses the observed peak can be tested by measuring the force spectrum as well as the motion.

The energy entering the damping element gives a further spectral check because high frequencies carry more velocity for the same displacement. The mean dissipated power of harmonic n is $c(n\Omega X_n)^2/2$. In the example the first harmonic dissipates about $9(10 \times 0.00382)^2/2 = 0.0066$ W, while the third dissipates $9(30 \times 0.0189)^2/2 = 1.45$ W. The third harmonic therefore dominates heating even more strongly than it dominates displacement. A sensor attached near a nodal point could suppress a local displacement reading, but the overall energy dissipation still has to be provided by the drive. Synchronous force and velocity measurements over a whole period can independently check this average input power.

A centred force–velocity recording also excludes a false average power component introduced by a sensor's DC offset.

Which speed excites an unbalanced rotor most strongly?

A rotating unbalance excites a single lateral mode with force $m_e e \omega^2$. Divide the harmonic displacement by $m_e e / m$. The resulting factor has r^2 in its numerator because the exciting force grows with speed.



The response tends toward one at high speed and peaks just above the undamped natural frequency for these damping ratios. Treating unbalance as a constant-amplitude shaker force predicts the wrong high-speed asymptote.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

Increase damping and compare the resonance neighbourhood with the high-speed asymptote.

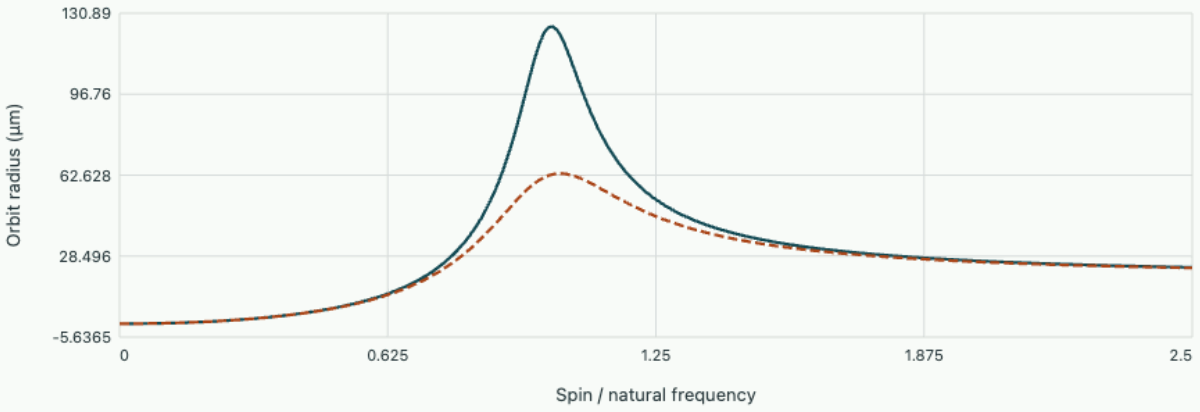
The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Damping ratio **0.08**

Unbalance / rotor mass (µm) **20**



Selected damping Twice damping

At resonance the amplitude is 125.000 µm; its high-speed limit is 20.000 µm because the excitation grows with speed squared.

Study the full derivation

Random base motion, spectral density and rms response · Vibration and identification in machinery

A support vibration is described by a power spectral density rather than a single sine-wave amplitude. Derive the relative-motion filter for a mounted instrument, calculate mean-square motion from a measured spectrum and distinguish a stationary rms prediction from a peak displacement guarantee.

Chapter 4 · Coupled oscillators and normal coordinates

Derive and interpret the two modes of a coupled machine, then decide which mode an applied load can excite.

FROM FIRST PRINCIPLES

Find natural modes from the free force balance

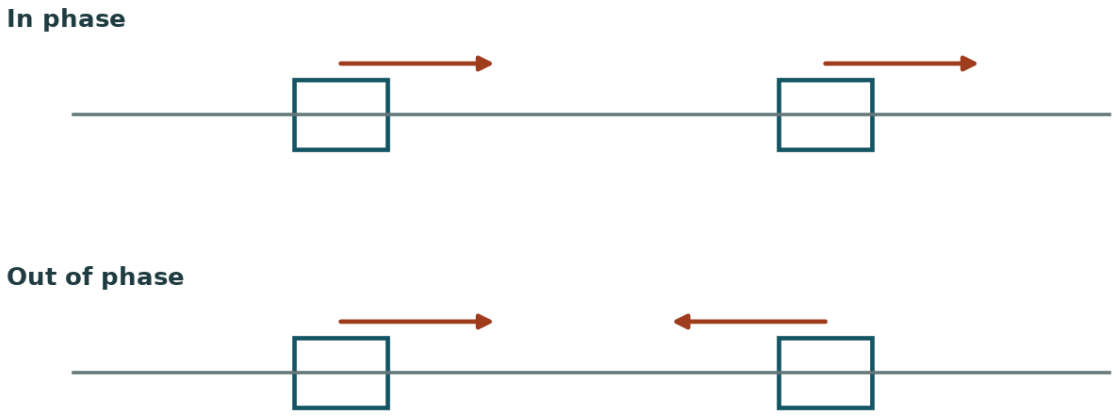
For small motions of a conservative system with several coordinates, linear inertia forces are represented by a positive-definite mass matrix M and linear restoring forces by a symmetric stiffness matrix K. Collect the free balance and try a motion in which the coordinate amplitudes stay in one fixed ratio as each oscillates at frequency ω.

$$M\ddot{\mathbf{x}} + K\mathbf{x} = 0,$$
$$\mathbf{x} = \boldsymbol{\phi}\cos(\omega t) \implies (K - \omega^2 M)\boldsymbol{\phi} = 0.$$

A nonzero shape $\boldsymbol{\phi}$ can solve these homogeneous equations only when the coefficient matrix is singular. The determinant condition gives candidate squared frequencies, and the nullspace at each frequency gives a mode shape up to scale.

$$\det(K - \omega^2 M) = 0,$$
$$K\boldsymbol{\phi}_j = \omega_j^2 M\boldsymbol{\phi}_j.$$

For identical masses linked by a spring, a common translation leaves that link unstretched and an opposite motion stretches it from both ends. This observation identifies common and differential shapes even before evaluating the determinant. Initial displacement and velocity select their amplitudes; the eigenproblem identifies the motions the structure can support.



Which spring deforms in each mode?
Assume equal end springs k, equal masses m and a central coupling spring k^c. A common displacement leaves the coupling spring at its original length. In the lower drawing each mass moves a distance x towards the other, compressing the central spring by 2x. Half a cycle later the same spring extends by 2x. The ratios of restoring force to displacement are k and k + $2k^c$, giving squared frequencies k/m and (k + $2k^c$)/m.

DERIVATION AND APPLICATIONS

Quantify the splitting of two weakly coupled modes

When two machines have the same uncoupled natural frequency, adding a weak connecting spring splits that frequency into two. Consider identical sliders of mass m each attached to ground by stiffness k and linked to each other by stiffness k_c. Neglect damping and choose displacements in the same direction from equilibrium. The coupling spring responds only to the displacement difference. Adding and subtracting the two force balances therefore gives separate equations for $s = (x_1 + x_2)/2$ and $d = (x_1 - x_2)/2$.

$$m\ddot{s} + ks = 0,$$

$$m\ddot{d} + (k + 2k_c)d = 0.$$

Release slider one at A and slider two at zero, both from rest. The two modal coordinates then both start at $A/2$. Write $\omega_s = \sqrt{k/m}$ and $\omega_d = \sqrt{(k + 2k_c)/m}$ and combine their solutions.

$$x_1 = \tfrac{A}{2}(\cos\omega_s t + \cos\omega_d t),$$

$$x_2 = \tfrac{A}{2}(\cos\omega_s t - \cos\omega_d t).$$

Introduce $\bar{\omega} = (\omega_s + \omega_d)/2$ and $\Delta\omega = \omega_d - \omega_s$. The sum-to-product identities separate the fast motion from slow modulation. When the split is small, the modulation resembles a repeated transfer of motion between the sliders.

$$x_1 = A\cos(\bar{\omega}t)\cos(\Delta\omega t/2),$$
$$x_2 = A\sin(\bar{\omega}t)\sin(\Delta\omega t/2).$$

The first minimum of the slider-one displacement envelope occurs at $t = \pi/\Delta\omega$. Its instantaneous velocity there need not vanish because the envelope itself is changing. The envelope timing remains identifiable from repeated peaks at weak coupling.

WORKED EXAMPLE

Time the exchange of motion

For $m = 1.00$ kg, $k = 100$ N/m and $k_c = 10.5$ N/m, the two frequencies are 10.0 and 11.0 rad/s. The first envelope minimum appears at 3.14 s and the first full recovery of its magnitude at 6.28 s. Releasing the first slider by 8.00 mm sets both modal initial amplitudes to 4.00 mm. Releasing both sliders together instead excites only the slower common motion.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Estimate coupling from a beat measurement

A second pair has mass 2.00 kg per slider and measured lower frequency 25.0 rad/s. Consecutive magnitude maxima of one slider's envelope occur 5.00 s apart. Thus

$$\Delta\omega = 2\pi/5.00 = 1.257\,\mathrm{rad/s},$$
$$k_c = m[(25 + 1.257)^2 - 25^2]/2 = 64.4\,\mathrm{N/m}.$$

The independently inferred grounding stiffness is $m\omega_s^2 = 1250$ N/m, making the connector about 5.2 percent as stiff.

Interactive calculation · Change the inputs of this example

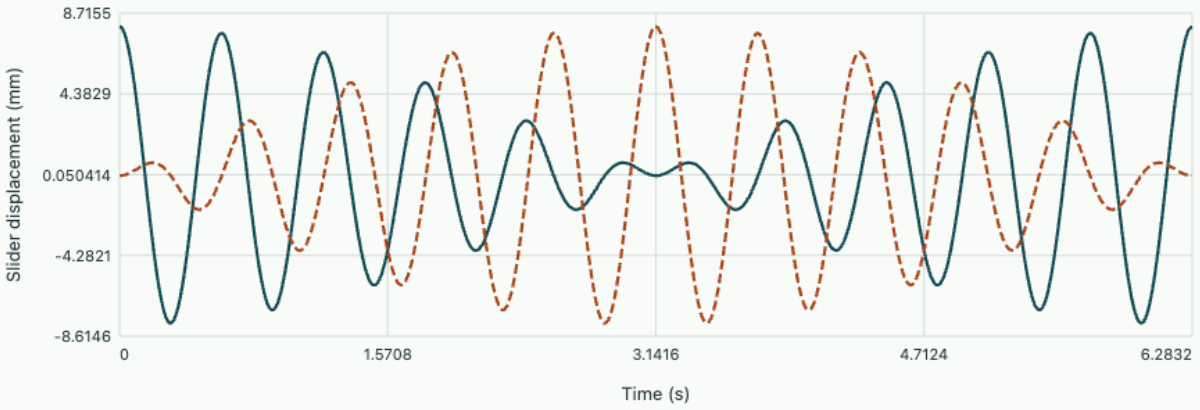
Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Coupling stiffness (N/m) **10.5**

Initial release (mm) **8**



Frequencies 10.000 and 11.000 rad/s; first envelope minimum at 3.142 s. Equal masses1 kg; grounded stiffness100 N/m.

Start from this chapter. A mode sensitive to rotary inertia carries appreciable rotational kinetic energy. Compare the fractional change in each eigenfrequency under the same imposed change of tip inertia.

OneSimulation laboratory · Separate the effect of mass from rotary inertia

Two masses allow two independent shapes

Two identical instruments sit on separate spring mounts and are connected by a flexible link. Pull both by the same amount and the link keeps its natural length. Pull them in opposite directions and the link stretches twice as much as either displacement. The two tests reveal distinct restoring mechanisms. A model with one equivalent spring and the sum of both masses would discard one of them.

Let x_1, x_2 be horizontal displacements from equilibrium. Each mass is m, each ground spring has stiffness k, and the coupling spring has stiffness k_c. With light springs, small displacements and no damping, the kinetic energy is $m(\dot{x}_1^2 + \dot{x}_2^2)/2$ and the elastic energy is $k(x_1^2 + x_2^2)/2 + k_c(x_1 - x_2)^2/2$. The coupling energy contains the difference of displacements because it is the extension between its ends.

Separate space from time

Taking the derivative of elastic energy with respect to each coordinate gives $m\ddot{x}_1 + (k + k_c)x_1 - k_c x_2 = 0$ and the symmetric second equation. Look for $\mathbf{x} = \boldsymbol{\phi}\cos\omega t$. Nonzero modal shapes solve $(\mathbf{K} - \omega^2\mathbf{M})\boldsymbol{\phi} = 0$. For this pair the determinant equation is

$$(k + k_c - m\omega^2)^2 - k_c^2 = 0.$$

Taking both signs of the square root gives

$$\omega_1^2 = k/m,$$

$$\omega_2^2 = (k + 2k_c)/m.$$

Substitution back yields shapes proportional to $(1, 1)$ and $(1, -1)$. For the first, the connector transmits no incremental force. For the second, the connector acts strongly. Reversing both signs of a plotted eigenvector or multiplying it by a nonzero number describes the same physical mode.

Predict a release test at both instruments

Take $m = 5.0$ kg, $k = 2000$ N/m and $k_c = 1500$ N/m. The two angular frequencies are 20.0 and 31.62 rad/s, or 3.18 and 5.03 Hz. Pull only the first instrument 4.0 mm and release from rest. The initial displacement vector equals $2(1, 1) + 2(1, -1)$ mm. Each modal amplitude starts at its corresponding value with zero initial velocity. Hence

$$x_1(t) = 2\cos(20t) + 2\cos(31.62t),$$

$$x_2(t) = 2\cos(20t) - 2\cos(31.62t)$$

in millimetres. At zero time the second displacement vanishes while the connector exerts 6.0 N on its mass. Its initial acceleration is 1.2 m/s² toward the first instrument’s displacement. This apparent transfer of motion needs neither forcing nor damping: modal displacements add and cancel differently at the two observation points.

Distinguish available modes from visible modes

Define average and relative coordinates $q_s = (x_1 + x_2)/2$ and $q_a = (x_1 - x_2)/2$. Adding or subtracting the original equations decouples them. Equal external forces excite only q_s; equal and opposite forces excite only q_a. Taking the signal $(x_1 + x_2)/2$ likewise hides the antisymmetric mode. A

missing spectral peak can therefore reveal the symmetry of a load or sensor arrangement, even though the corresponding eigenmode still exists.

If the two masses differ, the simple sum and difference cease to diagonalise the equations. The general route is to form the physical mass and stiffness matrices and solve the symmetric generalised eigenproblem. Distinct modes of an undamped conservative system are orthogonal with respect to mass, $\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_j = 0$. This weighted orthogonality is what permits a measured motion or distributed load to be projected into separate modal coordinates.

Predict which mode responds to adding rotary inertia

For a unit mode shape ϕ, the quotient $\omega^2 = \phi^T K \phi / (\phi^T M \phi)$ compares stored elastic energy with inertia. Increasing inertia at a coordinate matters most when that coordinate has appreciable motion in the mode. A frame with three tip coordinates offers a way to check this reasoning beyond the symmetric two-mass illustration.

Compare two changes to the same frame: double only the tip mass from 200 to 400 kg, then restore it and double only the tip rotary inertia from 60 to 120 kg m². Predict whether either change can raise a natural frequency for the same stiffness and compute the fractional shift of the third mode in each case. Use the negligible-member-mass row throughout. Check the frame frequency measurements.

Derive bounce and pitch modes of a supported machine

A long machine supported on two mounts can translate and rotate as a nearly rigid body even if there is negligible frame bending. Let the mass centre move vertically by z and rotate by small angle θ, with positive rotation raising a point to the right. Vertical spring deflections relative to ground are $z - a\theta$ on the left and $z + b\theta$ on the right. Measure displacements from static equilibrium. The kinetic energy is $(m\dot{z}^2 + I_G\dot{\theta}^2)/2$, and the spring energy is $[k_L(z - a\theta)^2 + k_R(z + b\theta)^2]/2$.

Differentiating potential energy gives stiffness terms $K_{zz} = k_L + k_R,\ K_{\theta\theta} = k_L a^2 + k_R b^2,\ K_{z\theta} = -k_L a + k_R b$. The mass matrix is diagonal because translation was measured at the centre of mass. A vertical translation produces a moment whenever the weighted mount locations fail to balance, and a pure rotation then also produces a vertical force by reciprocity. If $k_L a = k_R b$ the modes are pure translation and pure pitch. Equal spring rates alone do not guarantee decoupling when the lever arms differ.

For free motion solve $(K - \omega^2 M)\phi = 0$. In mixed coordinates, metres and radians label different displacement entries, so a numerical component ratio needs units or a stated length scale. Measured vertical motions at two actual sensor locations provide an unambiguous observable mode shape.

Separate a heaving mode from a rocking mode

Let $m = 200$ kg, $I_G = 40\,\mathrm{kg\,m^2}$, $a = b = 0.50$ m and mount rates $k_L = 30000,\ k_R = 50000\,\mathrm{N/m}$. The stiffness entries are 80000 N/m, 20000 N m/rad and 10000 N in the mixed term. Setting the

determinant to zero gives $(80000 - 200\lambda)(20000 - 40\lambda) - 10000^2 = 0$ with $\lambda = \omega^2$. Dividing through by 8000 yields

$$\lambda^2 - 900\lambda + 187500 = 0.$$

The squared frequencies are 327.5 and 572.5 s⁻², giving 2.88 and 3.81 Hz.

For the lower mode the first row implies

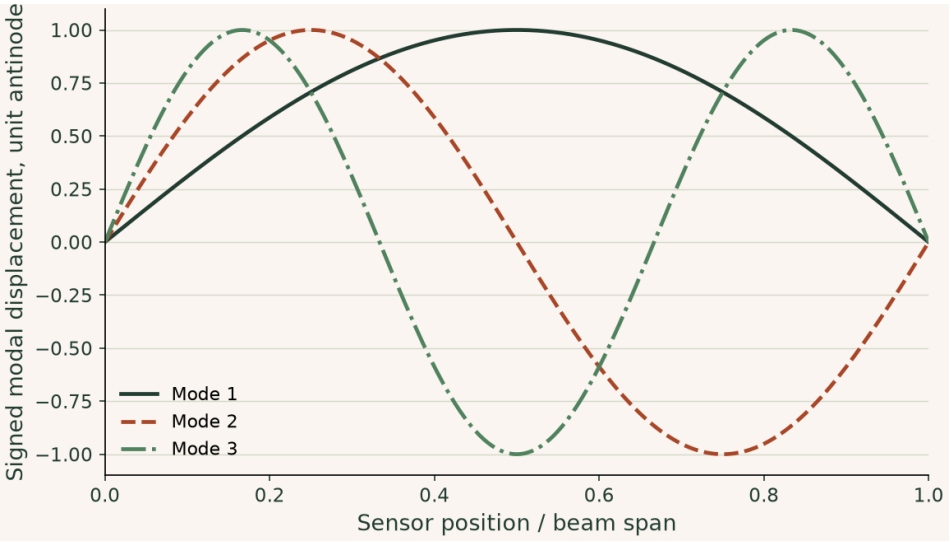
$$\theta/z = -(80000 - 200 \times 327.5)/10000 = -1.45\,\mathrm{rad/m}.$$

If mass-centre motion has 1 mm amplitude, the mount-point displacements are 1.725 and 0.275 mm, in phase but strongly unequal. The higher mode gives $\theta/z = +3.45\,\mathrm{rad/m}$, so the left point now moves oppositely to the right. An accelerometer placed close to a mode's nodal point can scarcely detect it although that mode remains physically present. For the low mode the vertical node lies about $1/1.45 = 0.69$ m to the right of the centre, beyond the right mount. Two well-separated measurement points make the rotational contribution identifiable.

A static load experiment can identify the stiffness coupling before a modal test. Apply a known vertical force F at the centre and no external moment. The pitch-equilibrium equation gives $\theta = -K_{z\theta}z/K_{\theta\theta} = -0.50z$ rad per metre for these mounts. Substitution into the vertical balance leaves effective static stiffness $80000 - 10000^2/20000 = 75000\,\mathrm{N/m}$. Under 150 N centre load, predicted motion is 2.00 mm and pitch −0.0010 rad, giving left and right mount readings of 2.50 and 1.50 mm. A centre gauge alone would only indicate an apparent softening, while the unequal pair exposes the rotational coupling that later appears in the two mode shapes.

Find a sensor position that can see a mode

Use the first three bending modes of an ideal simply supported uniform beam as a continuous analogue of coupled oscillators. A local displacement transducer reports a modal coordinate multiplied by the signed shape at the measurement location.



A midspan sensor misses the second mode even if excitation and damping would produce a large second modal coordinate. At one third span the third mode is invisible. Several sensor locations help distinguish a missing mode from a measurement node.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

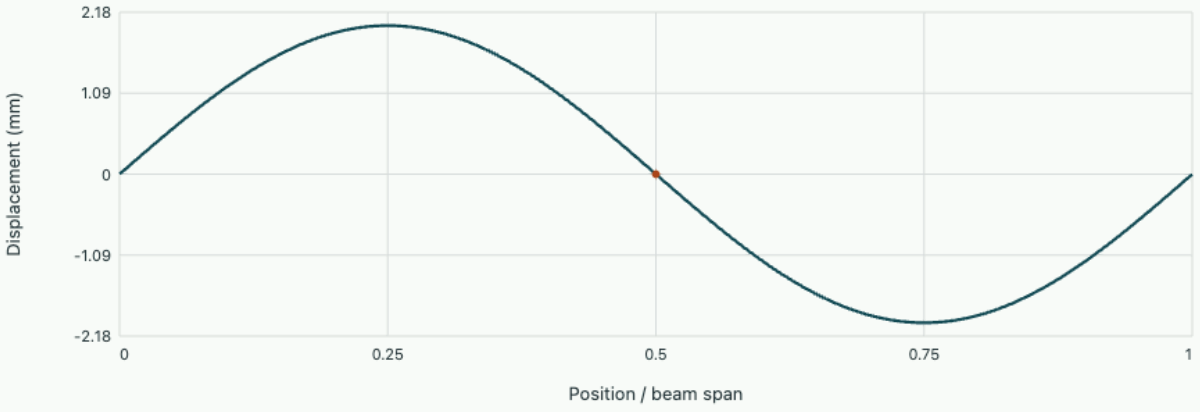
Move the sensor to a node and then change mode number. A zero local reading does not imply a stationary beam.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Mode number	2
Sensor position / span	0.5
Modal amplitude (mm)	2



The sensor reads 0.000 mm. Mode 2 has 1 internal displacement nodes. A zero reading can conceal nonzero modal energy.

Study the full derivation

Why a local damper couples otherwise separate modes · Vibration and identification in machinery

Undamped mode shapes diagonalise a symmetric mass and stiffness model. Adding a damper at one attachment does not generally leave modal equations separate. Derive the resulting coupling, test its magnitude and learn when a diagonal modal-damping approximation can hide an observable phase error.

Problem sheet 2 · Frequency response and modes

Use complex dynamic stiffness and modal symmetry to predict what a force and a sensor can observe.

1. Inspect a forced oscillator

A 2.0 kg mass on a 200 N/m spring and 4.0 N·s/m damper is driven at 10 rad/s by a 1.0 N peak sinusoidal force. (a) Find the displacement amplitude and phase. (b) Find average power. (c) Repeat the amplitude calculation at 5 rad/s with the same force.

▼ **Solve the phasor balance**

The natural angular frequency is 10 rad/s. At this speed the real dynamic stiffness vanishes, giving $|X| = 1/(4 \cdot 10) = 0.0250$ m and 90° displacement lag. The velocity peak is 0.250 m/s and average input power is

$$1(0.25)/2 = 0.125\,\mathrm{W}.$$

At 5 rad/s the dynamic stiffness is $150 + 20i$ N/m, hence

$$|X| = 1/\sqrt{150^2 + 20^2} = 6.61\,\mathrm{mm}.$$

Proximity to resonance has a larger effect than the static-force displacement alone suggests.

---

2. Invert measured modal frequencies

Two identical 4.0 kg masses have equal ground springs and a shared coupling spring. Their measured angular frequencies are 10 rad/s in phase and 16 rad/s in opposition. (a) Infer both stiffnesses. (b) Apply displacements +3 mm and –1 mm. Decompose this state into the two modes. (c) Predict the average displacement over time after release from rest.

▼ **Separate symmetric and antisymmetric coordinates**

The first mode gives

$$k = m\omega_s^2 = 400\,\mathrm{N/m}.$$

The other gives

$$k_c = (4(16)^2 - 400)/2 = 312\,\mathrm{N/m}.$$

The initial mean is 1 mm and the initial half-difference is 2 mm, corresponding to $1(1, 1) + 2(1, -1)$ mm. With zero initial velocities both modes start as cosines. Averaging the two signals removes the

antisymmetric term, leaving $1\,\mathrm{mm}\cos(10t)$. A single average sensor would miss a mode that is actually populated.

---

3. Choose the force pattern

Use the preceding system. Two actuators apply harmonic force amplitudes F and $-F$ simultaneously. (a) Find the drive of the mean coordinate and half-difference coordinate. (b) Which frequency is resonant? (c) If one actuator fails, which resonances can now appear?

▼ **Project the load into each mode**

Adding the two physical equations and dividing by two gives forcing $(F_1 + F_2)/2$ for the mean, which is zero. Subtracting gives $(F_1 - F_2)/2 = F$ for the half-difference. Only the antisymmetric mode is driven, at 16 rad/s. With one actuator operating, the force pattern decomposes as $(F, 0) = (F/2)(1, 1) + (F/2)(1, -1)$, so both 10 and 16 rad/s can produce resonant responses. Real damping limits amplitudes in a steady sweep.

Explore the governing equations after solving the sheet: Harmonic forcing, phase and dissipated power; Coupled oscillators and normal coordinates. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Estimate how a frame mode distributes its kinetic energy

For the highest tabulated mode, doubling tip translation mass lowers its frequency from 44.645 to 42.379 Hz. Doubling its rotary inertia instead lowers it to 34.791 Hz. Freeze the baseline mode shape for an estimate. Derive the fraction of its original modal mass associated with each changed contribution. Compare the estimates and state what the approximation omits.

▼ **Work through the solution**

The Rayleigh quotient is $\omega^2 = K_{\mathrm{mode}}/M_{\mathrm{mode}}$. For an unchanged shape, doubling one contribution M_i replaces the denominator with $M_{\mathrm{mode}} + M_i$. Thus

$$M_i/M_{\mathrm{mode}} = (f_0/f_i)^2 - 1.$$

Translation gives 0.110 and rotation 0.647. The much stronger rotary sensitivity is observable from the computed frequencies even if the frozen-shape estimates are imperfect. A finite change in the mass matrix can rotate the eigenvectors, so these fractions are estimates rather than an exact decomposition of modal kinetic energy.

Use the OneSimulation laboratory to check your answer

Vary the corresponding laboratory reference interactively. Compare the recalculated observable with the recorded OneSimulation runs and the approximation used in this exercise.

Chapter 5 · Base excitation and vibration isolation

Distinguish absolute motion, mount travel and transmitted force when choosing a support for a moving base.

FROM FIRST PRINCIPLES

Distinguish mount travel from the absolute motion of the mass

A moving foundation imposes a position rather than a force directly on a machine mass. Let $y(t)$ denote base displacement, $x(t)$ absolute mass displacement, and take positive directions alike. A spring of stiffness k and damper of coefficient c act through relative deformation and relative velocity. Newton's second law therefore gives

$$m\ddot{x} = -k(x - y) - c(\dot{x} - \dot{y}).$$

For steady sinusoidal base motion at frequency ω, replace each derivative by $i\omega$. Group absolute displacement terms on the left and prescribed motion on the right to obtain the transfer between their complex amplitudes.

$$\frac{X}{Y} = \frac{k + ic\omega}{k - m\omega^2 + ic\omega},$$
$$\frac{X - Y}{Y} = \frac{m\omega^2}{k - m\omega^2 + ic\omega}.$$

At low frequency the mass follows the base and mount travel approaches zero. At sufficiently high frequency absolute motion decreases, while the mount must accommodate nearly the full base displacement. A clearance sensor spanning the mount and an inertial sensor attached to the mass therefore respond to different quantities during the same isolation experiment.

DERIVATION AND APPLICATIONS

Choose damping by comparing two isolation frequencies

Damping can limit a resonant motion while increasing high-frequency vibration transmitted through a mount. Consider a machine mass m on a spring k and damper c in parallel, driven by a harmonic force on the mass with rigid support. Take $\omega_n = \sqrt{k/m}$,

$$\zeta = c/(2m\omega_n),$$
$$r = \omega/\omega_n.$$

The complex force transmitted into the support equals $(k + ic\omega)X$, where X is the steady displacement. Dividing by the applied force gives

$$T_F^2 = \left| \frac{F_{\text{base}}}{F_{\text{drive}}} \right|^2 = \frac{1 + 4\zeta^2 r^2}{(1 - r^2)^2 + 4\zeta^2 r^2}.$$

At a fixed nonzero frequency define $h = 4\zeta^2 r^2$. Differentiating in h leaves a numerator independent of damping. Its sign determines whether the transmitted fraction increases or decreases as damping is added.

$$\frac{\partial T_F^2}{\partial h} = \frac{(1 - r^2)^2 - 1}{[(1 - r^2)^2 + h]^2} = \frac{r^2(r^2 - 2)}{[(1 - r^2)^2 + h]^2}.$$

For $r < \sqrt{2}$, added damping lowers transmissibility; above $\sqrt{2}$ it raises it. At the boundary all damping choices give unity. A limit specified at resonance turns directly into a minimum damping ratio because the spring and inertial force cancel in the displacement denominator there.

$$T_F(1) = \frac{\sqrt{1 + 4\zeta^2}}{2\zeta},$$

$$\zeta_{\min} = \frac{1}{2\sqrt{T_{\lim}^2 - 1}} \quad (T_{\lim} > 1).$$

These results assume the same applied force amplitude across frequencies. A rotating imbalance has a force magnitude proportional to frequency squared, requiring that additional factor when predicting an absolute support load.

WORKED EXAMPLE

Trade a resonant limit against running isolation

A mount must keep resonant support force below three times the drive amplitude. The minimum damping ratio is

$$1/(2\sqrt{8}) = 0.1768.$$

At operating ratio $r = 3.00$ this choice gives $h = 1.125$ and $T_F = \sqrt{2.125/65.125} = 0.181$. A 50.0 N peak harmonic force then transmits 9.03 N at the operating frequency.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Determine whether more damping can meet a running target

Another design runs at $r = 2.00$. Its present $\zeta = 0.250$ gives $h = 1.00$ and $T_F = \sqrt{2/10} = 0.447$. To reach $T_F = 0.400$, solve $(1 + h)/(9 + h) = 0.160$, giving $h = 0.524$ and $\zeta = \sqrt{h}/(2r) = 0.181$. The reduction of damping improves the running value but raises the predicted resonant ratio to about 2.94.

Interactive calculation · Change the inputs of this example

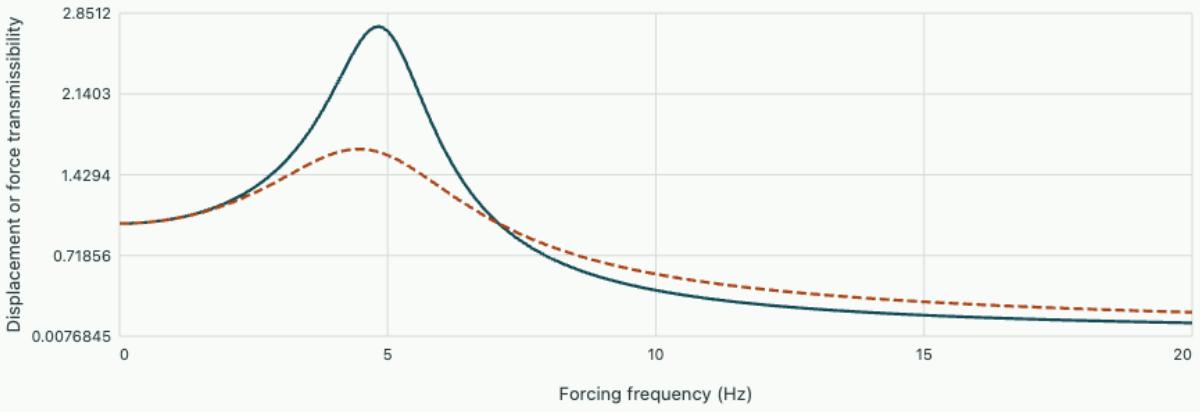
Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Damping ratio **0.2**

Natural frequency (Hz) **5**



Selected damping Twice damping

Both curves cross unity at 7.071 Hz. Extra damping suppresses resonance and increases the high-frequency transmission.

A moving foundation is a different input

Place a microscope on a spring-and-damper mount above a vibrating floor. An accelerometer on its body reports absolute acceleration, whereas a displacement gauge connected across the mount reports relative displacement. These can be very different. First write both coordinates in the same inertial frame. Let $y(t)$ be the motion of the floor and $x(t)$ the motion of the instrument. The spring force on the mass is $-k(x-y)$ and the damper force is $-c(\dot{x}-\dot{y})$.

For mass m, Newton's law therefore becomes

$$m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0.$$

A floor that moves sufficiently slowly carries the mass along with almost no relative motion. A floor moving very rapidly can move substantially beneath a nearly stationary mass. Whether the design is successful depends on which motion is constrained by the application.

Derive two transfer functions

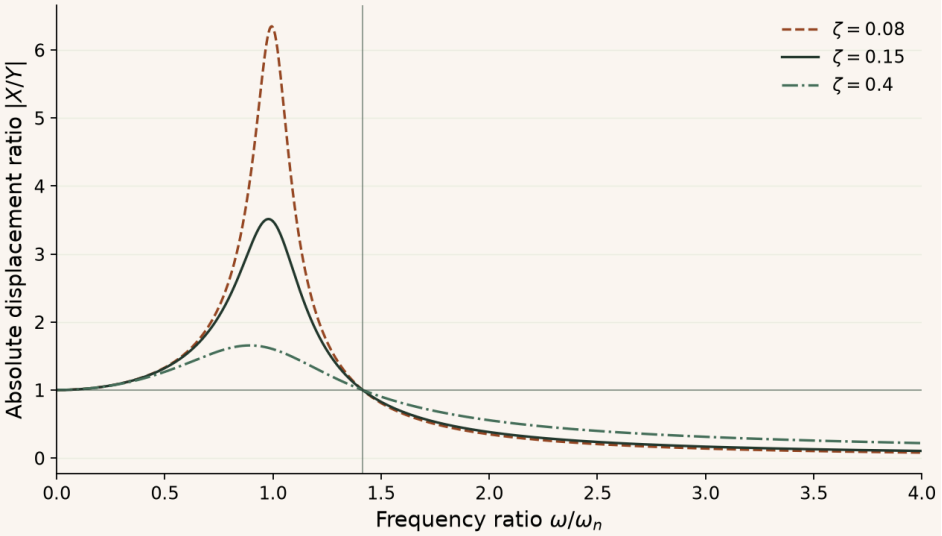
For harmonic input $y = \mathrm{Re}(Ye^{i\omega t})$, substitute $x = \mathrm{Re}(Xe^{i\omega t})$. Collect terms in X and Y. Writing $r = \omega/\sqrt{k/m}$ and $\zeta = c/(2\sqrt{km})$ gives

$$\frac{X}{Y} = \frac{1 + i2\zeta r}{1 - r^2 + i2\zeta r},$$

$$\frac{X - Y}{Y} = \frac{r^2}{1 - r^2 + i2\zeta r}.$$

Subtract the complex quantities before taking magnitudes. In general the magnitude of the relative displacement is not the difference of the absolute amplitudes, since the motions can have different phase angles. The acceleration ratio for harmonics equals $|X/Y|$ because both displacements acquire the same factor $-\omega^2$. That shortcut is inapplicable to arbitrary time records without decomposition into frequencies.

Compare the squares of numerator and denominator in the first ratio. Transmissibility falls below one when $(1 - r^2)^2 > 1$, or $r > \sqrt{2}$ for positive frequency. The crossing is independent of damping in this ideal model. Damping reduces the peak around resonance but provides a force path at high frequency, increasing the transmitted motion there compared with a lighter damper.



More damping suppresses the resonance peak but transmits more motion far above resonance. Read the vertical scale as an amplitude ratio.

Choose an isolation range and a clearance

Take $m = 80$ kg and desired undamped frequency $f_n = 4.0$ Hz. Then $k = m(2\pi f_n)^2 = 50.5$ kN/m. With $\zeta = 0.15$, the required viscous coefficient is $c = 2m(2\pi f_n)\zeta = 603$ N·s/m. Effective isolation begins above $4\sqrt{2} = 5.66$ Hz. A floor excursion of 0.30 mm amplitude at 12 Hz gives $r = 3$ and $2\zeta r = 0.90$. The absolute ratio is $\sqrt{1.81/64.81} = 0.167$, hence instrument motion is 0.050 mm. The relative amplitude is

$$9(0.30)/\sqrt{64.81} = 0.335\,\mathrm{mm}.$$

The smaller number is appropriate for motion performance and the larger for mount clearance.

At 4 Hz the same base displacement gives relative amplitude $0.30/(2\zeta) = 1.00$ mm. Before checking that a stop is far enough away, add the static spring compression $mg/k = 15.5$ mm if the mount is vertical and its available stroke is specified from the unloaded position.

Ask what a sensor can distinguish

Measure floor and instrument motion simultaneously with a shared clock. An isolated mass at high frequency can seem quiet despite substantial relative movement. Conversely, close to resonance both acceleration and support force may be amplified. For harmonic motion the force on the base is the spring-plus-damper force. Its magnitude equals $m\omega^2|X|$ when no other dynamic forces act on the mass. Comparing it with the measured acceleration is an independent check on force calibration and on the assumption of one moving coordinate.

Calculate what an accelerometer would report

A sensor bolted to the instrument measures absolute acceleration. With harmonic motion its amplitude is $\omega^2|X|$, even when the clearance requirement is governed by $|X - Y|$. For the 12 Hz condition, the 0.050 mm instrument displacement gives

$$(2\pi 12)^2(0.000050) = 0.284\,\mathrm{m/s^2}.$$

The floor acceleration is 1.71 m/s², so the acceleration transmissibility is the same 0.167 as the absolute displacement ratio at this frequency. Inferring spring force from the instrument’s absolute displacement would instead be wrong: the extension of the spring is the relative motion. The damper force also depends on relative velocity, and the two transmitted force components differ in phase.

Derive the response of a mounted accelerometer

An accelerometer is itself a base-excited oscillator. Let its casing follow the machine displacement $y(t)$ while an internal proof mass moves at $x(t)$. A spring and viscous element connect the two, and a

transducer detects relative motion $z = x - y$. Because damping and stiffness act on relative movement, Newton's law gives

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y}.$$

The instrument manufacturer may scale the output as estimated acceleration $\hat{a} = -\omega_n^2 z$. This scale is exact at zero frequency in the linear model, but both gain and phase change as frequency approaches the sensor resonance.

For harmonic casing acceleration with phasor A, the transfer from actual to indicated acceleration is

$$\frac{\hat{A}}{A} = \frac{1}{1 - r^2 + i2\zeta r},$$
$$r = \omega/\omega_n.$$

At small r, the proof mass is almost in force equilibrium with the spring and the output tracks acceleration. At large r, the proof mass is nearly inertially stationary and the relative displacement approximates negative casing displacement. This is the regime exploited by certain displacement-based seismic instruments. The very same equation permits two measurement principles, but their useful frequency ranges differ.

For $\zeta = 1/\sqrt{2}$ the squared magnitude denominator simplifies to $1 + r^4$. The leading magnitude error then appears at fourth order, although the phase still changes at first order. A sensor can consequently have an excellent amplitude calibration and a material timing error in the same frequency range.

Select a transducer for a base-isolation measurement

A vibration test uses an accelerometer with undamped frequency 100 Hz and $\zeta = 1/\sqrt{2}$. At a 30 Hz machine component, $r = 0.30$. The indicated magnitude is $1/\sqrt{1 + 0.3^4} = 0.9960$ times the true acceleration, whereas the lag is

$$\arctan[\sqrt{2}(0.30)/(1 - 0.30^2)] = 25.0^\circ.$$

An amplitude requirement of 1% is met, but a phase comparison between this sensor and an independently calibrated force transducer would carry a large offset. For an actual acceleration amplitude of $4.0\,\mathrm{m/s^2}$, proof-mass travel is $0.996(4)/(2\pi 100)^2 = 10.1\,\mu\mathrm{m}$, which also has to fit the transducer's linear travel range.

For the same damping, an amplitude error below 1% requires $1/\sqrt{1 + r^4} \geq 0.99$, or $r \leq 0.378$. A record extending to 50 Hz would thus require a resonance of at least about 132 Hz by magnitude alone. A 5° phase specification is much stricter: solving $\sqrt{2}r/(1 - r^2) = \tan 5^\circ$ gives $r \simeq 0.0616$, or resonance above about 812 Hz for the 50 Hz component. If two identical, well-matched sensors measure base and payload at the same frequency, much of their common transfer cancels in a ratio. It does not cancel in a phase comparison to another instrument with a different dynamics.

An instrument range may be specified in acceleration rather than proof-mass stroke. Near resonance the displacement per unit base acceleration increases according to the same frequency response. A sensor that remains linear at a given low-frequency acceleration can therefore approach its internal stop at a lower applied acceleration near resonance. Mechanical range and dynamic calibration must be checked together.

Interactive calculation · Change the inputs of this example

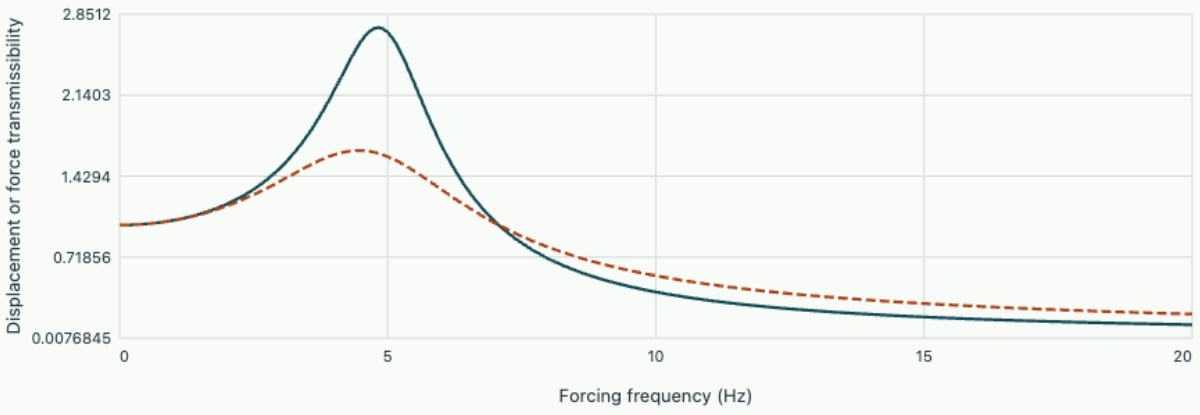
Change the parameters of this reference plot and compare its limiting cases. The fixed illustration remains beside the derivation; the adjustable calculation exposes the same dependence on geometry, material properties or loading.

Open this live calculation

Try the calculation

Damping ratio **0.2**

Natural frequency (Hz) **5**



Selected damping Twice damping

Both curves cross unity at 7.071 Hz. Extra damping suppresses resonance and increases the high-frequency transmission.

Study the full derivation

Random base motion, spectral density and rms response · Vibration and identification in machinery

A support vibration is described by a power spectral density rather than a single sine-wave amplitude. Derive the relative-motion filter for a mounted instrument, calculate mean-square motion from a measured spectrum and distinguish a stationary rms prediction from a peak displacement guarantee.

Chapter 6 · Impulse response and finite-duration loading

Construct the response to a short pulse and determine when an impulse approximation preserves the peak motion.

FROM FIRST PRINCIPLES

Build a transient response by accumulating elementary impulses

An impulse J is the time integral of applied force. Across an ideal instantaneous impulse the displacement of mass m stays continuous, while integrating Newton's law gives a velocity jump J/m. Finite spring and viscous forces have vanishing impulse over that ideal interval. For an initially stationary undamped mount with stiffness k, the following motion is free vibration with angular frequency $\omega_n = \sqrt{k/m}$ and the new initial velocity.

$$x(0^+) = 0,$$
$$\dot{x}(0^+) = J/m,$$
$$x(t) = \frac{J}{m\omega_n}\sin(\omega_n t).$$

Now divide a finite force trace into short intervals. The interval starting at s contributes impulse $F(s)\,ds$. For a linear time-invariant system its later displacement is the impulse solution delayed by s and scaled by that impulse. Summing over all earlier intervals yields

$$x(t) = \frac{1}{m\omega_n}\int_0^t F(s)\sin[\omega_n(t-s)]\,ds.$$

An existing initial displacement or velocity contributes a separate free response by superposition. If the pulse is very short relative to the vibration period, the sine factor changes little during it and the area of the pulse controls the subsequent motion.

DERIVATION AND APPLICATIONS

Obtain the residual vibration after a ramped force is removed

The time integral of a force sets momentum change only when the support forces have little time to act. For longer pulses a linear undamped oscillator retains information about the shape and timing of the load. Consider mass m, stiffness k, natural frequency $\omega = \sqrt{k/m}$ and zero initial motion. For a force acting only from zero to T, the post-pulse displacement follows the convolution with the impulse response.

$$x(t) = \frac{1}{m\omega}\int_0^T F(s)\sin[\omega(t-s)]ds, \qquad t \geq T.$$

Expand the sine of the difference into sine and cosine of t. The two integrals multiply quadrature components of free vibration. Their resultant magnitude is the modulus of a single Fourier integral at the natural frequency. For a symmetric triangular pulse centred at $T/2$, its area is $J = F_p T/2$ and its transform factorises because the triangle is a convolution of two equal rectangles of duration $T/2$.

$$A = \frac{1}{m\omega}\left|\int_0^T F(s)e^{-i\omega s}ds\right|,$$

$$\int_0^T F(s)e^{-i\omega s}ds = Je^{-i\omega T/2}\left[\frac{\sin(\omega T/4)}{\omega T/4}\right]^2.$$

Each rectangle contributes the average complex phase over its duration; the product of these averages produces the squared factor. The impulse estimate is recovered for a very short triangle, while certain longer pulses leave no residual motion in the ideal model.

$$\frac{A}{J/(m\omega)} = \left[\frac{\sin(\omega T/4)}{\omega T/4}\right]^2,$$

$$T_{\text{first zero}} = \frac{4\pi}{\omega}.$$

WORKED EXAMPLE

Compare a triangular pulse with its impulse limit

For $m = 2.00$ kg and $\omega = 20.0$ rad/s choose $T = \pi/10$ s and $F_p = 4.00$ N. Pulse area is 0.628 N s and the impulse estimate is 15.7 mm. Here $\omega T/4 = \pi/2$ and the squared factor is

$$4/\pi^2 = 0.405.$$

Actual residual amplitude is 6.37 mm. The cancellation occurs because different increments of force reach different phases of the oscillator.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Design a pulse with zero residual motion

Another lightly damped test rig is idealised as undamped at 5.00 Hz. The first zero requires a symmetric triangular pulse lasting two periods, $T = 0.400$ s. To deliver impulse 0.300 N s, choose $F_p = 2J/T = 1.50$ N. The mass can move during the pulse, but its final displacement and velocity are both zero in the ideal model. A measured ring-down identifies sensitivity to frequency error, damping or a distorted force shape.

Interactive calculation · Change the inputs of this example

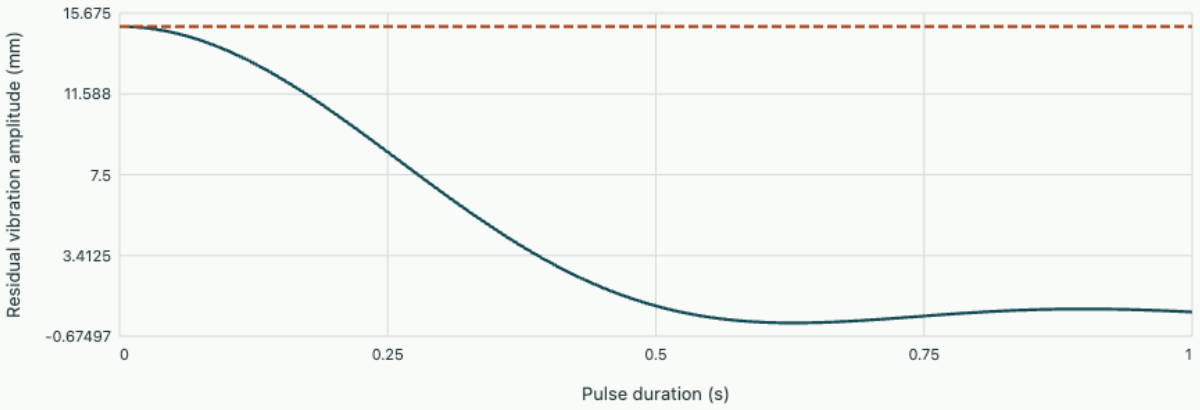
Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Impulse (N s) **0.6**

Natural frequency (rad/s) **20**



Triangular force pulse Impulse estimate

The first cancellation duration is 0.628 s. Mass2 kg; equal pulse area does not imply equal Fourier content at the natural frequency.

A short event and a long response

A latch strikes a mounted instrument for only a few milliseconds, but its motion persists after contact has ended. Peak force alone gives an incomplete description of that motion. Duration, impulse and the system's natural time scale all matter. Define impulse $J = \int F(t)\,dt$, with units N·s. Integrating $m\ddot{x} + c\dot{x} + kx = F$ over an ideal instantaneous event leaves $m[\dot{x}(0^+) - \dot{x}(0^-)] = J$. The position remains continuous since no finite mass can jump a finite distance in zero time under a finite impulse.

For an initially resting underdamped oscillator, the subsequent response follows from the two initial conditions $x(0^+) = 0$ and $\dot{x}(0^+) = J/m$. With $\alpha = c/(2m)$ and $\omega_d = \sqrt{k/m - \alpha^2}$, the displacement is $x = Je^{-\alpha t}\sin(\omega_d t)/(m\omega_d)$ for $t > 0$. The response to a unit impulse is the same expression with $J = 1$. Its units are displacement per unit impulse.

Build a solution from elementary impulses

Divide a general forcing record into short intervals at times s. Each interval contributes impulse $F(s)\,ds$ and creates a response delayed by s. Adding all contributions in a linear time-invariant system with zero initial state produces the convolution

$$x(t) = \int_0^t F(s)\,\frac{e^{-\alpha (t-s)}\sin[\omega_d (t-s)]}{m\omega_d}\,ds.$$

This construction makes causality explicit: later parts of the force record cannot influence the current response. Nonzero initial conditions add the free solution. In a multi-mode system each mode has its own impulse response, and the load’s location determines the force projection into it. Replacing the structure by its first mode may describe long-wave displacement well while missing a short, local acceleration peak.

Work a pulse of known duration

Neglect damping and apply constant force F_0 for $0 < t < t_p$. During the pulse integration gives

$$x = (F_0/k)[1 - \cos(\omega_n t)],$$
$$\dot{x} = F_0 \omega_n \sin(\omega_n t)/k.$$

After unloading the mass oscillates freely. Compute its amplitude from the end displacement and velocity:

$$A = \sqrt{x(t_p)^2 + \left[\dot{x}(t_p)/\omega_n\right]^2} = \frac{2F_0}{k}\left|\sin(\omega_n t_p/2)\right|.$$

Let $m = 4.0$ kg, $k = 1600$ N/m, $F_0 = 20$ N and $t_p = 0.050$ s. Then $\omega_n = 20$ rad/s and the impulse is 1.0 N·s. The formula predicts $A = 25\sin(0.50) = 12.0$ mm after unloading. Replacing the whole pulse by an instantaneous impulse gives

$$A_{\text{imp}} = J/(m\omega_n) = 12.5\,\text{mm}.$$

Some restoring force developed during the real pulse, so the impulse approximation is slightly high.

Find the criterion and its limitations

Divide the finite-pulse amplitude by the impulse approximation to obtain $|\sin z/z|$ with $z = \omega_n t_p/2$. Its small-z expansion is $1 - z^2/6 + \cdots$. At fixed impulse, a sufficiently short pulse therefore gives about the same low-mode amplitude regardless of duration. A shorter event has a broader frequency content, potentially increasing the importance of higher modes.

A pulse lasting exactly one natural period gives zero residual amplitude in the ideal undamped oscillator, although the displacement during the pulse peaks at $2F_0/k$. Checking only the post-event motion would

miss that maximum. In experiments record the force and response during contact as well as after it. Damping, uncertain pulse shape or nearby modes make exact cancellation unlikely, but the example shows why pulse duration must accompany a quoted peak force.

Compare pulses of equal impulse

A force transducer may report identical total impulse for a sharp tap and a broader pulse. For an undamped mode, the amplitude after the force has ended is the modulus of its frequency-weighted impulse divided by $m\omega_n$:

$$A = \left| \int F(t) e^{-i\omega_n t} dt \right| / (m\omega_n).$$

This follows by expanding the sine kernel of the convolution into two quadratures. For a rectangular pulse the ratio to the ideal instantaneous-impulse amplitude is $|\sin(\omega_n t_p/2)/(\omega_n t_p/2)|$. At a duration equal to half the natural period that factor is

$$2/\pi = 0.637.$$

Two force records with equal signed area can therefore have visibly different post-event amplitudes at one sensor if their durations differ. A short force pulse can consequently calibrate a low-frequency mode while underexciting a faster one in the same structure.

Derive an energy-consistent time update for a force record

A measured force history may be irregular enough that a closed-form convolution is inconvenient. The constant-average-acceleration Newmark update advances a linear oscillator one time step at a time. Let x_n, v_n, a_n be displacement, velocity and acceleration at t_n. Approximate the acceleration over the step through its endpoint average in the velocity update, and use the corresponding trapezoidal velocity update for displacement. With step h the relations are

$$v_{n+1} = v_n + \tfrac{h}{2}(a_n + a_{n+1}),$$

$$x_{n+1} = x_n + h v_n + \tfrac{h^2}{4}(a_n + a_{n+1}).$$

Enforce the actual equation $m a_{n+1} + c v_{n+1} + k x_{n+1} = F_{n+1}$ at the new time. Gathering unknown acceleration terms gives effective mass $m + ch/2 + kh^2/4$. The right side is $F_{n+1} - c(v_n + h a_n/2) - k(x_n + h v_n + h^2 a_n/4)$. Solve that scalar equation and back-substitute into both kinematic updates. For several coupled linear coordinates the same derivation gives a linear system with an effective matrix.

For undamped unforced systems, multiply the average endpoint equilibrium by $x_{n+1} - x_n = h(v_{n+1} + v_n)/2$. Differences of squares show that the discrete change in kinetic energy plus potential energy vanishes exactly. This property detects sign or unit errors in an implementation. It does not guarantee accurate phase at a coarse time step.

Check one step and distinguish stability from temporal resolution

Take $m = 2\,\mathrm{kg}$, $c = 0$, $k = 200\,\mathrm{N/m}$. Begin from $x_0 = 0.010\,\mathrm{m},\ v_0 = 0$ and release without applied force. Initial acceleration is

$$-kx_0/m = -1.00\,\mathrm{m/s^2}.$$

With $h = 0.10\,\mathrm{s}$, effective mass is

$$2 + 200(0.10)^2/4 = 2.50\,\mathrm{kg}.$$

Its right side is $-200[0.010 + 0.10^2(-1)/4] = -1.50\,\mathrm{N}$, giving $a_1 = -0.600\,\mathrm{m/s^2}$. The updates give

$$v_1 = -0.0800\,\mathrm{m/s}, x_1 = 0.00600\,\mathrm{m}.$$

The initial energy is

$$200(0.010)^2/2 = 0.0100\,\mathrm{J};$$

after one step it is

$$2(0.080)^2/2 + 200(0.006)^2/2 = 0.0100\,\mathrm{J}.$$

The exact displacement at 0.10 s is nevertheless $0.010\cos(1) = 0.00540\,\mathrm{m}$. The numerical oscillator advances through angle $2\arctan(\omega_n h/2) = 0.9273$ rad per step instead of 1.00 rad. Over many cycles this small period error accumulates. Halving the step is a direct temporal convergence test. For a measured force pulse, a second check is that its sampled impulse and timing converge as well; an accurate oscillator update cannot recover a force feature missed by the acquisition system.

It is possible to predict the amount of step refinement needed for a phase-sensitive comparison. For small $z = \omega_n h/2$, expand $2\arctan z = 2z - 2z^3/3 + \cdots$. Dividing the numerical angular advance by the true advance gives $1 - (\omega_n h)^2/12 + \cdots$. The leading relative frequency error therefore falls by a factor of four when the step is halved. A one-percent target suggests $\omega_n h \lesssim \sqrt{0.12} = 0.346$ for this single undamped mode. At 10 rad/s that means roughly 0.035 s or less, before checking the sampling of the force itself. This criterion addresses the fundamental frequency, not the fastest contributing mode in a multi-degree model. If a high-frequency mode contributes significantly at the measurement station, use its frequency to assess temporal phase resolution or test convergence of the complete measured signal directly.

Once these conditions are met, comparing the computed displacement and velocity at the end of loading also tests the predicted initial condition for the subsequent free ring-down.

Problem sheet 3 · Isolation and shock

Separate floor motion from mount travel, then distinguish peak response during and after an impact.

1. Choose an isolation frequency

A 100 kg platform must have $|X/Y| < 1$ for floor excitation at all frequencies from 8 Hz upward. (a) Find the upper limit on its natural frequency. (b) Choose 5.0 Hz and calculate k. (c) At 10 Hz and $\zeta = 0.20$, find absolute and relative amplitudes for a 0.50 mm base motion.

▼ **Apply the displacement transfer function**

For the specified inequality $f_n < 8/\sqrt{2} = 5.66$ Hz. At 5.0 Hz the stiffness is $100(2\pi 5)^2 = 98.7$ kN/m. At $r = 2$ and $2\zeta r = 0.8$, the common denominator magnitude is

$$\sqrt{9 + 0.64} = 3.105.$$

The absolute amplitude is $0.50\sqrt{1.64}/3.105 = 0.206$ mm, while relative amplitude is

$$0.50(4)/3.105 = 0.644\,\text{mm}.$$

Size the allowable mount travel using the latter value.

---

2. Compare impacts with the same impulse

An undamped 5 kg component has $\omega_n = 40$ rad/s. Two rectangular forces each provide 1.0 N·s. Pulse A lasts 0.010 s and pulse B 0.100 s. (a) Calculate each force. (b) Calculate the two residual amplitudes. (c) Compare them with the ideal impulse limit.

▼ **Keep impulse fixed while changing duration**

The forces are 100 and 10.0 N. The ideal limit is

$$J/(m\omega_n) = 5.0\,\text{mm}.$$

Applying the finite-duration correction $|\sin z/z|$ with $z = 0.20$ and 2.0 yields 4.97 and 2.27 mm. The longer force partly acts during a phase when the oscillator moves back. Its smaller residual response does not follow from impulse alone.

---

3. Check whether a stop is struck during a pulse

A 10 kg mass on a 1000 N/m spring starts at rest. A 5 N constant force lasts exactly one undamped period. A travel stop lies 8 mm from equilibrium in the force direction. (a) Find the period. (b) Find the

predicted motion at half and full period without the stop. (c) Does a zero residual response prove safe clearance?

▼ **Inspect the complete history**

The angular frequency is 10 rad/s, giving a period of 0.628 s. During the force $x = 5(1 - \cos 10t)/1000$ m. At half a period this is 10.0 mm, with zero velocity. At the full period both displacement and velocity vanish. The motion crosses the 8 mm stop beforehand at $\cos 10t = -0.6$, or $t = 0.221$ s. The real motion after contact must be recalculated with the additional force. A post-event sample would miss the predicted interference.

Explore the governing equations after solving the sheet: Base excitation and vibration isolation; Impulse response and finite-duration loading. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Chapter 7 · Velocities and accelerations in planar mechanisms

Use the rigid-body velocity and acceleration relations to connect an instantaneously known crank speed to a slider acceleration.

FROM FIRST PRINCIPLES

Differentiate a fixed link vector in a rotating body

For two material points A, B on a rigid planar link, their separation has constant length even while its direction changes. Let $\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A$ and angular velocity vector $\boldsymbol{\omega}$ be perpendicular to the plane. A small rotation by $d\theta$ sweeps a displacement perpendicular to the separation. Dividing by time gives the inertial derivative $\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$ and hence

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}.$$

Differentiate once more using the product rule. Change of angular speed provides the tangential acceleration contribution; change of the separation's direction gives the second cross product.

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

For planar rotation the final term is $-\omega^2 \mathbf{r}$, directed towards A. It remains even when angular speed is constant. These relations concern two fixed points on one rigid body. A slider moving along the rotating link changes the body-relative separation and contributes additional velocity and acceleration terms.

DERIVATION AND APPLICATIONS

Recover planar angular acceleration from two accelerometers

Velocities identify an instantaneous rotation centre, but a time-varying rotation is better characterised by both angular speed and angular acceleration. Mount two aligned accelerometers at points A and B on one rigid planar body. Assume the measured vectors have been expressed in common inertial axes and any gravity contribution cancels in their difference. The separation vector is $\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A$ with fixed length d. Rigid-body kinematics gives

$$\Delta\mathbf{a} = \mathbf{a}_B - \mathbf{a}_A = \alpha\, \mathbf{e}_z \times \mathbf{r} - \omega^2 \mathbf{r}.$$

The two terms point in perpendicular directions. Form the unit vector $\mathbf{e}_r = \mathbf{r}/d$ from A to B and its counterclockwise normal $\mathbf{e}_t = \mathbf{e}_z \times \mathbf{e}_r$. Projecting gives a radial signal proportional to squared angular speed and a tangential signal proportional to signed angular acceleration.

$$\Delta a_r = -d\omega^2,$$
$$\Delta a_t = d\alpha.$$

Thus no knowledge of the translational acceleration of the reference point is needed. An additional tangential relative-velocity measurement supplies the missing sense of rotation because the signed ω enters linearly.

$$|\omega| = \sqrt{-\Delta a_r / d},$$
$$\alpha = \Delta a_t / d,$$
$$\omega = (\mathbf{v}_B - \mathbf{v}_A) \cdot \mathbf{e}_t / d.$$

A positive radial acceleration difference is inconsistent with a rigid pair under the stated conventions. Flexible mounting, sensor misalignment or treating differently oriented sensor components as parallel can all spoil the projection.

WORKED EXAMPLE

Read the motion of a rotating link

At separation 0.200 m the measured acceleration difference is $-3.20\,\mathbf{e}_r + 0.600\,\mathbf{e}_t$ m/s². The link has speed magnitude $\sqrt{3.20/0.200} = 4.00$ rad/s and counterclockwise angular acceleration $0.600/0.200 = 3.00$ rad/s². A measured relative tangential velocity of -0.800 m/s identifies clockwise rotation. Its speed magnitude is decreasing at this instant.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Estimate a sensor spacing from a calibrated turn

A second instrument rotates steadily at 5.00 rad/s and records radial acceleration difference -3.75 m/s². Its effective sensor separation is

$$3.75/25.0 = 0.150\,\mathrm{m}.$$

A later measurement with the same instrument gives tangential difference -0.450 m/s², implying $\alpha = -3.00$ rad/s². This calibration uses steady rotation for the length scale and a distinct transient to determine angular acceleration.

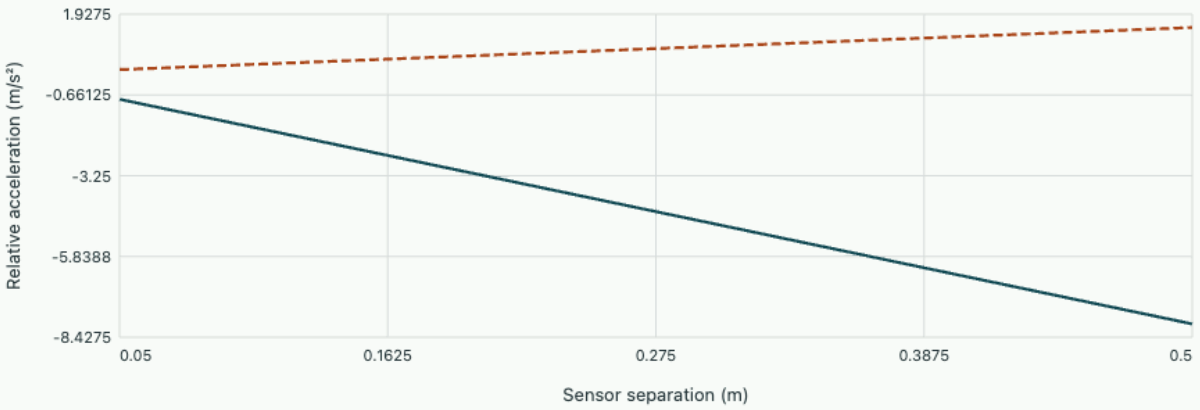
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Sensor separation (m)	0.2
Signed angular speed (rad/s)	-4
Angular acceleration (rad/s²)	3



At selected spacing, radial=-3.200 and tangential=0.600 m/s²; relative tangential velocity=-0.800 m/s supplies the rotation sign.

What can change between two points on a link?

The point at which a connecting rod drives a slider moves along a straight line although its other end moves around a crank circle. Writing every trajectory as a function of crank angle is possible, but often only one configuration matters. The rigidity constraint transfers instantaneous velocity and acceleration between points on a link. Choose a fixed inertial frame and use the positive out-of-plane direction for counterclockwise angles. Every vector component below refers to these fixed axes.

Write $r_B = r_A + r_{B/A}$. The vector between two body points keeps its length but changes direction. Its derivative is $\omega \times r_{B/A}$. Differentiating a second time introduces both the changing magnitude of angular velocity and the changing direction of the relative vector:

$$v_B = v_A + \omega \times r_{B/A},$$

$$a_B = a_A + \alpha \times r_{B/A} + \omega \times (\omega \times r_{B/A}).$$

The last term is $-\omega^2 r_{B/A}$ for planar motion. It points along the rod towards A, even when the rod has constant angular speed. Enforce constraints on the actual point B; a slider constrained horizontally has zero vertical velocity and zero vertical acceleration if its guide is straight and fixed.

A crank at its uppermost position

Place crank centre O at the origin and the slider along the positive horizontal axis. The crank length is 50 mm and the connecting rod length is 130 mm. At the configuration considered its pin is $A = (0, 50)\,\mathrm{mm}$ and slider pin $B = (120, 0)\,\mathrm{mm}$, since $120^2 + 50^2 = 130^2$. The crank rotates counterclockwise at constant 20 rad/s. Accordingly $v_A = (-1, 0)\,\mathrm{m/s}$ and $a_A = (0, -20)\,\mathrm{m/s^2}$.

The rod vector from A to B is $(0.120, -0.050)\,\mathrm{m}$. Its relative velocity is $\omega_r(0.050, 0.120)$. The vertical velocity constraint gives $0 = 0 + 0.120\omega_r$, so $\omega_r = 0$ at this instant and $v_B = (-1, 0)\,\mathrm{m/s}$. The rod is instantaneously translating. That description concerns velocity only. A moment later its two ends no longer have the same velocity.

Recover the slider force input

Since $\omega_r = 0$ here, the relative centripetal term for the rod vanishes. The angular-acceleration term gives $0 = -20 + 0.120\alpha_r$, hence $\alpha_r = 166.7\,\mathrm{rad/s^2}$. The slider's horizontal acceleration is

$$a_{Bx} = 0 + 0.050(166.7) = 8.33\,\mathrm{m/s^2}.$$

At this instant the slider is moving to the left while accelerating to the right. A bare slider mass of 3 kg needs a net force of 25.0 N to the right before guide friction or process forces are considered.

A useful independent check follows from the exact geometry $x_B = r\cos\theta + \sqrt{l^2 - r^2\sin^2\theta}$. At $\theta = \pi/2$ the first derivative with respect to angle is $-r$; multiplication by crank speed gives –1 m/s. A numerical second derivative must resolve the curvature of this path. Differentiating noisy measured positions twice can produce a large fictitious force, so comparing observed positions with this geometric constraint is often a better starting point.

Change the drive speed

Repeating the same configuration at twice the constant crank speed doubles every point velocity. Its centripetal terms and the constraint-induced angular acceleration scale with the square of that speed, so the predicted slider acceleration becomes 33.3 m/s². This scaling is one reason mechanism inertia loads can grow rapidly with production rate.

Differentiate a coordinate measured along a rotating guide

The rigid-body formula above assumes that the position vector joining the two points is fixed in their common body. A slider moving along a rotating guide violates that assumption. Introduce a guide-fixed unit vector e_r along the slot and its positive quarter-turn e_θ. At an inertially fixed pivot the slider's position is $r(t)e_r$; the basis changes according to $\dot{e}_r = \omega e_\theta,\ \dot{e}_\theta = -\omega e_r$. Differentiate both the scalar radial position and the changing basis at every step.

$$v = \dot{r}\,e_r + r\omega\,e_\theta,$$

$$a = (\ddot{r} - r\omega^2)e_r + (r\dot{\omega} + 2\dot{r}\omega)e_\theta.$$

The factor two in the transverse contribution has two origins. The outward relative velocity vector rotates with the guide, and the speed of the circumferential transport changes as the slider moves outwards. Each contributes $\dot{r}\omega$. This Coriolis term is an actual part of the acceleration measured in an inertial frame. In a rotating-frame force balance an apparent Coriolis force of opposite sign must be introduced only if the inertial acceleration has been removed from that balance. Counting both would double the effect.

If the guide's pivot itself translates, add its inertial acceleration to the expression. For an instantaneously stationary guide $\omega = 0$ eliminates centripetal and Coriolis terms, but a nonzero angular acceleration can still produce transverse acceleration $r\dot{\omega}$. This distinction parallels the translating rod in the first worked mechanism.

Determine drive and guide reactions on a rotating table

A 2.5 kg carriage slides outwards along a straight radial slot in a horizontal table. At the given instant $r = 0.30\,\mathrm{m}$, $\dot{r} = 0.20\,\mathrm{m/s}$, $\ddot{r} = 0.50\,\mathrm{m/s^2}$, $\omega = 4.0\,\mathrm{rad/s}$ counterclockwise, and $\dot{\omega} = 1.0\,\mathrm{rad/s^2}$. The radial acceleration is $0.50 - 0.30(4)^2 = -4.30\,\mathrm{m/s^2}$ and the transverse acceleration is

$$0.30(1) + 2(0.20)(4) = 1.90\,\mathrm{m/s^2}.$$

A frictionless slot supplies a 4.75 N transverse reaction in the positive rotational direction. The radial actuator must exert a net 10.75 N inward, even though the carriage is moving outwards and accelerating relative to the slot in that direction.

The transverse reaction produces a 1.425 N m moment about the centre. Independently, differentiating the angular momentum $mr^2\omega$ gives

$$m(2r\dot{r}\omega + r^2\dot{\omega}) = 1.425\,\mathrm{N\,m}.$$

A table drive that assumes fixed radius would include only $mr^2\dot{\omega} = 0.225\,\mathrm{N\,m}$, missing most of the required moment in this manoeuvre. Holding angular speed constant would remove this last term but leave the 1.20 N m moment associated with increasing inertia. A torque sensor at the turntable shaft therefore detects radial carriage motion even when no angular acceleration is commanded.

Mechanical power supplies an independent sign check. The rail reaction transmits positive power $4.75(r\omega) = 5.70$ W through the transverse motion at this instant. The inward radial actuator supplies $-10.75(0.20) = -2.15$ W. Their sum, 3.55 W, equals the instantaneous change of the point mass's translational energy, $m(\dot{r}\,a_r + r\omega a_\theta) = 2.5[0.20(-4.30) + 1.20(1.90)]$. The rail can supply power because it moves transversely with the table even though the contact has no work in the sliding direction.

Construct angular momentum when the rotation axis is not principal

A planar moment of inertia determines one scalar angular momentum. In three dimensions a balanced body can have angular momentum tilted relative to its angular velocity. About its centre of mass a particle at r has velocity $\omega \times r$. Integrate $r \times (\omega \times r)$ over mass and use the vector triple product. The result is

$$H_G = I_G \omega,$$

with tensor $I_G = \int [(r^T r)\mathbf{1} - r r^T]\, dm$. Its x-diagonal element is $\int (y^2 + z^2) dm$ and its xy element is $-\int xy\, dm$. Equal and opposite points cancel their first moments but add their products of inertia. Symmetry about a coordinate plane, rather than symmetry only about the centre, is sufficient to make the associated product vanish.

In a body-fixed basis rotating at ω, the external moment components obey $M_G = I_G \dot{\omega} + \omega \times (I_G \omega)$. This follows from differentiating H_G in an inertial frame: the body basis vectors rotate as well as the components. Even with constant body components of angular velocity, a transverse moment can be required to keep a nonprincipal axis fixed.

Model a compact instrument carrier as two equal 0.50 kg masses at $(0.20, 0.10, 0)$ m and its negative, neglecting the frame mass. Its tensor in kg m² has entries $I_{xx} = 0.010,\ I_{yy} = 0.040,\ I_{zz} = 0.050,\ I_{xy} = -0.020$. At $\omega = (20, 0, 0)$ rad/s, $H_G = (0.20, -0.40, 0)$ kg m²/s. Maintaining that rotation requires $M_G = (0, 0, -8.0)$ N m. There is no offset centre of mass to correct with a single static counterweight. Repositioning the masses to remove the product of inertia, or reorienting the shaft along a principal axis, eliminates this transverse couple in the model. A bearing reaction can reverse the sign convention for the load read by its support. Always state whether the reported moment acts on the rotor or on the frame.

Practise on sheet 4 · Predict the reaction moment from a tilted mass arrangement. Further reading · MIT · 16.07 Dynamics · Lecture L26, 3D Rigid Body Dynamics: The Inertia Tensor.

Chapter 8 · Impulse, angular momentum and impact

Find the motion immediately after a short off-centre impact by combining angular impulse with a local restitution condition.

FROM FIRST PRINCIPLES

Close a short collision with impulse and restitution

During a short collision, contact force can be large while the position changes little. Let $g^- = v_1^- - v_2^- > 0$ be the relative approach speed along a fixed impact normal for two freely translating masses. Ignore tangential impulse and external impulses during contact. A compressive contact impulse of magnitude J slows the first mass and speeds the second. Integrating each momentum balance gives

$$v_1^+ = v_1^- - J/m_1,$$
$$v_2^+ = v_2^- + J/m_2.$$

Momentum conservation alone leaves the impulse undetermined. A measured normal coefficient of restitution e supplies the separation speed as a fraction of the approach speed. Substitution then determines the impulse.

$$v_2^+ - v_1^+ = eg^-,$$
$$J = \frac{(1+e)g^-}{1/m_1 + 1/m_2}.$$

For a rigid body struck away from its centre the impulse also changes angular momentum by $\mathbf{r} \times \mathbf{J}$. The normal velocity used in restitution must then be evaluated at the contact point, combining centre translation and rotation. A restitution value is a contact model or measurement, whereas the impulse balances follow directly from mechanics.

Background · Force impulse

$$J = \int F\,dt$$

An impulse changes linear momentum over an event. A coefficient of restitution supplies a separate relative-velocity condition for an impact.

DERIVATION AND APPLICATIONS

Locate the impact point that leaves a pivot without an impulse

A rapid blow on a hinged body can set it rotating while producing zero instantaneous horizontal impulse at the hinge. Model a rigid pendulum initially at rest, with mass m, centre of mass distance a from its frictionless pivot, and moment of inertia I_O about the pivot. Apply a short tangential impulse J at distance b along the same radial line. Neglect the impulse from gravity during contact and take positive velocity along J. Angular impulse about the pivot removes the unknown hinge reaction.

$$I_O\omega^+ = Jb,$$
$$v_G^+ = a\omega^+.$$

Linear impulse along the impact direction includes both J and the unknown hinge impulse H. Substitution of angular speed therefore gives the signed hinge impulse as a function of the impact radius. A reaction in the opposite direction from J corresponds to $H < 0$.

$$mv_G^+ = J + H,$$
$$H = J\left(\frac{mab}{I_O} - 1\right).$$

Set $H = 0$ and use the parallel-axis theorem $I_O = I_G + ma^2$. The resulting radius locates the centre of percussion; it lies beyond the centre of mass for any extended body with $I_G > 0$.

$$b_{\mathrm{cp}} = \frac{I_O}{ma} = a + \frac{I_G}{ma}.$$

This result concerns the net hinge impulse during a short tangential event. Later hinge forces as the body swings need a separate force balance. A finite-duration or oblique impact also requires examining the other force components.

WORKED EXAMPLE

Locate the point on a laboratory pendulum

A uniform slender bar of length 0.900 m and mass 1.50 kg is hinged at one end. Its $a = 0.450$ m and $I_O = mL^2/3 = 0.405$ kg m², giving $b_{\mathrm{cp}} = 0.600$ m. A tangential impulse of 0.300 N s there produces

$$\omega^+ = 0.300(0.600)/0.405 = 0.444\,\mathrm{rad/s}.$$

Its centre momentum is $1.50(0.450)(0.444) = 0.300$ N s, accounting for all of the applied impulse.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Infer inertia from a zero-reaction test

A different hinged fixture has $m = 4.00$ kg and $a = 0.180$ m. An instrumented hinge shows vanishing tangential impulse at $b = 0.320$ m. This gives $I_O = mab = 0.2304$ kg m² and $I_G = 0.1008$ kg m². At $b = 0.160$ m, a 0.500 N s impulse should instead produce $H = 0.500(0.160/0.320 - 1) = -0.250$ N s, a direct sign and calibration check.

Interactive calculation · Change the inputs of this example

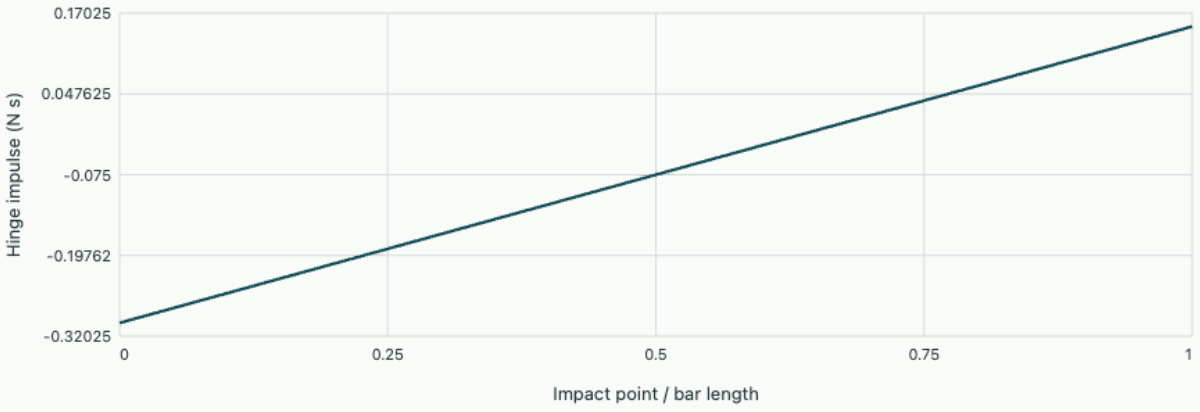
Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Bar length (m) **0.9**

Impulse (N s) **0.3**



Uniform end-hinged bar

Zero hinge impulse at b=2L/3=0.600 m. For mass1.5 kg the angular-speed jump there is 0.444 rad/s.

Does equal impulse leave equal vibration?

A force hammer and a soft contact pad can deliver the same total impulse in quite different times. The impulse fixes the change of momentum of a free mass. A mounted instrument also feels a spring force

during the contact. The instrument may have already started a substantial fraction of an oscillation while the applied force is still acting, allowing later parts of the pulse to cancel some of the earlier response.

Start an undamped oscillator at rest and apply a force $F(t)$ only between zero and T. Passing a small impulse $F(s)\,ds$ to the oscillator gives a later displacement $F(s)\sin[\omega(t-s)]\,ds/(m\omega)$. Integrating over arrival times for $t > T$ and collecting the coefficients of $\sin\omega t$ and $\cos\omega t$ gives the residual amplitude

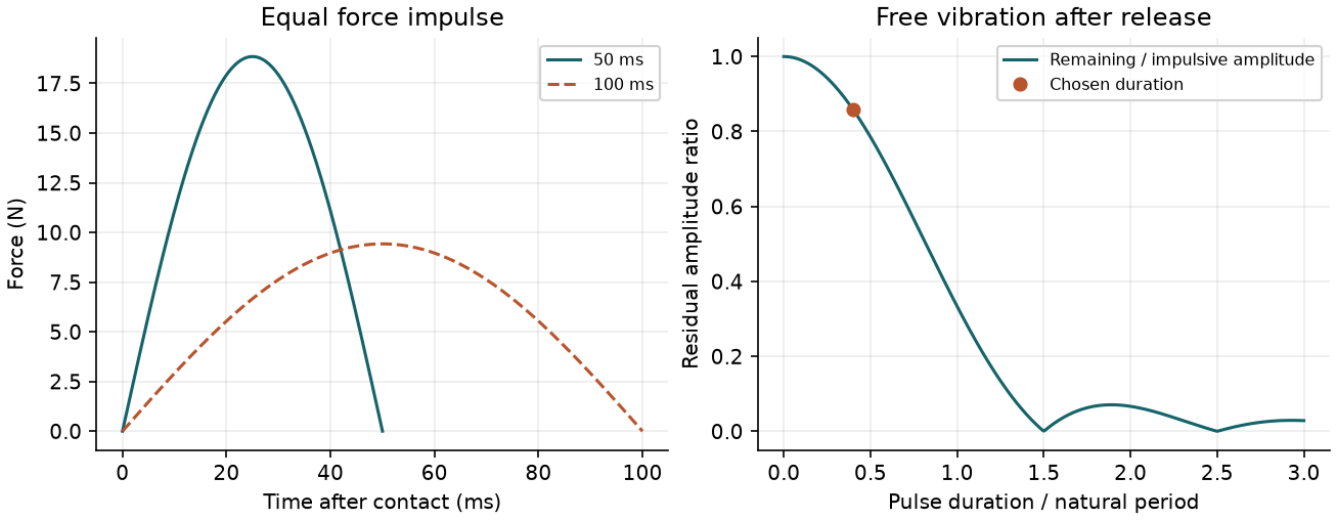
$$A = \frac{1}{m\omega}\left|\int_0^T F(s)e^{-i\omega s}\,ds\right|.$$

The relevant measure is the pulse spectrum at the oscillator frequency. Its zero-frequency value is the ordinary impulse. Their difference is negligible when the complex phase changes very little during contact. A wide-band impact model that fits only impulse can otherwise give the wrong residual vibration even when it closely predicts the overall force integral.

Choose a half-sine pulse $F(s) = J\pi\sin(\pi s/T)/(2T)$ with prescribed impulse J. Write the sine as a pair of exponentials, integrate each exponential and take the magnitude. With $r = \omega T$, this produces

$$A/[J/(m\omega)] = \pi^2|\cos(r/2)/(\pi^2 - r^2)|.$$

At $r = \pi$ the numerator and denominator both vanish. Expanding about π or integrating the sine pulse directly gives the finite limit $\pi/4$. The notebook evaluates a pair of sinc functions to avoid asking the computer to divide zero by zero.



The force integral stays fixed as the pulse broadens. The oscillator weights the individual parts by their arrival phase, so a full cancellation can occur after release.

For a 2 kg instrument with natural frequency 8 Hz and a pulse of 0.6 N s, an instantaneous impulse gives $A = 5.968$ mm. A 50 ms contact occupies 0.4 natural periods and leaves only 5.123 mm. In the notebook, compare the closed result with direct numerical integration of the complex forcing history. Halve the duration while preserving impulse and inspect the approach to the impulsive limit. The force peak must increase to conserve the area.

If an accelerometer is used to check the result, separate the interval during which the hammer is in contact from the ring-down after release. During free vibration the acceleration satisfies $\ddot{x} = -\omega^2 x$, and the measured acceleration amplitude divided by ω^2 gives the residual displacement amplitude. For the 50 ms example this predicts about 12.94 m/s² during the ring-down. During contact the force term $F(t)/m$ also contributes to the acceleration. Applying the same conversion to the largest acceleration in the whole record could therefore assign the forcing pulse to the free vibration. A force transducer identifies the release time independently.

At a duration of 1.5 natural periods, the remaining oscillation of this model vanishes. Does this make such a pulse equivalent to a force that never acted?

▼ **Work through the solution**

At $r = 3\pi$ the cosine numerator is zero and the denominator is finite. Positive parts of the pulse delivered at different phases cancel in the *final* modal state. During contact there is displacement, stored spring energy and a transient force in the support. A specimen containing a second mode will weight the same pulse at another frequency. A cancellation for one mode says little about the maximum support displacement during contact or the other mode’s final amplitude.

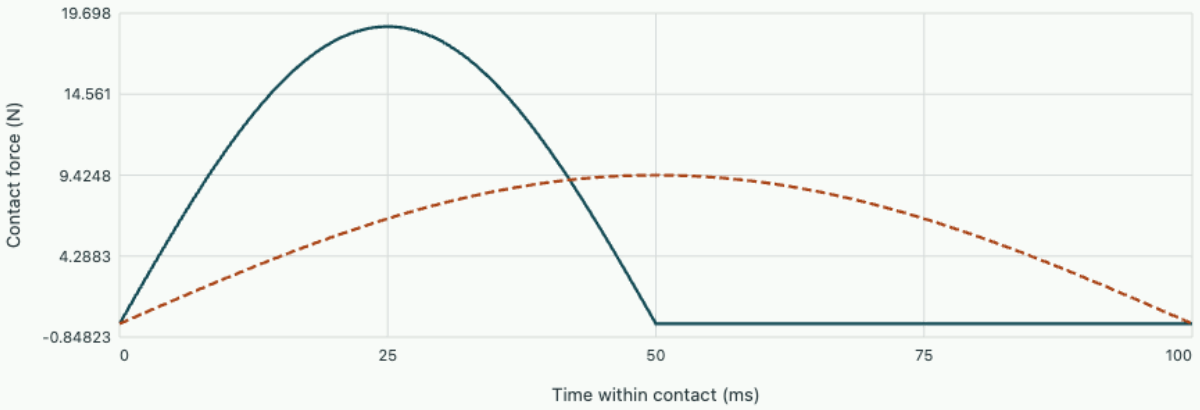
Interactive calculation · Half sine force

Vary the notebook inputs and compare both plotted observables. The same equations generate the browser curves and the notebook reference; fixed material, geometry and model assumptions are stated in the readout.

Open this live calculation

Try the calculation

Impulse (N s)	**0.6**
Natural frequency (Hz)	**8**
Pulse duration (ms)	**50**



Mass2 kg. Residual amplitude 5.123 mm; instantaneous-impulse limit 5.968 mm. The force integral remains 0.600 N s.

Download the editable notebook for this example

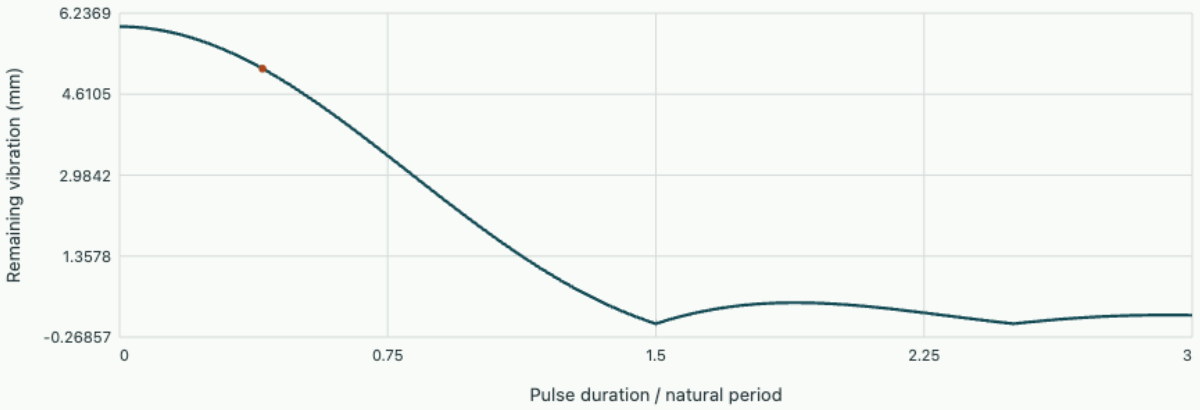
Interactive calculation · Half sine residual

Vary the notebook inputs and compare both plotted observables. The same equations generate the browser curves and the notebook reference; fixed material, geometry and model assumptions are stated in the readout.

Open this live calculation

Try the calculation

Impulse (N s)	0.6
Natural frequency (Hz)	8
Pulse duration (ms)	50



Mass2 kg. Residual amplitude 5.123 mm; instantaneous-impulse limit 5.968 mm. The force integral remains 0.600 N s.

Download the editable notebook for this example

Open editable notebook Download .ipynb

Select a time interval and a reference point

For a short collision, a finite change of velocity occurs over a time in which positions change very little. Integrate Newton’s law over that interval. A force that is harmless in an ordinary static calculation can deliver a large impulse if sustained long enough. Conversely, gravity often contributes very little over a millisecond. Compare its estimated impulse $mg\Delta t$ with the contact impulse before neglecting it.

For planar rigid-body motion the impulse–momentum balances about the mass centre G and about an inertially fixed point O are

$$m(v_G^+ - v_G^-) = \sum J,$$
$$H_O^+ - H_O^- = \sum r_{i/O} \times J_i.$$

The superscripts denote states just before and after contact. A hinge can deliver an unknown impulse but exerts no moment impulse about its own centre when modelled as an ideal pin. Choosing that point can eliminate unknowns when calculating angular speed. Kinetic energy is generally lost to deformation, contact vibration and other processes. It cannot supply a conservation equation without an additional elastic-impact assumption.

Apply the contact condition at the points that meet

For a smooth frictionless collision choose the normal from the incoming body towards the target. The separation speed along that normal equals the specified coefficient of restitution e times the approach speed. Evaluate velocities of the actual points in contact; the target's mass-centre speed can be different if the force also rotates it. For a contact point at distance a along a pinned bar and normal perpendicular to the bar, the normal contact velocity immediately after impact is $a\omega^+$.

Take a small slider of mass 1.0 kg moving at 3.0 m/s into a stationary slender uniform bar of mass 3.0 kg, length 0.60 m, pinned at one end. It strikes at $a = 0.40\,\mathrm{m}$, perpendicular to the bar. The bar inertia about the pin is

$$I_O = m_b L^2/3 = 0.360\,\mathrm{kg\,m^2}.$$

Let J be the positive impulse delivered to the bar. The slider exits at $3 - J/1$, and the bar receives angular speed Ja/I_O.

Close the collision and audit its energy

With $e = 0.50$, the restitution condition is $a(Ja/I_O) - (3 - J) = 0.50(3)$, giving $J = 4.5/[1 + 0.4^2/0.36] = 3.115\,\mathrm{N\,s}$. Thus $\omega^+ = 3.46\,\mathrm{rad/s}$ and the slider rebounds slowly at –0.115 m/s. The bar's contact point recedes at 1.385 m/s, exactly 1.50 m/s faster than the slider after separation.

The initial kinetic energy was 4.50 J. The final energy is

$$\tfrac{1}{2}(1)(0.115)^2 + \tfrac{1}{2}(0.360)(3.46)^2 = 2.16\,\mathrm{J}.$$

The energy loss is physically consistent with the specified restitution. The pin impulse need not vanish although its moment impulse vanished. The bar's centre moves at $0.30(3.46) = 1.04\,\mathrm{m/s}$; its linear momentum gain happens to equal about 3.12 N s for this impact position, very close to the contact impulse. Indeed $a = I_O/(m_b L/2) = 0.40\,\mathrm{m}$ is the centre-of-percussion location, for which the ideal transverse pin impulse vanishes exactly.

Determine whether a rough impact sticks or slides

The smooth collision model leaves tangential velocities unaffected. A rough planar impact can also exchange tangential impulse and rotational momentum. Consider a solid disc of radius R striking a fixed horizontal surface at its lowest point. Take horizontal right and vertical up as positive; angular speed is counterclockwise positive. Just before impact let centre velocities be u^-, v^- and angular speed ω^-. The horizontal speed of its contact point is $g_t^- = u^- + R\omega^-$. A rightward tangential impulse J_t produces both a centre velocity increase J_t/m and a positive angular change RJ_t/I_G.

The post-impact contact velocity is $g_t^+ = g_t^- + (1/m + R^2/I_G)J_t$. For the horizontal surface the normal impulse passes through the centre; restitution requires $J_n = -m(1+e)v^-$ for $v^- < 0$. The tangential impulse needed to eliminate slip is

$$J_t^{\text{stick}} = -g_t^- / (1/m + R^2/I_G).$$

Under a simple Coulomb impulse approximation compare this required magnitude with μJ_n. If it lies inside the bound the contact can finish without tangential slip. Otherwise take the limiting friction impulse opposing the pre-impact relative slip. This elementary model assumes no change in slip direction before the limiting impulse is reached, negligible external impulse, and no tangential elastic restitution. Using only the disc centre’s velocity to form the stick condition would miss its rotational compliance.

Compare two surfaces with the same normal restitution

A solid disc has $m = 2.0\,\text{kg}$, $R = 0.10\,\text{m}$,

$$I_G = mR^2/2 = 0.010\,\text{kg}\,\text{m}^2.$$

Its centre approaches at $u^- = 3.0,\ v^- = -2.0\,\text{m/s}$, with zero initial spin. For $e = 0.50$, $J_n = 6.0\,\text{N s}$ and $v^+ = 1.0\,\text{m/s}$. The inverse tangential effective mass is

$$1/2 + 0.10^2/0.010 = 1.50\,\text{kg}^{-1}.$$

Thus $J_t^{\text{stick}} = -2.0\,\text{N s}$ and the friction coefficient needed is at least $2/6 = 0.333$.

On a surface with $\mu = 0.20$, $J_t = -1.2\,\text{N s}$, giving $u^+ = 2.4\,\text{m/s}$ and $\omega^+ = -12\,\text{rad/s}$. The contact-point speed remains $2.4 - 1.2 = 1.2\,\text{m/s}$ to the right, consistent with the chosen sliding direction. On a surface with $\mu = 0.50$, the sticking impulse is achievable. It gives $u^+ = 2.0\,\text{m/s}$, $\omega^+ = -20\,\text{rad/s}$, and zero instantaneous tangential contact speed. The disc still leaves the surface vertically because $v^+ = 1.0$. Here sticking means zero relative tangential velocity at separation and does not mean a permanently attached body. The original 13 J of kinetic energy drops to 7.48 J in the sliding case and 7.00 J in the sticking case, consistent with frictional dissipation and a normal restitution below unity.

A high-speed camera that measures only the centre trajectory would determine the change in centre momentum and hence the tangential impulse, provided other horizontal impulses are negligible. Marking the disc edge permits an independent angular check: in the sliding case the ratio $I_G|\Delta\omega|/(Rm|\Delta u|)$ is

$$0.010(12)/[0.10(2)(0.60)] = 1.$$

A substantial measured deviation could indicate an additional contact couple, a second impact point or uncertainty in inertia. Measuring spin is particularly important near the transition to sticking because centre velocity by itself cannot establish that the tangential contact speed has vanished.

A marked rim therefore provides an observable that a centre-tracking experiment alone cannot supply.

Problem sheet 4 · Mechanism kinematics and impact

Recover constrained point accelerations and close two impact balances with appropriate contact conditions.

A point on a rotating arm

Point A on a planar body has velocity (0.8, 0.1) m/s and acceleration (1, −2) m/s². Point B is 0.30 m to the right and 0.10 m above A. The body rotates counterclockwise at 4 rad/s with clockwise angular acceleration 3 rad/s². Find the velocity and acceleration of B. Which term survives at constant angular speed?

▼ **Reveal the worked solution**

The relative velocity is $4(-0.10, 0.30) = (-0.4, 1.2)$, hence $v_B = (0.4, 1.3)\,\mathrm{m/s}$. The relative tangential acceleration is $(-3)(-0.10, 0.30) = (0.3, -0.9)$; the centripetal term is $-16(0.30, 0.10) = (-4.8, -1.6)$. Adding A gives $a_B = (-3.5, -4.5)\,\mathrm{m/s^2}$. Even with zero angular acceleration the centripetal term remains.

Observe a collision

On a horizontal line a 2 kg cart moving at 3 m/s collides with a stationary 1 kg cart. The restitution coefficient is 0.60. Find their final velocities, impulse on the target and energy lost. State what allows linear momentum conservation here.

▼ **Reveal the worked solution**

Horizontal external impulse is negligible. Thus

$$2v_1 + v_2 = 6,$$
$$v_2 - v_1 = 0.60(3) = 1.8.$$

Solving gives

$$v_1 = 1.4,\ v_2 = 3.2\,\mathrm{m/s}.$$

Target impulse is 3.2 N s. Initial energy is 9 J and final energy $1.4^2 + 3.2^2/2 = 7.08\,\mathrm{J}$, giving loss 1.92 J.

A particle sticks to a pinned bar

A 0.20 kg clay mass strikes perpendicular to a 0.50 m slender bar at 4 m/s and sticks to its free end. The bar mass is 1.2 kg and its other end is pinned. Find final angular speed and energy lost. Why is angular momentum about the pin the convenient balance?

▼ **Reveal the worked solution**

The incoming angular momentum is $0.20(4)(0.50) = 0.40\,\mathrm{kg\,m^2/s}$. Combined inertia about the pin is

$$1.2(0.50)^2/3 + 0.20(0.50)^2 = 0.150\,\mathrm{kg\,m^2}.$$

Thus $\omega = 2.667\,\mathrm{rad/s}$. Final energy is $0.150(2.667)^2/2 = 0.533\,\mathrm{J}$, versus 1.60 J initially. The ideal pin impulse has no moment about the pin, although it can change total linear momentum.

Predict the reaction moment from a tilted mass arrangement

Two 0.25 kg trim masses sit at $(0.12, 0, 0.08)$ m and its negative relative to a shaft centre. The shaft rotates steadily about positive z at 30 rad/s. Find I_{xz}, angular momentum, required external moment on the rotor and the moment at twice the speed.

▼ **Show the worked solution**

Here $I_{xz} = -2(0.25)(0.12)(0.08) = -0.0048\,\mathrm{kg\,m^2}$, and $I_{zz} = 2(0.25)(0.12)^2 = 0.0072\,\mathrm{kg\,m^2}$. Therefore $H = (-0.144, 0, 0.216)$ kg m²/s. The torque $(0, 0, 30) \times H = (0, -4.32, 0)$ N m. Doubling speed doubles angular momentum and its rotation rate, so the required moment magnitude grows fourfold to 17.28 N m. The frame feels the equal and opposite moment.

Revisit chapter 7 · Construct angular momentum when the rotation axis is not principal.

Explore the governing equations after solving the sheet: Velocities and accelerations in planar mechanisms; Impulse, angular momentum and impact. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Notebook practice · Does equal impulse leave equal vibration?

Chapter 9 · Static and couple balance of rotors

Represent rotating imbalance as a vector and determine the corrections needed to cancel both net force and rocking moment.

FROM FIRST PRINCIPLES

Represent a rotating offset by a force and moment phasor

A rigid rotor carries a small mass m at eccentricity e from its shaft axis. At constant angular speed ω, its radial acceleration has magnitude $e\omega^2$. The shaft supplies the inward force and receives an equal outward reaction whose direction rotates with the shaft. Encode the offset orientation by phase ϕ and the unbalance by $U = me\,e^{i\phi}$. Complex notation retains both perpendicular force components.

$$F_x + iF_y = \omega^2 U e^{i\omega t}.$$

For several masses spinning together, force superposition makes the net phasor proportional to $\sum U_j$. Axially separated forces can also produce a rotating couple even when the net force vanishes. Taking moments about a reference plane weights each unbalance by its axial distance z_j.

$$\sum_j U_j = 0,$$

$$\sum_j z_j U_j = 0.$$

These are the two rigid-rotor balance conditions. Changing the reference plane adds a multiple of the first sum to the second, so once net force vanishes the couple balance is independent of that choice. Flexible rotor modes at operating speed require additional information.

DERIVATION AND APPLICATIONS

Separate the imbalance force and couple at the bearings

A rotor can have little net rotating force yet cause large opposed bearing reactions. Treat the shaft as rigid with bearings at axial positions $z = 0$ and $z = L$. Assume speed ω is constant and sufficiently below the first bending resonance for a rigid load distribution. Represent each unbalance by the complex vector $U_j = m_j e_j e^{i\phi_j}$. The physical rotating force on the support is the real part of the phasor times $e^{i\omega t}$. Summing forces and taking moments about the first support gives the transmitted load phasors F_A, F_B.

$$F_A + F_B = \omega^2 \sum_j U_j,$$

$$L F_B = \omega^2 \sum_j z_j U_j.$$

The moment equation weights each unbalance by axial position. Solving for the individual support loads identifies the force distribution even for a resultant whose direction rotates with the shaft. The weights can be negative for overhung masses.

$$F_B = \frac{\omega^2}{L}\sum_j z_j U_j,$$

$$F_A = \omega^2 \sum_j (1 - z_j/L) U_j.$$

Conversely, two synchronised bearing measurements supply both the summed unbalance and its first axial moment. Their relative phases must be retained, since adding magnitude readings would erase the signature of a rotating couple.

$$U_{\mathrm{sum}} = \frac{F_A + F_B}{\omega^2},$$

$$M_U = \sum_j z_j U_j = \frac{L F_B}{\omega^2}.$$

With $U_{\mathrm{sum}} = 0$, equal and opposite support phasors can remain. Their mount loads grow with squared speed even though the instantaneous vector sum vanishes.

WORKED EXAMPLE

Predict bearing reactions from a pure couple

Take $L = 0.600$ m, $\omega = 100$ rad/s, and equal opposite unbalances of magnitude 200 g mm in planes at 0.100 and 0.500 m. Taking the first unbalance as positive,
$M_U = 0.000200(0.100 - 0.500) = -8.00 \times 10^{-5}$ kg m². The bearing load $F_B = -1.33$ N and $F_A = +1.33$ N. A single accelerometer measuring common support translation could miss the rotating moment.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Identify the plane of a single residual mass

In a different machine, in-phase bearing load amplitudes are 3.00 N and 1.00 N at A, B respectively, $\omega = 200$ rad/s and $L = 0.800$ m. The inferred net unbalance is $4.00/200^2 = 1.00 \times 10^{-4}$ kg m, or 100 g mm. For a single residual plane its equivalent position is

$$L F_B/(F_A + F_B) = 0.200\,\mathrm{m}.$$

A pair of distributed errors may produce the same two rigid-body measurements.

Interactive calculation · Change the inputs of this example

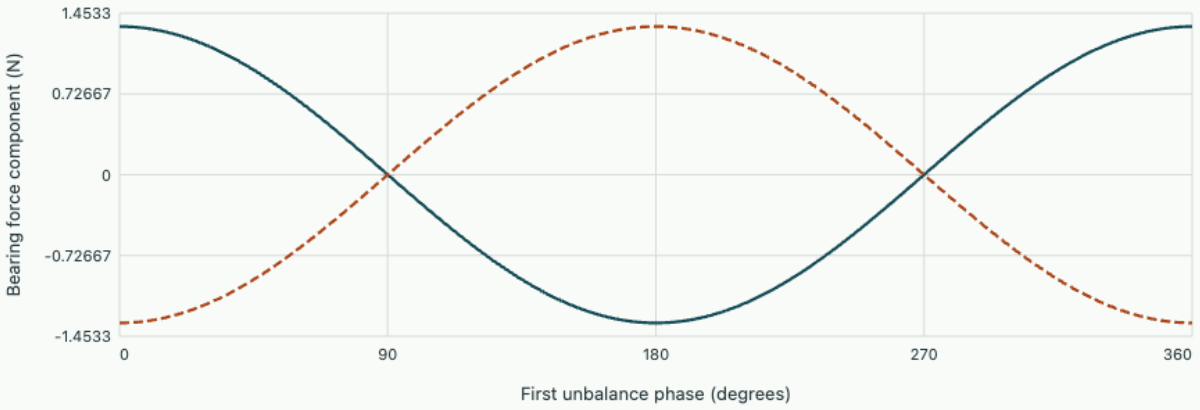
Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Unbalance magnitude (g mm) **200**

Spin speed (rad/s) **100**



Equal opposite unbalances at0.10 and0.50 m on a0.60 m span give opposite bearing-force amplitudes 1.333 N. Net force is zero while the rotating couple persists.

Translate a mass offset into a periodic bearing load

A rotor can have its total mass centre on the shaft axis and still shake its housing. The reason is spatial: two equal and opposite centrifugal force vectors in separate transverse planes form a rotating couple. First describe one small excess mass m at radial offset e and clock angle ϕ. At constant angular speed ω, its required radial acceleration has magnitude $e\omega^2$. The equal and opposite reaction transmitted to the rotating assembly has magnitude $me\omega^2$.

Represent its radial unbalance by a complex number $U = me(\cos\phi + i\sin\phi)$. This is shorthand for two transverse components at a reference shaft orientation, not a complex physical force. Multiplication by $e^{i\omega t}$ rotates all vectors together. Since both ω^2 and this rotation are common to all masses, cancellation can be carried out using U alone.

$$\sum U_j = 0,$$
$$\sum z_j U_j = 0.$$

Here z is axial location from a chosen reference plane. The second condition cancels the rotating transverse moment; its vector direction includes a common quarter-turn from the cross product that drops out of the zero condition. If the first condition holds, shifting the axial origin leaves the second condition unchanged.

Split one known unbalance between two planes

A rigid rotor has an 18 g excess mass at radius 40 mm, angle 0°, and axial location 120 mm between two correction planes at $z_A = 0$ and $z_B = 300\,\mathrm{mm}$. Its unbalance is $720\,\mathrm{g\,mm}$ along the reference direction. Moment balance gives $300 U_B = -120(720)$, hence $U_B = -288\,\mathrm{g\,mm}$. Force balance then gives $U_A = -432\,\mathrm{g\,mm}$. Both corrections point at 180° because the original mass lies between the planes.

At correction radius 60 mm these vectors require added masses 4.80 g in plane B and 7.20 g in plane A. If material is removed instead, the equivalent removed masses lie at the original 0° clock position. At 1800 rpm the original force amplitude was $0.000720(188.5)^2 = 25.6\,\mathrm{N}$. Doubling speed quadruples that force, yet the same mass corrections still balance the assumed rigid rotor.

Identify what a single-plane test can miss

Consider instead two unbalances $+400$ and $-400\,\mathrm{g\,mm}$ in planes 200 mm apart. Their resultant force is zero at every speed. On ideal low-friction supports there is no preferred heavy side in a static test. Their couple magnitude at 1800 rpm is $0.000400(0.200)(188.5)^2 = 2.84\,\mathrm{N\,m}$, sufficient to generate equal opposing reactions at bearings. Measurements in two support planes can distinguish this pattern from a net shaking force. Near a flexible rotor’s bending critical speed, shaft deflection changes the mass distribution and response phase; the rigid-body correction formulas then require a modal or measured influence-coefficient model.

Interpret correction accuracy

An angular placement error also leaves a residual vector. A correction of the correct magnitude set one degree away leaves a residual approximately $U\pi/180$ for small angle, since the tip of the vector follows a circle. At large operating speed even this residual produces a force scaling with the square of speed.

Derive a balance correction from a trial mass

An assembled rotor rarely reveals the angular location of its excess mass directly. A once-per-revolution vibration measurement supplies an amplitude and phase referred to a shaft mark. At fixed operating speed and linear support conditions, represent the measured complex vibration V by $V = \alpha U$, where U is the complex unbalance in one plane and α is the response per unit unbalance. This coefficient includes bearing compliance, dynamic amplification and the sensor's phase lag. The phase of largest displacement is therefore not necessarily the angle where correction mass should be added.

Record the original vibration V_0, add a known trial unbalance U_t and measure V_t at the same speed with the same phase reference. Subtraction cancels the original signal and gives

$$\alpha = (V_t - V_0)/U_t,$$

$$U_c = -V_0/\alpha.$$

Remove the trial before installing the calculated total correction. If the trial remains attached, install only $U_c - U_t$ as further correction. Subtract complex vectors, not their magnitudes. A trial too small to resolve above measurement noise makes the inferred phase and correction unreliable; an unnecessarily large trial can take the rotor outside the measured linear range. A repeat baseline after removing the mass can expose a changing thermal or mounting condition.

Convert measured shaft phase into a physical correction

At fixed speed a laser reports an original synchronous displacement $V_0 = (30 + 40i)\ \mu\mathrm{m}$. Add a 5.0 g trial mass at a 40 mm radius at the reference angle, so $U_t = 200\,\mathrm{g\,mm}$. The new reading is $V_t = (50 + 60i)\ \mu\mathrm{m}$. The influence coefficient is $(20 + 20i)/200 = 0.10 + 0.10i$ micrometres per g mm. The required correction is

$$U_c = -\frac{30 + 40i}{0.10 + 0.10i} = -(350 + 50i)\,\mathrm{g\,mm}.$$

Its magnitude is $353.6\,\mathrm{g\,mm}$ and angle is 188.1° from the trial reference in the same positive rotational convention. With a correction radius of 50 mm, add about 7.07 g after removing the trial. Multiplying the correction vector by the measured coefficient gives $-30 - 40i$ micrometres, cancelling the baseline in the linear model. If the trial must stay, the extra vector is $-550 - 50i$ g mm. Installing the full correction in addition to the trial would leave its trial response as a predictable residual.

Single-plane correction minimises one measured response at one speed. For a long rotor with a separate couple unbalance, a response measured at one support may vanish while the opposite support continues to vibrate. Measuring both supports under trial masses in two planes produces a two-by-two complex influence matrix. Nearly proportional measured columns mean that the two trial planes cannot be distinguished reliably at those sensor locations, a conditioning issue that no amount of arithmetic precision can cure.

The initial amplitude in the example is 50 micrometres; the trial amplitude is 78.1 micrometres. Their scalar difference, 28.1 micrometres, happens to be close to the actual complex response increment's

magnitude of 28.3 micrometres, but the scalar comparison contains no correction angle. Even apparently favourable numbers cannot replace the shaft reference.

A balanced rotor can load its bearings during a turn

After a rotor is balanced, its spin angular momentum still has to change direction whenever its supporting structure turns. Consider an axisymmetric flywheel rotating rapidly at ω_s about unit vector e_s, with polar inertia I_s. If its axis is steered at angular rate Ω, then $\dot{e}_s = \Omega \times e_s$. Isolate the spin part of angular momentum and differentiate in an inertial frame:

$$H_s = I_s \omega_s e_s,$$

$$M_s = I_s \dot{\omega}_s e_s + \Omega \times H_s.$$

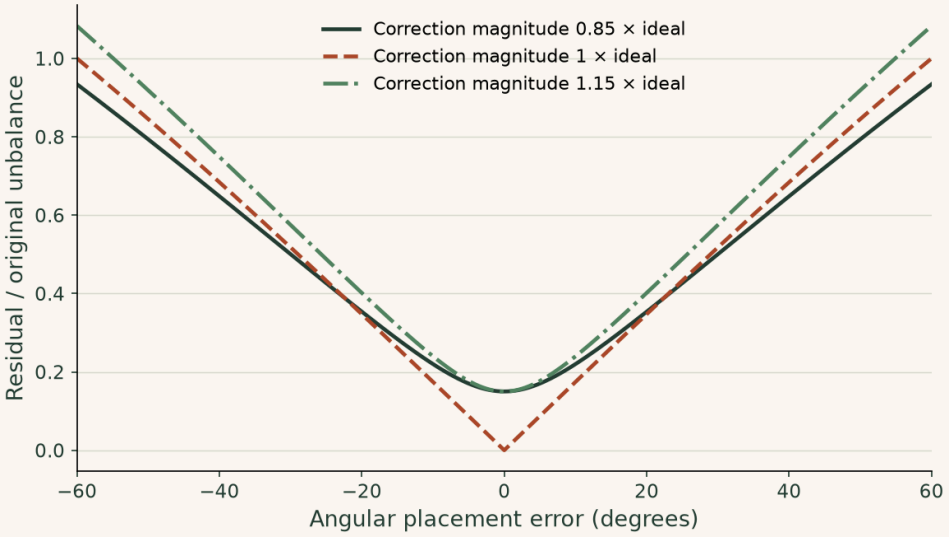
The first term spins the rotor up or down. The second turns its axis without changing its spin energy to leading order. For steady orthogonal steering the moment magnitude is $I_s |\omega_s \Omega|$. Components of Ω parallel to the spin axis do not turn the angular momentum. For slow regular precession of a fast rotor this spin-dominated relation also gives $\Omega \simeq M / H_s$ for a transverse applied moment. Rapid steering acceleration, strong nutation or significant transverse angular momentum requires the full tensor equations.

A horizontally mounted flywheel on a turntable has $I_s = 0.12\,\mathrm{kg\,m^2}$, spins in the positive x direction at 3000 rpm and is turned about positive z at 0.40 rad/s. Angular momentum is $0.12(3000)(2\pi/60) = 37.7\,\mathrm{kg\,m^2/s}$ along x. Since $e_z \times e_x = e_y$, the bearings must exert 15.1 N m along positive y on the rotor. A 0.30 m support separation along x requires a pair of equal opposite vertical bearing forces of 50.3 N to deliver this moment, in addition to weight and ordinary radial loads. Reversing the steering reverses the couple immediately at a given speed. A rigid imbalance force at fixed shaft speed does not reverse just because steering changes direction. That distinction is a practical way to diagnose the origin of a load measured on a moving platform.

Practise on sheet 5 · Balance the steering moment of a two-flywheel instrument. Further reading · MIT · 16.07 Dynamics · Lecture L30, 3D Rigid Body Dynamics: Tops and Gyroscopes, steady precession.

How accurately must a balance mass be placed?

A single-plane correction should cancel the original unbalance vector. If its magnitude ratio is k and its angle differs from perfect opposition by δ, vector addition leaves the normalised magnitude $\sqrt{1 + k^2 - 2k\cos\delta}$.



Even an otherwise exact correction leaves 0.174 of the original unbalance for a 10° angular error. A magnitude tolerance creates a nonzero floor at zero angle. The plotted ratio also scales the residual centrifugal force at any chosen speed.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

Change magnitude error and spin speed. Identify the residual force that remains even with perfect angular placement.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Correction / ideal magnitude	1
Original unbalance (g mm)	100
Spin speed (rpm)	3000



With zero placement error, residual force is 0.000 N. Doubling speed multiplies every force by four.

Study the full derivation

Critical speed, synchronous whirl and two different rotor centres · Vibration and identification in machinery

A stationary probe observes motion of a rotating shaft surface, while a balance calculation follows an eccentric mass centre. Derive both trajectories from one flexible-rotor model. Their difference explains the apparent recentring of a rotor at high speed and why a single vibration amplitude cannot identify unbalance and damping.

Chapter 10 · Energy coordinates and Lagrange’s equation

Derive a coupled translation–rotation equation with a rolling constraint and identify which reaction disappears from the energy balance.

FROM FIRST PRINCIPLES

Derive the equation of motion using one independent coordinate

An ideal geometric constraint ties several motions together. Choose an independent coordinate q that automatically satisfies it and express kinetic energy $T(q,\dot{q})$ and potential energy $V(q)$ in that coordinate. In the principle of virtual work, ideal constraint reactions do no work along admissible small motions. The balance of inertial, conservative and applied virtual work gives Lagrange's equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = Q_{\mathrm{nc}},$$

$$\delta W_{\mathrm{nc}} = Q_{\mathrm{nc}}\,\delta q.$$

For a rigid cylinder rolling on a stationary surface without slip, take centre displacement $q = x$ and angular speed $\dot{x}/R$. Its translation and rotation contribute to kinetic energy; a horizontal spring attached at the centre supplies the potential.

$$T = \tfrac{1}{2}(m + I_G/R^2)\dot{x}^2,$$

$$V = \tfrac{1}{2}kx^2, \qquad (m + I_G/R^2)\ddot{x} + kx = Q_{\mathrm{nc}}.$$

Static friction supplies the contact reaction required for rolling. The contact point has zero instantaneous velocity, so that reaction contributes zero power here. The solution must still demand no more friction than the surface can supply.

DERIVATION AND APPLICATIONS

Derive the effective stiffness of a pendulum on an accelerating cart

A continuously accelerating support changes both the equilibrium direction and the small-oscillation frequency of a hanging load. Model a point mass m on a massless rigid link of length ℓ with an ideal hinge. Let the support move horizontally with imposed coordinate $X(t)$ and measure angle θ from the downward vertical towards positive horizontal displacement. Ignore drag. With mass coordinates $x = X + \ell\sin\theta$ and $y = -\ell\cos\theta$, the kinetic and potential energies are

$$T = \tfrac{1}{2}m[\dot{X}^2 + 2\dot{X}\ell\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2],$$

$$V = -mg\ell\cos\theta.$$

Differentiate $\partial T/\partial\dot\theta$ with respect to time and subtract $\partial T/\partial\theta$. The velocity cross terms involving $\dot X\dot\theta$ cancel, leaving a forcing term set by support acceleration, independent of support speed.

$$\ell\ddot\theta + g\sin\theta + \ddot X\cos\theta = 0.$$

For constant support acceleration a, a stationary angle obeys $\tan\theta_0 = -a/g$. Choose the hanging equilibrium. Expanding the restoring term about θ_0 gives the effective gravitational acceleration along this new direction and the frequency of small oscillations.

$$\theta = \theta_0 + \eta,$$

$$\ddot\eta + \frac{\sqrt{g^2 + a^2}}{\ell}\eta = 0,$$

$$\omega_n = \sqrt{\frac{\sqrt{g^2 + a^2}}{\ell}}.$$

A positive cart acceleration tips the pendulum backwards. The first correction to angle is linear in a/g, while the first relative correction to frequency is quadratic. The exact equilibrium result holds at finite a; the frequency expression requires small oscillations about it.

WORKED EXAMPLE

Predict a suspended sensor's direction

A cart accelerates at 3.00 m/s² carrying a pendulum of length 0.600 m. Its steady angle is

$$-\arctan(3.00/9.81) = -17.0^\circ.$$

The effective acceleration is 10.26 m/s², giving $\omega_n = 4.13$ rad/s and period 1.52 s. A sensor that treats the pendulum direction as vertical during this manoeuvre would carry the full angular bias.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Infer acceleration from a steady deflection

In another rig a 0.400 m pendulum settles 8.00° backwards after weak damping removes its initial swing. Constant acceleration is

$$9.81\tan 8.00^\circ = 1.38\,\mathrm{m/s^2}.$$

The effective gravity is $9.81/\cos 8.00^\circ = 9.91$ m/s², predicting a small-oscillation period $2\pi\sqrt{0.400/9.91} = 1.26$ s for weak damping. The angle is much more sensitive to modest acceleration than the period.

Interactive calculation · Change the inputs of this example

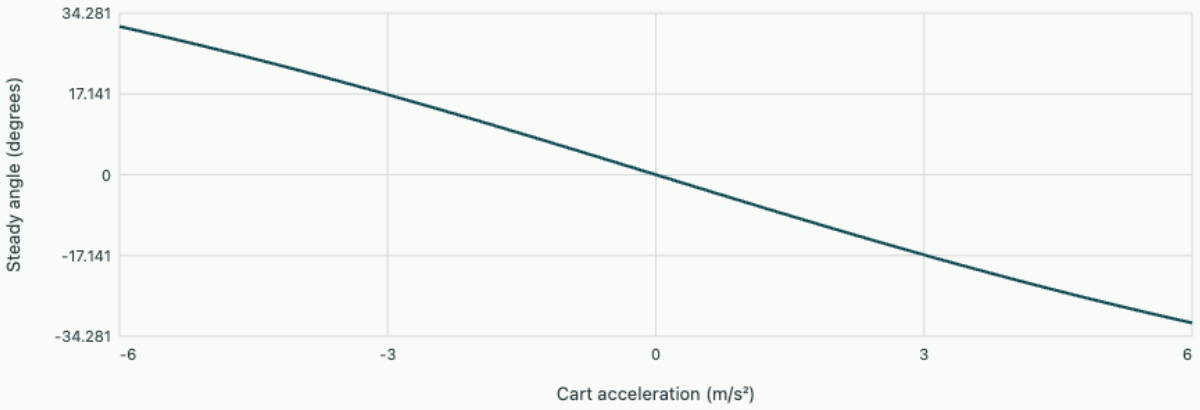
Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Cart acceleration (m/s²) **3**

Pendulum length (m) **0.6**



Effective-gravity alignment

Selected equilibrium angle -17.004°; small oscillation frequency 4.135 rad/s.

Let geometry remove a reaction

A cylinder rolls along a horizontal rail while a spring attached at its centre pulls it towards equilibrium. The rail friction is essential for rolling, but an ideal contact point is instantaneously at rest and the friction force does no work. One can write separate translational and rotational equations with an unknown friction force. Alternatively choose a coordinate that satisfies rolling from the beginning. The reduced energy then contains the rotational inertia without introducing a contact reaction.

Take the rightward displacement x of the cylinder centre as independent coordinate and clockwise angle as positive. On the fixed surface the no-slip condition gives $x = R\theta$, up to a reference offset. A body of total mass m and centre inertia I_G has kinetic energy from both translation and rotation. A centre spring with stiffness k has elastic potential $kx^2/2$.

$$T = \tfrac{1}{2}(m + I_G/R^2)\dot{x}^2,$$
$$V = \tfrac{1}{2}kx^2.$$

Connect energy to a dynamical equation

The virtual work of a horizontal force F applied at the centre is $F\delta x$. In this coordinate its generalised force is simply F. A torque M acting clockwise would do $M\delta\theta = (M/R)\delta x$ and would enter as a force M/R. With a centre damper c, the dissipative force is $-c\dot{x}$. Lagrange's equation for this system becomes

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = F - c\dot{x}, \qquad (m + I_G/R^2)\ddot{x} + c\dot{x} + kx = F.$$

For coordinates whose inertia depends on position, retain both derivatives involving T. The differentiation of $\partial T/\partial \dot{x}$ then generates a term proportional to $\dot{x}^2$, part of which cancels $\partial T/\partial x$. Simply multiplying a configuration-dependent effective mass by acceleration would miss the remaining term.

Predict a frequency and check rolling

A solid cylinder has $m = 8.0\,\mathrm{kg}$, $R = 0.10\,\mathrm{m}$, and $I_G = mR^2/2 = 0.040\,\mathrm{kg\,m^2}$. The effective mass seen by the spring is

$$8 + 0.040/0.10^2 = 12.0\,\mathrm{kg}.$$

With $k = 480\,\mathrm{N/m}$ and negligible damping the natural frequency is

$$\sqrt{480/12}/(2\pi) = 1.007\,\mathrm{Hz}.$$

Release it from rest at $x = +0.060\,\mathrm{m}$. Its initial acceleration is

$$-480(0.060)/12 = -2.40\,\mathrm{m/s^2}.$$

Writing only the centre's translation and ignoring rotation would overpredict the acceleration magnitude.

Reconstruct the previously eliminated contact force from translation. The spring pulls leftwards with 28.8 N and the centre accelerates leftwards with $m\ddot{x} = -19.2\,\mathrm{N}$. Friction must therefore point rightwards with 9.6 N to supply the appropriate angular acceleration. The coefficient of static friction must be at least $9.6/(8 \times 9.81) = 0.122$ at release. For a smaller coefficient, the assumed rolling constraint fails and the model needs separate rotational and translational coordinates together with a slipping contact law.

Compare two cylinders with identical mass

A thin-walled hoop of the same mass and rolling radius has $I_G = mR^2$. With the original spring its effective mass doubles the translational mass, giving natural frequency $\sqrt{480/16}/(2\pi) = 0.872\,\mathrm{Hz}$. A no-slip ring-down can therefore distinguish two bodies whose masses and outer dimensions are identical but whose internal mass distributions differ.

Use a cyclic coordinate to reduce coupled motion

Mount a pendulum on a cart that can roll horizontally on a level frictionless guide. Let cart displacement be x, bob angle θ from downward vertical, length l, bob mass m and remaining cart mass M. The bob position is $(x + l\sin\theta, -l\cos\theta)$. Differentiating and squaring its speed gives a cross term $2l\dot{x}\dot{\theta}\cos\theta$ because cart and pendulum velocities need not be perpendicular. The full energy terms are

$$T = \tfrac{1}{2}(M+m)\dot{x}^2 + ml\cos\theta\,\dot{x}\dot{\theta} + \tfrac{1}{2}ml^2\dot{\theta}^2,$$

$$V = mgl(1-\cos\theta).$$

Horizontal position itself does not occur in either expression. It is a cyclic coordinate, so $\partial T/\partial\dot{x} = (M+m)\dot{x} + ml\cos\theta\,\dot{\theta}$ is conserved if no external horizontal force acts. This is horizontal momentum written in generalised coordinates. The pendulum equation from Lagrange's equation is

$$l\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0.$$

For small swings from rest, differentiate the momentum relation and linearise to obtain

$$\ddot{x} = -ml\ddot{\theta}/(M+m).$$

Substituting gives

$$[lM/(M+m)]\ddot{\theta} + g\theta = 0.$$

Free horizontal motion makes the small-swing frequency larger than for a clamped cart, by $\sqrt{(M+m)/M}$. This result reflects motion about the system's horizontally constrained centre of mass. The extra degree of freedom also permits uniform horizontal translation at zero frequency. Its presence does not imply an additional oscillatory mode.

Predict opposing cart travel in a release test

Let $M = 3.0$ kg, $m = 1.0$ kg and $l = 0.60$ m. Clamp the cart: the oscillation period would be $2\pi\sqrt{0.60/9.81} = 1.55\,\mathrm{s}$. Release its horizontal guide and the period becomes

$$2\pi\sqrt{0.60(3)/(9.81\times 4)} = 1.35\,\mathrm{s}.$$

At release the bob is at $\theta_0 = 10^\circ$ and both bodies are at rest. Because horizontal momentum remains zero, integrate the exact relation to obtain

$$(M+m)x + ml\sin\theta = \text{constant}.$$

As the bob first crosses the vertical, the cart has shifted $ml\sin 10^\circ/(M+m) = 0.0260\,\text{m}$ to the right from its starting point.

At the vertical the potential-energy decrease is

$$mgl(1-\cos 10^\circ) = 0.0894\,\text{J}.$$

Substituting $\dot{x} = -ml\dot{\theta}/(M+m)$ into the kinetic energy at this instant gives effective angular inertia $ml^2M/(M+m) = 0.270\,\text{kg}\,\text{m}^2$. Thus the angular speed magnitude is $\sqrt{2(0.0894)/0.270} = 0.814\,\text{rad/s}$, and the cart speed magnitude is $0.15(0.814) = 0.122\,\text{m/s}$. The frequency estimate used a small-angle approximation; the centre-of-mass and energy results at the vertical used the actual release angle. A horizontal cable pulling on the cart or measurable wheel friction would change the conserved momentum and should be measured before attributing deviations solely to pendulum nonlinearity.

The two-sensor experiment can measure conservation without estimating acceleration. Track the cart position and bob angle from video, then plot $x + [ml/(M+m)]\sin\theta$. For the chosen masses its value should stay constant at the initial offset despite the cart and pendulum both moving. A steadily sloping line indicates nonzero initial horizontal momentum; a curved departure suggests an external force or tracking bias. This separation is useful because rolling resistance can be weak enough to leave the first measured period nearly unchanged while producing visible drift over several cycles. The conserved combination is specific to the inertial laboratory coordinate, so a camera fixed on the moving cart needs a separate record of cart displacement.

Problem sheet 5 · Balancing and constrained motion

Separate the net and couple unbalance of a rotor and recover the forces required by ideal rolling.

Add two angular unbalances

A 10 g mass at radius 30 mm and angle 0° shares a plane with an 8 g mass at radius 50 mm and angle 90°. Find the mass and angular position to add at radius 40 mm. Compute the original force amplitude at 1200 rpm.

▼ **Reveal the worked solution**

The unbalance is $(300, 400)\,\mathrm{g\,mm}$ with magnitude 500 g mm and angle 53.13°. The correction opposes it at 233.13°, and its mass is

$$500/40 = 12.5\,\mathrm{g}.$$

In SI units the force amplitude is

$$0.0005(2\pi 1200/60)^2 = 7.90\,\mathrm{N}.$$

Correct an overhung imbalance

An imbalance of 600 g mm at angle 0° lies 50 mm beyond plane B. A and B are 200 mm apart. Find both correction vectors, then give the required added masses at respective radii of 25 and 50 mm. Why do the corrections point in different directions?

▼ **Reveal the worked solution**

With A as origin, $200U_B = -250(600)$ gives $U_B = -750\,\mathrm{g\,mm}$. Force balance gives $U_A = +150\,\mathrm{g\,mm}$. Add 15 g at 180° in B and 6 g at 0° in A. A correction between the planes cannot reproduce the larger moment arm of the overhang without an opposing correction to recover the required resultant.

Roll a ring down a slope

A thin hoop of mass 4 kg rolls without slipping down a 12° slope. Derive its acceleration, the friction force magnitude and minimum coefficient of static friction. Assume no rolling resistance.

▼ **Reveal the worked solution**

The energy-coordinate effective mass is

$$m + I/R^2 = 2m.$$

Gravity supplies $mg\sin 12^\circ$, so

$$a = g\sin 12^\circ/2 = 1.020\,\mathrm{m/s^2}.$$

Translational balance requires uphill friction $mg\sin 12^\circ - ma = 4.08\,\mathrm{N}$. With normal load $mg\cos 12^\circ$, the minimum coefficient is

$$\tan 12^\circ/2 = 0.106.$$

The friction supplies torque while its instantaneous contact point is at rest.

Balance the steering moment of a two-flywheel instrument

Two coaxial flywheels spin oppositely at 2400 rpm while their frame turns at 0.30 rad/s about a perpendicular axis. Their polar inertias are 0.080 and 0.060 kg m². Find the residual couple. What speed of the smaller wheel cancels the couple if the larger wheel retains its speed?

▼ **Show the worked solution**

The signed net angular momentum magnitude is

$$(0.080 - 0.060)(2400)(2\pi/60) = 5.03\,\mathrm{kg\,m^2/s}.$$

The moment is $0.30(5.03) = 1.51\,\mathrm{N\,m}$. Its direction follows the turn rate crossed into the larger wheel's spin. Cancelling angular momentum requires $0.060 n_{\mathrm{small}} = 0.080(2400)$, giving 3200 rpm in the opposing sense. Equal speeds alone cancel only if the polar inertias also match. Changes of either spin speed during the turn add separate axial torques.

Revisit chapter 9 · A balanced rotor can load its bearings during a turn.

Explore the governing equations after solving the sheet: Static and couple balance of rotors; Energy coordinates and Lagrange's equation. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Chapter 11 · Modal participation and absorber tuning

Project external forcing onto vibration modes and determine when a vibration absorber suppresses displacement at a selected frequency.

FROM FIRST PRINCIPLES

Project a physical force onto a vibration mode

For small undamped motions a structure obeys $M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{f}(t)$ with symmetric mass and stiffness matrices. Its normal modes satisfy

$$K\boldsymbol{\phi}_j = \omega_j^2 M\boldsymbol{\phi}_j.$$

Expand displacement in these mode shapes, using time-dependent coefficients q_j. Premultiply the result by the transpose of one mode.

$$\mathbf{x} = \sum_j \boldsymbol{\phi}_j q_j,$$

$$\sum_j (\boldsymbol{\phi}_i^{\mathsf{T}} M\boldsymbol{\phi}_j\, \ddot{q}_j + \boldsymbol{\phi}_i^{\mathsf{T}} K\boldsymbol{\phi}_j\, q_j) = \boldsymbol{\phi}_i^{\mathsf{T}}\mathbf{f}.$$

For distinct frequencies, symmetry makes different modes orthogonal under both matrices. The corresponding off-diagonal products vanish, leaving a separate scalar oscillator for the selected mode.

$$m_i\ddot{q}_i + m_i\omega_i^2 q_i = Q_i(t),$$

$$m_i = \boldsymbol{\phi}_i^{\mathsf{T}} M\boldsymbol{\phi}_i,$$

$$Q_i = \boldsymbol{\phi}_i^{\mathsf{T}}\mathbf{f}.$$

The force projection has a direct interpretation as work per unit modal displacement. A symmetric load can have zero projection onto an antisymmetric mode despite a large total force. The displacement at a particular sensor is obtained only after adding its components from all excited modes.

Background · Modal participation

$$Q_j = \phi_j^{\mathsf{T}} f$$

A mode is excited according to its overlap with the force. Always state how its mode vector and modal mass were normalised.

DERIVATION AND APPLICATIONS

Determine how a point force appears in collocated modal measurements

The displacement measured at the force location can have a zero between two structural resonances. Consider an undamped linear structure with two relevant modes, distinct natural frequencies $\omega_1 < \omega_2$ and modal masses $m_j > 0$. Apply a harmonic point force of complex amplitude F at coordinate a and measure displacement at the same coordinate. For modal shape component $\phi_j(a)$, the generalised force is $\phi_j(a)F$, and reconstruction at the sensor supplies a second factor of that component.

$$q_j = \frac{\phi_j(a)F}{m_j(\omega_j^2 - \omega^2)},$$

$$H_{aa} = \frac{x_a}{F} = \frac{A_1}{\omega_1^2 - \omega^2} + \frac{A_2}{\omega_2^2 - \omega^2},$$

$$A_j = \frac{\phi_j(a)^2}{m_j}.$$

The residues A_j are positive when both modes are visible. Between the two poles the first denominator is negative and the second positive. Set the sum to zero and multiply through by both denominators to find the antiresonance.

$$A_1(\omega_2^2 - \omega_z^2) + A_2(\omega_1^2 - \omega_z^2) = 0,$$

$$\omega_z^2 = \frac{A_1\omega_2^2 + A_2\omega_1^2}{A_1 + A_2}.$$

A zero in the measured displacement can therefore result from cancellation of two finite modal displacements. Its position also reveals their relative participation without requiring the absolute force calibration.

$$\frac{A_1}{A_2} = \frac{\omega_z^2 - \omega_1^2}{\omega_2^2 - \omega_z^2}.$$

Separated force and response locations can give signed residues and change the interlacing. With damping a finite minimum commonly replaces the exact zero, so using its frequency as an exact antiresonance is an approximation.

WORKED EXAMPLE

Predict the dip of a two-mode stage

For $\omega_1 = 10.0$ rad/s, $\omega_2 = 20.0$ rad/s and $A_1 = 2A_2$, the squared zero frequency is $(2 \times 400 + 100)/3 = 300$ s⁻², hence $\omega_z = 17.3$ rad/s. At that frequency the first contribution is

$2A_2/(100-300) = -0.0100A_2$ and the second is

$$A_2/(400-300) = +0.0100A_2.$$

The sensor is stationary while both modes respond.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Recover a relative participation from a measured minimum

A different collocated test gives resonances at 30.0 and 50.0 Hz and a negligible-damping minimum at 40.0 Hz. Ratios of squared angular frequencies can equally be expressed through squared frequencies in hertz. The residue ratio is

$$(40^2 - 30^2)/(50^2 - 40^2) = 7/9 = 0.778.$$

With equal modal masses the first shape magnitude at the force point is $\sqrt{7/9} = 0.882$ of the second.

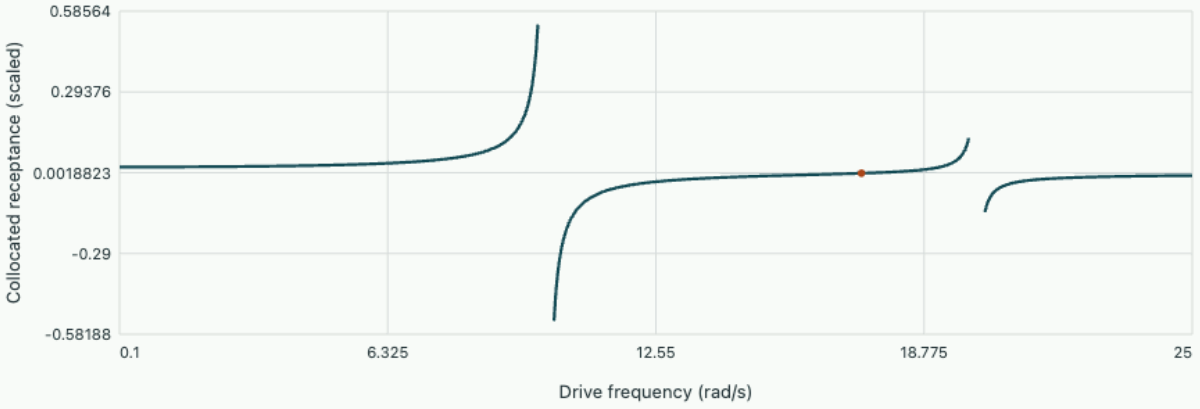
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

First modal participation / second	2
First natural frequency (rad/s)	10
Second natural frequency (rad/s)	20



Two undamped modal terms | Antiresonance

The cancellation lies at 17.321 rad/s between the two poles. Small neighbourhoods of the undamped poles are omitted from the curve.

Start from this chapter. Use the frequency shifts to identify which mode is most sensitive to an added rotary inertia. A forcing amplitude would also need a participation factor. The numerical experiment here computes free modes.

OneSimulation laboratory · Separate the effect of mass from rotary inertia

Recover motion from mode amplitudes

Knowing a structure's natural frequencies does not yet tell you which frequencies a particular actuator will excite. For a linear undamped system write $M\ddot{x} + Kx = f(t)$ and let ϕ_j be a real mode shape solving $K\phi_j = \omega_j^2 M\phi_j$. Assume a complete set of independent modes and expand the displacement

$x = \sum_j \phi_j q_j$. Multiplying the equation by ϕ_i^T uses mass and stiffness orthogonality to remove other modal coordinates.

$$m_i \ddot{q}_i + k_i q_i = \phi_i^T f(t),$$

$$m_i = \phi_i^T M \phi_i,$$

$$k_i = \omega_i^2 m_i.$$

Multiplying a mode shape by a constant changes the numerical modal mass and generalised force, while the reconstructed physical motion remains invariant. Thus a tabulated mode shape needs its normalisation if it is used to compute a physical response. A forcing pattern whose projection vanishes excites none of that mode from rest, even if the excitation frequency equals its natural frequency.

Select a mode by choosing the actuator pattern

Suppose two identical masses 2 kg each have mode shapes $(1, 1)$ and $(1, -1)$, with frequencies 10 and 20 rad/s. Both modal masses are 4 kg. A 12 N harmonic force on the first mass projects 12 N onto each mode in this normalisation. At a forcing frequency of 5 rad/s the undamped steady modal amplitudes are

$$12/[4(10^2 - 5^2)] = 0.0400\,\mathrm{m},$$

$$12/[4(20^2 - 5^2)] = 0.00800\,\mathrm{m}.$$

Recombination gives first and second physical amplitudes 0.0480 and 0.0320 m.

Now distribute the same total force as 6 N on each mass in phase. The first projection stays 12 N and the second becomes $6 - 6 = 0$. The masses now both move 0.0400 m. Relocating or phasing actuators can therefore change observed vibration without changing any mass, spring or natural frequency.

Cause the primary displacement to vanish

Attach an auxiliary mass m_a by spring k_a to a primary mass m_p grounded by spring k_p. A harmonic force of amplitude F acts on the primary mass. Seek displacement amplitudes X_p, X_a. The absorber equation is

$$-k_a X_p + (k_a - m_a \omega^2) X_a = 0.$$

At $\omega^2 = k_a/m_a$ this enforces $X_p = 0$. The primary equation then gives

$$X_a = -F/k_a.$$

The absorber's spring force balances the excitation while the primary mass is stationary in this ideal steady response.

For $m_a = 0.50\,\mathrm{kg}$, a measured forcing frequency of 20 Hz requires

$$k_a = m_a(2\pi 20)^2 = 7896\,\mathrm{N/m}.$$

An 8 N forcing amplitude gives absorber travel amplitude 1.01 mm. The two coupled resonances bracket this tuned antiresonance; startup or a change of shaft speed can pass through them. Damping limits the resonant peaks but generally prevents the exact cancellation obtained from these undamped equations.

OneSimulation · How can two resonant modes cancel the primary displacement?

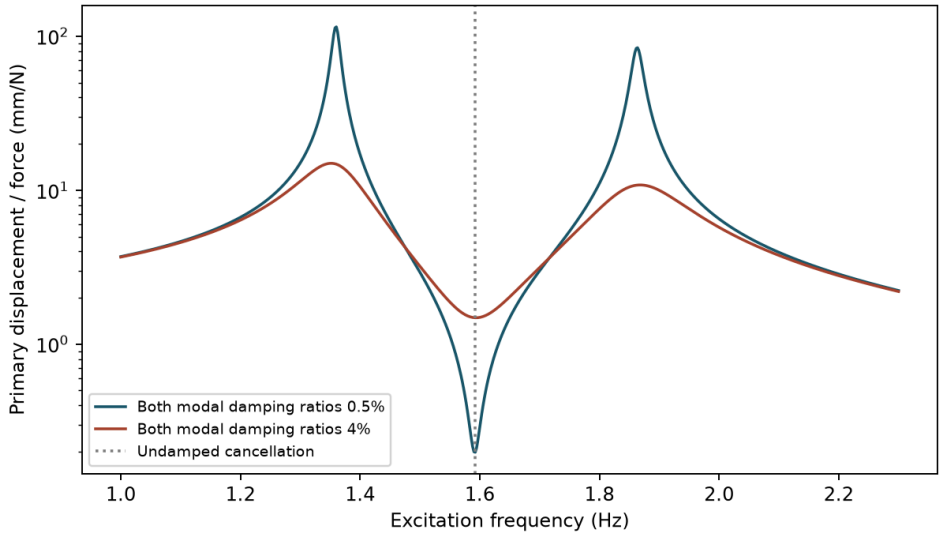
Apply a sinusoidal force to a 5 kg primary mass supported by a 500 N/m spring. An attached 0.5 kg absorber connects through a 50 N/m spring. At the undamped tuning frequency the absorber equation forces the primary displacement to zero. The full coupled system still has two distinct natural frequencies. With mass-normalised modes and the force and displacement at coordinate 1, the receptance is

$$\frac{X_1}{F_1} = \sum_j \frac{\phi_{1j}^2}{\omega_j^2 - \omega^2 + 2i\zeta_j\omega_j\omega}.$$

OneSimulation obtains the eigenpairs from the mass-scaled stiffness. The eigenfrequencies are 1.36 and 1.86 Hz. Between these frequencies the two undamped real contributions have opposite signs. They cancel at 1.59 Hz. The code checks this zero and independently verifies the static compliance. Increase the common modal damping ratio and inspect the notch: losses shift the modal phases, so the exact zero becomes a finite minimum.

```
"""A force on the primary mass excites two modes of a tuned absorber."""
import matplotlib.pyplot as plt
import numpy as np
from onesim.numerics.linear.matrices.dense_linear_algebra import symmetric_eigensystem

mass = np.array([5., 0.5])  # kg, primary and absorber
support, spring = 500., 50.  # N/m, support and connecting spring
stiffness = np.array([[support+spring, -spring], [-spring, spring]])
scale = np.diag(1/np.sqrt(mass))
omega2, shapes = symmetric_eigensystem(scale @ stiffness @ scale)
modes = scale @ shapes
residues = modes[0]**2      # applied force and measured motion both at mass 1
tuned = np.sqrt(spring/mass[1])
assert abs(np.sum(residues/(omega2 - tuned**2))) < 1e-12
assert np.isclose(np.sum(residues/omega2), 1/support)

frequency = np.linspace(1.0, 2.3, 600)  # Hz
omega = 2*np.pi*frequency[:, None]
fig, ax = plt.subplots(figsize=(7, 4), constrained_layout=True)
for damping, colour in [(0.005, "#19596a"), (0.04, "#a5442e")]:
    denominator = omega2 - omega**2 + 2j*damping*np.sqrt(omega2)*omega
    response = np.sum(residues / denominator, axis=1)
    ax.semilogy(frequency, 1000*abs(response), color=colour,
                label=f"Both modal damping ratios {100*damping:g}%")

ax.axvline(tuned/(2*np.pi), color="0.5", linestyle=":", label="Undamped cancellation")
ax.set(xlabel="Excitation frequency (Hz)", ylabel="Primary displacement / force (mm/N)")
```

```
ax.legend(fontsize=8)
fig.savefig("machine-dynamics.png", dpi=170)
```



A low reading at the primary mass can coexist with substantial absorber motion. The plotted response and static-limit check both refer to the force application point.

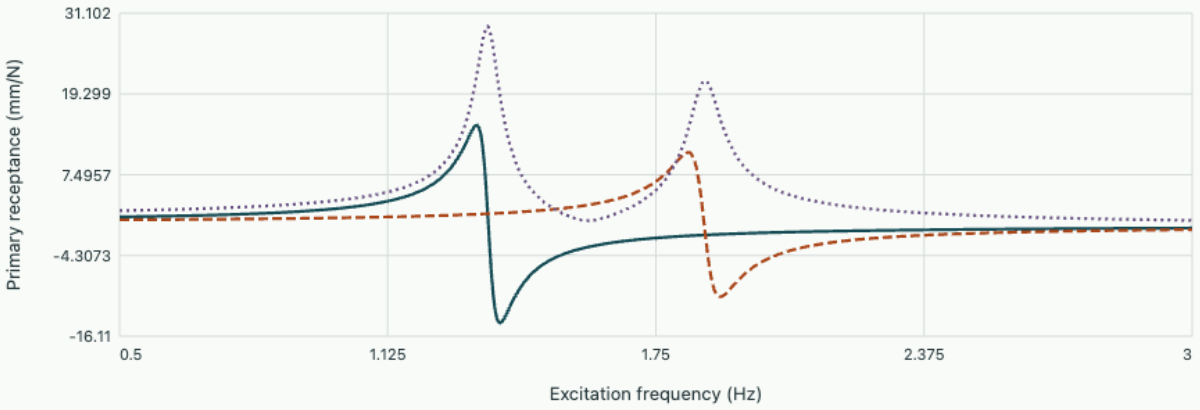
Download the Python example

Change the inputs of this exact example and compare the recalculated curves with the recorded OneSimulation output above. The browser evaluates the stated analytical or modal reference.

Open this live calculation

Try the calculation

Common modal damping ratio	0.02
Absorber mass (kg)	0.5
Absorber spring (N/m)	50



Natural frequencies 1.360, 1.863 Hz. At undamped tuning 1.592 Hz, receptance is 0.77774 mm/N. Static compliance 2.000 mm/N. A common modal damping ratio modifies each modal phase; it is distinct from a physical dashpot attached only to the absorber.

Choose where to observe cancellation

At the tuned absorber frequency a sensor on the primary reports zero ideal steady displacement, while a sensor on the attached mass records finite motion. A force transducer between the bodies measures the full applied excitation in the opposing phase. Moving the sensor can therefore turn the same test from an apparent null into a readily measurable response, without altering the underlying eigenfrequencies.

Project a spatially distributed load onto a flexible member

A beam or string has motion distributed along its length. Its modal forcing is determined by a spatial integral, analogous to the vector dot product for discrete coordinates. For a taut uniform string of length L, linear density μ and constant tension H, small transverse displacement $w(x,t)$ obeys $\mu w_{tt} - H w_{xx} = p(x,t)$. The tension term follows from the difference in vertical tension components on a short segment. Assume both ends fixed in displacement and neglect bending stiffness, damping and the change of tension caused by stretching. The modes are $\phi_n = \sin(n\pi x/L)$, with frequencies $\omega_n = (n\pi/L)\sqrt{H/\mu}$.

Expand $w = \sum \phi_n q_n$, multiply by ϕ_j and integrate over the length. The mass integral is

$$m_j = \mu \int_0^L \phi_j^2 dx = \mu L/2.$$

Integration by parts of the tension term gives the modal stiffness $k_j = m_j \omega_j^2$, with the fixed-end boundary terms vanishing. The modal input becomes

$$Q_j = \int_0^L p \phi_j dx.$$

Under a uniform harmonic load amplitude p_0,

$$Q_j = \frac{p_0 L}{j\pi}[1 - (-1)^j].$$

Even modes have zero generalised force because their positive and negative portions cancel exactly. A point load at x_0 instead yields

$$Q_j = F \phi_j(x_0).$$

Moving an actuator or changing a load footprint can thus suppress a resonance by symmetry without changing the material, tension or natural frequency.

Estimate a cable’s displacement from its first few modes

A laboratory cable spans 2.0 m under 200 N tension and has linear density 0.50 kg/m. Its wave speed is $20\,\mathrm{m/s}$, so $f_1 = 20/(2L) = 5.0\,\mathrm{Hz}$ and $m_j = 0.50\,\mathrm{kg}$. Apply a distributed harmonic pressure-equivalent line load $p_0 = 0.40\,\mathrm{N/m}$ at 2.0 Hz. For mode one, $Q_1 = 2(0.40)(2)/\pi = 0.509\,\mathrm{N}$, and its dynamic stiffness is

$$0.50[(2\pi 5)^2 - (2\pi 2)^2] = 414.5\,\mathrm{N/m}.$$

Its modal amplitude is 1.23 mm.

The third mode has $Q_3 = 0.170\,\mathrm{N}$, $f_3 = 15$ Hz and amplitude $0.170/\{0.50[(2\pi 15)^2 - (2\pi 2)^2]\} = 0.0390\,\mathrm{mm}$. At the midpoint $\phi_1 = +1$ while $\phi_3 = -1$, so the two-term prediction is approximately 1.19 mm. The next odd mode contributes with the opposite sign again, but is far stiffer at this drive frequency. In the static limit an independent check follows from the parabolic string solution $w(L/2) = p_0 L^2/(8H) = 1.00\,\mathrm{mm}$. If a strong measured response appears at the symmetric load’s even-mode resonance, investigate load asymmetry, asymmetric support stiffness or a disturbed initial condition before assigning a nonzero ideal modal projection.

The spatial convergence pattern can be estimated before adding many terms. For odd mode number n well above the forcing frequency, the uniform-load projection decreases as $1/n$ while modal stiffness increases as n^2. Each displacement contribution therefore decreases roughly as $1/n^3$. A point actuator

at midspan instead gives modal projections of essentially alternating constant magnitude for odd modes, producing only a $1/n^2$ displacement decay. Hence a truncation that accurately predicts the response to a distributed flow load can be less accurate for an impulsive local tap. Strain or slope predictions may require still more terms because a spatial derivative introduces an additional factor of mode number. Choose the retained modes for the actual load and observed quantity.

The uniform cable problem admits an independent harmonic calculation without modal truncation. Set $w(x,t) = W(x)\cos\Omega t$. The governing equation gives

$$W'' + \kappa^2 W = -p_0/H,$$

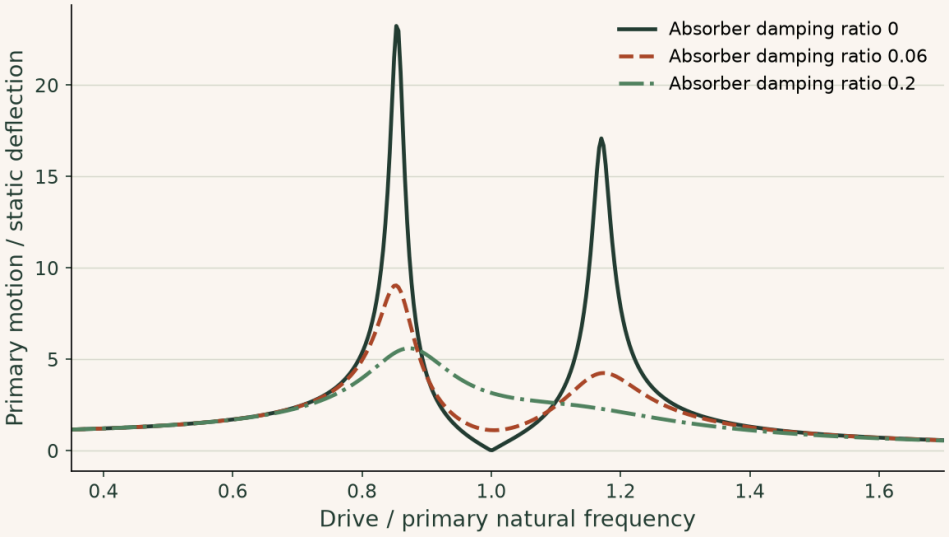
$$\kappa = \Omega\sqrt{\mu/H}.$$

Symmetry about midspan and fixed ends select $W(x) = [p_0/(\mu\Omega^2)]\{\cos[\kappa(x - L/2)]/\cos(\kappa L/2) - 1\}$, provided the denominator is nonzero. For the test above $\kappa L/2 = 0.6283$ and $W(L/2) = 0.40[1/\cos(0.6283) - 1]/[0.50(2\pi 2)^2] = 1.196\,\mathrm{mm}$. This direct boundary-value result confirms both the signs and the scale of the first two odd-modal contributions.

As the excitation approaches the first natural frequency, $\kappa L/2$ approaches $\pi/2$ and the undamped steady expression becomes singular. An actual time record may instead show progressively growing amplitude before damping or geometric effects intervene. At very low frequency expand $\sec q \simeq 1 + q^2/2$; substituting $q = \Omega L\sqrt{\mu/H}/2$ recovers the static midpoint displacement without subtracting two nearly equal numerical values. The modal and spatial calculations thus furnish independent checks in the static, off-resonant and resonant limits.

What happens to the notch of a tuned absorber?

Attach an absorber of mass 0.10 times the primary mass, tuned to the primary’s undamped frequency. Include 2.5% primary damping. Solve the complex two-mass force balance for primary displacement under harmonic forcing; the absorber spring and dashpot act on relative motion.



Without absorber damping the primary displacement vanishes at the tuned frequency, with response peaks on both sides. Absorber damping fills the notch and changes those adjacent peaks. An installation with variable drive speed must check the entire band.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

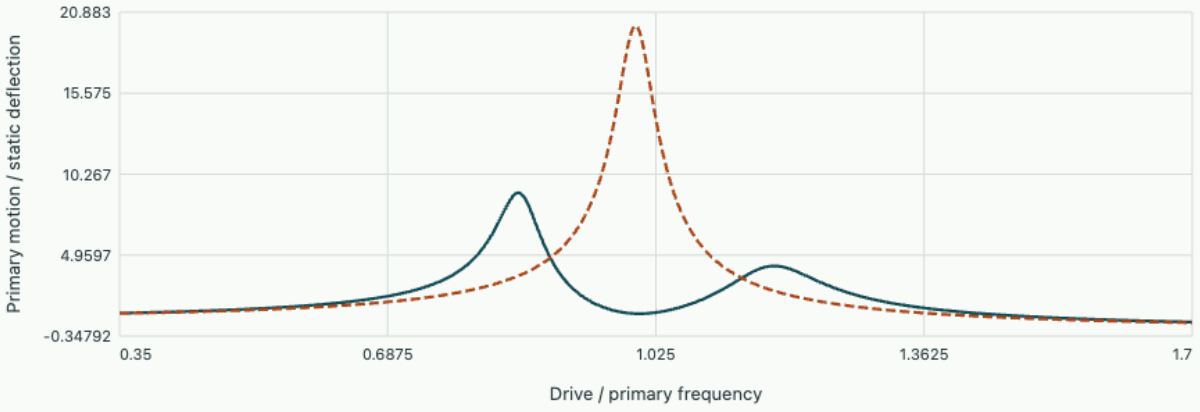
Detune the absorber, then add damping. Compare the cancellation notch with both adjacent peaks.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Absorber / primary mass	**0.1**
Absorber damping ratio	**0.06**
Absorber / primary frequency	**1**



At the primary natural frequency, amplification is 1.125. A damped or mistuned absorber fills the exact undamped cancellation notch.

Study the full derivation

Why a local damper couples otherwise separate modes · Vibration and identification in machinery

Undamped mode shapes diagonalise a symmetric mass and stiffness model. Adding a damper at one attachment does not generally leave modal equations separate. Derive the resulting coupling, test its magnitude and learn when a diagonal modal-damping approximation can hide an observable phase error.

Chapter 12 · Finite-amplitude oscillations and model validity

Use conservation of energy and a frequency approximation to quantify when a small-angle pendulum model loses timing accuracy.

FROM FIRST PRINCIPLES

Derive the pendulum period and identify the small-angle assumption

A point mass m suspended on a massless rigid link of length ℓ feels a gravitational torque $-mg\ell\sin\theta$ about the hinge. Assume a stationary pivot and negligible drag. Angular momentum balance gives an exact nonlinear equation. Replacing $\sin\theta$ by θ only for small angles in radians gives a linear oscillator and its predicted period.

$$m\ell^2\ddot{\theta} = -mg\ell\sin\theta,$$

$$T_{\text{small}} = 2\pi\sqrt{\ell/g}.$$

To track finite amplitude, release from rest at angle $A < \pi$ and use mechanical energy conservation. At any intermediate angle, loss of potential energy equals gained rotational kinetic energy.

$$\tfrac{1}{2}m\ell^2\dot{\theta}^2 = mg\ell(\cos\theta - \cos A),$$

$$T = 4\sqrt{\frac{\ell}{2g}}\int_0^A \frac{d\theta}{\sqrt{\cos\theta - \cos A}}.$$

The factor four uses the symmetry of the four quarter cycles. The mass cancels even at large amplitude. The exact restoring torque grows more slowly with deflection than its small-angle approximation, so finite-amplitude swings take longer. Measured variation of period with release angle directly tests the linearisation.

DERIVATION AND APPLICATIONS

Derive the period of an oscillator with purely cubic stiffness

A spring with almost no linear stiffness near equilibrium can have a period set primarily by release amplitude. Consider mass m on a horizontal guide with restoring force $-\beta x^3$ for $\beta > 0$. Ignore damping and assume the fitted force law holds between $-A$ and $+A$. Release from rest at $x = A$. Integrating the force law gives a quartic potential, and conservation of energy provides the speed anywhere on the inward leg.

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{4}\beta x^4 = \tfrac{1}{4}\beta A^4,$$

$$|\dot{x}| = \sqrt{\frac{\beta}{2m}}\sqrt{A^4 - x^4}.$$

By symmetry the time from an extreme position to the centre is one quarter of a full cycle. Write $dt = |dx|/|\dot{x}|$ and substitute $u = x/A$ in the integral. This isolates the amplitude dependence without guessing a sinusoidal shape.

$$T = \frac{4}{A}\sqrt{\frac{2m}{\beta}}\int_0^1 \frac{du}{\sqrt{1-u^4}}.$$

The dimensionless integral evaluates numerically to 1.31103. Its endpoint singularity is integrable because the speed near the turning point vanishes as the square root of the distance remaining. The resulting period and angular frequency are

$$T = \frac{7.4163}{A}\sqrt{\frac{m}{\beta}},$$

$$\omega = \frac{2\pi}{T} = 0.8472\,A\sqrt{\frac{\beta}{m}}.$$

The positive cubic restores the mass for displacements of either sign, but its tangent stiffness at zero is zero. The full nonlinear dynamics therefore predicts a period diverging with decreasing amplitude. Even a weak residual linear spring would eventually dominate in that limit.

WORKED EXAMPLE

Predict the timing of two releases

Take $m = 0.500$ kg, $\beta = 2.00 \times 10^5$ N/m³ and $A = 20.0$ mm. The predicted period is

$$7.4163\sqrt{0.500/(2.00\times10^5)}/0.0200 = 0.586\,\mathrm{s}.$$

Reducing the release to 10.0 mm doubles the predicted period to 1.17 s. At the larger amplitude the centre-crossing speed is

$$A^2\sqrt{\beta/(2m)} = 0.179\,\mathrm{m/s}.$$

Change the inputs and inspect this calculation below

SECOND APPLICATION

Infer stiffness from a measured period

A second fixture of mass 0.800 kg completes a cycle in 0.750 s after release at 15.0 mm. The implied coefficient is $\beta = m[7.4163/(AT)]^2 = 3.48 \times 10^5$ N/m³. Its predicted restoring-force magnitude at release is

$$\beta A^3 = 1.17\,\mathrm{N}.$$

An independent static force measurement there tests the dynamic estimate without assuming a sinusoidal motion.

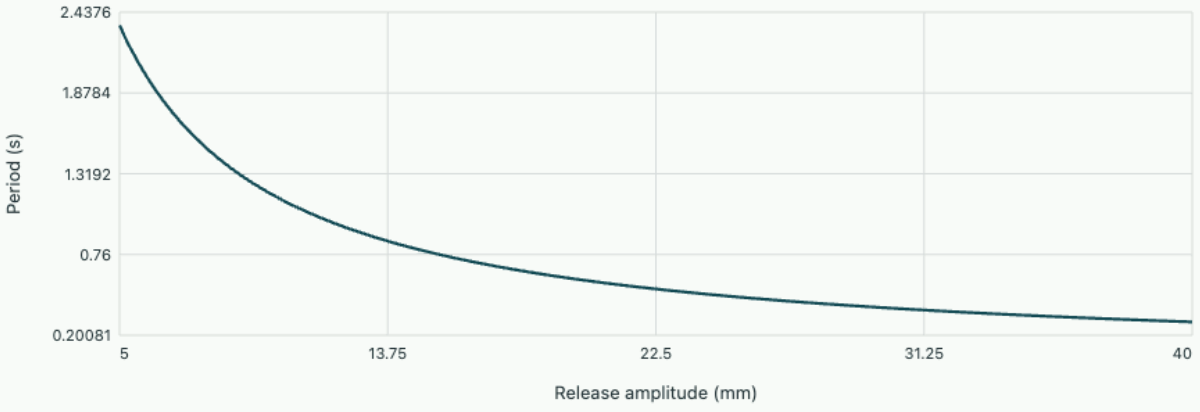
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Mass (kg)	**0.5**
Cubic force coefficient (kN/m³)	**200**



Energy-integral period

At20 mm amplitude, period=0.586 s. This pure cubic spring has no nonzero small-amplitude linear frequency.

Retain the actual restoring moment

A pendulum gives a practical test of a linear model because its length and release angle are easy to set while its period can be measured over many cycles. Approximate the moving body first by a point mass m at distance l from a frictionless pivot. With θ measured from vertically downwards, gravity produces torque $-mgl\sin\theta$. Dividing the rotational balance by ml^2 leaves $\ddot{\theta} + (g/l)\sin\theta = 0$. The linear small-angle frequency $\omega_0 = \sqrt{g/l}$ follows only after replacing $\sin\theta$ by θ in radians.

The leading correction has a definite sign:

$$\sin\theta \simeq \theta - \theta^3/6.$$

At a large positive angle the restoring force magnitude is smaller than the linear extrapolation. The pendulum should take longer to complete its swing. A sign test like this is useful before calculating a period correction.

Derive an integral valid at finite amplitude

For release from rest at amplitude $A < \pi$, conservation of mechanical energy gives

$$\tfrac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta) = mgl(1-\cos A).$$

During one quarter of the orbit the angle travels between zero and A. Substitute $\sin(\theta/2) = \sin(A/2)\sin\psi$ into the time integral. Cancelling common square roots gives

$$T = 4\sqrt{l/g}\int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - k^2\sin^2\psi}},$$

$$k = \sin(A/2).$$

For modest amplitudes expand the integrand as $1 + \tfrac{1}{2}k^2\sin^2\psi + \cdots$. The average of $\sin^2\psi$ over the quarter interval is one half. Therefore $T/T_0 \simeq 1 + k^2/4 \simeq 1 + A^2/16$ through second order in amplitude. The oscillation period depends on energy, while in the linear spring model all amplitudes share the same period.

Estimate a measurable timing error

Take $l = 0.80\,\mathrm{m}$, $g = 9.81\,\mathrm{m/s^2}$ and release at $A = 30^\circ = 0.5236\,\mathrm{rad}$. The small-angle period is

$$T_0 = 2\pi\sqrt{0.80/9.81} = 1.794\,\mathrm{s}.$$

The leading correction is 1.71%, giving about 1.825 s. Over ten full oscillations the predicted delay relative to the small-angle model is roughly 0.31 s. Reducing the release angle to 15° reduces the leading relative correction by a factor of four.

A timing series from a damped experimental pendulum may show the period drifting towards T_0 as its amplitude decays. Measuring only the first release angle while averaging many later periods mixes different amplitudes. To test the finite-amplitude law, use consecutive same-direction crossings to measure each local period and estimate its associated amplitude from neighbouring turning points. The correction above is an expansion around zero amplitude. As A approaches the inverted equilibrium, the exact integral grows without bound and a low-order polynomial is no longer an adequate approximation.

Check the scale of the neglected term

Retaining one further term in the integrand gives $1 + \tfrac{1}{2}k^2\sin^2\psi + \tfrac{3}{8}k^4\sin^4\psi$. The quarter-interval average of the fourth power of sine is $3/8$, so $T/T_0 \simeq 1 + k^2/4 + 9k^4/64$. At 30° this gives 1.01738, while the leading expansion in amplitude gave 1.01713. The small difference is measurable only if the uncertainty in pendulum length, release angle and crossing times supports the comparison.

Derive an amplitude-dependent resonance relation

A pendulum softens as its amplitude grows because the sine restoring law rises more slowly than its linear tangent. A mechanical flexure can instead harden through geometry. A simple modal model is $m\ddot{x} + c\dot{x} + kx + \beta x^3 = F\cos\omega t$. For positive β the restoring stiffness increases with excursion. Assume one dominant harmonic and take $x \approx A\cos(\omega t - \phi)$. Expanding the cubic gives

$$x^3 = A^3[3\cos(\omega t - \phi) + \cos(3\omega t - 3\phi)]/4.$$

Keeping only the fundamental turns the cubic term into an amplitude-dependent in-phase force.

Balance in-phase and quadrature components against the input, square and add to eliminate ϕ. The first-harmonic amplitude estimate satisfies

$$F^2 = A^2\left[\left(k + \tfrac{3}{4}\beta A^2 - m\omega^2\right)^2 + (c\omega)^2\right].$$

In a lightly damped free motion the same approximation gives a backbone frequency $\omega_b^2 \approx [k + (3/4)\beta A^2]/m$. This is an amplitude-dependent estimate of conservative oscillation frequency, not a new eigenvalue of the linearised equilibrium. At very small amplitude it recovers $\sqrt{k/m}$, while a negative cubic coefficient tilts the relation towards lower frequencies. If the cubic approximation is only a local expansion, do not extrapolate it until it predicts an unphysical loss of global stiffness.

Identify a hardening spring from two release amplitudes

A test component has moving mass 1.0 kg and small-amplitude stiffness 400 N/m. Low-level ring-down gives $\omega_0 = 20\,\mathrm{rad/s}$. At measured oscillation amplitude 20 mm the dominant free frequency is approximately 22 rad/s. Under the weakly nonlinear one-harmonic model,

$\beta \simeq 4m(22^2 - 20^2)/[3(0.020)^2] = 2.80 \times 10^5\,\mathrm{N/m^3}$. At 10 mm the inferred backbone is $\sqrt{400 + 0.75(2.8 \times 10^5)(0.010)^2} = 20.52\,\mathrm{rad/s}$, a direct third reading against which to test the fit.

Now drive near 22 rad/s with viscous coefficient $c = 0.80\,\mathrm{N\,s/m}$. At the 20 mm backbone amplitude the spring and inertia fundamentals cancel, so maintaining this motion takes forcing $F = c\omega A = 0.352\,\mathrm{N}$ in quadrature with displacement. The neglected cubic third-harmonic force amplitude is

$$\beta A^3/4 = 0.56\,\mathrm{N};$$

its displacement effect can nevertheless be small because the oscillator is driven far from its fundamental resonance at 66 rad/s. A separate mode near that frequency would invalidate this simplification. In a forced frequency sweep the amplitude equation can admit several positive roots near resonance. Its algebra does not establish their stability; slow upward and downward sweeps, each with sufficient settling time, can reveal different stable responses and jump transitions.

An initial static measurement tests the sign of the fitted coefficient independently. At a displacement of 20 mm the linear restoring force is

$$400(0.020) = 8.0\,\mathrm{N};$$

the cubic contribution is

$$2.8 \times 10^5 (0.020)^3 = 2.24\,\mathrm{N}.$$

The predicted static force is therefore 10.24 N, and the tangent stiffness is

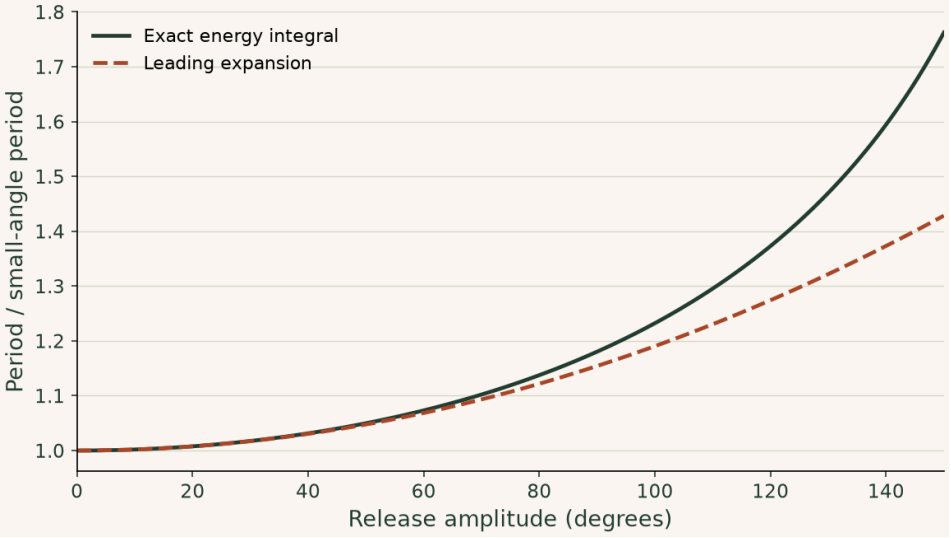
$$400 + 3(2.8 \times 10^5)(0.020)^2 = 736\,\mathrm{N/m}.$$

This tangent stiffness applies to small vibrations around a held biased equilibrium. Using it directly for an oscillation centred at zero would overestimate the measured amplitude-dependent frequency because the instantaneous stiffness changes during each cycle. Static bias and free oscillation amplitude are physically different tests of the same nonlinear restoring law.

Repeating a decay test around a deliberately shifted equilibrium distinguishes the two experimentally.

Set an amplitude limit for a pendulum-based timing model

For a rigid pendulum released from rest the exact period ratio is $2K(\sin^2(\theta_0/2))/\pi$, where the complete elliptic integral here uses its parameter rather than its modulus. Compare it with the first amplitude correction $1 + \theta_0^2/16$, with angle in radians.



At 90° the exact period is about 1.180 times the linear prediction; the leading expansion gives 1.154. The series is useful at modest amplitudes, but it cannot reproduce the divergence on approach to an upright release.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

Increase the maximum release angle and compare the energy integral with the first series correction. Changing length rescales both periods.

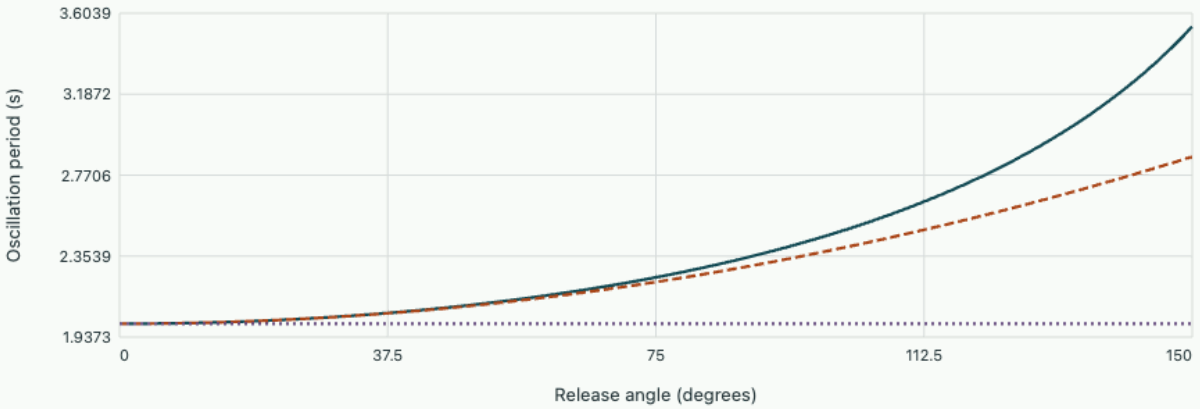
The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Maximum release angle (degrees) **150**

Pendulum length (m) **1**



Energy integral via AGM · First amplitude correction · Small-angle limit

At 150° the exact period is 3.535 s, 76.22% above the linear estimate.

Problem sheet 6 · Modal selectivity and finite amplitude

Determine whether a force excites a mode and relate an oscillator's amplitude to the accuracy of a period measurement.

Excite an antisymmetric mode

Two masses of 1.5 kg have symmetric and antisymmetric mode shapes (1,1) and (1,–1) at natural frequencies 12 and 18 rad/s. Apply harmonic forces (+6,–6) N at 6 rad/s. Find both modal amplitudes and the physical displacement amplitudes with their relative phase.

▼ **Reveal the worked solution**

Both modal masses are 3 kg. The symmetric projection is zero, and the antisymmetric projection is 12 N. Hence its amplitude is

$$12/[3(18^2 - 6^2)] = 0.01389\,\mathrm{m}.$$

The physical amplitudes are +13.89 and –13.89 mm in a common phase convention. The cancellation of the symmetric force projection determines the response symmetry.

Set a working stroke

An auxiliary mass moves against a spring attached to a primary machine. An undamped absorber must cancel an 18 N harmonic force at 25 Hz while its own amplitude stays at or below 1.5 mm. Find minimum spring stiffness and corresponding tuned mass. Does tuning ensure a small response at all speeds?

▼ **Reveal the worked solution**

At exact cancellation $X_p = 0$ and $|X_a| = F/k_a$. Thus

$$k_a \geq 18/0.0015 = 12000\,\mathrm{N/m}.$$

At the limiting stiffness the tuned mass is

$$12000/(2\pi 25)^2 = 0.486\,\mathrm{kg}.$$

The coupled modes give resonances away from the cancellation frequency. A varying-speed drive can cross these frequencies.

Estimate a release-angle limit

A pendulum experiment tolerates an amplitude-dependent period change of 0.5%. Use the leading finite-amplitude correction to estimate its largest release angle. For a 1.0 m pendulum released at half

that angle, estimate the accumulated timing shift over 40 cycles.

▼ **Reveal the worked solution**

The leading fraction is $A^2/16$. Setting it to 0.005 gives

$$A = 0.2828\,\mathrm{rad} = 16.2^\circ.$$

Halving the angle reduces the leading correction to 0.00125. The small-angle period is 2.006 s, giving $40(2.006)(0.00125) = 0.100\,\mathrm{s}$ delay. Precision beyond the retained order would require a higher-order term or the exact integral.

Specify an isolation mount from two measurements

A 40.0 kg instrument is mounted on springs. Free-vibration peaks of the same sign are 8.0 and 4.0 mm four periods apart. The four periods take 1.60 s. Estimate the equivalent stiffness and damping. A bench beneath the instrument moves sinusoidally at 10 Hz with 0.50 mm amplitude. Predict absolute instrument motion and relative mount travel.

▼ **Worked solution**

The logarithmic decrement per period is ln(2)/4 = 0.1733 and the damping ratio is $\delta/\sqrt{(4\pi^2+\delta^2)} = 0.0276$. The damped period is 0.40 s, giving $\omega_d = 15.71$ rad/s. The undamped frequency $\omega_n = \omega_d/\sqrt{(1-\zeta^2)}$ is 15.71 rad/s to the displayed precision. The equivalent $k = m\omega_n^2 \approx 9.88$ kN/m and $c = 2\zeta m\omega_n \approx 34.7$ N s/m.

At $r = \omega/\omega_n \approx 4.00$ the absolute-motion transmissibility is $\sqrt{[1+(2\zeta r)^2]}/\sqrt{[(1-r^2)^2+(2\zeta r)^2]} \approx 0.0683$. The instrument thus moves about 0.034 mm. The base-relative displacement amplitude instead uses r^2 in the numerator and is about 1.067 times the input, or 0.533 mm.

These two answers size different things. The instrument sees the small absolute vibration while the spring must accommodate roughly the bench excursion. Near resonance the measured low damping would permit much larger travel; a mount chosen only from the 10 Hz test could bottom during startup. The single-mode fit also assumes the measured peaks lie within a linear range.

Explore the governing equations after solving the sheet: Modal participation and absorber tuning; Finite-amplitude oscillations and model validity. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Chapter 13 · How a longitudinal pulse reflects from a change in rod section

A tap on one end of a rod reaches a receiver after a finite delay. Two rods joined end to end can even return a pulse before the receiver sees a far-end reflection. Derive propagation speed and distinguish the displacement amplitude from the energy carried by a pulse.

From the cut force to a travelling disturbance

Measure the small axial displacement $u(x,t)$ from a stress-free reference. In a slender homogeneous rod of area A, tensile force is

$$N = EAu_x.$$

A slice of length dx and mass $\rho A\,dx$ accelerates because its right and left forces differ by $N_x dx$. For constant properties, cancelling A gives

$$\rho A u_{tt} = (EAu_x)_x,$$

$$u_{tt} = c^2 u_{xx},$$

$$c = \sqrt{E/\rho}.$$

Either $u = f(x-ct)$ or $u = g(x+ct)$ satisfies this equation by differentiation. Curves of constant phase move right or left with speed c. Long-wavelength assumptions suppress lateral inertia and radial modes. Rod area cancels from wave speed but remains in the force carried at given particle velocity. For a right-moving pulse $v = u_t = -cu_x$, so tensile force $N = -Zv$, with mechanical impedance $Z = \rho A c = EA/c$. The sign reverses for a left-moving wave.

Match velocity and force at a bonded junction

Let amplitudes I, R and T denote incident, reflected and transmitted particle velocity at the junction. A massless, perfectly bonded interface requires equal velocities and transmitted force. Apply the preceding sign convention separately to each travelling direction. The two equations determine two unknowns.

$$I + R = T,$$

$$-Z_1 I + Z_1 R = -Z_2 T,$$

$$\frac{R}{I} = \frac{Z_1 - Z_2}{Z_1 + Z_2},$$

$$\frac{T}{I} = \frac{2Z_1}{Z_1 + Z_2}.$$

The instantaneous power transported to the right is $-Nv = Zv^2$ for a single right-moving wave. For the same pulse shape the reflected energy fraction is $(R/I)^2$ and the transmitted fraction is

$(Z_2/Z_1)(T/I)^2$. They sum to one. A transmitted velocity amplitude exceeding the incident amplitude can thus be compatible with conservation of energy.

A stepped rod and a pulse-echo observation

Join two aluminium rods of areas 80 and 160 mm², $E = 70$ GPa and $\rho = 2700$ kg/m³. Both have $c = 5092$ m/s, but the second impedance is twice the first. An incident velocity of 0.30 m/s produces a reflected −0.10 m/s and a transmitted 0.20 m/s. The reflected pulse carries one ninth of the energy. Reversing the experiment makes the reflection positive and the transmitted velocity 0.40 m/s for the same incident amplitude. The two energy fractions are unchanged on reversal.

A receiver at the tap detects a reflected pulse 0.40 ms after launch, before any far-end reflection contaminates it. The return path covers the distance twice, so the inferred junction is $c(0.00040)/2 = 1.02$ m away. Measure pulse velocity rather than strain when comparing the signs to the plot. Incident tensile force has sign opposite incident velocity with the convention used here.

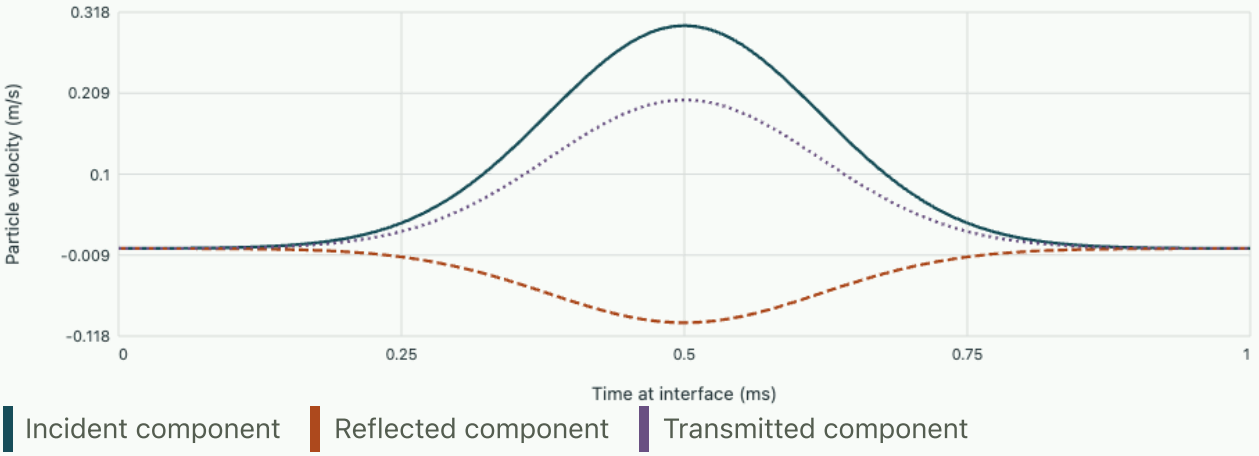
Keep the incoming pulse fixed and change the area ratio from 2 to 0.5. Determine which velocity component reverses sign and compare reflected energy fractions. Explain the difference by writing power balance for the two experiments.

Open this experiment online

Try the calculation

Second / first section area **2**

Incident peak velocity (m/s) **0.3**



Peak reflected velocity is -0.100 m/s and transmission is 0.200 m/s. The reflection carries 11.1% of the incident energy.

Tutorial · infer a second cross-section

On a uniform material an isolated junction returns 25% of the incident velocity amplitude with the same sign. Infer the area ratio and transmitted energy fraction. How could an energy-only measurement leave the geometry ambiguous?

▼ **Show the inverse relation**

Let $z = Z_2/Z_1 = A_2/A_1$. Then $0.25 = (1 - z)/(1 + z)$, so $z = 0.60$. The transmitted velocity is $1.25I$. Its energy fraction is

$$0.60(1.25)^2 = 0.9375.$$

Measuring only the reflected energy gives 0.0625 and loses the sign of the amplitude. A thickening by a factor $1/0.60$ would give the same energy reflection and the opposite sign in the velocity trace.

Further reading · Bower, Applied Mechanics of Solids · solutions to problems involving vibrations and wave propagation

Chapter 14 · Why the frequencies of a bending beam are not equally spaced

A distributed beam can bend into infinitely many shapes. Derive the relation between wavelength and oscillation frequency, then compare the measured response of a beam with the mode family predicted by its supports.

Combine transverse force, moment and curvature

Take a straight uniform beam with small transverse deflection $w(x,t)$, density ρ, area A and second moment of area I about the bending axis. Euler–Bernoulli kinematics keeps cross-sections perpendicular to the centreline. The bending resistance is proportional to curvature EIw_{xx}. For a load-free part of the beam the gradient of moment supplies shear and the gradient of shear accelerates the mass. With restoring forces opposing deflection, the resulting equation is

$$\rho A\, w_{tt} + EI\, w_{xxxx} = 0.$$

The fourth spatial derivative calls for four boundary conditions. At a simple support the displacement vanishes and the bending moment is zero, hence

$$w = w_{xx} = 0.$$

A clamp instead fixes displacement and slope. Interchanging a pin and a clamp changes the allowed shape family even if the span and material remain fixed.

Obtain the simply supported series

Try $w = W\sin(kx)\cos(\omega t)$. Two spatial derivatives bring down $-k^2$, and four give $+k^4$. Substitution balances the inertia and stiffness terms. The conditions at x=0 hold for every k in this sine family. At x=L the sine must vanish, yielding

$$\omega^2 = \frac{EI}{\rho A}k^4,$$

$$k_n = \frac{n\pi}{L},$$

$$f_n = \frac{n^2\pi}{2L^2}\sqrt{\frac{EI}{\rho A}}.$$

A long travelling sinusoid has phase velocity $\omega/k = k\sqrt{EI/(\rho A)}$. The group velocity $d\omega/dk$ is twice that value in this model. A localized pulse contains many wavelengths, which can therefore separate as it propagates. Predictions at wavelengths near the thickness require accounting for rotary inertia and shear deformation.

Choose sensor positions as carefully as the frequency range

Use a 1 m steel strip 20 mm wide and 4 mm thick with $E = 200$ GPa and $\rho = 7800$ kg/m³. For bending about the thin direction $I = bh^3/12$ and $I/A = h^2/12$. The first three simply supported frequencies are roughly 9.18, 36.7 and 82.7 Hz. Doubling the width alone doubles stiffness and mass together and leaves these frequencies unchanged. Doubling thickness doubles each frequency. Shortening the span by 20% increases all three by $1/0.8^2 = 1.56$.

At the centre a displacement pickup sees $\sin(n\pi/2)$. It sees the first mode at its antinode and misses the second mode entirely. At the quarter point it sees that second mode strongly. An absent spectral peak can therefore result from excitation or sensing at a node. Move the measuring point before adjusting a fitted damping value to explain its absence.

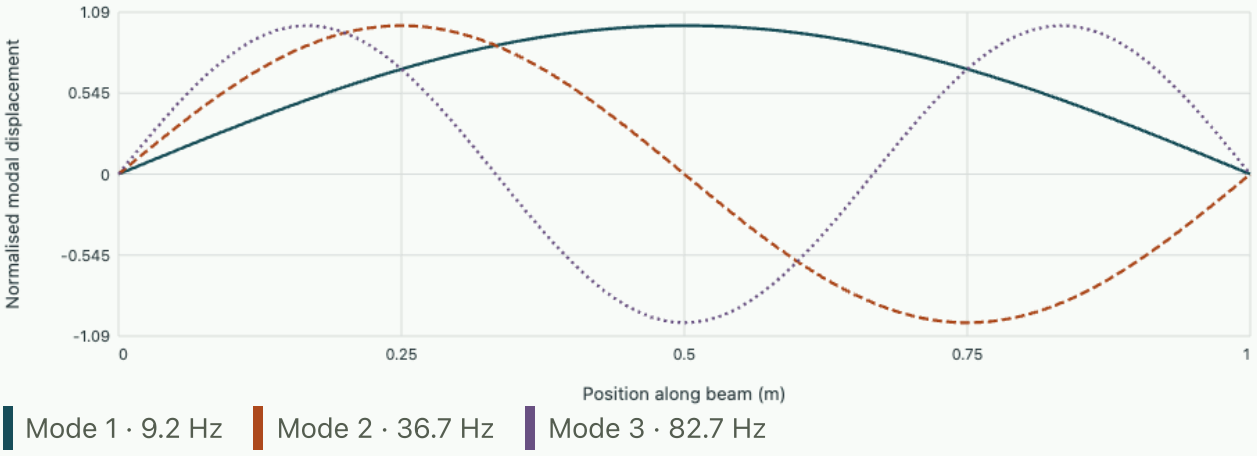
Find a common node of the first and third modes and a location that suppresses the second mode alone. Change the thickness and then the span. Use the resulting frequencies to test whether each scaling is linear or quadratic.

Open this experiment online

Try the calculation

Span (m) 1

Thickness (mm) 4



At midspan the second mode has a displacement node. The three frequencies are 9.18, 36.74, 82.66 Hz.

A related OneSimulation calculation

For a frame whose geometry and end payload preclude the simply supported sine family, compare the first three computed modes after changing the tip mass. Test whether all frequencies scale by the same factor.

Compare frame modes with OneSimulation

Tutorial · distinguish distributed bending from a tensioned string

A simply supported specimen has prominent peaks at 12 Hz and 108 Hz with the receiver at midspan. Infer which bending modes these could be and predict the frequency and receiver location for the missing mode. What would the third-to-first frequency ratio be for an ideal string?

▼ **Show the mode test**

The ratio $108/12 = 9$ agrees with n^2 for $n = 3$. The second bending mode should lie near $4(12) = 48$ Hz. Measure at one quarter span and excite away from its nodes. An ideal uniform string has $\omega = ck$ and a third-to-first ratio of 3. Checking both the midspan node and the shifted pickup spectrum distinguishes these models more effectively than a single peak.

Further reading · MIT 2.003SC · lecture 27 on vibration of continuous structures

Chapter 15 · Estimating a cantilever frequency from energy

A clamped beam carries a small sensor at its tip. Its precise mode shape is unknown. Use the static deflection shape to approximate the fundamental frequency and see why concentrating a small mass at the tip has a large effect.

Choose a displacement field that respects the support

Write $w(x,t) = q(t)\phi(s)$ with $s = x/L$, taking q as tip displacement. Choose the shape from a uniform cantilever under a static transverse tip force. The bending moment varies linearly from root to tip. Integrating curvature twice with $w(0) = w'(0) = 0$, then normalizing at $s = 1$, gives

$$\phi = s^2(3 - s)/2.$$

The shape satisfies both essential clamp conditions. Its free-tip shear need not match the exact unloaded vibration mode. The energy formulation can use such admissible trial shapes.

Background · Rayleigh quotient

$$R(u) = u^{\mathsf{T}} K u / (u^{\mathsf{T}} M u)$$

For symmetric stiffness and positive mass this estimates squared angular frequency. A kinematically admissible displacement is required for a useful structural estimate.

Compute effective stiffness and mass from the same field

The strain energy density of a slender beam in bending is $\frac{1}{2} E I w_{xx}^2$. Its translational kinetic energy is $\frac{1}{2}\rho A w_t^2$ per unit length. Changing variables from x to s introduces different powers of L into the two integrals. Include a point mass M at the tip, neglecting its rotary inertia.

$$U = \frac{EIq^2}{2L^3}\int_0^1 (\phi'')^2\, ds,$$

$$K = \frac{\dot{q}^2}{2}\left[\rho A L \int_0^1 \phi^2\, ds + M\right].$$

Since $\phi'' = 3(1 - s)$, the stiffness integral is 3. Squaring $s^2(3 - s)/2$ and integrating $(9s^4 - 6s^5 + s^6)/4$ gives $33/140$. The Euler–Lagrange equation for q is consequently a single oscillator equation with

$$\omega_R^2 = \frac{3EI/L^3}{(33/140)\rho A L + M}.$$

For M=0 this gives the dimensionless coefficient $\sqrt{140/11} = 3.5675$. Solving the exact unloaded cantilever condition $\cosh\beta\cos\beta = -1$ gives

$$\beta_1^2 = 3.5160.$$

The estimate is high by about 1.5%. The Rayleigh minimum principle explains that direction of error for conservative systems with admissible trial shapes. Restricting motion to a single shape restricts the set over which the energy quotient can attain its minimum.

Choose a sensor after estimating its dynamic effect

Take $EI = 10$ N m², $L = 0.50$ m and total beam mass 0.40 kg. The tip stiffness is 240 N/m and the distributed effective mass is only 0.0943 kg. The estimate gives 8.03 Hz without a sensor and 6.49 Hz with a 0.05 kg tip sensor. The sensor weighs one eighth of the beam, yet it has over half the effective mass associated with the chosen tip coordinate.

Place the same sensor at half-span instead. There $\phi(0.5) = 0.3125$. Its kinetic energy contributes $M\phi(0.5)^2 = 0.00488$ kg to the effective mass, so the estimate rises to 7.83 Hz. The sensor measures a smaller local motion as it perturbs the frequency less. Compare that trade-off at different attachment positions with the plot.

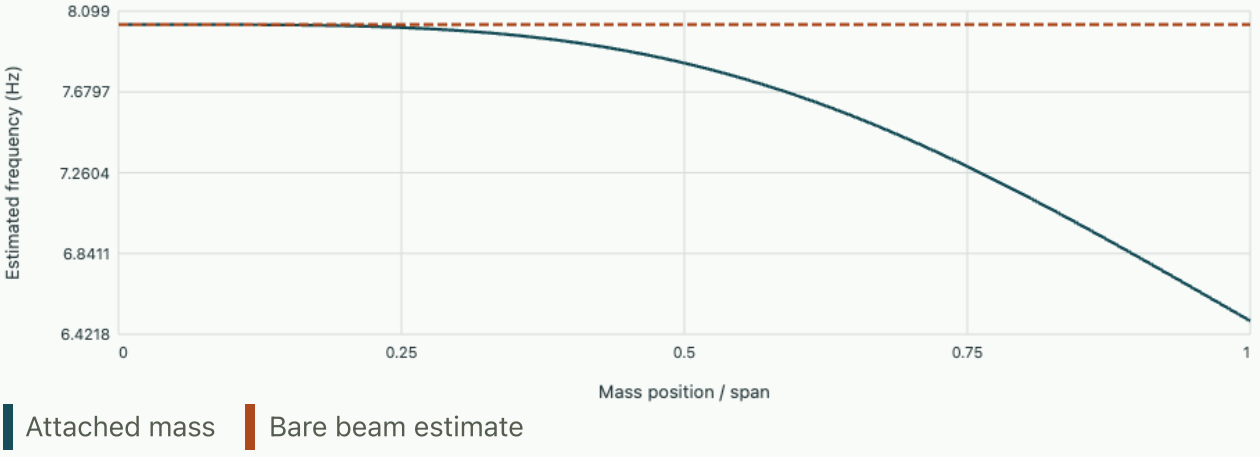
Move along the position axis with a fixed sensor mass. Identify the interval in which frequency is most sensitive to changing attachment position. Double the sensor mass and check whether the change in frequency doubles at the tip.

Open this experiment online

Try the calculation

Attached mass (kg) **0.05**

Bending rigidity (N m²) **10**



The estimate is 6.49 Hz with the sensor at the tip and 7.83 Hz at half-span. The bare-beam estimate is 8.03 Hz.

A related OneSimulation calculation

The course frame model has several modes and an independently variable payload rotary inertia. Compare a pure translation-mass change with a rotary-inertia change; the scalar point-mass estimate above addresses the first mechanism.

Open the OneSimulation frame eigenvalue experiment

Tutorial · estimate an unknown attached mass

The same bare beam is fitted with a small tip part and resonates at 6.0 Hz. Estimate the attached mass. Give one reason the inferred mass could differ from a scale reading even when the frequency measurement is accurate.

▼ **Show the mass estimate**

The required effective mass is

$$240/(2\pi \cdot 6)^2 = 0.1689\,\mathrm{kg}.$$

Subtract 0.0943 kg for the beam to obtain about 0.0746 kg. Clamp compliance would reduce the effective stiffness and frequency, which this fit would interpret as extra tip mass. The trial shape and omission of rotary inertia also limit the estimate for a part with appreciable extent.

Further reading · Bower, Applied Mechanics of Solids · Rayleigh–Ritz methods

Chapter 16 · Why a radial slide presses sideways on a rotating guide

A low-friction carriage moves along a radial slot in a horizontal turntable. It can move at constant speed relative to the guide and still require a changing force. Keeping track of the moving basis exposes both acceleration components.

Differentiate the direction vectors

Let $r(t)$ be displacement from the turntable centre along the guide and $\theta(t)$ the angle of the guide. In fixed Cartesian components the radial unit vector is $(\cos\theta, \sin\theta)$ and the tangential vector is $(-\sin\theta, \cos\theta)$. Differentiating shows $\dot{\mathbf{e}}_r = \dot\theta \mathbf{e}_\theta$ and $\dot{\mathbf{e}}_\theta = -\dot\theta \mathbf{e}_r$. Apply the product rule to the position $r\mathbf{e}_r$ to obtain the velocity and then differentiate both terms.

$$\mathbf{v} = \dot r \mathbf{e}_r + r\dot\theta \mathbf{e}_\theta,$$

$$\mathbf{a} = (\ddot r - r\dot\theta^2)\mathbf{e}_r + (r\ddot\theta + 2\dot r\dot\theta)\mathbf{e}_\theta.$$

One contribution to $2\dot r\dot\theta$ comes from differentiating the radially directed velocity as its basis turns. The other comes from differentiating the magnitude of the tangential velocity $r\dot\theta$. For positive rotation and outward travel, both point in the positive tangential direction. The expression already gives the acceleration in an inertial frame. Multiplying by mass gives the net real force required on the carriage.

Check sideways force with angular momentum and energy

Hold the angular speed at Ω, and drive the radial position as $r = r_0 + ut$ with constant u. Gravity and support cancel vertically. The net radial drive force is $-mr\Omega^2$ and the guide must exert a side force $2mu\Omega$. The resulting torque about the axis is $2mru\Omega$. This equals the derivative of the angular momentum $mr^2\Omega$, providing an independent sign and factor-of-two check.

$$P_r + P_\theta = (-mr\Omega^2)u + (2mu\Omega)(r\Omega) = mru\Omega^2 = \frac{d}{dt}\left[\frac{m}{2}(u^2 + r^2\Omega^2)\right].$$

The side force does work in the laboratory since its contact point moves tangentially. Describing a smooth slot as doing zero work without specifying the frame would lose this energy transfer. The radial actuator absorbs power during outward constant-speed travel.

Forces and power on two passes

A 0.40 kg carriage moves from $r = 0.10$ m outwards at 0.20 m/s, with $\Omega = 5$ rad/s. At $t = 0.50$ s it reaches 0.20 m. Its radial force is −2.0 N, tangential force is +0.80 N and net mechanical power is

$$-2(0.20) + 0.80(1) = 0.40\,\mathrm{W}.$$

Setting u to zero removes the side force, while the guide still requires a radial restraint to maintain circular travel.

On an inward pass at −0.20 m/s, evaluated at the same radius, the radial force remains −2.0 N. The side force reverses and the turntable now receives angular momentum from the carriage. The magnitude of each force depends on a different combination of radius, radial speed and angular speed. Use the trace to distinguish the constant side load from the changing inward force.

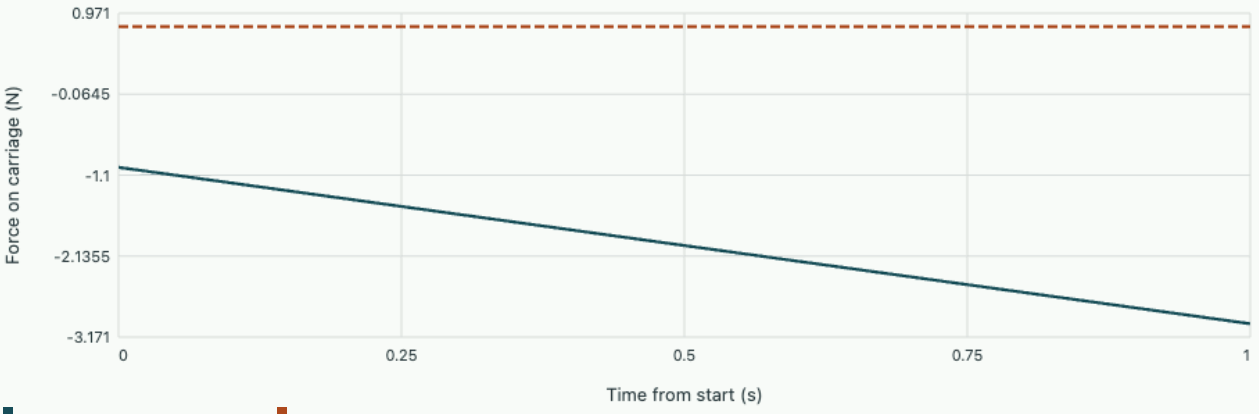
Set radial speed to zero before changing angular speed. Restore outward motion and predict how doubling angular speed changes the two force components at the same radius. Use mechanical power as a separate sign check.

Open this experiment online

Try the calculation

Angular speed (rad/s) **5**

Outward speed (m/s) **0.2**



At 0.50 s the radial and tangential forces are -2.00 N and 0.80 N. The net mechanical power is 0.40 W.

Tutorial · reconstruct angular acceleration from a side load

At one instant the carriage is at 0.20 m, with mass 0.40 kg, $\dot{r} = 0.10$ m/s and $\dot{\theta} = 4$ rad/s. A tangential force transducer measures +0.48 N. Estimate $\ddot{\theta}$ and explain what a rigidly fixed carriage would read at the same instant.

▼ **Show the reconstruction**

The tangential acceleration is

$$0.48/0.40 = 1.20\,\mathrm{m/s^2}.$$

Subtract $2(0.10)(4) = 0.80$ m/s² to isolate $r\ddot{\theta} = 0.40$ m/s². Thus $\ddot{\theta} = 2.0$ rad/s². With no radial sliding the second term disappears and the side load falls to $0.40(0.20)(2.0) = 0.16$ N. The radial centripetal component points along the guide and does not enter this reading.

Further reading · MIT 2.003SC · velocity, acceleration and rotational motion

Chapter 17 · Steady gyroscopic precession and its two possible rates

A spinning axisymmetric rotor supported at one point can keep a constant tilt under gravity. Predict the rate at which its axis turns, examine the fast-spin approximation and determine when slow steady precession is possible.

Identify the changing direction of angular momentum

Let the centre of mass be a distance l along the symmetry axis from a fixed pivot. The symmetry axis is tilted by θ from the upward vertical. The moment of inertia about that axis is I_3, and about either transverse axis through the pivot it is I_1. Define ω_3 as the total component of angular velocity along the symmetry axis. When the spin angular momentum dominates,

$$\mathbf{H} \simeq I_3 \omega_3 \mathbf{e}_3.$$

Under uniform precession at rate Ω, its tip traces a horizontal circle of radius $I_3\omega_3 \sin\theta$.

$$|\dot{\mathbf{H}}| \simeq \Omega I_3 \omega_3 \sin\theta,$$

$$|\boldsymbol{\tau}_g| = mgl \sin\theta,$$

$$\Omega_{\text{slow}} \simeq \frac{mgl}{I_3\omega_3}.$$

The tilt factors cancel for nonvertical orientations. Gravity changes the direction of angular momentum while doing zero work in a steady precession at fixed height. Reversing the spin changes the direction of slow precession. The support force has zero moment about the pivot.

Keep the angular momentum due to precession

For a regular precession at fixed tilt the transverse angular velocity is the part of $\Omega \mathbf{e}_z$ perpendicular to $\mathbf{e}_3$. Thus $\mathbf{H} = I_1 \Omega \mathbf{e}_z + (I_3\omega_3 - I_1\Omega\cos\theta)\mathbf{e}_3$. The vertical first term is constant. The rotating second term has derivative $\Omega \mathbf{e}_z \times \mathbf{H}$. Equating it to the gravitational torque yields

$$I_1 \cos\theta\, \Omega^2 - I_3\omega_3\Omega + mgl = 0.$$

For $0 < \theta < 90^\circ$ and positive spin, two positive real rates exist only if $D = (I_3\omega_3)^2 - 4I_1 mgl\cos\theta \geq 0$. The smaller root is most accurately evaluated as $2mgl/(I_3\omega_3 + \sqrt{D})$, avoiding subtraction of nearly equal numbers at high spin. The larger root corresponds to a different initial angular velocity. Releasing a rotor without providing an appropriate initial precession generally adds nutation.

A tabletop rotor as its spin slows

Take $m = 0.50$ kg, $l = 0.060$ m, $I_1 = 0.0030$ kg m², $I_3 = 0.0020$ kg m² and $\theta = 60^\circ$. At $\omega_3 = 100$ rad/s the first estimate gives 1.47 rad/s. The quadratic gives 1.49 rad/s for the slow branch and 131.8 rad/s for the fast branch. The 1% discrepancy in the slow result is consistent with a small transverse angular momentum.

At $\omega_3 = 30$ rad/s the approximation gives 4.91 rad/s and the exact slow rate is 5.72 rad/s. The two rates meet at a spin of roughly 21.0 rad/s. Below this value there is no constant-tilt solution for the prescribed angle and inertias. Plotting the discriminant or using the real branches avoids reporting a fictitious real rate after the branches disappear.

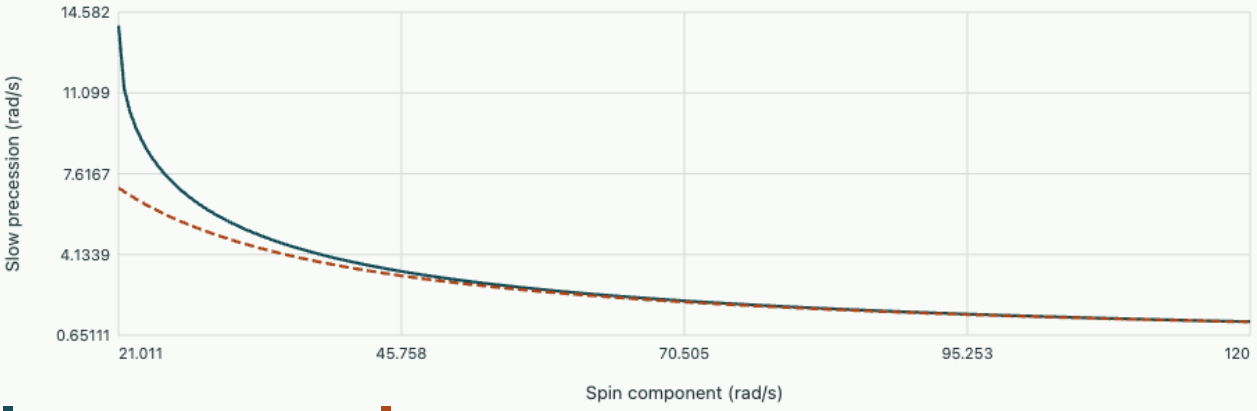
Compare the approximation and exact branch near the minimum admissible spin, then at high spin. Raise the tilt toward a right angle. Predict whether the minimum spin increases or falls before checking the new domain of the curve.

Open this experiment online

Try the calculation

Tilt from vertical (°) **60**

Moment arm (mm) **60**



For this tilt the two regular-precession branches first exist at spin 21.0 rad/s. At 100 rad/s the exact slow rate is 1.488 rad/s.

Tutorial · infer a moment arm from observed motion

For the same inertias and mass, measure $\omega_3 = 50$ rad/s, $\Omega = 3.0$ rad/s and $\theta = 60^\circ$. Estimate l from the full relation. Calculate the bias if transverse angular momentum is omitted.

▼ **Show the torque estimate**

The full torque parameter is $mgl = I_3\omega_3\Omega - I_1\cos\theta\,\Omega^2 = 0.300 - 0.0135 = 0.2865$ N m. The arm is

$$l = 0.2865/(0.50 \cdot 9.81) = 58.4\,\mathrm{mm}.$$

Keeping only the first term gives 61.2 mm, overestimating the arm by about 4.7%. Measuring steady precession over several cycles helps distinguish that systematic error from fluctuations caused by nutation.

Further reading · MIT 2.003SC · angular momentum and rotating rigid bodies

Study the full derivation

Critical speed, synchronous whirl and two different rotor centres · Vibration and identification in machinery

A stationary probe observes motion of a rotating shaft surface, while a balance calculation follows an eccentric mass centre. Derive both trajectories from one flexible-rotor model. Their difference explains the apparent recentring of a rotor at high speed and why a single vibration amplitude cannot identify unbalance and damping.

Chapter 18 · Why Coulomb damping loses a fixed amplitude each half-cycle

A sliding carriage on a spring can stop away from zero displacement. Determine each turning point without time stepping, then identify the experimentally observable difference between Coulomb and viscous damping.

Write a different equation for each direction of motion

A mass m slides horizontally against a linear spring k. During motion take friction magnitude $F_k > 0$, with force opposing velocity. At rest allow any friction up to the static limit $F_s \geq F_k$. The moving equation is

$$m\ddot{x} + kx = -F_k \operatorname{sgn}(\dot{x}).$$

Release the mass at $x = A > 0$ with zero speed. If $kA \leq F_s$ static friction can hold it there. If $kA > F_s$ it starts moving left. While it moves left, friction points right. Shift the coordinate by $a = F_k/k$ to obtain a homogeneous oscillator:

$$m\ddot{x} + k(x - a) = 0,$$

$$x(t) = a + (A - a)\cos(\omega_n t),$$

$$\omega_n = \sqrt{k/m}.$$

At $t = \pi/\omega_n$ the next turning point is $x_1 = 2a - A$. Test static force balance there before starting another sliding segment. If the mass continues, the sign of friction changes and the next cosine is centred on $-a$. The time for each complete sliding half-cycle is constant despite the decreasing amplitudes. An integration routine that uses only $\operatorname{sgn}(0) = 0$ can generate spurious chatter near the final state.

Recover the amplitude drop from work done by friction

For a full crossing from a positive peak A to a negative peak $-B$, with $A, B > 0$, friction dissipates $F_k(A + B)$. The change in spring energy is $\frac{1}{2}k(A^2 - B^2)$. Cancel the travelled distance to obtain

$$A - B = \frac{2F_k}{k},$$

$$A_j - A_{j+2} = \frac{4F_k}{k}$$

for successive magnitudes on opposite sides and successive positive peaks, respectively, while sliding continues. An ordinary viscous oscillator instead has an approximately fixed ratio between successive positive peaks. A mixed friction-damping experiment can exhibit both effects. An ideal Coulomb model also permits a final equilibrium anywhere in the static-friction band $[-F_s/k, F_s/k]$.

Predict a finite stopping time and reconstruct a friction force

Take $m = 1.0$ kg, $k = 100$ N/m, $F_k = 0.50$ N, $F_s = 0.65$ N and $A = 40$ mm. The shift is 5.0 mm and each sliding half-period is 0.314 s. Starting at +40 mm the successive turns are −30, +20, −10 and 0 mm. The mass sticks at the fourth turn after 1.257 s. A release from +43 mm instead reaches −33, +23, −13 and +3 mm. Since $100(0.003) = 0.30$ N lies below 0.65 N, the residual displacement persists under the assumptions.

Suppose a different test with $k = 200$ N/m shows successive positive peaks of 35, 27 and 19 mm. The 8 mm decrement gives

$$F_k = k(0.008)/4 = 0.40\,\mathrm{N}.$$

A viscous fit would infer logarithmic decrements 0.260 and 0.351 from adjacent pairs. Applying a small constant external force after the final stop can help identify static breakaway independently of the moving-friction estimate.

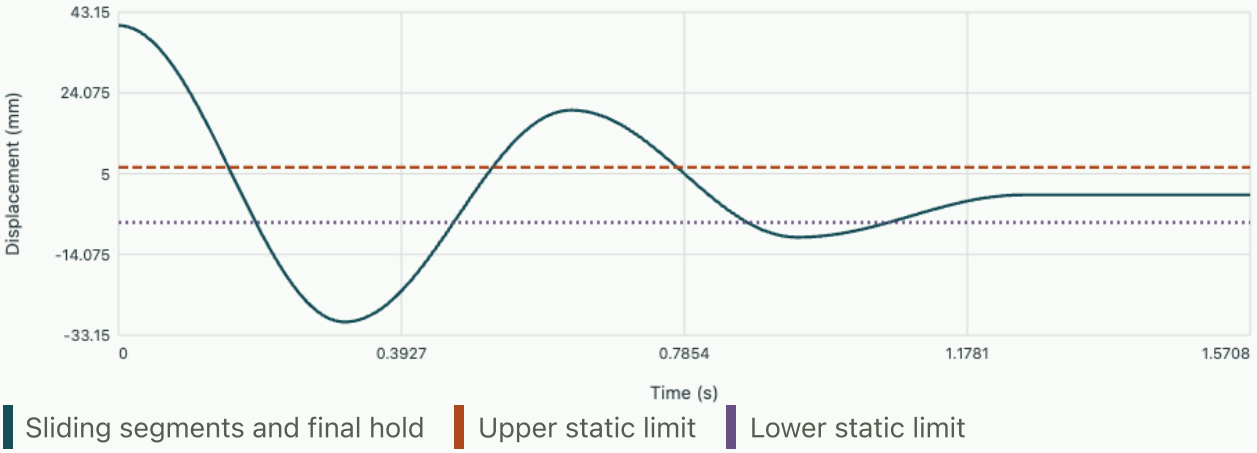
Compare releases at 40 and 43 mm. Predict the final resting position from the sequence of turning points. Then reduce the release below the static threshold; explain why an initial energy above zero need not imply any motion.

Open this experiment online

Try the calculation

Release displacement (mm) **40**

Sliding friction (N) **0.5**



With static friction 1.30 times sliding friction, sticking begins at 1.257 s and 0.00 mm. A full sliding half-cycle lasts 0.314 s.

Tutorial · predict the last turning point even without crossing zero

Use the original mass, stiffness and friction values but release at +8 mm. Does it start, cross zero, and then reverse velocity? Give the final displacement and dissipated energy.

▼ **Follow the event conditions**

The initial spring force 0.80 N exceeds 0.65 N so it moves left. The next turn is

$$2a - A = 10 - 8 = +2\,\mathrm{mm}.$$

It has not crossed zero. At that turn spring force is 0.20 N so it sticks. Friction work is $0.50(0.008 - 0.002) = 0.0030$ J, equal to $\frac{1}{2}(100)(0.008^2 - 0.002^2)$. Blindly subtracting a positive amplitude decrement would have assigned an impossible negative magnitude.

Further reading · University of Alberta, Vibrations and Sound · viscous and Coulomb damping

Chapter 19 · When the third harmonic matters more than the fundamental

A machine applies a short force once per cycle. Its repetition rate looks safely below a structural resonance, but one of its harmonics can lie close to that resonance. Build the forcing spectrum from a pulse and apply a complex response coefficient to each term.

Integrate over the part of the period where force is present

Let an even rectangular force have peak P, period T and on-time dT centred on zero, where $0 < d < 1$. Use an even periodic continuation. Its mean is Pd. Sine coefficients vanish by symmetry. With $\Omega = 2\pi/T$, integrate the cosine terms over the active interval and exploit periodicity:

$$F(t) = Pd + \sum_{n=1}^{\infty} a_n \cos(n\Omega t),$$

$$a_n = \frac{2}{T}\int_{-dT/2}^{dT/2} P\cos(n\Omega t)\,dt = \frac{2P}{n\pi}\sin(n\pi d).$$

The coefficients are signed. A negative coefficient adds a phase reversal; taking its magnitude before summing the time response loses that information. The mean drives a static offset. When nd is an integer, that harmonic vanishes even though the force still has abrupt edges. At a jump the Fourier series converges to the mean of the adjacent values. These details matter when reconstructing force but can have much smaller effects on the filtered displacement.

Solve each harmonic separately and keep phase

For $m\ddot{x} + c\dot{x} + kx = F(t)$ the steady particular solution follows from $H(i\omega) = 1/(k - m\omega^2 + ic\omega)$. Define $\omega_n = \sqrt{k/m}$,

$$\zeta = c/(2\sqrt{km}),$$

$$r_j = j\Omega/\omega_n.$$

The jth response contribution is the real part of $a_j H(ij\Omega)e^{ij\Omega t}$. Thus

$$x(t) = \frac{Pd}{k} + \sum_{j=1}^{\infty} \frac{a_j}{k}\,\frac{(1 - r_j^2)\cos(j\Omega t) + 2\zeta r_j \sin(j\Omega t)}{(1 - r_j^2)^2 + (2\zeta r_j)^2}.$$

Far above resonance each force coefficient is attenuated roughly by another j^{-2} in displacement. Acceleration weights a displacement coefficient by j^2 again, so an adequate displacement truncation can be inadequate for acceleration. Start-up transients are omitted because the expression is the settled periodic response for positive damping.

Compare harmonics before selecting an operating speed

Take $P = 10$ N, $d = 0.25,\ k = 1000$ N/m, natural frequency 12 Hz, damping ratio 0.04, and repetition frequency 4 Hz. The input mean is 2.50 N and $a_1 = 4.50$ N while $a_3 = 1.50$ N. The first harmonic has $r = 1/3$, so its displacement amplitude is about 5.06 mm. The third is exactly resonant and gives

$$1.50/(1000 \cdot 0.08) = 18.8\,\mathrm{mm}.$$

A single harmonic based on the largest force coefficient therefore misses the dominant response.

Choose a new duty $d = 1/3$ while leaving peak force and repetition frequency unchanged. Now $\sin(3\pi d) = 0$, so the resonant third harmonic disappears. The static offset rises from 2.50 to 3.33 mm, and the first two harmonics remain. Holding mean force fixed instead would require rescaling P. A small timing error can reintroduce the third harmonic, especially important when damping is low. Move the duty around one third and compare the full signal with its fundamental-only prediction.

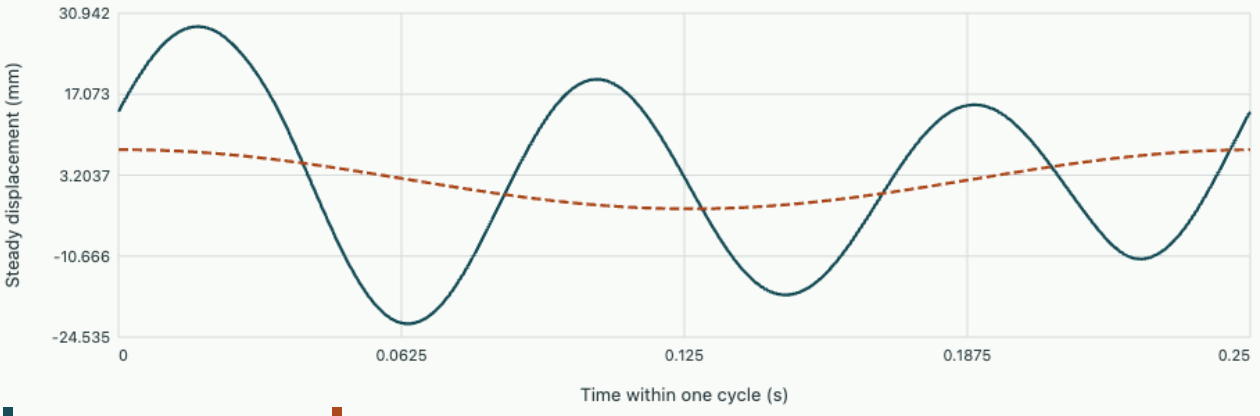
Use 4 Hz and tune the duty as close as the slider allows to one third. Read the residual third harmonic and compare the waveform to the fundamental-only trace. Next shift repetition frequency and distinguish removing the forcing coefficient from moving away from resonance.

Open this experiment online

Try the calculation

Pulse duty fraction **0.25**

Repetition frequency (Hz) **4**



The static offset is 2.50 mm. The third force harmonic is 1.501 N and its displacement amplitude is 18.76 mm at 12.00 Hz.

Tutorial · estimate sensitivity to an on-time error

Near $d = 1/3$ let the duty increase by 0.010 while keeping P fixed. Estimate the newly excited third force harmonic to first order and the resulting displacement near exact resonance. Can this signed

contribution be added directly to the static peak displacement?

▼ **Differentiate the coefficient, then use the transfer function**

From $a_3 = 2P\sin(3\pi d)/(3\pi)$, its derivative is $2P\cos(3\pi d) = -20$ N per unit duty at $d = 1/3$. A change of 0.010 gives about −0.20 N. At resonance the displacement amplitude is

$$0.20/[1000(0.08)] = 2.50\,\mathrm{mm}.$$

The sign sets its phase, and the dynamic transfer adds a quarter-cycle lag at resonance. The largest combined displacement requires summing instantaneous contributions with their correct phases, rather than adding every harmonic magnitude as if their crests coincided.

Further reading · MIT 18.03SC · differential equations with periodic input and resonance

Study the full derivation

Parametric resonance under periodic stiffness modulation · Vibration and identification in machinery

A carriage supported by a spring whose stiffness changes periodically can gain oscillation energy with no additive force at its natural frequency. Compare this mechanism with an ordinary forced resonance, estimate the first instability band and learn how to measure growth before nonlinear saturation invalidates the prediction.

Chapter 20 · What a passenger and a tyre feel on the same road

Two vehicles encounter the same nearly sinusoidal series of road undulations. In one vehicle the occupant feels little vibration, yet its contact force varies substantially. The other transmits a more noticeable body acceleration. Build a two-degree-of-freedom model from force balances, calculate the two responses to a prescribed road geometry, and test which changes in suspension damping improve one measurement at the expense of another.

Choose coordinates around a loaded equilibrium

Represent one vehicle corner by a sprung mass m_s and an unsprung mass m_u. The first collects the share of the body supported at this corner. The second collects the wheel assembly and the portion of the suspension moving with it. Let x_s and x_u be upward displacements from static equilibrium. A suspension spring k_s and damper c connect the masses. A tyre spring k_t connects the wheel coordinate to the road height $y(t)$. The increments in the tyre force describe deviations about a substantial static force.

Draw the two masses separately. When the body sits momentarily above the wheel relative to its equilibrium separation, the suspension force on the body pulls down. The same force acts upward on the wheel. The tyre pushes upward on the wheel when the road rises more than the wheel. Applying Newton's law to each free body gives

$$m_s\ddot{x}_s = -k_s(x_s - x_u) - c(\dot{x}_s - \dot{x}_u),$$

$$m_u\ddot{x}_u = k_s(x_s - x_u) + c(\dot{x}_s - \dot{x}_u) - k_t(x_u - y).$$

Gravity has disappeared because the coordinates measure changes from an equilibrium that already carries weight. To test continued tyre contact, restore the static force at the end. Replacing the input by a force $k_t y$ on the wheel is an algebraic convenience. The tyre still extends between two moving positions, and its incremental force still depends on their difference. Overlooking this distinction predicts a changing contact load even for an arbitrarily slow rigid translation of the whole vehicle.

For fixed road height, multiply the equations by their respective velocities and sum. The spring terms form the time derivatives of $k_s(x_s - x_u)^2/2$ and $k_t(x_u - y)^2/2$. The remaining internal power is $-c(\dot{x}_s - \dot{x}_u)^2$. With a moving road the contact point additionally supplies $k_t(y - x_u)\dot{y}$. This check detects a damper force accidentally applied with the same sign to both masses. It also identifies the mechanical source of energy in the driven oscillation.

Determine the bounce and wheel modes before adding damping

Set $c = 0$ and $y = 0$. For a displacement vector $\hat{x}$ varying as $e^{i\omega t}$, the homogeneous equations become

$$(K - \omega^2 M)\hat{x} = 0.$$

Here $M = \mathrm{diag}(m_s, m_u)$ and the stiffness matrix has diagonal entries $k_s, k_s + k_t$ and off-diagonal entry $-k_s$. Nonzero motion requires the determinant to vanish. Expanding the determinant produces a quadratic in ω^2:

$$m_s m_u \omega^4 - \big[m_s(k_s + k_t) + m_u k_s\big]\omega^2 + k_s k_t = 0.$$

The two roots are not determined by assigning one spring rigidly to each mass. In the lower mode the wheel follows part of the body motion and the tyre deflects slightly. In the upper mode the wheel moves strongly, while the much heavier body has only a small counter-motion. The approximate estimates $\sqrt{k_s/m_s}$ and $\sqrt{(k_s + k_t)/m_u}$ are reasons to look in two frequency bands. The exact eigenvectors determine whether the chosen sensor has enough motion in each band to reveal the mode.

Use $m_s = 250$ kg, $m_u = 35$ kg, $k_s = 15$ kN/m and $k_t = 160$ kN/m. Mass-normalising the stiffness by $M^{-1/2}$ gives an ordinary real symmetric eigenproblem. Its eigenvectors must be mapped back by $M^{-1/2}$ to recover displacement ratios. The frequencies are 1.178 and 11.26 Hz. The ratios x_s/x_u are approximately 11.54 and -0.012. The opposite sign in the second ratio does not imply that the body strikes the wheel. It identifies a phase difference of half a cycle in a small-displacement eigenmode.

Open editable notebook Download .ipynb

Project spatial geometry onto a temporal response

Suppose the road profile is $y(s) = Y\cos(2\pi s/\lambda)$, where s is the distance travelled along the road. At constant speed v, substitute $s = vt$. The resulting temporal angular frequency is

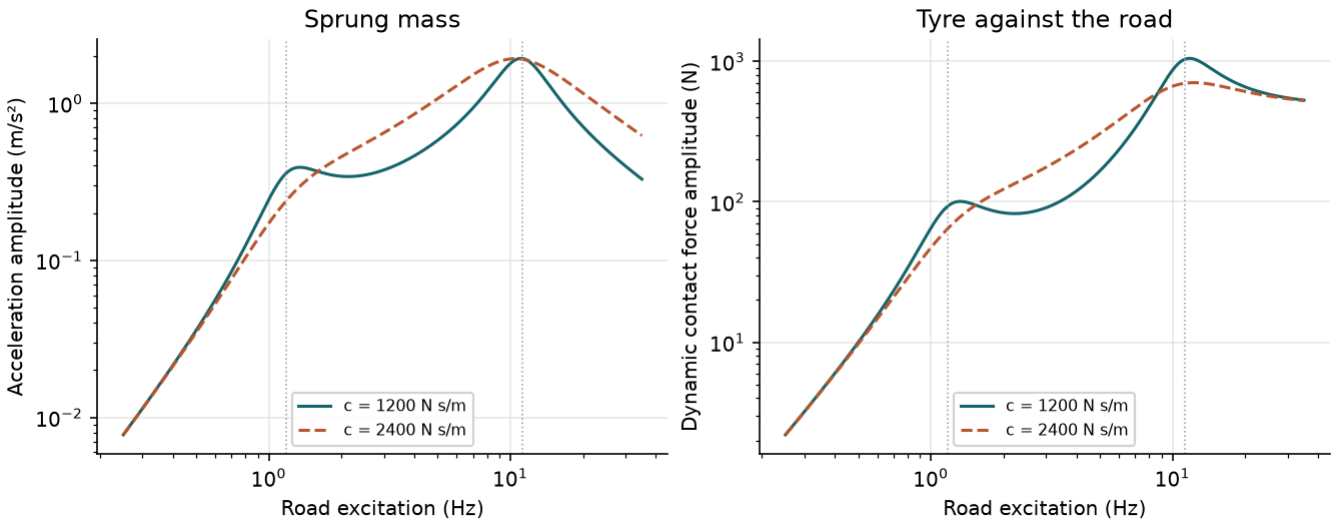
$$\omega = 2\pi v/\lambda.$$

A road with 4 m spacing at 12 m/s therefore supplies a 3 Hz harmonic excitation. Comparing two travel speeds on the same road involves moving along a frequency-response curve with a fixed height amplitude. This differs from holding the input acceleration amplitude fixed while sweeping frequency on a shaker.

Introduce $S = k_s + i\omega c$, $A = S - m_s\omega^2$ and $B = S + k_t - m_u\omega^2$. The complex amplitudes satisfy $AX_s - SX_u = 0$ and $-SX_s + BX_u = k_t Y$. Invert this two-by-two system once and retain the phase of each result:

$$\frac{X_s}{Y} = \frac{k_t S}{AB - S^2},$$
$$\frac{X_u}{Y} = \frac{k_t A}{AB - S^2}.$$

The body accelerometer reads the amplitude $\omega^2|X_s|$. The dynamic tyre force amplitude is $k_t|Y - X_u|$. Subtract the complex displacements before taking an absolute value. The difference between their magnitudes cannot in general recover the extension of the tyre. For example, two signals of equal amplitude and opposite phase differ by twice that amplitude at the peak. Treating them as equal simply because their magnitudes match would predict no force change.



The vertical markers locate the undamped modal frequencies. A body-mounted accelerometer emphasises different parts of the response from a force measurement at the road. Damping changes both curves.

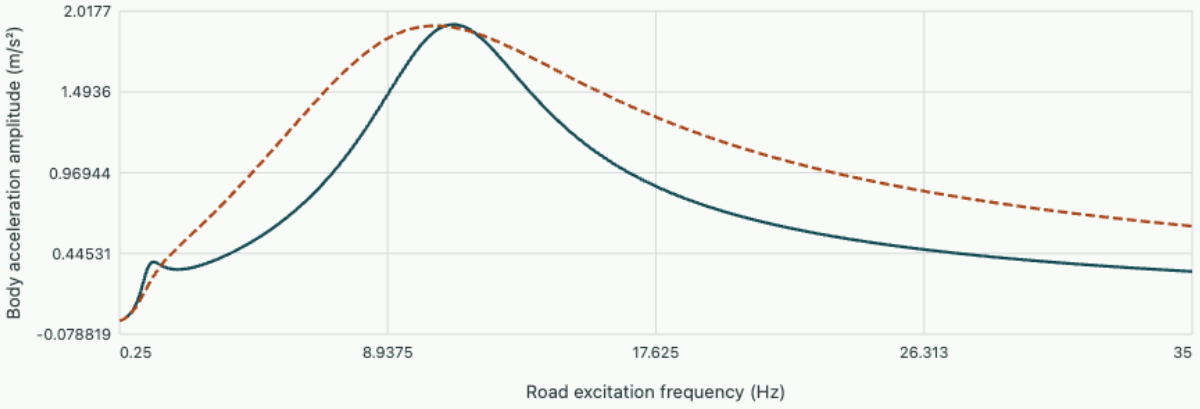
Interactive calculation · Quarter acceleration

Change the physical and numerical inputs of the notebook model. Compare the updated curves and balance checks with the recorded notebook figures. Each plot uses the same equations and assumptions documented in that notebook; its readout names any analytical or reduced-model reference.

Open this live calculation

Try the calculation

Sprung mass (kg)	250
Suspension damping (N s/m)	1200
Road amplitude (mm)	3



Selected suspension Double damping

At3 Hz, body acceleration 0.376 m/s² and dynamic tyre force 90.8 N. Unsprung mass35 kg; suspension15 kN/m; tyre160 kN/m. Linear contact is assumed.

Download this example's editable notebook

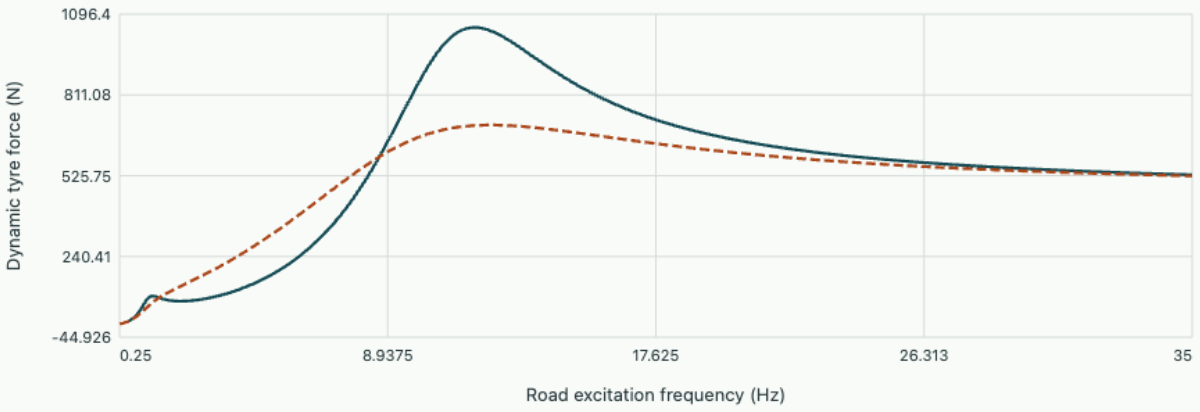
Interactive calculation · Quarter contact

Change the physical and numerical inputs of the notebook model. Compare the updated curves and balance checks with the recorded notebook figures. Each plot uses the same equations and assumptions documented in that notebook; its readout names any analytical or reduced-model reference.

Open this live calculation

Try the calculation

Sprung mass (kg)	250
Suspension damping (N s/m)	1200
Road amplitude (mm)	3



At3 Hz, body acceleration 0.376 m/s² and dynamic tyre force 90.8 N. Unsprung mass35 kg; suspension15 kN/m; tyre160 kN/m. Linear contact is assumed.

Download this example's editable notebook

For $Y = 3$ mm and $c = 1200$ N s/m, the 3 Hz input produces an acceleration amplitude of 0.376 m/s² and a dynamic tyre force amplitude of 90.8 N. An instrument reporting root mean square acceleration for this pure sinusoid will show 0.266 m/s². When comparing an experimentally averaged spectrum with the notebook, identify the amplitude convention before interpreting a factor of $\sqrt{2}$ as a model error.

Ask three separate questions about damping

Near the lower resonance, relative suspension motion is substantial. A damper dissipates energy there and reduces the high resonant response of a lightly damped body. Well above that first mode, the connecting damper also transmits a force proportional to the relative velocity. Increasing c can therefore

increase the body acceleration in a band where the spring alone would isolate motion reasonably well. Near wheel hop the same change can again reduce a peak. Nobody needs to decide which damping is universally best to interpret these effects. A specification must identify the input spectrum, the observables and the frequency range.

At the slow end both displacements follow the road, so the dynamic contact increment tends to zero. At very high frequency the wheel scarcely follows the rapidly varying road height and the dynamic force approaches $k_t Y$ for fixed Y. For a mathematical harmonic input this limit is finite. A real tyre averages a sufficiently short spatial wavelength over its contact patch; the ideal point input will then overestimate the transmitted excitation. This is one reason to keep a spatial description of the road alongside the temporal response.

The notebook evaluates the transfer functions over a frequency sweep, and an independent complex matrix solve checks the selected point. It also offers the mass-normalised eigensolve through OneSimulation when installed. Run the low and high frequency limits before using a visually plausible peak as evidence of a mode. Sensor placement, a numerator zero and damping can all shift or suppress an observed maximum.

Combine two measured spatial wavelengths

A single sinusoid is rarely a complete description of a test track. Imagine that a survey identifies one displacement component of amplitude 3 mm at a wavelength of 4 m and a second component of amplitude 1 mm at a wavelength of 1 m. At a steady speed of 12 m/s these components excite the model at 3 and 12 Hz, respectively. The smaller road displacement is close to the upper mode. Its contribution to the dynamic tyre load can therefore exceed that of the larger displacement component. Amplitude alone gives no ordering until the spatial wavelengths and speed are known.

Apply the complex frequency response separately to each component. To obtain a time trace, multiply each response by its measured road phase factor, add the complex oscillations with their own time dependence, and take the real part. For a record containing an integer number of periods of both harmonics the cross term integrates to zero. If the two body acceleration amplitudes are A_1, A_2, the root mean square reading is $\sqrt{(A_1^2 + A_2^2)/2}$. Peak acceleration requires the relative phase as well. This distinction also matters for a minimum contact force because the most negative extrema of the two components may occur at different times. A short record starting at an arbitrary road position loses the exact cancellation of the cross term. For such a record, compute the time history and average its square over the actual measurement window.

Exercises in measurement and scaling

1. At what speed does a 4 m spatial wavelength excite each undamped natural frequency of the example?

▼ **Work through the solution**

Multiply $v = \lambda f$ by each frequency. The speeds are 4.71 and 45.0 m/s. These values locate the modal frequencies on a speed axis. The maximum of the damped body acceleration need not occur at exactly

either speed because the response has a frequency-dependent numerator and the ordinate includes an acceleration factor.

---

2. How large a road amplitude would drive the minimum tyre force to zero at 3 Hz in this linear calculation?

▼ **Work through the solution**

The static corner force is

$$(250 + 35)9.81 = 2796\,\mathrm{N}.$$

The force increment is proportional to road amplitude; a 3 mm input produces 90.8 N. Extrapolation gives

$$3(2796/90.8) = 92.4\,\mathrm{mm}.$$

This is a diagnostic extrapolation. Large motion can reach travel limits, change the effective spring rates or invalidate the small-displacement tyre model well before that value.

---

3. A data logger reports very little upper-mode body displacement. Is that evidence that the wheel remains in steady contact?

▼ **Work through the solution**

Use the upper-mode displacement ratio. A 1 mm wheel modal amplitude corresponds to only about 0.012 mm of body motion. The contact force involves the wheel displacement relative to the road, and therefore cannot be bounded well by this single body displacement measurement. Measure the wheel or contact force directly and include the static load when interpreting the minimum.

Compare a suspension with another input channel

The University of Michigan suspension modelling tutorial includes a controlled force between the masses. Add equal and opposite actuator forces to the two balances and identify which response denominator stays the same for an open-loop prescribed force.

Problem sheet 7 · A sensor changes the resonance it is meant to measure

An accelerometer is attached to a slender cantilever near a periodically driven machine. Use an energy model to estimate the shifted mode, predict how large the forced vibration becomes, and choose a second measurement that distinguishes mass loading from damping.

1 · Identify the modal stiffness and effective mass

A clamped beam has $L = 0.50$ m, $EI = 10$ N m² and distributed mass 0.40 kg. Take tip displacement as generalized coordinate and use $\phi(s) = s^2(3 - s)/2$, $s = x/L$. Calculate effective stiffness, mass and the bare frequency. Why does inserting the entire beam mass as a lump at the tip fail?

▼ **Integrate the assumed shape**

The integrals are

$$\int_0^1 (\phi'')^2 ds = 3,$$

$$\int_0^1 \phi^2 ds = 33/140.$$

Hence $k = 3EI/L^3 = 240$ N/m, $m_{\text{eff}} = (33/140)(0.40) = 0.0943$ kg and $f = \sqrt{k/m_{\text{eff}}}/(2\pi) = 8.03$ Hz. Each distributed mass contribution scales with the square of its local velocity relative to the tip. Concentrating all 0.40 kg at the tip would predict only 3.90 Hz.

---

2 · Update the model before interpreting the spectrum

Mount a 30 g sensor first at the tip and then halfway along the beam. Predict both frequencies. At which position will the accelerometer signal be smaller for the same modal tip amplitude?

▼ **Compute two different mass participations**

At the tip the full sensor mass participates, giving $m_{\text{eff}} = 0.1243$ kg and 6.99 Hz. At half-span $\phi = 0.3125$, so the correction to effective mass is $0.030(0.3125)^2 = 0.00293$ kg and the predicted frequency is 7.91 Hz. Half-span acceleration at the same frequency and modal amplitude is only 31.25% of the tip acceleration. In a real run, moving the sensor also shifts the resonance, so account for the changed response when comparing raw accelerometer amplitudes.

Set attached mass to 0.030 kg and compare half-span with the tip

3 · Convert an operating speed into vibration amplitude

Leave the sensor at the tip. A measured once-per-revolution harmonic supplies 0.20 N peak generalized force. Use modal damping ratio 0.040. Calculate the tip response amplitude at 420 rpm and at 300 rpm using the loaded frequency. Monitor displacement as well as acceleration.

▼ **Evaluate the frequency response**

Peak displacement is $(F_0/k)/\sqrt{(1-r^2)^2+(2\zeta r)^2}$ with $r = f_{\text{force}}/f_n$. At 420 rpm the forcing is 7.00 Hz, essentially on the 6.99 Hz mode. The predicted amplitude is 0.0104 m. At 300 rpm the forcing is 5.00 Hz and the prediction drops to 0.00169 m. At 7 Hz the 10.4 mm amplitude corresponds to about 20.1 m/s² acceleration. The large amplification warrants checking the assumed linear spring and damping against amplitude-dependent measurements.

---

4 · Use a ring-down as an independent observation

After removing harmonic forcing, two successive positive displacement peaks are 8.0 and 6.2 mm. Estimate damping from their ratio. How could you tell whether shifting a peak after moving the sensor resulted primarily from damping or inertia?

▼ **Use logarithmic decrement before comparing resonances**

The decrement is $\delta = \ln(8.0/6.2) = 0.255$, hence

$$\zeta = \delta/\sqrt{4\pi^2+\delta^2} = 0.0405.$$

At that damping the damped free frequency is $f_d = f_n\sqrt{1-\zeta^2}$, only about 0.08% below the undamped frequency. The roughly 13% increase predicted when moving this sensor from tip to half-span is much larger. Measure decay ratios and peak periods in both configurations and compare the mass-weighted mode prediction with the period shift.

Problem sheet 8 · Can a vibration absorber create a quiet mounting point?

A precision instrument sits on a flexible support and is disturbed by a narrowband harmonic force. A smaller mass is attached to the instrument through a spring and damper. Determine when the mounting point becomes still, what happens elsewhere in the system and what a frequency sweep must show before calling the absorber effective.

Find the force which must balance at a quiet primary mass

Let the primary mass and support stiffness be $M = 10$ kg and $K = 40{,}000$ N/m. Attach a secondary mass $m = 1.0$ kg through a spring k, with the external harmonic force F applied only to the primary. First neglect damping. Write the coupled complex displacement equations. What absorber frequency makes the primary exactly stationary under a finite external force?

▼ **Derive a zero of the measured transfer function**

For displacements X, Y measured in the laboratory, the secondary spring force depends on $Y - X$. Under harmonic forcing the equations have diagonal coefficients $K + k - M\omega^2$ and $k - m\omega^2$ and both off-diagonal coefficients equal $-k$. Eliminating Y gives

$$\frac{X}{F} = \frac{k - m\omega^2}{(K + k - M\omega^2)(k - m\omega^2) - k^2}.$$

At $\omega = \sqrt{k/m}$ the numerator vanishes but the denominator is $-k^2$. The secondary equation enforces $X = 0$, and the primary balance gives $Y = -F/k$. The quiet coordinate remains in equilibrium because the secondary mass moves enough to create an opposing force. The entire structure is not quiet. To target the original primary frequency choose $k/m = K/M = 4000$ per second squared, giving $k = 4000$ N/m and a target frequency 10.07 Hz.

Locate both resonances introduced by the absorber

The primary response has a zero at the desired speed, but an operating machine also passes through neighbouring frequencies. Find both undamped natural frequencies and compare with the original primary natural frequency. How would a very slow steady sweep differ from a rapid ramp?

▼ **Solve the coupled eigenproblem before interpreting its zeros**

With $r = \omega/\sqrt{K/M}$, mass ratio $\mu = m/M$ and matched tuning, the determinant gives

$$r^4 - (2 + \mu)r^2 + 1 = 0.$$

At $\mu = 0.10$ the two frequency ratios are 0.8543 and 1.1705, corresponding to 8.60 and 11.78 Hz. Their product is one. The original single resonance has become two normal modes separated by an

antiresonance in the primary response. Each mode generally exhibits primary motion; its eigenvector follows from either homogeneous equation.

At vanishing damping, exact indefinitely sustained forcing at either pole has no bounded stationary amplitude. A finite-duration sweep samples transient build-up that depends on sweep rate and initial conditions, so the largest measured amplitude can shift away from the steady-state pole. Compare forward and reverse sweeps at decreasing sweep rates and use a measured damping value before assigning a peak height to a mode.

Add damping and check the stroke of the secondary mass

At 10 N force amplitude calculate the ideal secondary travel at the target frequency. Now include primary damping ratio 0.03 and secondary damping ratio 0.04 relative to its own mass and spring. Compare the graph's primary and relative displacement predictions at exact tuning.

▼ **Distinguish absolute and relative motion in the damper**

The static primary deflection is $F/K = 0.25$ mm and ideal absorber travel at the antiresonance is

$$|F/k| = 2.50\,\mathrm{mm}.$$

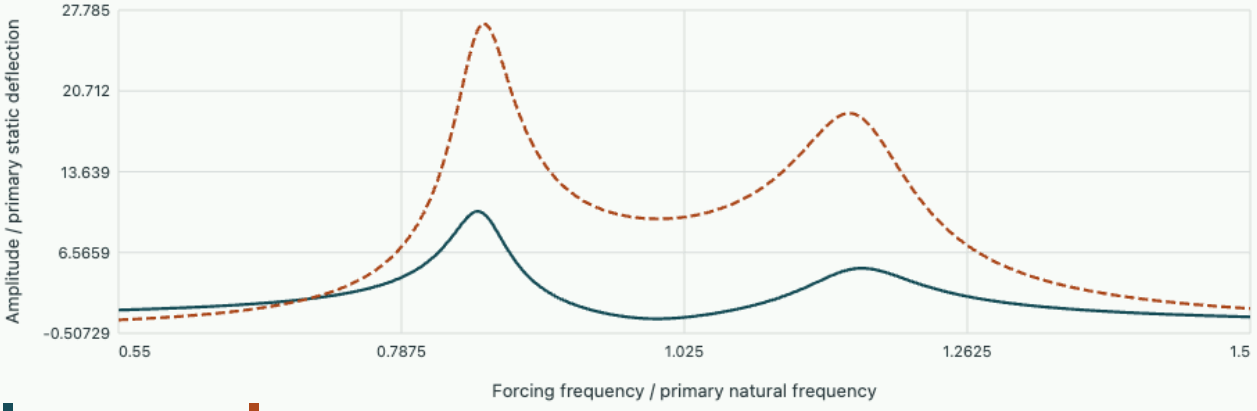
The physical secondary damper sees relative motion $Y - X$. In the harmonic solution replace k by $k + i\omega c_a$ in the coupling terms and add $i\omega c_p$ to the primary diagonal, using the positive-exponential convention for this calculation. The previous exact numerator zero is then complex and the primary response at a real forcing frequency stays finite.

At the stated default, the graph gives primary motion about 0.76 times static deflection or 0.19 mm and relative travel about 9.51 times static deflection or 2.38 mm. With an available stroke of only 2.0 mm in either direction the design would exceed its range even near the intended quiet point. The mechanical stop would invalidate the linear calculation and could introduce harmonics. A damped absorber redistributes bandwidth and peak heights at the cost of a shallower ideal notch.

Open this experiment online

Try the calculation

Absorber / primary mass	0.1
Absorber tuning ratio	1
Absorber damping ratio	0.04



At the bare primary frequency the primary amplitude is 0.76 times static deflection. Relative absorber travel is 9.51 times static deflection.

Design a frequency and displacement verification

The excitation frequency can vary by five percent and the absorber spring by ten percent after temperature changes. Which measurements would test whether a useful operating band remains? Explain why reducing primary response at only one speed cannot establish this.

▼ **Sweep both operating and material parameters**

Because absorber tuning is proportional to the square root of stiffness at fixed mass, a ten percent increase in stiffness raises its natural frequency by approximately 4.9 percent. Move the tuning control accordingly, then examine primary amplitude and relative travel over the complete forcing interval. Record worst primary amplitude and secondary stroke separately. An apparent optimum that falls outside the stroke limit is not an implementable steady-state result.

Measure primary and secondary displacement with a common timing reference. Their relative phase near the notch should produce the opposing spring force required by the primary balance. Use a calibrated applied-force measurement as well: an excitation system whose delivered force falls near the resonance can imitate successful suppression. A comparison with the absorber locked, measured at matched actual force and with the same sensor mass, provides an independent reference. Add further structural modes or frequency-dependent support damping only if residuals show that the two-coordinate approximation no longer explains the measured band.

NPTEL Mechanical Vibrations · principles of passive vibration control and absorber design

Notebook practice · What a passenger and a tyre feel on the same road

Vibration and identification in machinery

Connect modes, moving components and measured frequency responses. Investigate how forcing, damping and the observation point change an inferred dynamic model.

1. ## Critical speed, synchronous whirl and two different rotor centres

   A stationary probe observes motion of a rotating shaft surface, while a balance calculation follows an eccentric mass centre. Derive both trajectories from one flexible-rotor model. Their difference explains the apparent recentring of a rotor at high speed and why a single vibration amplitude cannot identify unbalance and damping.

2. ## Parametric resonance under periodic stiffness modulation

   A carriage supported by a spring whose stiffness changes periodically can gain oscillation energy with no additive force at its natural frequency. Compare this mechanism with an ordinary forced resonance, estimate the first instability band and learn how to measure growth before nonlinear saturation invalidates the prediction.

3. ## Random base motion, spectral density and rms response

   A support vibration is described by a power spectral density rather than a single sine-wave amplitude. Derive the relative-motion filter for a mounted instrument, calculate mean-square motion from a measured spectrum and distinguish a stationary rms prediction from a peak displacement guarantee.

4. ## Why a local damper couples otherwise separate modes

   Undamped mode shapes diagonalise a symmetric mass and stiffness model. Adding a damper at one attachment does not generally leave modal equations separate. Derive the resulting coupling, test its magnitude and learn when a diagonal modal-damping approximation can hide an observable phase error.

5. ## Which damping law produced the free-decay trace?

   A carriage slides on a guide and a spring pulls it back to its resting position. Two dampers can be adjusted to give the same first peak reduction. Predict the following turning points and decide which measurement separates viscous loss from dry sliding friction.

6. ## From beam curvature to a mass matrix and a frequency measurement

   A coarse cantilever model often predicts its first resonance accurately while putting the third far too high. Derive both matrices from the same displacement field and inspect how each mode converges.

7. ## Which approximation shifted the resonance?

   A simulation of a pulsed cantilever contains three separate approximations. Measure the effect of spatial interpolation, retaining finitely many modes and representing a changing force on a time

grid.

Vibration and identification in machinery · 1. Critical speed, synchronous whirl and two different rotor centres

A stationary probe observes motion of a rotating shaft surface, while a balance calculation follows an eccentric mass centre. Derive both trajectories from one flexible-rotor model. Their difference explains the apparent recentring of a rotor at high speed and why a single vibration amplitude cannot identify unbalance and damping.

Choose the observed coordinate before writing a force

A symmetric disk of mass m sits at the centre of a massless flexible shaft with equal lateral stiffness k in two directions and viscous damping c_d acting on its geometric centre. Its mass centre lies a distance e from that point. Assume constant spin Ω, small lateral deflections, simple supports and negligible gyroscopic tilt. Write the complex displacement of the shaft centre as $z = x + iy$. The mass centre has displacement $z + e e^{i\Omega t}$. Its acceleration enters Newton's law while bearing restoring and damping forces depend on the shaft-centre motion:

$$m\ddot{z} + c_d\dot{z} + kz = me\Omega^2 e^{i\Omega t}.$$

Gravity merely shifts the static position for a horizontal isotropic linear shaft and can be subtracted before fitting the synchronous response. Set $\omega_n = \sqrt{k/m}$, $\zeta = c_d/(2m\omega_n)$ and $r = \Omega/\omega_n$. The steady response divided by eccentricity is

$$\frac{Z}{e} = \frac{r^2}{1 - r^2 + 2i\zeta r},$$

$$\frac{Z_{\mathrm{mass}}}{e} = 1 + \frac{Z}{e} = \frac{1 + 2i\zeta r}{1 - r^2 + 2i\zeta r}.$$

At small r the shaft hardly moves while the mass centre describes its geometric eccentricity. Near resonance the shaft amplitude rises and lags the unbalance reference by about a quarter turn. At large r shaft motion approaches amplitude e in opposite phase to the eccentricity, leaving the mass centre comparatively close to the bearing line. A shaft probe does not then read zero merely because the mass centre recentres.

Measure amplitude and phase with a once-per-revolution reference

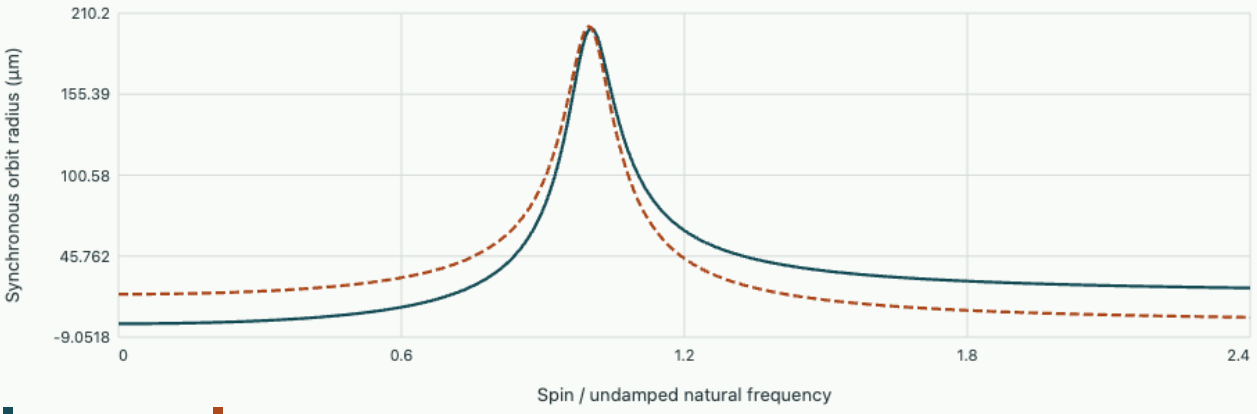
For $m = 2.0$ kg and $k = 80$ kN/m the undamped frequency is 200 rad/s or 1910 rpm. At $\zeta = 0.05, e = 20$ micrometres the shaft-centre orbit radius at $r = 1$ is 200 micrometres. A peak-to-peak probe reading across one axis would be 400 micrometres for a perfectly circular orbit, while an rms sinusoidal axis reading is 141 micrometres. State which amplitude is reported before comparing test and theory.

Open the live experiment

Try the calculation

Eccentricity (µm) **20**

Damping ratio **0.05**



Shaft centre Mass centre

At unit speed ratio the shaft-centre radius is 200.0 µm and its lag behind unbalance is 90°. The mass centre and shaft centre are different points.

The amplitude peak of a force proportional to speed squared occurs at $r = 1/\sqrt{1 - 2\zeta^2}$ when damping is sufficiently small, slightly above the undamped frequency. A constant-force resonance has its displacement peak slightly below. Near a fast run-up the steady expression need not apply because the rotor crosses the resonance faster than the transient can settle. Compare speed sweeps in both directions and hold the drive at representative speeds to distinguish sweep-rate memory from speed-dependent material stiffness.

Recover a complex influence coefficient from a trial mass

At a fixed operating speed write the measured synchronous probe response as a complex number relative to the once-per-turn mark. In linear operation a small calibrated trial unbalance adds a second complex response proportional to its mass-radius product. Subtract the baseline response as a vector; subtracting its magnitude would discard the phase information needed to identify the influence coefficient. The shaft centre model predicts a different phase as the test speed passes through resonance, so the coefficient belongs to the specific spin speed and instrument reference.

For an illustrative optical test the baseline response is 30 micrometres at zero reference phase. With the trial contribution present the response is 30 plus 20 imaginary micrometres in the same coordinates. Its displayed magnitude is 36.1 micrometres. The change of displayed magnitude is only 6.1 micrometres, while the actual trial response is 20 micrometres in quadrature. An estimate based on the two magnitudes would understate the response by more than a factor of three and lose its orientation. A convenient test of linearity is to repeat with the same trial pattern rotated through half a turn. To first order the complex change should reverse sign while the unchanged baseline remains.

When a probe sees a small geometric eccentricity of the actual shaft surface, its synchronous electrical reading includes this runout even at a very low speed where dynamic shaft bending is small. A slow-roll measurement recorded with the same phase mark provides an estimate of that vector. Subtract it in complex coordinates and test whether the resulting low-speed bending tends towards the square-of-speed prediction.

Tutorial · Infer damping and eccentricity separately

An isotropic test has known critical frequency. At exactly that speed the measured shaft radius is 150 micrometres; at high speed the radius tends to 15 micrometres. Estimate eccentricity and damping ratio. Why would one measurement at resonance be insufficient?

▼ **Use the two distinct asymptotes**

The high-speed magnitude tends to e, giving 15 micrometres if geometric surface runout has been removed. At $r = 1$, $Z/e = 1/(2\zeta)$; hence

$$\zeta = 15/(2 \cdot 150) = 0.050.$$

A resonance amplitude alone determines only the ratio e/ζ. High-speed data from a second bending mode or a noncircular shaft surface can violate the asymptote. An independently measured free-decay damping provides another way to separate the parameters without relying on a high-speed plateau.

Tutorial · Interpret orbit shape from two probes

Two calibrated orthogonal probes have synchronous amplitudes 100 and 60 micrometres with phase difference ninety degrees. What is the orbit and is that consistent with the simple isotropic response to one rotating unbalance?

▼ **Reconstruct the signed two-dimensional motion**

The trajectory is an ellipse with semiaxes 100 and 60 micrometres if those are peak amplitudes. Ideal isotropic stiffness and damping driven by a circular rotating imbalance give a circular synchronous orbit, so investigate support anisotropy, probe gains, cross-coupled bearings or another directional forcing. Swapping electronics and rotating probe orientation can separate an instrument gain difference from a physical ellipse. The sign of phase difference relative to the keyphasor determines whirl sense and cannot be recovered from the two amplitudes alone.

Tutorial · Locate a second peak without calling every peak critical

speed

A run-up shows one feature at a fixed shaft speed and a second one whenever a twice-per-turn component crosses the same structural frequency. How would an order–frequency map distinguish them and which terms are missing from the derived model?

▼ Track spectral order and physical frequency

The basic run-up forcing follows the one-per-turn line $f = \Omega/(2\pi)$; the second harmonic follows $2f$. If both excite the same fixed structural frequency at different speeds, their crossings align horizontally in physical-frequency coordinates but at separate shaft speeds. Noncircular runout, misalignment or other periodic effects can create higher shaft orders. The one-eccentric-disk equation contains only the synchronous term, so it cannot independently generate the second harmonic. Measure the forcing reference, examine phase and repeat after a calibrated trial mass. A changed one-per-turn response with a nearly unchanged twice-per-turn line helps distinguish the sources.

NPTEL Rotor Dynamics · Jeffcott rotor model and synchronous response

Vibration and identification in machinery · 2. Parametric resonance under periodic stiffness modulation

A carriage supported by a spring whose stiffness changes periodically can gain oscillation energy with no additive force at its natural frequency. Compare this mechanism with an ordinary forced resonance, estimate the first instability band and learn how to measure growth before nonlinear saturation invalidates the prediction.

A varying stiffness supplies work even at zero additive force

Let $k(t) = k_0[1 + h\cos(\Omega t)]$ with $|h| \ll 1$, and use a linear damper. The free coordinate obeys $\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2[1 + h\cos\Omega t]x = 0$, $\omega_0^2 = k_0/m$. Differentiating its instantaneous mechanical energy $m\dot{x}^2/2 + k(t)x^2/2$ and using the equation gives

$$\dot{E} = -c\dot{x}^2 + \dot{k}x^2/2.$$

Modulating stiffness can therefore do positive or negative work depending on the oscillation phase. For an exactly zero initial displacement and velocity the ideal equation stays at zero; real noise or a deliberate small initial motion supplies a seed.

The squared displacement has a component at twice the oscillation frequency. A stiffness variation near twice ω_0 can preferentially raise stiffness at large displacement and lower it near a zero crossing, supplying net energy over a cycle. This identifies the principal parametric band but does not determine its threshold. A nonzero damping loss must first be overcome.

Background · At what point in a cycle should the stiffness increase to feed motion?

Imagine a spring whose stiffness can switch between two nearby values fast enough that displacement and velocity stay continuous during a switch. The instantaneous change in mechanical energy equals one half of the stiffness change times displacement squared. Increase stiffness at the two displacement extrema and decrease it at the two zero crossings. For an almost sinusoidal motion of amplitude A, the leading injected energy per cycle is

$$\Delta E_{\mathrm{pump}} \simeq 2\left(\tfrac{1}{2}\Delta k\, A^2\right) - 2\left(\tfrac{1}{2}\Delta k\, 0^2\right) = \Delta k\, A^2.$$

Reverse the switching phase and the net transfer changes sign. This accounting explains both the importance of phase and the two stiffness changes of the same sign per mechanical period. The ideal switches describe a square modulation; the numerical threshold for sinusoidal modulation must instead use its own Fourier amplitude and the damping loss.

Find slow growth and detuning near twice resonance

Write the motion as slowly varying cosine and sine quadratures at half the modulation frequency. Substitute into the oscillator equation, drop second derivatives of slow amplitudes and average out

terms oscillating near three times the basic frequency. With detuning $\Delta = \Omega/2 - \omega_0$ and retaining first orders in small damping, modulation and detuning, the two coupled quadratures have exponents

$$s_{\pm} = -\zeta\omega_0 \pm \sqrt{(h\omega_0/4)^2 - \Delta^2}.$$

The square-root term is real sufficiently near the principal band. Net growth requires its positive branch to exceed damping. At exact tuning the approximate threshold is $h > 4\zeta$; away from tuning the unstable interval narrows according to $\Delta^2 < (h\omega_0/4)^2 - (\zeta\omega_0)^2$. The calculation assumes the damping, relative modulation and fractional detuning are small. For large modulation use the periodic linear system and compute its Floquet multipliers directly.

For $\omega_0 = 100$ rad/s, $\zeta = 0.015$ and $h = 0.10$, exact-tuning growth is $2.5 - 1.5 = 1.0$ per second. The unstable half-width in $\Omega/2$ is 2.0 rad/s, hence in the actual drive Ω it is 4.0 rad/s. A one-percent change of pump frequency near 200 rad/s already changes half-frequency detuning by 1 rad/s. Distinguish the drive-frequency width from oscillation-frequency width when comparing an experimental scan.

Compare this finite instability band with the bounded steady response of a damped rotating force

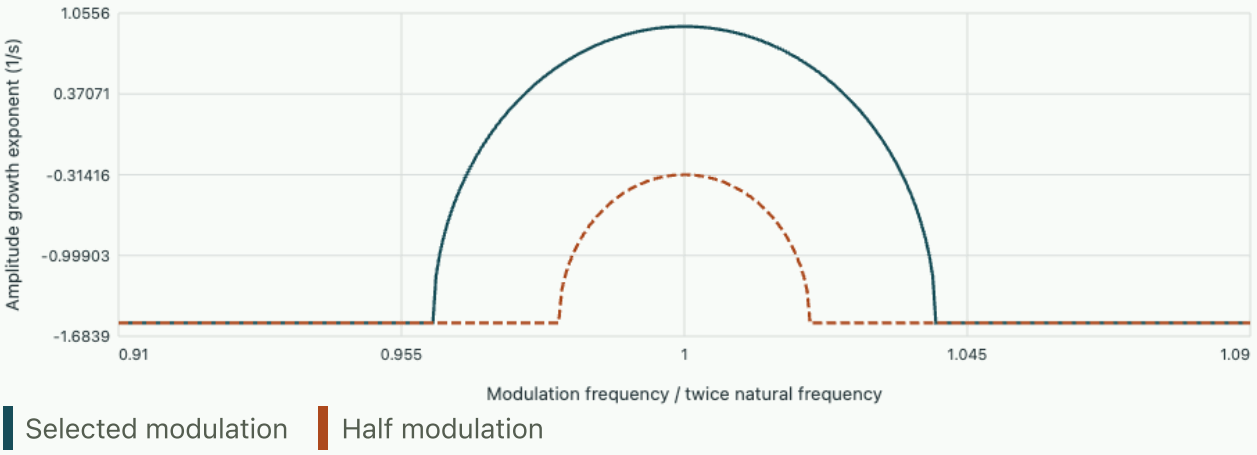
Distinguish the real part of a growth exponent from detuning phase

Outside the principal resonance tongue the square root in the approximate exponent becomes imaginary. The oscillation phase then drifts relative to half the modulation frequency, while the real part remains the viscous decay rate at this order. A graph showing only the real part therefore develops a flat negative floor. This floor means exponential decay of the envelope, not that the instantaneous displacement has stopped oscillating. At the tongue boundaries the larger real exponent crosses zero; inside the band the early envelope increases if the initial condition contains the growing quadrature.

Open the live experiment

Try the calculation

Natural frequency (Hz)	**10**
Stiffness modulation fraction	**0.16**
Damping ratio	**0.025**



The leading-order exact-tuning exponent is 0.942 per second. The predicted unstable band is 19.38 to 20.62 Hz.

At a natural frequency of ten hertz, modulation fraction 0.16 and damping ratio 0.025, the exact-tuning early exponent is about 0.942 per second. An amplitude doubles in approximately 0.736 seconds. Halving the modulation gives a negative exponent at the centre, even though the stiffness is still being varied periodically. Compare measured amplitude over many oscillation periods but stay within the early linear regime. Starting the drive with a phase almost entirely in the initially decaying quadrature can delay the emergence of the growing component and make a short time record look stable.

A direct numerical check uses a pair of independent initial states. Integrate the original periodic oscillator for one modulation period from each initial state, then place their final states in the columns of a two-by-two transition matrix. Repeating the evolution multiplies by this same matrix. A multiplier of magnitude greater than one means that a suitable initial state grows from period to period. The logarithm of its magnitude divided by the period estimates a growth rate without assuming weak detuning. Agreement with the simpler displayed equation should improve near the primary band when damping and modulation both decrease.

Tutorial · Determine the threshold from a measured decay

With modulation off, a freely released oscillator decays in amplitude as $e^{-0.80t}$ and has natural frequency 80 rad/s. Estimate the damping ratio and minimum small periodic stiffness variation required near exact principal tuning.

▼ **Translate an amplitude decay into a pump threshold**

The measured decay rate is $\zeta\omega_0$, so $\zeta = 0.010$ and the threshold is about $h = 0.040$, a four-percent relative modulation amplitude. If the measured quantity were energy rather than amplitude, its exponential rate would be twice as large for the same damping. A stiffness swing quoted peak-to-peak is also twice the h amplitude in the stated equation. Both conventions must be resolved before claiming agreement to a factor of two.

Tutorial · Predict whether a detuned pump grows

For the numerical 100 rad/s oscillator driven with $h = 0.10$, compare modulation frequencies 202 and 205 rad/s. Estimate growth or decay of the dominant quadrature.

▼ **Use half the drive detuning**

At 202, $\Delta = 1$, giving $s_+ = -1.5 + \sqrt{6.25 - 1} = 0.791$ per second, hence growth. At 205, $\Delta = 2.5$ and the leading square root vanishes, giving decay at 1.5 per second at this approximation level. Starting in a predominantly decaying quadrature can delay visibly growing motion even inside the band. Record a long enough early linear window and vary initial phase rather than inferring stability from one quiet short trace.

Tutorial · Use a one-period map beyond weak modulation

Describe a numerical procedure that determines linear stability without fitting an arbitrary initial envelope. How can it be checked in the limit of zero periodic modulation?

▼ **Propagate independent displacement and velocity states**

Integrate two independent initial states, for example unit displacement with zero velocity and vice versa, through one pump period. Place their final states in the columns of a monodromy matrix. An eigenvalue of magnitude greater than one implies linear growth with repeated periods. Its magnitude sets the growth rate through its logarithm divided by the period. Refine the integration time step until both multipliers converge. The determinant should be $e^{-2\zeta\omega_0 T}$ from the trace of the state matrix. With $h = 0$ and underdamping both multipliers have magnitude $e^{-\zeta\omega_0 T}$. A nonlinear spring or motion stops ultimately limit real growing amplitudes; linear multipliers predict onset and early growth, not the final oscillation level.

NPTEL Mechanical Vibrations · parametrically excited systems and stability

Vibration and identification in machinery · 3. Random base motion, spectral density and rms response

A support vibration is described by a power spectral density rather than a single sine-wave amplitude. Derive the relative-motion filter for a mounted instrument, calculate mean-square motion from a measured spectrum and distinguish a stationary rms prediction from a peak displacement guarantee.

Start from moving-base force balance

Let $y(t)$ be base displacement, $x(t)$ the instrument's absolute displacement and $z = x - y$ relative travel inside its mount. Newton's law for the mass is $m\ddot{x} = -c(\dot{x} - \dot{y}) - k(x - y)$. Therefore

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -\ddot{y}.$$

In the frequency domain the transfer magnitude squared from measured base acceleration to relative travel is

$$|H_{za}(\omega)|^2 = \frac{1}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}.$$

A base accelerometer's physical units are metres per second squared. If it reports in multiples of g, convert amplitude by 9.81 and spectral density by 9.81^2. A displacement spectrum is not obtained simply by treating base acceleration as an applied displacement; the numerator and units differ. At sufficiently low frequency a constant base acceleration produces a quasi-static relative deflection $-a/\omega_n^2$, which checks the zero-frequency transfer.

A variance is an area over a stated frequency convention

For a stationary linear response and one-sided base acceleration spectrum $S_a(f)$ per hertz, the relative mean-square motion is $\int_0^\infty |H_{za}(2\pi f)|^2 S_a(f)\,df$. The integral treats distinct frequency components as uncorrelated under stationarity. If a two-sided density per radian per second is used instead, both frequency integration and normalisation must change consistently. The integral of an input spectrum over its own stated axis must reproduce the independently measured acceleration variance.

A convenient benchmark assumes a constant one-sided spectrum S_0 across all frequencies relevant to the mode. Integrating the squared response gives

$$\langle z^2 \rangle = \frac{S_0}{8\zeta\omega_n^3}.$$

With $\omega_n = 100$ rad/s, $\zeta = 0.05$ and $S_0 = 0.040\ (\mathrm{m/s^2})^2/\mathrm{Hz}$, rms relative motion is 0.316 mm. A real finite passband and nonwhite input should be integrated from its measured spectrum; extending a nominal constant outside its measured range is defensible only if the omitted contribution is small. An ideal white acceleration input over all frequencies has infinite acceleration variance, yet this relative displacement filter yields a finite response variance. It serves as a response benchmark, not a physically realisable unlimited-band source.

Compare relative mount travel with the harmonic isolation and transmissibility chapter

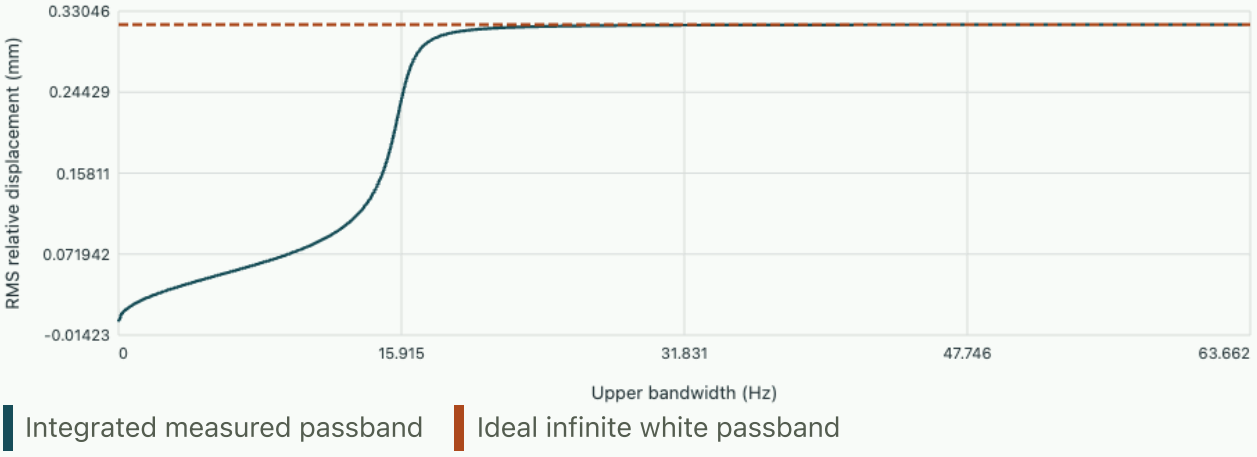
Integrate the spectrum before reporting a single motion amplitude

The experiment accumulates output variance from zero frequency up to a movable upper bandwidth. Most of the increase occurs close to the mode when damping is small. Below that region the support acceleration can contain measurable energy but the relative mount travel stays small; far above resonance the displacement response to acceleration drops quickly. Read the asymptote as a numerical integration check. A response moving toward the predicted limit does not imply that a real excitation spectrum remains flat beyond the accelerometer bandwidth.

Open the live experiment

Try the calculation

Natural angular frequency (rad/s)	100
Damping ratio	0.05
One-sided acceleration PSD ((m/s²)²/Hz)	0.04



The ideal white-input limit is 0.316 mm rms; stopping the integration at 63.7 Hz gives 0.316 mm.

With a natural frequency near sixteen hertz and damping ratio 0.05, the modal half-power width is roughly 1.6 hertz. A one-second record gives frequency spacing around one hertz, leaving very few independent locations across the resonance. A longer record improves frequency resolution. Dividing the record into many shorter segments and averaging their estimates reduces scatter while broadening the resolvable features. State both record duration and segment length and inspect how changing them affects the estimated cumulative variance.

A narrow deterministic tone superposed on random acceleration requires care. Its line energy is distributed across nearby bins according to the window; a density value at the tallest bin changes with resolution even when the physical tone amplitude does not. Integrate the line over its full width or fit a

sinusoid in the time domain. Then combine its mean-square output with the background variance if the two contributions have zero cross correlation. For example, a statistically independent 0.20 mm rms line and 0.30 mm rms broadband contribution produce about 0.361 mm rms total. Adding the two rms amplitudes would yield 0.50 mm, corresponding to a very different assumption about phase and dependence.

A final consistency check uses a short time-domain record with an independently measured travel sensor. Remove a known static offset, compute its root mean square and compare with the integral over the sensor's actual bandwidth. Compare the same frequency interval in both records before attributing a discrepancy to modal damping.

Tutorial · Verify a finite-band low-frequency calculation

A base spectrum is flat at 0.020 acceleration-squared per hertz between 0 and 2 Hz, negligible elsewhere. For a mount with $\omega_n = 200$ rad/s, estimate base rms acceleration and relative displacement rms. Why does the white-through-resonance formula fail here?

▼ Apply the low-frequency transfer over the measured band

Base variance is $0.020(2) = 0.040$, giving rms 0.20 m/s². All excited frequencies are well below the mount resonance near 31.8 Hz, so rms relative travel is approximately $0.20/200^2 = 5.0$ micrometres. The broadband result would assume input energy near the resonant amplification peak that this spectrum explicitly lacks. Direct numerical quadrature with the exact transfer checks the small low-frequency approximation.

Tutorial · Distinguish rms from a time-dependent peak

A zero-mean Gaussian displacement record has the 0.316 mm rms value of the broadband example. What does a three-rms clearance mean at a single time, and why does it not provide that same probability of never touching the stop over a long run?

▼ Include time correlation and observation duration

At a random stationary time a Gaussian value lies outside plus or minus three standard deviations with probability about 0.0027. Over many samples or many correlation times there are repeated opportunities for an excursion. Independent observations would give probability $1 - (1 - 0.0027)^N$ of at least one exceedance, whereas actual filtered vibration has correlated values. Peak predictions need the observation duration and a model of frequency content or crossings. Mechanical stops introduce nonlinearity after contact and invalidate the original Gaussian linear response thereafter.

Tutorial · Identify insufficient frequency resolution

A lightly damped resonance at 20 Hz has damping ratio 0.01. A measured transfer and input spectrum are tabulated every 1 Hz. Estimate its half-power bandwidth and design a quadrature and measurement check.

▼ Resolve the resonance before integrating its area

For a lightly damped isolated mode the full bandwidth is approximately $2\zeta f_n = 0.40$ Hz. A one-hertz grid can miss most of the peak or place a single anomalously high bin at its centre. Use a much finer frequency step or fit a justified complex modal response, then integrate the product with the measured input spectrum. Increase acquisition duration to obtain true frequency resolution; interpolating sparse measured bins alone cannot reveal an unmeasured narrow input line. Compare predicted rms with a time-domain record and account for window leakage and accelerometer noise.

NPTEL Elements of Mechanical Vibration · random vibration, spectral density and system response

Vibration and identification in machinery · 4. Why a local damper couples otherwise separate modes

Undamped mode shapes diagonalise a symmetric mass and stiffness model. Adding a damper at one attachment does not generally leave modal equations separate. Derive the resulting coupling, test its magnitude and learn when a diagonal modal-damping approximation can hide an observable phase error.

Transform the entire equation, including the damper

For two degrees of freedom write $M\ddot{x} + C\dot{x} + Kx = f$, with symmetric positive definite M, K and passive nonnegative C. Choose real undamped eigenvectors as columns of Φ, mass normalised so $\Phi^{\mathsf{T}} M \Phi = I$ and $\Phi^{\mathsf{T}} K \Phi = \operatorname{diag}(\omega_1^2, \omega_2^2)$. Setting $x = \Phi q$ gives transformed damping $C_m = \Phi^{\mathsf{T}} C \Phi$. This matrix is generally full. Orthogonality established for mass and stiffness is not automatically orthogonality for an independently located dissipative element.

A Rayleigh model $C = aM + bK$ is a sufficient condition for diagonal modal damping, with coefficients $a + b\omega_j^2$. It is not necessary in every multimode setting, nor does fitting two measured damping ratios prove the actual physical damping follows this law across other modes. An isolated dashpot to ground at coordinate one produces $C = c e_1 e_1^{\mathsf{T}}$, hence $(C_m)_{jk} = c\phi_{1j}\phi_{1k}$. If both modes move appreciably at that attachment, off-diagonal damping is unavoidable.

Compute a forced response in both modal coordinates

Let $M = I$ kg and let symmetric and antisymmetric mass-normalised vectors $(1,1)/\sqrt{2}$, $(1,-1)/\sqrt{2}$ have natural frequencies 10 and 14 rad/s. A dashpot of 4 N s/m on coordinate one gives modal damping matrix with every entry 2 per second. At forcing frequency ω, the complex dynamic stiffness has diagonal entries $\omega_j^2 - \omega^2 + 2i\omega$ and off-diagonal entries $2i\omega$. Apply a modal force only in the first coordinate. The second equation now requires

$$\frac{Q_2}{Q_1} = -\frac{2i\omega}{\omega_2^2 - \omega^2 + 2i\omega}.$$

At $\omega = 10$ rad/s its magnitude is

$$20/\sqrt{96^2 + 20^2} = 0.204.$$

A diagonal modal-damping approximation would predict exactly zero second modal response. Both models dissipate energy and can give similar-width first-mode peaks, yet their reconstructed physical coordinates differ in phase and amplitude. At larger mode spacing or lower damping the off-diagonal transfer becomes smaller, supplying a quantitative rather than categorical justification for ignoring it.

Compare a single circular harmonic response with an instrument observing two coupled modal coordinates

Check passivity independently of fitted modal damping ratios

With external force zero the mechanical energy derivative is $-\dot{x}^{\mathsf{T}} C \dot{x}$. For the single grounded dashpot this is $-4\dot{x}_1^2 \leq 0$. In modal coordinates it is $-2(\dot{q}_1 + \dot{q}_2)^2$, including a cross term. Dropping that term assigns different instantaneous dissipation to particular velocity combinations. When $\dot{q}_1 = -\dot{q}_2$, the physical damped coordinate is momentarily stationary and true instantaneous loss is zero; a diagonal approximation incorrectly predicts positive loss. Subsequent forces can move that coordinate, so a lossless instant does not imply an undamped eigenmode.

Interpret the second modal coordinate in an experiment

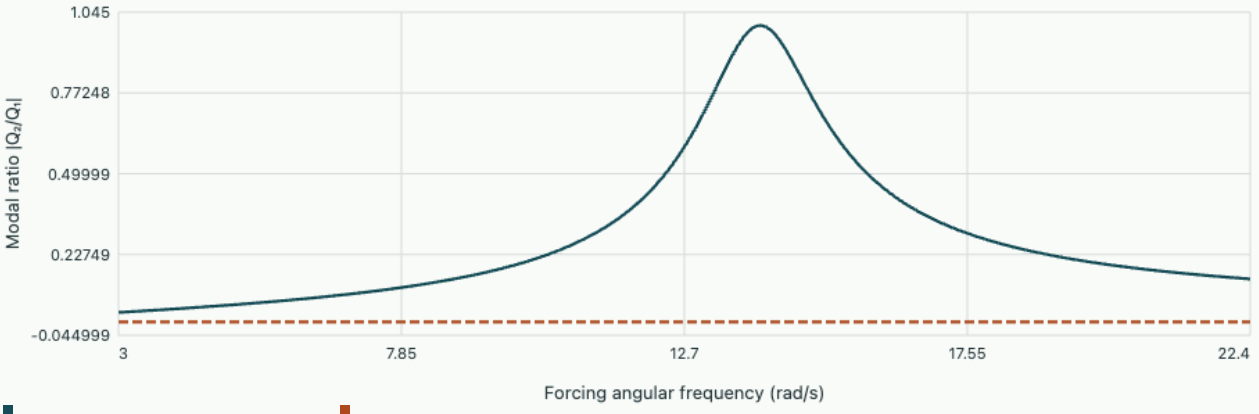
A laboratory cannot normally measure the modal coordinates directly. For the symmetric two-mass example, mount matched accelerometers on the two masses and compute sum and difference using a common clock. For steady harmonic response both measured accelerations share the factor of minus frequency squared relative to their displacements. The ratio of the difference channel to the sum channel therefore recovers the same modal displacement ratio, provided the sensors use the same sign convention and gain. A unit gain error or a cable polarity reversal changes the inferred cross-response even if the physical system is stationary.

Open the live experiment

Try the calculation

Local dashpot (N s/m) **4**

Second modal frequency (rad/s) **14**



At the first natural frequency of 10 rad/s, the second-to-first modal amplitude ratio is 0.204. At the undamped second frequency the ratio is 1.00.

Apply equal-phase forces to the two masses using a known load-sharing beam. That forcing is purely symmetric in the ideal model. An independently measured small antisymmetric force contaminates the nominally unforced modal coordinate. Shift the local damper to the opposite mass and repeat at the

same frequency. The off-diagonal terms of the modal damping matrix reverse sign while the diagonal terms do not. The induced antisymmetric response should reverse phase relative to the symmetric one; an unchanged spurious force is less likely to reproduce that reversal. The sign information is absent from the displayed ratio magnitude and must be preserved in the experimental phase records.

A useful limiting case sets the forcing frequency equal to the undamped second natural frequency. Its real diagonal stiffness then vanishes and the local-damping term directly balances the coupling from the first modal coordinate. In this example their ratio has unit magnitude. A large ratio can occur even when the applied modal force has no second component. This observation follows from the coupled equations at that frequency and does not imply a universal mode shape or a one-to-one transfer of energy between modes.

Tutorial · Find a damping placement that restores modal separation

Keep the same grounded dashpot on coordinate one and add another identical dashpot on coordinate two. Compute the new modal matrix and dissipation. What happens if the added dashpot is only half as large?

▼ **Add physical contributions before transforming**

Equal grounded dampers give $C = 4I$, so $C_m = 4I$. Each mode then has amplitude damping rate two per second and no damping coupling. If the second is only 2 N s/m, C_m has diagonal entries three and off-diagonal entries one. Common and differential combinations have different physical loss. Symmetric placement removes the off-diagonal term because the two undamped shapes have opposite relative signs on the second coordinate.

Tutorial · Reconstruct both modes from two physical probes

At one frequency the calibrated complex sensor readings are $X_1 = 1.0$ mm and $X_2 = 0.8 + 0.2i$ mm. Determine Q_1, Q_2 and assess whether a second mode is detectable even though no second peak has been observed.

▼ **Transform complex amplitudes without discarding phase**

In this normalisation $Q_1 = (X_1 + X_2)/\sqrt{2} = (1.8 + 0.2i)/\sqrt{2}$ and $Q_2 = (X_1 - X_2)/\sqrt{2} = (0.2 - 0.2i)/\sqrt{2}$ mm in the corresponding modal coordinate units. The magnitude ratio is approximately 0.156. A single-magnitude sensor cannot perform this separation, and two magnitude-only readings lose the relative phase. Compare this ratio with a noise and gain-mismatch estimate before declaring it physical. A weakly present mode need not appear as a visually isolated peak at the measured driving frequency.

Tutorial · Determine the true damped poles

How should the complete free-decay problem be solved when modal damping is non-diagonal? Give a diagnostic distinguishing a fitted damping error from an incorrectly specified mass or stiffness.

▼ Use a state-space or quadratic eigenproblem

Seek $x = ve^{st}$, obtaining $(s^2M + sC + K)v = 0$. Solve the quadratic eigenproblem or convert it to first-order state space with displacement and velocity. Damped eigenvectors can be complex even though all physical matrices are real, and complex poles occur in conjugate pairs. Compare both measured phases and decay rates at multiple sensors, then repeat after altering the dashpot while leaving masses and elastic supports unchanged. Frequency shifts or mode-shape changes present already with the damper removed suggest an underlying mass or stiffness error. A fit to one resonance width alone cannot identify the full matrix.

Adhikari · Optimal complex modes and an index of damping non-proportionality, Cambridge research paper

Vibration and identification in machinery · 5. Which damping law produced the free-decay trace?

A carriage slides on a guide and a spring pulls it back to its resting position. Two dampers can be adjusted to give the same first peak reduction. Predict the following turning points and decide which measurement separates viscous loss from dry sliding friction.

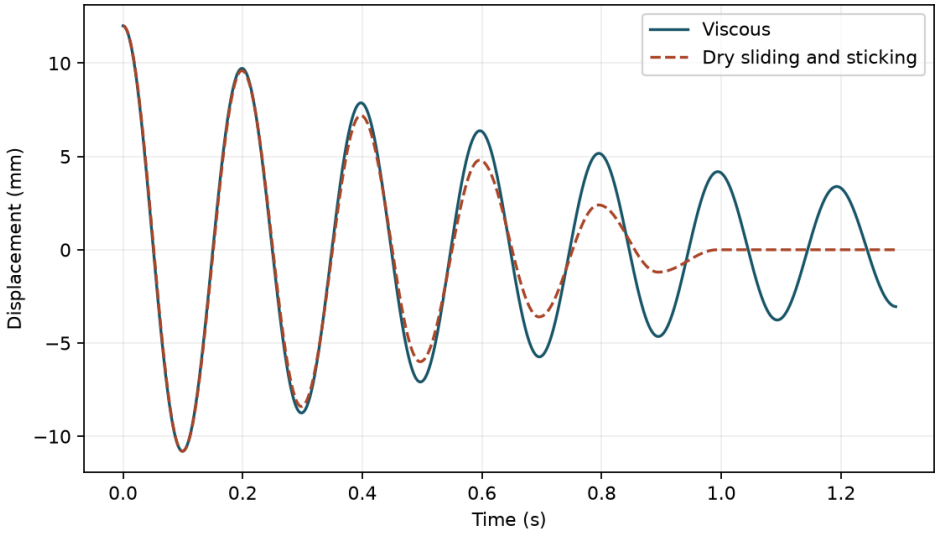
Use the turning points to identify the energy loss mechanism

For a viscous damper the free equation is $m\ddot{x} + c\dot{x} + kx = 0$. In the underdamped regime let $\omega_n = \sqrt{k/m}$, $\zeta = c/(2\sqrt{km})$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$. Released from $x(0) = A_0$ with zero velocity, the displacement is $A_0 e^{-\zeta\omega_n t}[\cos\omega_d t + (\zeta\omega_n/\omega_d)\sin\omega_d t]$. Turning points occur every π/ω_d. Their absolute sizes have a constant ratio $e^{-\pi\zeta/\sqrt{1-\zeta^2}}$.

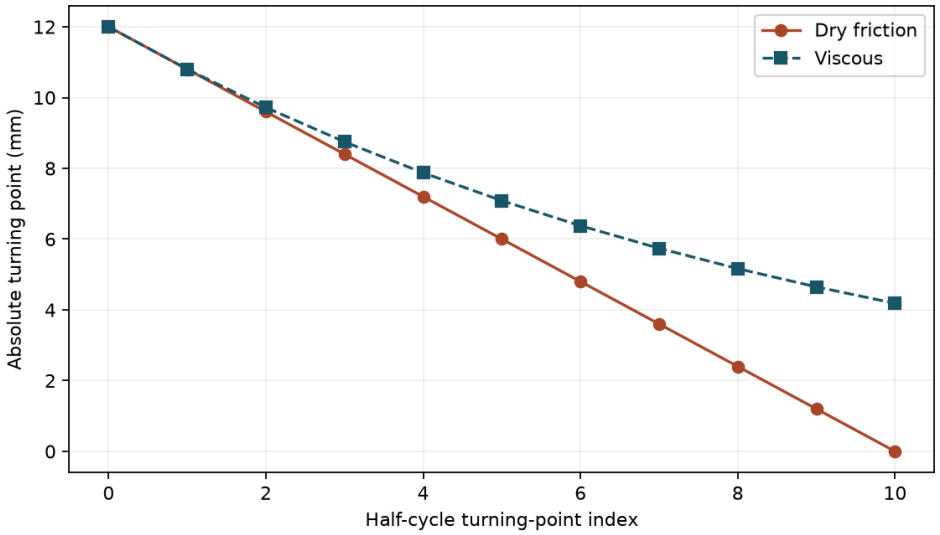
Now let a dry guide exert a constant opposing force of magnitude F. During a sliding half-cycle its sign remains fixed. Starting from a positive peak with $kx > F$, the spring accelerates the mass to the left while friction acts to the right. Subtract the shifted equilibrium F/k from the coordinate. The remainder obeys an undamped oscillator equation, so the next turning point after π/ω_n is $2F/k - x$. For a release from the other side, replace F/k by its signed counterpart. The magnitudes of successive peaks fall by $2F/k$ while the mass still crosses the origin. At a turning point where $|kx| \le F$, static friction can support the spring and the body sticks there. Here the static threshold is idealised as equal to the sliding force.

Match the first peak and predict the next three

Let $m = 1$ kg, $k = 1000$ N/m, $F = 0.6$ N and $A_0 = 12$ mm. Dry friction subtracts 1.2 mm of magnitude per sliding half-cycle. The first return is at –10.8 mm. A viscous model can reproduce its magnitude if its half-period peak ratio is 0.9. Solving $\delta = \pi\zeta/\sqrt{1-\zeta^2}$ for $\delta = -\ln(0.9)$ gives $\zeta = 0.0335$. Its fourth half-cycle peak is $12(0.9)^4 = 7.87$ mm, whereas the dry-friction prediction is 7.20 mm. A single peak pair cannot distinguish the models. Several peaks well above the instrument resolution can.



The sliding model is evaluated on exact half-cycle segments. A plateau is physical when the static force threshold can hold the carriage.



Plotting amplitude against peak count makes an almost missed difference in the first displacement trace easier to measure.

Exercise · estimate sliding friction

A 1000 N/m guide returns through peak magnitudes 7.2, 6.0 and 4.8 mm. Estimate its friction and predict the following magnitude.

▼ **Work through the solution**

The decrement per half-cycle is 1.2 mm. The corresponding force is $k(0.0012)/2 = 0.60$ N. The next alternating peak magnitude is 3.6 mm, still well above the sticking threshold of 0.6 mm.

Exercise · check the last interval

A positive turning point is 0.9 mm and $F/k = 0.6$ mm. Will the mass reach a negative turning point?

▼ **Work through the solution**

Sliding restarts because $0.9 > 0.6$. The next position is $2(0.6) - 0.9 = 0.3$ mm, still positive. Since $0.3 < 0.6$, static friction can hold it there. Extrapolating a positive linear amplitude envelope through zero would lose the actual final position.

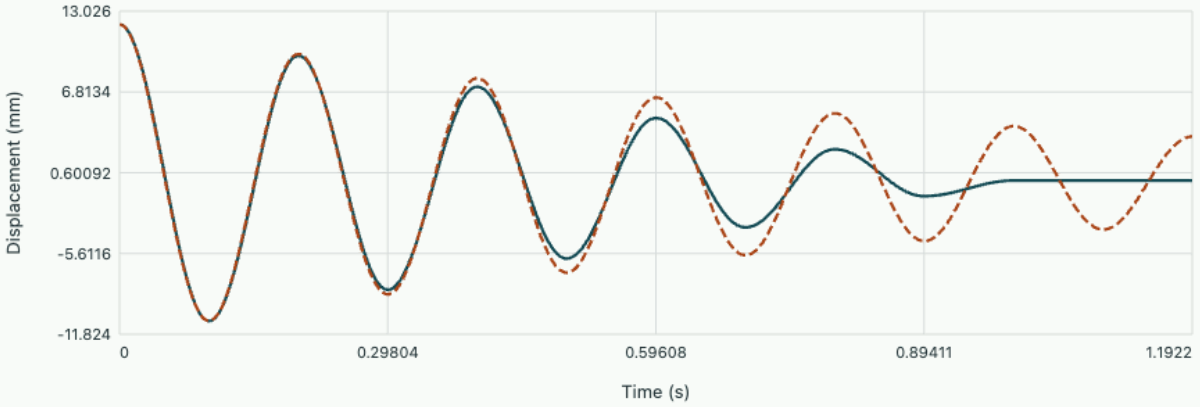
Interactive calculation · Dry vs viscous

Change the physical and numerical inputs of the notebook model. Compare the updated curves and balance checks with the recorded notebook figures. Each plot uses the same equations and assumptions documented in that notebook; its readout names any analytical or reduced-model reference.

Open this live calculation

Try the calculation

Release amplitude (mm)	**12**
Sliding friction (N)	**0.6**
Stiffness (N/m)	**1000**



Matched viscous damping ratio 0.0335. Dry friction stops after 10 half cycles at 0.000 mm; the viscous model approaches zero asymptotically. Mass1 kg.

Download this example's editable notebook

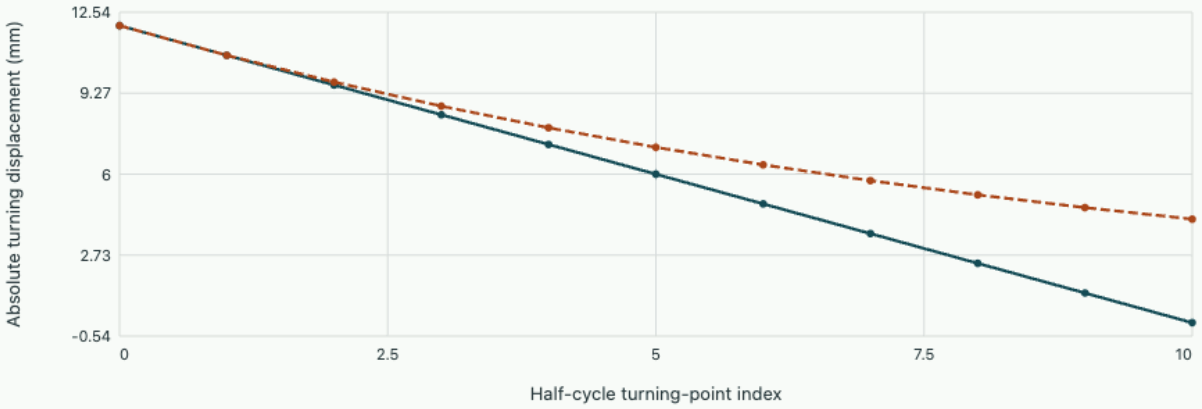
Interactive calculation · constant loss versus fractional loss

Compare turning points from the same piecewise sliding solution and matched viscous oscillator. Continue until dry friction reaches its sticking condition.

Open this live calculation

Try the calculation

Release amplitude (mm)	12
Sliding friction (N)	0.6
Stiffness (N/m)	1000



Coulomb, terminating in stick Matched viscous decay

Dry sliding loses 1.200 mm in turning amplitude per complete half cycle, until sticking at 0.000 mm. The viscous sequence loses a fixed fraction 0.100 instead. Its index counts its own slightly longer damped half cycles.

Download the editable notebook and compare both observables

Open editable notebook Download .ipynb

Vibration and identification in machinery · 6. From beam curvature to a mass matrix and a frequency measurement

A coarse cantilever model often predicts its first resonance accurately while putting the third far too high. Derive both matrices from the same displacement field and inspect how each mode converges.

Turn distributed kinetic and strain energy into coordinates

Take a uniform slender beam at rest, with transverse displacement $w(x,t)$ and small rotation. Neglect cross-section rotary inertia, shear compliance and prestress. An element of length h stores strain energy $\int EI(w'')^2 dx/2$ and kinetic energy $\int \rho A \dot{w}^2 dx/2$. Continuity of both displacement and slope at the interfaces prevents a spurious concentrated curvature. A cubic polynomial has four coefficients, exactly enough to interpolate the displacement and slope at each end.

Put $s = x/h$ and $q_e = (w_1, \theta_1, w_2, \theta_2)^T$. Apply the four endpoint conditions to a cubic, or verify them directly, to obtain $w = N q_e$ with

$$N = (1 - 3s^2 + 2s^3,\ h(s - 2s^2 + s^3),\ 3s^2 - 2s^3,\ h(-s^2 + s^3)).$$

Substitution into the energy integrals gives the element stiffness and consistent mass. The word consistent records that the same shape functions transport mass and displacement.

$$K_e = \int_0^h EI (N'')^T N'' \, dx,$$

$$M_e = \int_0^h \rho A N^T N \, dx.$$

For example, $N_1'' = (-6 + 12s)/h^2$, so $K_{11} = 12EI/h^3$. Integrating N_1^2 gives $M_{11} = 156\rho A h/420$. Mass couplings to rotations carry one additional length factor, and rotation–rotation terms carry two. These dimensions catch sign and assembly errors which a plausible first eigenvalue can hide.

Recover the vibration equation and its boundary conditions

Scatter-add each element into both global matrices at its shared nodal indices. A clamped root imposes $w_0 = \theta_0 = 0$. Delete those rows and columns from both matrices. The free end has no applied transverse force or moment, which the weak form already enforces as natural conditions. Lagrange's equations give $M\ddot{q} + Kq = 0$. A harmonic ansatz gives $K\phi = \omega^2 M \phi$. Eigenvectors from different frequencies are orthogonal in the mass inner product. Their overall signs remain arbitrary.

Take a one-metre steel cantilever, 30 mm wide and 10 mm deep, $E = 210$ GPa and $\rho = 7800$ kg/m³. The analytical clamped-free roots solve $\cosh\beta\cos\beta = -1$. Their frequencies scale as $\beta^2\sqrt{EI/(\rho A)}/(2\pi L^2)$. The test provides three independent references and a direct check on units and boundary conditions.

Interactive experiment · Rotate a beam mode and vary length, thickness and phase

The live plot evaluates the analytical clamped-free mode. The spatial view gives the beam its rectangular cross-section so that the modal centreline can be inspected from different directions. Move the phase slider through one cycle; the displayed frequency gives the physical time corresponding to that cycle.

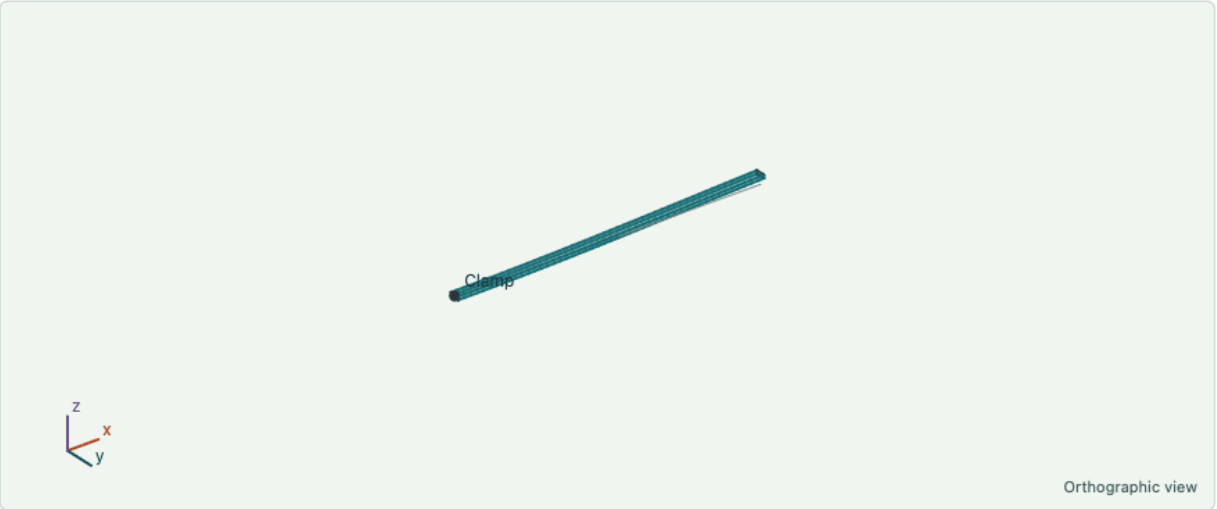
$$\phi(x) = \cosh(\beta x/L) - \cos(\beta x/L) - \frac{\cosh\beta + \cos\beta}{\sinh\beta + \sin\beta}\left[\sinh(\beta x/L) - \sin(\beta x/L)\right].$$

Scale this shape to the chosen tip amplitude and multiply it by the cosine of the phase. Double the length and check the factor of four in period; double the depth and check the factor of two in frequency. Modes two and three introduce internal nodes which a coarse mesh may miss. The interactive view uses the analytical slender-beam solution; the OneSimulation notebook below independently assembles and solves the discrete mass–stiffness eigenproblem. The selected amplitude illustrates a linear mode and is not a load-response prediction.

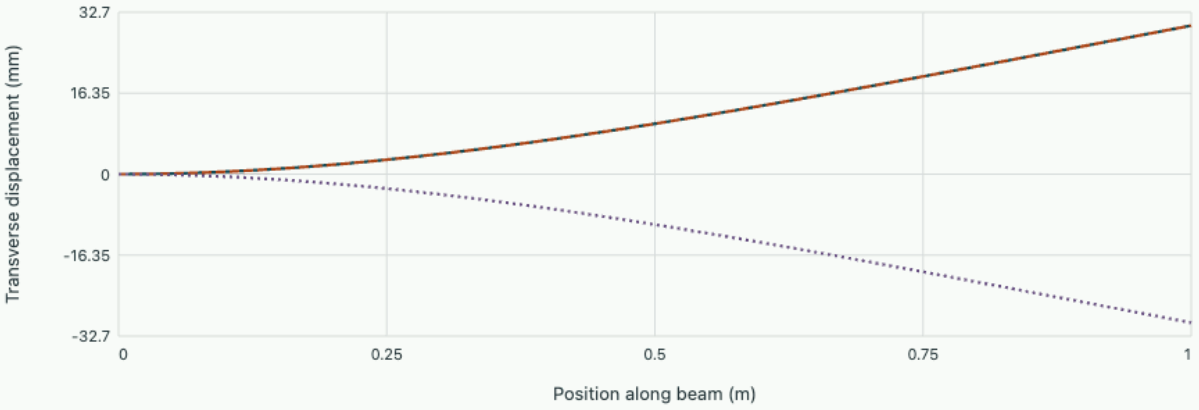
Open this experiment online

Try the calculation

Mode number	1
Beam length (m)	1
Beam depth (mm)	10
Tip amplitude (mm)	30
Oscillation phase (degrees)	0



A three-dimensional extrusion of the analytical beam mode. Width is 30 mm; depth and modal phase are adjustable. The section is drawn without the small rotation correction. Frequency and centreline follow the slender-beam model; the companion notebook checks the numerical eigenproblem. Axes: x = x (m); y = y (m); z = z (m).



Mode 1: 8.3819 Hz; period 119.305 ms. The analytical Euler–Bernoulli mode has a clamped root and a free tip. Shape amplitude is an illustration of the linear mode, not a predicted forced response.

Continue in the OneSimulation notebook · Download its Python model

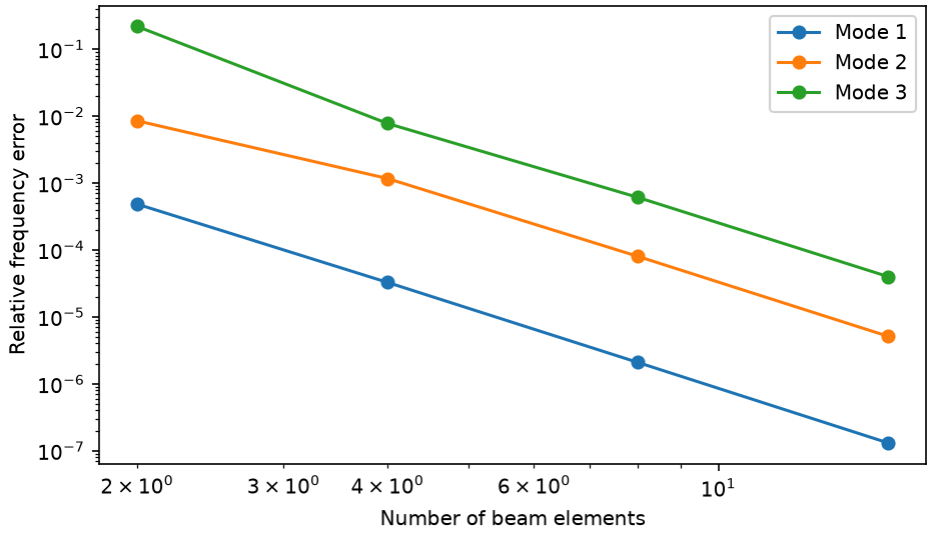
OneSimulation experiment · Convergence of the first three cantilever frequencies under mesh refinement

Run the notebook with OneSimulation 0.2.0, NumPy, SciPy and Matplotlib. Change the input data and compare your run with the recorded result.

```
"""Assemble a cantilever from Hermite elements; validate its first three modes."""
import numpy as np
import matplotlib.pyplot as plt
from onesim.numerics.linear import SparseSymmetricEigenSolver

length, width, depth = 1.0, 0.03, 0.01  # m
E, density = 210e9, 7800.0
area, inertia = width*depth, width*depth**3/12

def beam_frequencies(elements):
    h = length/elements
    k = E*inertia/h**3*np.array([[12,6*h,-12,6*h],[6*h,4*h*h,-6*h,2*h*h],
                                 [-12,-6*h,12,-6*h],[6*h,2*h*h,-6*h,4*h*h]])
    m = density*area*h/420*np.array([[156,22*h,54,-13*h],[22*h,4*h*h,13*h,-3*h*h],
                                     [54,13*h,156,-22*h],[-13*h,-3*h*h,-22*h,4*h*h]])
    size = 2*(elements+1)
    mass, stiffness = np.zeros((size,size)), np.zeros((size,size))
    for element in range(elements):
        dofs = np.arange(2*element,2*element+4)
        mass[np.ix_(dofs,dofs)] += m
        stiffness[np.ix_(dofs,dofs)] += k
    values, modes = SparseSymmetricEigenSolver()(mass[2:,2:], stiffness[2:,2:],
                                                 mode_count=3, target=0.0)
    assert np.allclose(modes.T @ mass[2:,2:] @ modes, np.eye(3), atol=1e-8)
    return np.sqrt(values)/(2*np.pi)

beta = np.array([1.875104069,4.694091133,7.854757438])
exact = beta**2*np.sqrt(E*inertia/(density*area))/(2*np.pi*length**2)
counts = [2, 4, 8, 16]
frequencies = np.array([beam_frequencies(count) for count in counts])
print("2 elements [Hz]", np.round(frequencies[0], 4))
print("16 elements [Hz]", np.round(frequencies[-1], 4))
print("analytical [Hz]", np.round(exact, 4))
```

Reference run

```
2 elements [Hz] [  8.386   52.9742 179.1685]
16 elements [Hz] [  8.3819  52.5288 147.0871]
analytical [Hz] [  8.3819  52.5285 147.0813]
```



Convergence of the first three cantilever frequencies under mesh refinement

Download editable notebook · Download Python script

Explain the disagreement between modes

With two elements the first mode is 8.3860 Hz against 8.3819 Hz analytically, an error of about 0.05%. The third mode is 179.1685 Hz against 147.0813 Hz, about 22% high. A conforming displacement space restricts admissible shapes and typically overestimates Ritz frequencies when integration is exact. Shorter spatial wavelengths require more independent shape degrees of freedom. The sixth significant digit of the first mode cannot certify the third.

Why attaching a sensor at a node changes the prediction

An accelerometer of mass m_s at coordinate x_s adds $m_s N(x_s)^T N(x_s)$ to the element mass. It shifts a mode strongly where that mode moves, while its first-order effect at a true modal node vanishes. A location that resolves mode one may be almost blind to mode two.

Exercises on model assembly

Suppress the root deflection but release its rotation. Which diagnostic should fail?

▼ **Worked solution**

A beam with one pinned end and a free end can rotate rigidly with $w = ax$. This mode has zero curvature and stiffness. The smallest eigenvalue should tend to zero. The remaining list must not be compared in order with cantilever roots.

If the beam thickness doubles while width and length stay fixed, predict the frequency change without remeshing.

▼ **Worked solution**

Inertia scales as thickness cubed and area as thickness, so $\sqrt{I/A}$ doubles. All Euler–Bernoulli frequencies double within the slender-beam regime. A thick-beam test can depart systematically owing to shear and rotary inertia.

Test whether a calculated eigenvector is numerically valid without a known exact frequency.

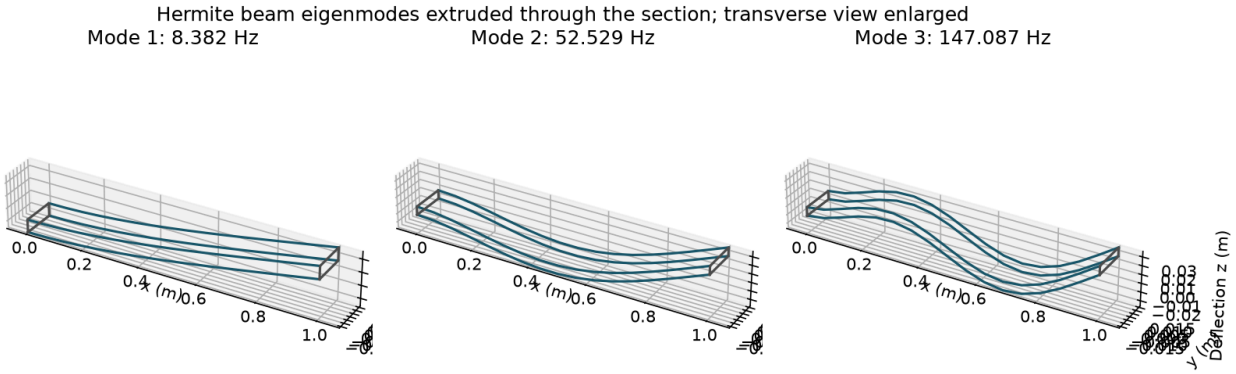
▼ **Worked solution**

Form $K\phi - \omega^2 M\phi$ and scale its norm by $\|K\phi\| + \omega^2\|M\phi\|$. A small residual certifies the discrete eigenpair. Only refinement or physical data test whether the discrete system is adequate.

Further reading · Rao, Mechanical Vibrations, 6th ed., Chapters 7, 8 and 12.

Spatial notebook output · inspect the same solved model in3D

The downloadable notebook now also draws the solved member forces or discrete beam eigenvectors in three dimensions. Its spatial figure below is produced by the actual OneSimulation run.



Recorded spatial output; the beam view enlarges transverse dimensions to make its mode shape legible.

Download the executed notebook with3D output

Vibration and identification in machinery · 7. Which approximation shifted the resonance?

A simulation of a pulsed cantilever contains three separate approximations. Measure the effect of spatial interpolation, retaining finitely many modes and representing a changing force on a time grid.

Project the driven system onto measured modes

Use mass-normalized modes $\Phi^T M \Phi = I$ from the preceding unit. Let a force history $p(t)$ act through a fixed load vector b and measure $y = c^T q$. If modal damping ratios ζ_j are an adequate damping model, projection with $q \simeq \Phi_r z$ gives independent second-order oscillators. Their measured frequency response is

$$\frac{Y(i\Omega)}{P(i\Omega)} = \sum_{j=1}^{r} \frac{(c^T \phi_j)(\phi_j^T b)}{\omega_j^2 - \Omega^2 + 2i\zeta_j \omega_j \Omega}.$$

A mode appears only if the force excites it and the instrument sees it. In a collocated force/displacement pair the numerator is a square. At zero frequency the omitted contributions are then nonnegative and sum to a missing compliance. Test this with a static solve $c^T K^{-1} b$. At a noncollocated sensor the residues can have either sign, especially near an antiresonance.

Treat excitation bandwidth as part of the experiment

A rectangular pulse of duration T has a transform proportional to $T\,\mathrm{sinc}(\Omega T/2)$ with $\mathrm{sinc}(u) = \sin(u)/u$. Its first zero is at $f = 1/T$, though higher-frequency lobes remain. A 5 ms pulse therefore has its main lobe up to 200 Hz, encompassing all three modes of the previous example. A first-mode fit can match the ring-down long after impact while missing the initial peak acceleration. Smoothing the pulse rise changes the high-frequency content of the physical loading.

Distinguish stability from phase fidelity

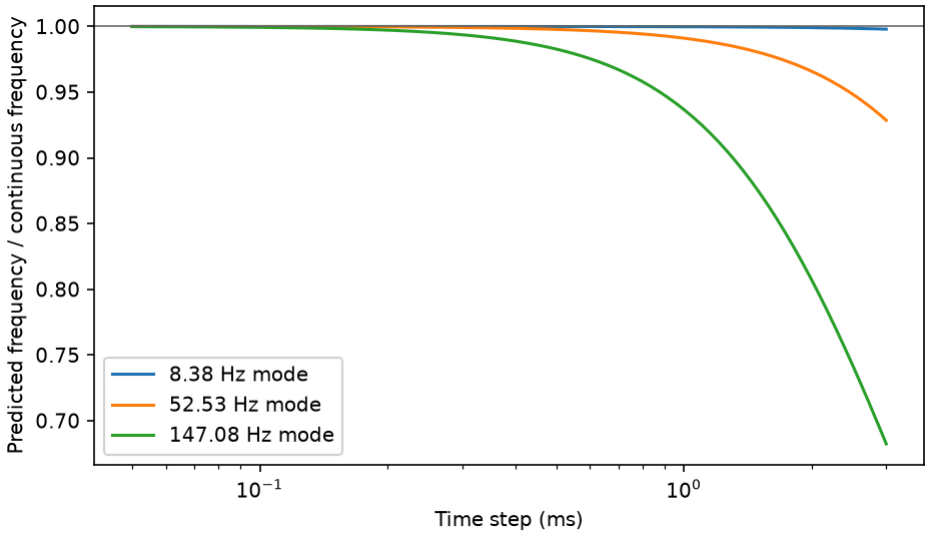
For an undamped mode $\ddot{z} + \omega^2 z = 0$, trapezoidal integration of the two-component first-order system maps its continuous eigenvalue $i\omega$ to the multiplier $(1 + i\omega h/2)/(1 - i\omega h/2)$. Its modulus is one for every step h, but the numerical frequency is

$$\widetilde{\omega} = \frac{2}{h}\arctan(\omega h/2) = \omega - \frac{\omega^3 h^2}{12} + O(h^4).$$

For mode three at 147.08 Hz and $h = 1$ ms, $\widetilde{f}$ is about 137.8 Hz. The signal can remain bounded indefinitely and still be badly out of phase. Halving the step brings it to about 144.5 Hz. At fixed h, refining the mesh alone cannot remove this phase lag. Conversely, integrating the two-element third frequency with an extremely fine time grid merely reproduces the wrong spatial model more faithfully.

How to choose a diagnostic sequence

First converge spatial eigenfrequencies using a modal solver. Next compare the static compliance and measured frequency response when another mode is retained. Finally halve the time step with fixed spatial and modal spaces and a fixed, explicitly interpolated force signal. This keeps the three modeling decisions identifiable.



Trapezoidal time stepping preserves the amplitude of each free mode but retards the phase, particularly for higher frequencies.

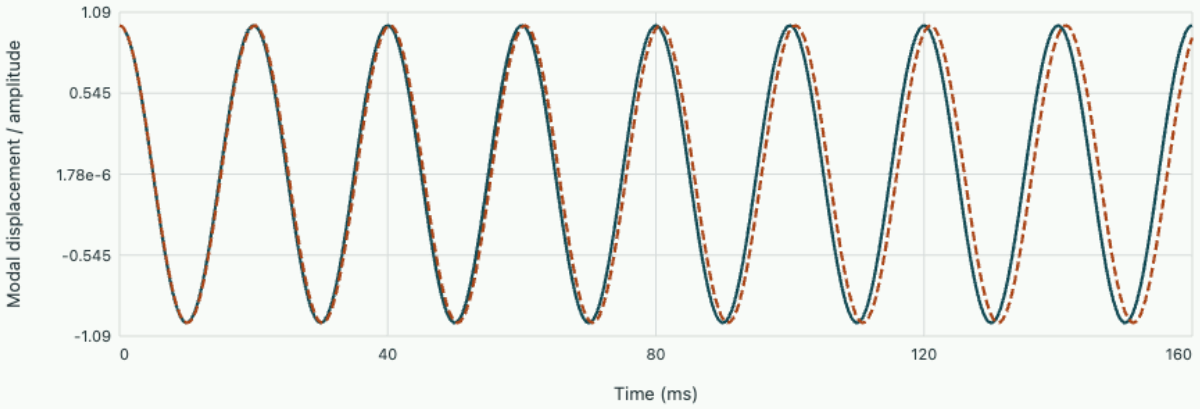
Interactive calculation · Resolve phase error over repeated oscillations

Increase time step at a fixed modal frequency, then compare a higher mode. Trapezoidal integration retains amplitude for this free undamped oscillator while slowing its phase.

Open this live calculation

Try the calculation

Mode frequency (Hz)	50
Time step (ms)	1
Observe for cycles	8



Numerical frequency 49.595 Hz versus 50 Hz. Relative frequency error -0.811%. The energy-conserving amplitude does not certify phase accuracy.

Exercises on model discrimination

Two modes have frequencies 10 and 40 Hz and equal collocated modal residues. How much of their combined static compliance disappears when the second is omitted?

▼ **Worked solution**

The ratio of second to first is $(10/40)^2 = 1/16$. The second contributes $1/17 = 5.88\%$ of the combined compliance, but it can dominate a narrowband force near 40 Hz.

A colleague matches the measured first frequency by adjusting Young's modulus. Can the two-element third-frequency error also be fixed this way?

▼ **Worked solution**

All frequencies scale with $\sqrt{E}$. Matching the first applies one common scale near one. The third-to-first ratio remains wrong. Use spatial refinement or revise the beam assumptions before interpreting a fitted modulus.

Further reading · Rao, Mechanical Vibrations, 6th ed., Chapters 6–8 and 11–12.

Coordinates, invariants and integration

Derive a coordinate description for a loaded mechanism, follow its energy through a cycle and assess numerical drift before assigning a measured decay to damping.

1. Constraints and Lagrange’s equations
2. Hamiltonian motion in phase space
3. Principal axes and torque-free precession
4. Nonlinear stiffness and amplitude-dependent period
5. Read a period from the area enclosed by an orbit
6. Parametric resonance and Floquet growth
7. Störmer–Verlet and a discrete invariant
8. Implicit midpoint, reversibility and the phase of an oscillator
9. Locating a grazing event between time steps
10. Virtual work and the forces hidden by a constraint
11. Euler–Poinsot motion, torque-free nutation and the intermediate axis
12. The heavy symmetric top, precession branches and the sleeping state
13. Canonical perturbations, small denominators and resonant capture

Constraints and Lagrange’s equations

Classical mechanics

Eliminate an unknown tension with a generalised coordinate and identify how a pulley’s rotational inertia changes an acceleration.

FROM FIRST PRINCIPLES

Derive the one-coordinate Lagrange equation from virtual work

For point masses with fixed, smooth ideal constraints, describe all positions as $\mathbf{r}_i(q)$. An allowed instantaneous virtual displacement is

$$\delta\mathbf{r}_i = (\partial_q \mathbf{r}_i)\delta q.$$

Assume constraint reactions do zero total virtual work. Newton's equations projected along this common displacement therefore eliminate the unmeasured reactions:

$$\sum_i m_i \mathbf{a}_i \cdot \partial_q \mathbf{r}_i = \sum_i \mathbf{F}_i^{\text{applied}} \cdot \partial_q \mathbf{r}_i = Q.$$

Use $K = \sum_i m_i \mathbf{v}_i^2/2$ and $\mathbf{v}_i = (\partial_q \mathbf{r}_i)\dot{q}$. Since $\partial_{\dot{q}} \mathbf{v}_i = \partial_q \mathbf{r}_i$, differentiate $\partial_{\dot{q}} K$ in time. The product term involving $d(\partial_q \mathbf{r}_i)/dt$ equals $\partial_q K$. Subtracting it leaves

$$\sum_i m_i \mathbf{a}_i \cdot \partial_q \mathbf{r}_i = \frac{d}{dt}\frac{\partial K}{\partial \dot{q}} - \frac{\partial K}{\partial q}.$$

If the applied forces derive from a potential $U(q)$ then $Q = -\partial_q U$. Moving the potential term gives the equation for $L = K - U$:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

The coordinate choice can eliminate a tension without assuming that tension vanishes. A rolling constraint requires the virtual displacements to obey its no-slip condition.

Background · Action

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$$

Compare nearby admissible paths with the specified endpoint data. Stationarity gives a local equation of motion. In SI the action has units of joule seconds.

DERIVATION AND APPLICATIONS

How does a moving pulley change the inertia seen by a load?

A string constraint fixes acceleration ratios before tensions are known. Consider a mass M carried by a light movable pulley. One end of a single inextensible string is fixed to the ceiling. It passes around the movable pulley, over a fixed pulley of radius R and moment of inertia I, and down to a free counterweight m. Use downward coordinate q for the movable assembly and downward coordinate y for the counterweight. The fixed pulley rotates freely about its axle. Assume a massless string, no slip, and constant lengths of the wrapped arcs.

The moving pulley changes the lengths of two vertical segments. Differentiate the length constraint to determine both the counterweight speed and the rim speed of the fixed pulley:

$$\begin{aligned} 2q + y &= \text{constant}, \\ \dot{y} &= -2\dot{q}, \\ |\omega| &= 2|\dot{q}|/R. \end{aligned}$$

Write all kinetic and potential energies in terms of the single coordinate. Downward displacement reduces gravitational potential, and raising the counterweight contributes with the opposite sign:

$$\begin{aligned} K &= \tfrac{1}{2}\big(M + 4m + 4I/R^2\big)\dot{q}^2, \\ U &= (-M + 2m)gq + \text{constant}. \end{aligned}$$

The Lagrange equation equates the rate of change of generalized momentum to the generalized force. The velocity ratios enter the inertia squared, whereas the displacement ratios enter gravitational work linearly:

$$(M + 4m + 4I/R^2)\ddot{q} = (M - 2m)g.$$

All terms multiplying acceleration have units of mass. An ideal fixed pulley with a heavy rim still slows the motion, even though its centre does not translate. At $M = 2m$ the gravitational driving force vanishes

for every value of I.

WORKED EXAMPLE

Predict a descending load and an ascending counterweight

Change the inputs and investigate this derivation.

Set $M = 1.20$ kg, $m = 0.400$ kg, $R = 0.0500$ m and $I = 2.50 \times 10^{-4}$ kg m². The rotational contribution is 0.400 kg and the effective mass is 3.20 kg. The driving force is $0.400g = 3.924$ N, giving $\ddot{q} = 1.23$ m/s² downward and counterweight acceleration 2.45 m/s² upward. A release from rest moves the load 0.153 m in 0.500 s if sufficient travel is available.

SECOND APPLICATION

Measure a pulley inertia from a reversed acceleration

Change the inputs and investigate this derivation.

With the same M and R, change the counterweight to $m = 0.800$ kg. A camera records upward load acceleration 0.8175 m/s². The signed numerator is now -3.924 N, so the measured effective mass is 4.80 kg. Subtracting $M + 4m = 4.40$ kg leaves $4I/R^2 = 0.400$ kg, recovering $I = 2.50 \times 10^{-4}$ kg m² to measurement precision. The reversal tests both the sign of gravitational work and the factor of two in the length constraint.

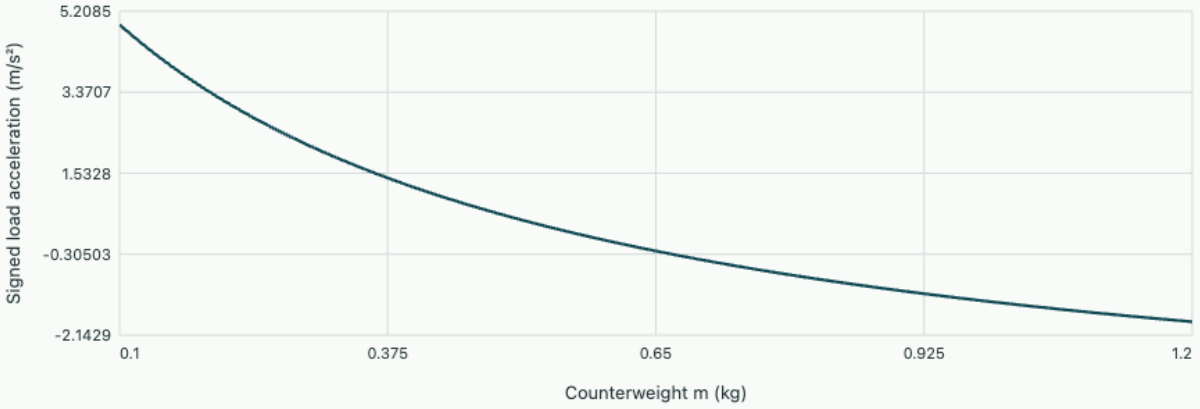
Try it · Pulley

Vary the physical parameters in the derived relation. Use the units and assumptions in the readout to connect the curves to the two worked examples. A parameter study can test the forward prediction; the same relation can then be inverted to infer a parameter from a measurement.

Open this live calculation

Try the calculation

Descending load M (kg)	**1.2**
Pulley rotational equivalent 4I/R² (kg)	**0.4**



Constrained pulley mechanism

The counterweight moves twice as far with opposite sign. Driving mass M−2m and effective inertia M+4m+4I/R² explain the reversal at m=M/2.

Which parts of a vertical loop need a two-sided constraint?

A bead of mass 100 g moves in a frictionless circular groove of radius R. It starts at the bottom with speed v_0. At angle θ from the downward vertical, energy conservation gives

$$v^2 = v_0^2 - 2gR(1 - \cos\theta).$$

The inward radial equation is

$$N - mg\cos\theta = mv^2/R.$$

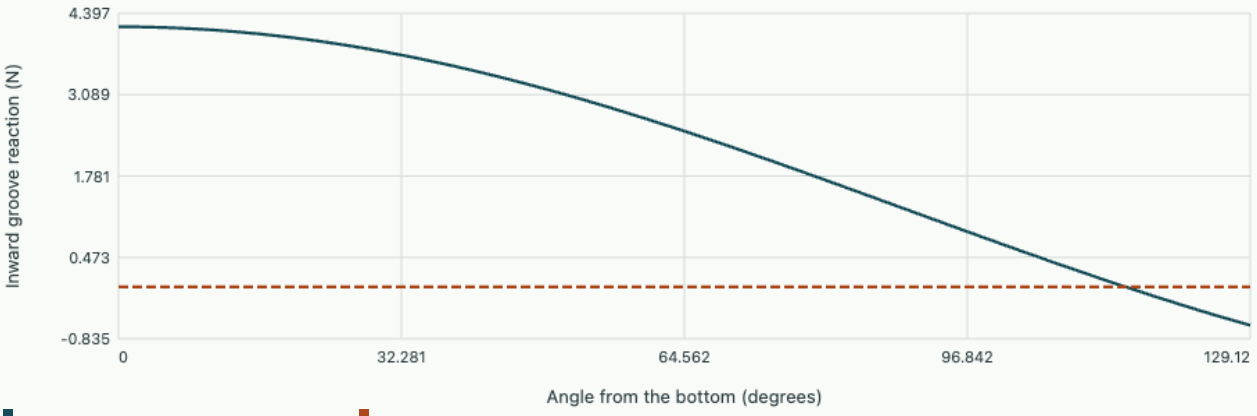
The sign of the groove force N is measurable. A two-sided groove can reverse this force; a bead held only against the inside of a smooth rail would lose contact where the required inward force first becomes negative.

Open this experiment online

Try the calculation

Bottom speed (m/s) **4**

Loop radius (m) **0.5**



Reachable angle 129.12°. Inward groove reaction there -0.619 N. One-sided loop threshold v_0 = 4.9523 m/s.

The graph ends at the first turning point if the energy cannot reach the top. Compare the speed needed just to reach the height of the top with the speed needed to maintain one-sided contact there. The former has $v_0^2 = 4gR$; setting $N = 0$ at the top gives $v_0^2 = 5gR$ for the latter. A conserved integral of motion and a constraint inequality answer different experimental questions.

Count independent motions

Two hanging masses and a massive pulley appear to introduce two translations, a rotation and several forces. Yet an inextensible taut string and rolling contact without slip leave just one independent motion. Let mass m_2 descend a distance q while mass m_1 rises by the same distance. For a fixed axle and pulley radius R, the rotation is $\theta = q/R$ relative to its initial value. Assume the masses stay clear of the pulley, the string is light, the axle friction is negligible and the pulley's moment of inertia about the axle is I.

A constraint relates permitted displacements before any force balance is written. The fact that both ends move the same distance follows from length conservation. It does not require equal tensions. A massive accelerating pulley needs a net torque, so replacing the tensions by a common value would remove its angular acceleration from the model. This example is a useful test of when a constraint allows us to remove coordinates and when it also allows us to remove forces.

Background · How can an ideal constraint transfer energy?

A bead slides without friction on a rail whose shape and position can change with time. An admissible virtual displacement compares nearby positions at the same time. It is tangent to the rail, so the normal reaction does no virtual work. The actual laboratory velocity also contains the motion of the rail itself. Write a point on the rail as $\mathbf{r}(q,t)$ and take its time derivative.

$$P_{\text{rail}} = \mathbf{N}\cdot\dot{\mathbf{r}} = \mathbf{N}\cdot\left(\frac{\partial \mathbf{r}}{\partial q}\dot{q} + \frac{\partial \mathbf{r}}{\partial t}\right) = \mathbf{N}\cdot\frac{\partial \mathbf{r}}{\partial t}.$$

A horizontal rail lifted vertically at constant speed u gives the simplest check. The bead's vertical acceleration is zero and the normal force is mg. The rail supplies power mgu, exactly the rate at which gravitational potential energy rises. The constraint can be ideal while an external actuator supplies the energy.

Construct the stationarity equation

For ideal time-independent constraints the reaction forces do no total work on allowed virtual displacements. Form the Lagrangian $\mathcal{L} = K - U$ using the independent coordinates. Consider two paths with the same initial and final values of q, differing by a small variation $\delta q(t)$. The first variation of the action is $\int (\partial_q \mathcal{L}\,\delta q + \partial_{\dot q}\mathcal{L}\,\delta\dot q)\,dt$. Integrate the second term by parts. The endpoint term vanishes and stationarity for arbitrary interior variations requires

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q} - \frac{\partial \mathcal{L}}{\partial q} = 0.$$

This equation supplies one independent balance for each independent coordinate. If an applied nonconservative force does work $\delta W = Q\,\delta q$, its generalised force Q enters the right-hand side. For an angular coordinate Q has units of torque; for this translational coordinate it has units of force. It must be calculated using the same sign choice as the coordinate.

Find a conserved quantity from a continuous symmetry

Constraints identify which coordinates can vary independently. A symmetry asks a different question: which small change of those coordinates leaves the action unchanged? Let $q_i \mapsto q_i + \varepsilon \xi_i(q,t)$ at fixed time and $p_i = \partial L/\partial \dot q_i$. Along a solution the Euler–Lagrange equations turn the variation into a total derivative,

$$\frac{\delta L}{\varepsilon} = \sum_i \left(\dot p_i \xi_i + p_i \dot \xi_i\right) = \frac{d}{dt}\sum_i p_i \xi_i.$$

If the left side vanishes for every path, $Q = \sum_i p_i \xi_i$ is conserved on solutions. More generally, a change $\delta L = \varepsilon\, dB/dt$ changes only endpoint terms in the action, and $Q = \sum_i p_i \xi_i - B$ is conserved. This is Noether's theorem for this fixed-time transformation. Separate conservation of mechanical energy requires time-translation symmetry.

Consider a bead of mass m on a smooth horizontal table. A light string attached to the bead passes through a central hole. Pulling the string prescribes the radius $r(t)$. Its tension has no azimuthal component. Rotation of the entire trajectory through a constant azimuth leaves $L = m(\dot r^2 + r^2\dot\phi^2)/2$ unchanged. Therefore $mr^2\dot\phi$ is constant even when a pulling device supplies work. For $m = 0.250$ kg, $r_i = 0.400$ m and $\dot\phi_i = 3.00$ rad/s, $Q = 0.120$ kg m²/s. Halving the radius raises angular speed to 12.0 rad/s. With zero radial speed at each endpoint, kinetic energy rises from 0.180 to 0.720 J. A central force can do work while exerting no torque about the centre.

Add every energy contribution

Both hanging masses move at speed $|\dot q|$. The pulley contributes rotational kinetic energy $I(\dot q/R)^2/2$. Raising the first mass increases its gravitational potential, while lowering the second decreases it. Omitting an arbitrary constant gives

$$U = (m_1 - m_2) g q.$$

The full kinetic energy is

$$K = (m_1 + m_2 + I/R^2)\dot q^2/2.$$

Substitution yields

$$(m_1 + m_2 + I/R^2)\ddot q = (m_2 - m_1) g.$$

The quantity I/R^2 enters like an additional translational mass. It stores kinetic energy without changing the gravitational driving force. A constant opposing axle torque of magnitude τ_f would give a generalised resisting force τ_f/R while the second mass descends. Applying a single constant sign across a reversal would incorrectly model dry friction.

Verify the reactions by restoring the constraints

Take $m_1 = 1.00$ kg, $m_2 = 1.50$ kg, $R = 0.100$ m and $I = 0.0050$ kg·m². The effective mass is 3.00 kg and the unbalanced weight is 4.905 N. Thus $\ddot q = 1.635$ m/s². From rest, a descent of 0.300 m produces a speed $\sqrt{2(1.635)(0.300)} = 0.990$ m/s. The drop in potential energy is 1.472 J, equal to the gain in translational and rotational energy together.

Now isolate each hanging body. Upward acceleration on the lighter side requires

$$T_1 = m_1(g + \ddot q) = 11.45\,\mathrm{N}.$$

The heavier side gives

$$T_2 = m_2(g - \ddot{q}) = 12.26\,\mathrm{N}.$$

The tension difference times radius is about 0.082 N·m. The pulley independently requires

$$I\ddot{q}/R = 0.0818\,\mathrm{N\,m}.$$

Agreement checks both the direction convention and the contribution of pulley inertia. The energy description solved for observable motion first. The separate body balances recovered forces that a mount or a string must actually withstand.

A useful check is to let one inertia become dominant. If the pulley's moment of inertia is made very large while its radius and both hanging masses stay fixed, the effective inertia in the denominator grows and acceleration approaches zero. The two string tensions then approach their respective hanging weights, which can remain unequal despite the tiny acceleration. The finite difference supplies torque to a very large rotational inertia. Equality of tension throughout a string that wraps around a massive pulley is therefore an additional physical assumption that must be justified.

Rolling without slipping and the force enforcing it

A constraint can remove an unknown contact force from the equation of motion while leaving a definite physical condition for its realization. Consider a cylinder of mass m, radius R and moment of inertia I about its centre, rolling straight down a plane inclined at angle α. Let x increase down the plane, and take the spin angle θ positive for forward rolling. A stationary contact point on the plane requires

$$\dot{x} = R\dot{\theta}.$$

In this straight-line geometry the velocity constraint integrates directly to $x - R\theta = \text{constant}$. General rolling on a surface with changing direction requires more care because the analogous constraints need not integrate to coordinate relations.

Use x as the sole coordinate. The centre falls through $x\sin\alpha$ and kinetic energy contains both translation and rotation. The reduced Lagrangian is

$$L = \tfrac{1}{2}(m + I/R^2)\dot{x}^2 + mgx\sin\alpha.$$

The Euler–Lagrange equation gives

$$\ddot{x} = g\sin\alpha/(1 + I/(mR^2)).$$

For a body released from rest, the speed follows by integrating $\dot{x}\,d\dot{x}/dx = \ddot{x}$. Different mass distributions at fixed radius and mass therefore arrive at different times even if their initial gravitational energies are equal. Some of that energy appears in spin, whose contribution grows with I.

To determine the unknown friction, return to the original translational and rotational balances. If f points up the plane,

$$m\ddot{x} = mg\sin\alpha - f,$$

$$I\ddot{\theta} = fR.$$

Combining with the rolling constraint gives

$$f = (I/R^2)\ddot{x}.$$

The absence of f from the reduced Lagrangian came from zero virtual work under an admissible rolling displacement. It did not imply zero force.

Decide whether a solid cylinder will actually roll

A uniform solid cylinder has $I = mR^2/2$. Let $m = 2.00$ kg, $R = 0.100$ m and $\alpha = 30.0^\circ$. Predicted acceleration is

$$9.81(0.5)/(1.5) = 3.27\,\mathrm{m/s^2}.$$

The required upward friction is $(m/2)\ddot{x} = 3.27$ N, while the normal force is

$$mg\cos 30^\circ = 17.0\,\mathrm{N}.$$

Static friction can maintain rolling only for $\mu_s \geq f/N = 0.192$. This comparison tests an assumption that could not be checked from the acceleration equation alone.

After travelling 1.00 m from rest a rolling cylinder reaches 2.56 m/s and spins at 25.6 rad/s. Translational energy is 6.54 J and rotational energy is 3.27 J, accounting for the 9.81 J released by gravity. In the limit $I/(mR^2) \to 0$, nearly all energy goes into translation, acceleration tends to $g\sin\alpha$ and the required friction tends to zero. A surface with a measured coefficient below the threshold instead permits slipping. The translation and rotation must then be solved independently using the sliding contact law.

Rolling tests can be run without knowing the cylinder’s mass distribution in advance. Obtain acceleration from two light gates at known positions and use $I/(mR^2) = g\sin\alpha/\ddot{x} - 1$ to infer a dimensionless inertia. Repeat at a shallower inclination. A genuine moment of inertia has the same inferred value in both runs, whereas slipping can appear only above a limiting angle. The most direct test tracks a painted mark on the rim as well as the centre. Admissible rolling requires the increase in centre displacement to equal radius times accumulated spin angle in each time interval.

Hamiltonian motion in phase space

Classical mechanics

Construct a Hamiltonian, read a phase portrait and distinguish bounded motion from rotation at a separatrix.

FROM FIRST PRINCIPLES

Construct Hamilton's equations by exchanging velocity for momentum

Begin with a smooth Lagrangian $L(q, \dot{q}, t)$ for a conservative mechanical system. Assume the momentum definition $p = \partial_{\dot{q}} L$ can locally be inverted to give $\dot{q}$ as a function of (q, p, t). This regularity is what lets position and momentum serve as independent state coordinates. Define the Hamiltonian and differentiate the product before substituting any equations of motion:

$$H = p\dot{q} - L,$$
$$dH = \dot{q}\,dp + p\,d\dot{q} - L_q\,dq - L_{\dot{q}}\,d\dot{q} - L_t\,dt.$$

The two $d\dot{q}$ terms cancel by the momentum definition. Comparing the coefficients of the independent differentials gives

$$H_p = \dot{q},$$
$$H_q = -L_q.$$

Hamilton's principle previously gave the Lagrange equation $\dot{p} = L_q$, so

$$\dot{q} = \partial_p H,$$
$$\dot{p} = -\partial_q H,$$
$$\partial_t H = -\partial_t L.$$

Along a solution the q and p contributions to dH/dt cancel. For a time-independent Hamiltonian H is constant, so planar trajectories lie on its level sets. In the elementary case $L = m\dot{q}^2/2 - U(q)$, the transformation yields

$$H = p^2/(2m) + U(q).$$

Background · Generalised momentum

$$p_i = \partial L / \partial \dot{q}_i$$

Its units depend on the coordinate. Conjugacy to an angle gives an angular momentum. Conservation requires the associated coordinate to be cyclic in the chosen Lagrangian.

DERIVATION AND APPLICATIONS

How Hamiltonian flow preserves phase-space area

An energy contour shows where a single trajectory may lie. To learn what the dynamics does to nearby initial conditions, follow a small patch of the canonical (q, p) plane. Consider one degree of freedom and a twice continuously differentiable Hamiltonian. Let the system advance by a short time δt. To first order the coordinate map is a displacement along the Hamiltonian velocity field:

$$q' = q + H_p\,\delta t,$$
$$p' = p - H_q\,\delta t.$$

The area of a sufficiently small parallelogram changes by the determinant of this map's derivative. Differentiate with respect to both initial coordinates. The product of off-diagonal elements enters only at second order:

$$\det \frac{\partial(q', p')}{\partial(q, p)} = (1 + H_{pq}\delta t)(1 - H_{qp}\delta t) + H_{pp}H_{qq}\delta t^2 = 1 + O(\delta t^2).$$

For the exact continuous map its area rate is the divergence of the phase velocity. Equality of the mixed second derivatives gives

$$\frac{1}{A}\frac{dA}{dt} = \partial_q \dot{q} + \partial_p \dot{p} = H_{pq} - H_{qp} = 0$$

for an infinitesimal material area A. Integrating adjacent patches gives the same conclusion for a finite region. The outline can stretch and rotate. Even explicit time dependence of H does not change this volume result. A friction force imposed independently of the Hamiltonian can change it. The second-order remainder above belongs to a short expansion; repeatedly taking finite steps with an approximate rule can accumulate a spurious change.

WORKED EXAMPLE

Track a cloud in a harmonic trap

Change the inputs and investigate this derivation.

Prepare a small rectangular set of initial states, 2.00 mm wide in position and 0.0100 kg m/s wide in momentum. Its area is 2.00×10^{-5} J s. In a trap of mass 0.500 kg and frequency $\omega = 4.00$ rad/s, the exact quarter-period map is $q' = p/(m\omega)$, $p' = -m\omega q$. The new widths are 5.00 mm and 0.00400 kg m/s. Their product retains the original area even though neither width is conserved.

SECOND APPLICATION

Diagnose a numerical phase portrait

Change the inputs and investigate this derivation.

An explicit Euler step in the same trap uses $q' = q + p\delta t/m$ and $p' = p - m\omega^2 q\delta t$. Its determinant is exactly $1 + \omega^2\delta t^2$. At $\delta t = 0.0500$ s it expands area by 1.04 per step, giving $1.04^{50} = 7.11$ times the initial area after 50 steps. A graph of the growing cloud could look like physical heating. The determinant exposes a discretization error without requiring an exact trajectory for each point.

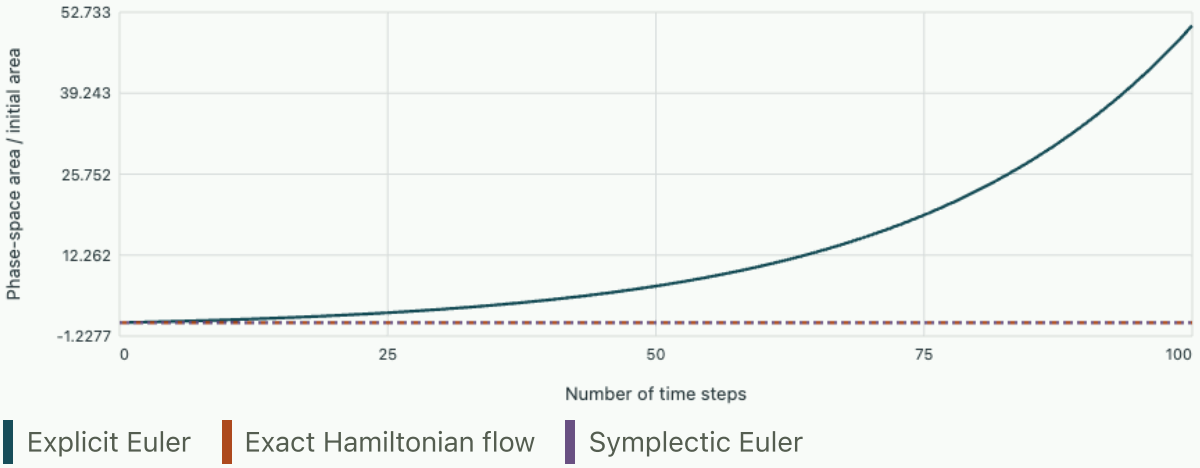
Try it · Phase area

Vary the physical parameters in the derived relation. Use the units and assumptions in the readout to connect the curves to the two worked examples. A parameter study can test the forward prediction; the same relation can then be inverted to infer a parameter from a measurement.

Open this live calculation

Try the calculation

Scaled step ωΔt **0.2**



Euler determinant 1+(ωΔt)² = 1.04. A symplectic scheme preserves area but still needs a step-convergence check for phase accuracy.

Position needs a conjugate momentum

The position of a pendulum at one instant does not say which way it will swing. A state needs both a configuration coordinate and a momentum. In Lagrangian mechanics define the canonical momentum by $p = \partial\mathcal{L}/\partial\dot{q}$. It can have different physical dimensions depending on the coordinate. For a particle on a line it has units of linear momentum. For a pendulum angle it has units of angular momentum. The pair (q, p) forms a two-dimensional phase space for one degree of freedom.

Provided the relation between p and $\dot{q}$ can be inverted, define $H(q, p, t) = p\dot{q} - \mathcal{L}(q, \dot{q}, t)$ with the velocity expressed in terms of q, p. Take a differential. The terms proportional to $d\dot{q}$ cancel by the definition of p. Using Lagrange's equation $\partial_q\mathcal{L} = \dot{p}$ gives

$$dH = \dot{q}\,dp - \dot{p}\,dq - \partial_t\mathcal{L}\,dt.$$

Matching the coefficients produces Hamilton's equations

$$\dot{q} = \partial_p H,$$
$$\dot{p} = -\partial_q H.$$

For a system with ordinary quadratic kinetic energy, a velocity-independent potential and fixed constraints, $H = K + U$. Its change along a solution is $dH/dt = \partial_t H$ because the two terms containing the derivatives with respect to q, p cancel. A time-independent Hamiltonian confines the trajectory to a level set.

Use Poisson brackets to test conservation

Hamilton's two equations specify how q_i and p_i change. To follow an observable $A(q, p, t)$, apply the chain rule and substitute $\dot{q}_i = \partial H/\partial p_i$ and $\dot{p}_i = -\partial H/\partial q_i$. This defines the Poisson bracket,

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, H\},$$
$$\{A, B\} = \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \right).$$

Since $\{q_i, p_j\} = \delta_{ij}$, a candidate change of phase-space coordinates must preserve these brackets to be canonical. A vanishing bracket with H certifies conservation for an observable having no explicit time dependence. It is a local calculation and does not require solving an orbit.

For a particle in a two-dimensional trap set $H = (p_x^2 + p_y^2)/(2m) + (k_x x^2 + k_y y^2)/2$ and $L_z = x p_y - y p_x$. The momentum derivatives in the bracket cancel the two kinetic contributions. The remaining coordinate terms give $\dot{L}_z = (k_x - k_y) x y$, exactly the torque $x F_y - y F_x$. With $k_x = 4.00$ N/m, $k_y = 6.00$ N/m and instantaneous position $(0.030, 0.040)$ m, angular momentum decreases at 2.40×10^{-3} N m. At a point on either axis the instantaneous derivative vanishes even for the anisotropic trap. For a conservation law holding on every trajectory the expression must vanish throughout phase space, as it does when $k_x = k_y$.

Draw energy contours before integrating

A rigid pendulum of mass m and length ℓ has $p = m\ell^2\dot{q}$ and $H = p^2/(2m\ell^2) + mg\ell(1 - \cos q)$. At the bottom $q = 0, p = 0$ is a stable equilibrium. Low-energy contours are ellipses to leading order because the cosine potential is quadratic there. Each closed contour returns to its original phase point after one full oscillation. Points of greatest $|p|$ are fastest and occur at the bottom.

The top $q = \pi, p = 0$ has energy $2mg\ell$. Its contour separates swinging trajectories from trajectories that keep rotating in one direction. This separating curve is the separatrix. Since angles differing by 2π label the same physical position, the full portrait lives on a cylinder. A rotation winds around that cylinder. As a trajectory approaches the separatrix, it spends a long time near the unstable top, so the small-angle period no longer estimates its return time accurately.

Compare thresholds and phase velocities

For $m = 0.50$ kg, $\ell = 0.80$ m and $g = 9.81$ m/s², the potential barrier is 7.85 J. A start from the bottom at 5.0 rad/s supplies kinetic energy $0.5(0.50)(0.80)^2(5.0)^2 = 4.00$ J. At the turning point $\cos q = 1 - 4.00/(0.50 \times 9.81 \times 0.80) = -0.0194$. The maximum angle is about 91.1°. Energy classifies this as a swing even though a small-angle expansion would be inaccurate at the turning point.

Continuous rotation requires bottom speed above $\sqrt{4g/\ell} = 7.00$ rad/s for this ideal pendulum. At precisely that value the system approaches the upper equilibrium asymptotically. This statement assumes a rigid link that can sustain compression. A pendulum made from a flexible string may lose tension before reaching the top, in which case the assumed circular constraint fails.

Check a numerical phase portrait

For any smooth one-degree Hamiltonian the divergence of the phase velocity is

$$\partial_q \dot{q} + \partial_p \dot{p} = H_{pq} - H_{qp} = 0.$$

A small patch of initial states can shear and stretch while preserving its phase-space area. Viscous damping would instead contract the patch. Numerical time integration can also produce apparent contraction or growth, so refining the step is essential before interpreting a spiral as evidence of friction.

For the harmonic oscillator, ordinary forward Euler gives

$$q' = q + \Delta t\, p/m,$$
$$p' = p - \Delta t\, kq.$$

The determinant of this linear update is $1 + \Delta t^2 k/m$, larger than one at every step. The spurious outward spiral follows from the numerical map alone. Updating momentum first and then using the new momentum for the position gives determinant one. That area property preserves the geometric structure, though it does not by itself guarantee accurate phase for a coarse step. Keep both geometry and timing in view when judging numerical trajectories.

The area inside a closed harmonic-oscillator contour provides another numerical test. Its semiaxes are $\sqrt{2E/k}$ in position and $\sqrt{2mE}$ in momentum. Their product times π gives $2\pi E/\omega$. Increasing oscillator energy therefore expands the area in direct proportion. A phase portrait plotted using velocity replaces every momentum value by its value divided by mass, changing numerical areas and their units. Specify the vertical coordinate before comparing this diagnostic between models with different masses.

A slowly changing spring transports a phase-space area

Suppose an oscillator's spring stiffness can be varied externally during the motion. Its Hamiltonian $H = p^2/(2m) + m\omega(t)^2 q^2/2$ now depends explicitly on time, so ordinary energy conservation does not apply. At an instantaneous point on a trajectory Hamilton's equations make the terms involving $\dot{p}$ and $\dot{q}$ cancel in dH/dt. The remaining change is

$$dE/dt = \partial H/\partial t = m\omega\dot{\omega}q^2.$$

Energy is supplied when the imposed parameter raises the potential at the current displacement.

Take $|\dot{\omega}| \ll \omega^2$ so that many cycles occur while the frequency changes by an appreciable fraction. Over one cycle ω and E may then be treated as nearly fixed. Because average potential and kinetic energies of a harmonic oscillator are equal,

$$\langle q^2 \rangle = E/(m\omega^2).$$

Inserting this mean into the work rate gives the averaged relation

$$\frac{dE}{dt} \simeq \frac{E}{\omega}\dot{\omega}, \qquad \frac{d}{dt}\left(\frac{E}{\omega}\right) \simeq 0.$$

The quantity $J = E/\omega$ is the area enclosed by the instantaneous constant-energy ellipse divided by 2π. Calling it an adiabatic invariant states both the approximation and the observable. A slowly changing system migrates between contours of different energies in a way that preserves this action to leading order. This mechanical meaning of adiabatic concerns a timescale ratio. The derivation does not assume heat exchange, dissipation or a thermodynamic temperature.

Compare a slow compression with a sudden change

A 0.500 kg oscillator has initial angular frequency 20.0 rad/s and amplitude 0.0400 m. Its energy is $m\omega_i^2 A_i^2/2 = 0.160$ J and initial action is 0.00800 J s. Raise the frequency smoothly to 40.0 rad/s over many periods. Action conservation predicts $E_f = 0.320$ J and final amplitude $\sqrt{2E_f/(m\omega_f^2)} = 0.0283$ m. The external control supplies work even as the maximum displacement decreases.

For comparison, change the stiffness instantaneously at a turning point, retaining the same initial and final frequencies. Position and momentum are continuous, but the potential energy at that position quadruples immediately. The new energy is 0.640 J and the amplitude remains 0.0400 m. Make the same switch at the zero crossing instead and the energy initially remains 0.160 J because the potential vanishes there. These two outcomes distinguish a rapid change tied to an oscillator phase from the nearly phase-independent slow ramp. Recording both position and velocity after the ramp allows energy, action and phase to be assessed separately.

A time-dependent Hamiltonian still carries each small phase-space area element into one of equal area, even during a sudden change of curvature. This exact conservation for an ensemble is distinct from slow conservation of the contour area assigned to a single orbit. A group of uniformly distributed initial phases illustrates the difference: immediately after a sudden spring switch all points occupy the same locations and enclose their original region, but their new energies depend on phase. The switch maps the original single-energy contour onto points that subsequently occupy many different energy contours of the new Hamiltonian.

Principal axes and torque-free precession

Classical mechanics

Use the inertia tensor to distinguish rotation about a principal axis from the precession of a freely rotating symmetric body.

FROM FIRST PRINCIPLES

Build the inertia tensor from the momenta of particles in a rigid body

Choose the centre of mass as origin and consider a rigid body of fixed mass distribution. Its instantaneous internal velocity field is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

Starting from angular momentum as the moment of linear momentum, integrate over small masses. The vector triple-product identity separates the part parallel to $\boldsymbol{\omega}$ from a direction-dependent subtraction:

$$\mathbf{L} = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, dm = \int [r^2 \boldsymbol{\omega} - (\mathbf{r} \cdot \boldsymbol{\omega})\mathbf{r}]\, dm.$$

Reading off coefficients of the components of $\boldsymbol{\omega}$ defines the symmetric inertia tensor. It is diagonal only in appropriately chosen principal axes:

$$L_i = \sum_j I_{ij} \omega_j,$$

$$I_{ij} = \int (r^2 \delta_{ij} - r_i r_j)\, dm.$$

Torque gives the laboratory time derivative of angular momentum. In a body-fixed principal basis the tensor components are constant but the basis rotates. Applying the rotating-vector derivative yields, for the first component,

$$\tau_1 = I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_2 \omega_3.$$

Cyclic permutation gives the others. Even zero torque can change the body-frame components of angular velocity. Alignment of angular velocity and angular momentum holds about a principal direction, or about every direction for spherical inertia.

DERIVATION AND APPLICATIONS

Find a pivot that gives a specified physical-pendulum period

An extended body hanging from a pivot rarely oscillates like a point at its centre of mass. Take a rigid body of mass M with centre of mass a distance d below a horizontal, frictionless axis. Let I_C be its moment of inertia about the parallel axis through the centre of mass. Measure angle θ from vertically downward. For each mass element, decompose its perpendicular displacement from the pivot into the centre-of-mass offset and a displacement relative to the centre. The cross term integrates to zero by definition of the centre:

$$I_O = \int r_\perp^2 \, dm = I_C + M d^2.$$

Gravity is equivalent to a force Mg at the centre, so its torque depends only on d. Angular momentum about the fixed pivot is $I_O \dot{\theta}$ along the rotation axis. The angular equation, followed by its small-angle limit, is

$$(I_C + M d^2)\ddot{\theta} = -M g d \sin\theta,$$
$$\omega^2 = \frac{M g d}{I_C + M d^2}.$$

Define the radius of gyration $k_C = \sqrt{I_C / M}$. A simple pendulum with the same period has effective length $\ell_{\text{eff}} = g/\omega^2$. Unlike the actual distance d, this length includes the cost of rotating the mass distribution about its centre:

$$T = 2\pi \sqrt{\frac{\ell_{\text{eff}}}{g}},$$
$$\ell_{\text{eff}} = d + \frac{k_C^2}{d}.$$

Moving the pivot toward the centre eventually increases the period because the torque vanishes faster than the inertia. Differentiating the effective length gives $d\ell_{\text{eff}}/dd = 1 - k_C^2/d^2$, so the shortest period occurs at $d = k_C$ if the apparatus admits that pivot.

WORKED EXAMPLE

Compare two pivot positions on a uniform bar

Change the inputs and investigate this derivation.

A thin 0.900 m uniform bar oscillates in a vertical plane. Integration of $x^2\,dm$ along its length gives

$$k_C^2 = L^2/12 = 0.0675\,\text{m}^2.$$

Pivoting at the end gives $d = 0.450$ m and $\ell_{\mathrm{eff}} = 0.600$ m, hence $T = 1.55$ s for $g = 9.81$ m/s². Pivoting 0.150 m from the centre also gives

$$0.150 + 0.0675/0.150 = 0.600\,\mathrm{m}.$$

The two periods agree even though the gravitational torque coefficients differ by a factor of three.

SECOND APPLICATION

Infer an unknown inertia from equal periods

Change the inputs and investigate this derivation.

Two distinct pivot distances $d_1 = 0.0800$ m and $d_2 = 0.180$ m give the same measured period for a 0.500 kg body. Equating the effective lengths and canceling $d_1 - d_2$ yields

$$k_C^2 = d_1 d_2 = 0.0144\,\mathrm{m}^2.$$

Thus $I_C = 0.00720$ kg m², and the common effective length is

$$d_1 + d_2 = 0.260\,\mathrm{m}.$$

A predicted period of 1.02 s checks the inference independently of the inertia measurement.

Try it · Physics lab reversible pendulum

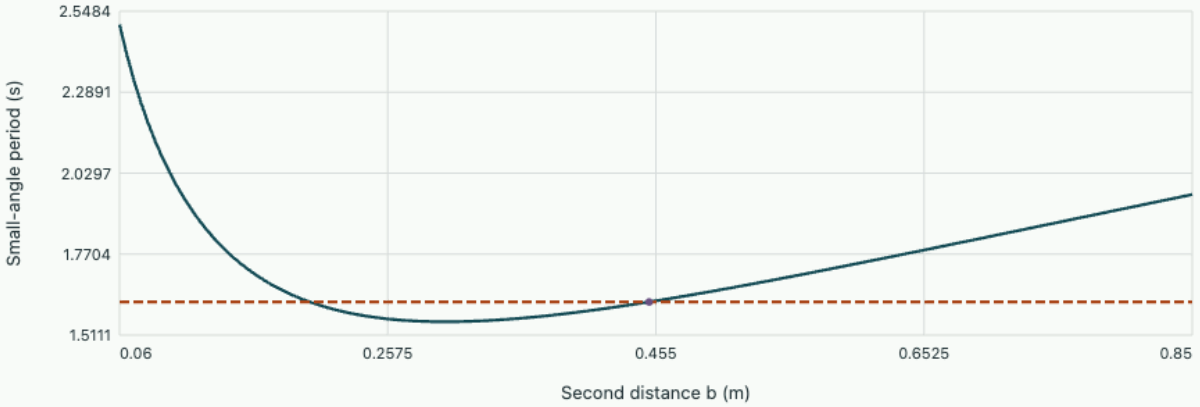
Vary the physical parameters in the derived relation. Use the units and assumptions in the readout to connect the curves to the two worked examples. A parameter study can test the forward prediction; the same relation can then be inverted to infer a parameter from a measurement.

Open this live calculation

Try the calculation

Radius of gyration (m) **0.3**

Second pivot distance (m) **0.45**



First period 1.6173 s; second 1.6173 s. The other matching distance is 0.45 m.

Distinguish loss of orientation from loss of angular momentum

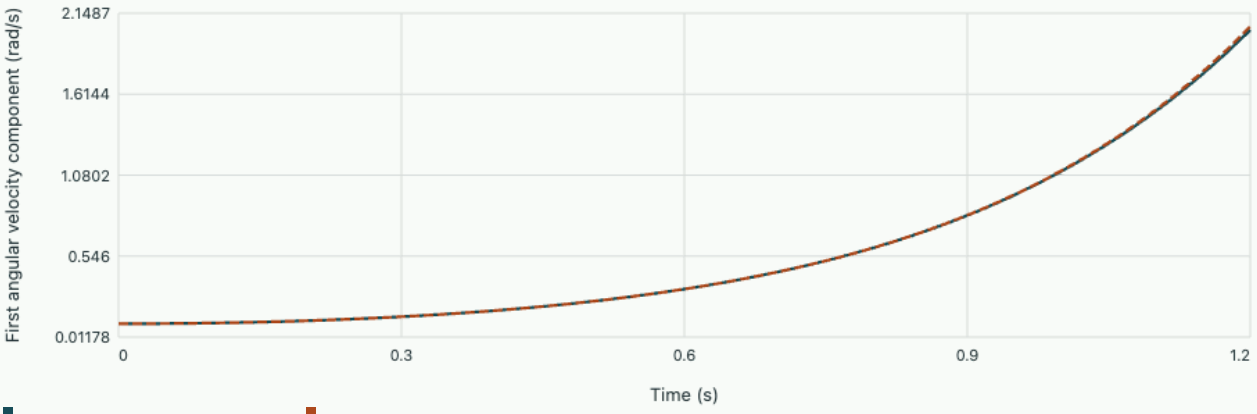
Release a body spinning almost about its intermediate principal axis, with moments $0.030,\ 0.040,\ 0.050\,\mathrm{kg\,m^2}$. The first transverse angular velocity component can grow while both energy and angular momentum magnitude stay constant. The experiment compares all three torque-free Euler equations with the growing and decaying modes of their linearisation. Vary the spin and the initial departure from the middle axis. The downloadable OneSimulation run includes both conserved quantities alongside the three components.

Open this experiment online

Try the calculation

Nominal middle-axis speed (rad/s) **12**

Initial transverse speed (rad/s) **0.1**



Initial linear growth rate 3.0984 s⁻¹. With zero initial transverse slope the predicted first doubling time is 0.42505 s.

Download the OneSimulation model

In your OneSimulation environment, run the model with these inputs.

```
python spectra-and-orbits.py rotor --spin 12 --seed 0.1
```

Use the worked instrument problem to derive the doubling time for two different preparations. A single image showing a complete flip is a much less precise rate measurement than the early time series.

Describe rotation around a moving centre

Take a rigid body with a freely moving centre of mass and no applied torque about that centre. Remove the uniform translation and label material positions $\mathbf{r}$ relative to the centre. Each material point has velocity $\boldsymbol{\omega} \times \mathbf{r}$. Integrating $\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, dm$ gives

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega},$$

where $I_{ij} = \int (r^2 \delta_{ij} - r_i r_j) dm$. The inertia tensor is symmetric, so a set of orthogonal body axes diagonalises it.

In its principal axes the rotational kinetic energy is $\sum_i I_i \omega_i^2 / 2$. This expression follows by integrating the kinetic energy of individual mass elements. A body containing mass far from an axis usually has large moment of inertia about that axis. Unless the components seeing different moments vanish, angular momentum and angular velocity point in different directions. A body can therefore rotate freely even when the angular velocity changes direction relative to painted marks on its surface.

Background · Where does the parallel-axis term come from?

Compare the kinetic energies for rotation about two parallel axes, one through the centre of mass and one displaced a perpendicular distance d. For a mass element write the perpendicular position about the second axis as $\boldsymbol{\rho} + \mathbf{d}$, where $\boldsymbol{\rho}$ is measured from the centre of mass. Expanding its squared distance produces a mixed term. The mass-weighted mean of $\boldsymbol{\rho}$ vanishes by the definition of the centre.

$$I_{\text{shifted}} = \int |\boldsymbol{\rho} + \mathbf{d}|^2 \, dm = I_{\text{CM}} + Md^2.$$

For a uniform thin rod of length L rotating about a transverse axis through an end, the shift adds $ML^2/4$ to $ML^2/12$. The result is $ML^2/3$. If instead the new axis tilts, each mass element has a different perpendicular projection. Then apply the inertia tensor to the new axis direction before making a parallel shift.

Apply the moving-frame derivative

The laboratory angular momentum is constant in the absence of torque. Its components in the body basis satisfy $(d\mathbf{L}/dt)_{\text{body}} + \boldsymbol{\omega} \times \mathbf{L} = 0$. For axial symmetry put $I_1 = I_2 = I_\perp$ and call I_3 the moment about the symmetry axis. Then $\dot{\omega}_3 = 0$,

$$\dot{\omega}_1 = -(I_3 - I_\perp)\omega_2\omega_3/I_\perp,$$
$$\dot{\omega}_2 = (I_3 - I_\perp)\omega_1\omega_3/I_\perp.$$

These last two equations describe uniform rotation of the pair of transverse components in body coordinates.

The signed angular rate of that rotation is $\Omega_b = (I_3 - I_\perp)\omega_3/I_\perp$. A body-mounted sensor sees its transverse spin vector execute a circle, even though a laboratory observer sees a constant angular-momentum vector. For a spherical inertia tensor the transverse components also become constant. The possible motion then simplifies because $\mathbf{L}$ is parallel to $\boldsymbol{\omega}$ for any spin direction.

Predict what two observers see

The symmetry-axis unit vector $\mathbf{e}_3$ has laboratory derivative $\dot{\mathbf{e}}_3 = \boldsymbol{\omega} \times \mathbf{e}_3$. Express the angular velocity as $\mathbf{L}/I_\perp$ plus a term parallel to $\mathbf{e}_3$, which drops out of the cross product. The result is

$$\dot{\mathbf{e}}_3 = (\mathbf{L}/I_\perp) \times \mathbf{e}_3.$$

Thus in the laboratory the symmetry axis sweeps around fixed $\mathbf{L}$ at rate $L/I_\perp$ and constant tilt. This rate differs from the spin-vector rotation observed in body coordinates because the two measurements track different vectors relative to different frames.

Let $I_\perp = 0.020$ kg·m², $I_3 = 0.030$ kg·m², $\omega_3 = 40$ rad/s and initial transverse angular speed 6.0 rad/s. The body-frame rate is 20 rad/s. The magnitude of momentum is $\sqrt{(0.020 \times 6)^2 + (0.030 \times 40)^2} = 1.206$ kg·m²/s. Laboratory precession is about 60.3 rad/s. The tilt obeys $\tan\theta = 0.120/1.20 = 0.10$, giving 5.71°. Two cameras recording these distinct angular rates need not disagree on the mechanics.

For three distinct moments the two transverse equations no longer form this simple uniform circle. The stability of individual principal-axis spins then becomes an additional question. Recording both energy and angular momentum helps discriminate conservative tumbling from external bearing drag.

One can also infer a moment of inertia without opening the rotor. Measure angular momentum by applying a calibrated torque for a known duration and integrating the torque impulse. During free motion, track the body axis with an external camera and find its precession frequency about the fixed angular momentum. For the symmetric rotor their ratio estimates the transverse moment I_1. A second measurement of the axial component of angular momentum and spin gives I_3.

Balance torque and angular momentum in a heavy top

Support a symmetric top at one fixed point on its symmetry axis, a distance ℓ below its centre of mass when upright. Unlike free rotation about the centre, gravity now exerts a torque about the pivot. Use principal moments about the pivot, $I_1 = I_2$ and I_3, and let θ be the tilt from upward vertical. In steady precession θ is constant, the azimuth increases at rate Ω, and the body's total angular-velocity component along its symmetry axis is $\omega_3 = \dot\psi + \Omega\cos\theta$.

The rotational kinetic energy in Euler angles is $I_1(\dot\theta^2 + \dot\phi^2\sin^2\theta)/2 + I_3(\dot\psi + \dot\phi\cos\theta)^2/2$. Insert $U = mg\ell\cos\theta$ into the θ equation, differentiating before imposing constant tilt. For $\sin\theta \neq 0$ this yields

$$I_1\cos\theta\,\Omega^2 - I_3\omega_3\Omega + mg\ell = 0.$$

The slower root for rapid spin tends to $mg\ell/(I_3\omega_3)$. This is the rate required for gravity to turn the nearly axial angular momentum. The full quadratic distinguishes that rate from a second, fast precession and gives the spin threshold for a tilted steady solution when $\cos\theta > 0$. Initial conditions outside such a solution produce a varying tilt, called nutation.

For $m = 0.200$ kg, $\ell = 0.0450$ m, $I_1 = 0.00180$ kg m², $I_3 = 0.000800$ kg m², $\omega_3 = 200$ rad/s and $\theta = 30^\circ$, the slow-root approximation is

$$0.0883/0.160 = 0.552\,\mathrm{rad/s}.$$

Substituting the quadratic coefficients gives 0.555 rad/s, a precession period of about 11.3 s. The other root is about 102 rad/s. A camera measuring only rotation of a mark painted on the body cannot generally separate axial spin, precession and nutation without also tracking the axis.

Stability of rotation about a principal axis

A symmetric rotor can precess steadily. A body with three different principal moments has a striking additional possibility. Suppose $I_1 < I_2 < I_3$ and no torque acts about the centre of mass. In body coordinates Euler's equations can be written $I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3$, with cyclic permutations for the other components. A pure rotation about any one principal axis is an exact solution. Stability asks what happens to a small transverse disturbance of that solution, a question about neighbouring trajectories rather than the exact rotation itself.

Consider nominal rotation about axis 2 at rate Ω. Write $\omega_1 = u$ and $\omega_3 = w$, both initially small. The change of ω_2 is a product of two small quantities and can be discarded to first order. The first and third equations give

$$\dot{u} = -(I_3 - I_2)\Omega w / I_1,$$
$$\dot{w} = -(I_2 - I_1)\Omega u / I_3.$$

Differentiating the first and substituting the second produces

$$\ddot{u} = \Omega^2 \frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}\, u.$$

A positive coefficient gives hyperbolic solutions with an exponentially growing component for a generic disturbance. Repeating the calculation about the smallest or the largest moment reverses the sign of the product, giving bounded oscillatory disturbances. Energy and angular momentum remain conserved throughout. A nearby angular-velocity direction can change substantially while moving along the intersection of the two invariant surfaces.

Estimate the onset of a visible tumble

Take moments $I_1 = 0.020$, $I_2 = 0.030$ and $I_3 = 0.040$ kg m² and a nominal intermediate-axis rotation $\Omega = 20.0$ rad/s. The small-perturbation growth rate is $20.0\sqrt{0.010 \times 0.010 / (0.020 \times 0.040)} = 7.07$ s⁻¹. Prepare an initial transverse component of 0.020 rad/s in the growing eigenmode. Reaching 0.200 rad/s then takes $\ln(10)/7.07 = 0.326$ s, still within the small-angle range. A perturbation prepared with zero initial derivative instead contains both growing and decaying contributions and follows a hyperbolic cosine initially.

These estimates predict the early departure, not the duration of a full flip. When transverse components approach the nominal spin, the nonlinear terms that preserved the dominant spin to first order can no longer be dropped. As two adjacent moments approach equality the growth rate tends to zero, recovering the absence of this isolated intermediate-axis instability in a symmetric rotor. A motion-capture experiment that records all three body components can test the exponential interval and independently check whether external torques or aerodynamic drag have changed the conserved angular momentum.

There is also a camera-independent prediction for the orientation of the early disturbance in body coordinates. A growing solution has $\dot{u} = \lambda u$. From the first linearized equation this requires

$$w/u = -I_1\lambda/[(I_3 - I_2)\Omega].$$

In the numerical example that ratio is -0.707: the small components point in opposite signed directions. A disturbance with the reverse sign projects partly onto the decaying solution. The eventual growth rate can be the same while the visibility of the first few cycles differs greatly. Recording both components avoids assigning a disagreement in initial conditions to the underlying instability rate.

Nonlinear stiffness and amplitude-dependent period

Classical mechanics

Derive the leading frequency shift of a symmetric nonlinear oscillator and test whether a measured amplitude shift is consistent with a cubic spring.

FROM FIRST PRINCIPLES

Derive the first amplitude correction to a spring's frequency

A symmetric elastic restoring force has no quadratic displacement term; the first correction to a linear spring can therefore be cubic. Suppose a conservative oscillator satisfies $m\ddot{x} + kx + \alpha x^3 = 0$ with $k > 0$ and $|\alpha A^2/k| \ll 1$ at measured peak amplitude A. Leading motion is approximately $A\cos\Omega t$. Cube this motion to find which frequencies enter the perturbing force:

$$x^3 \simeq A^3[\tfrac{3}{4}\cos\Omega t + \tfrac{1}{4}\cos 3\Omega t].$$

To maintain a bounded periodic solution, its fundamental component must balance at first order. Otherwise the remaining fundamental force would resonantly grow the correction. Combine the inertia and restoring-force coefficients at $\cos\Omega t$:

$$-m\Omega^2 A + kA + \tfrac{3}{4}\alpha A^3 \simeq 0, \qquad \Omega \simeq \sqrt{k/m}\left(1 + \frac{3\alpha A^2}{8k}\right).$$

The third-harmonic component also distorts the waveform, but reinserting that order-α distortion into the cubic force affects the fundamental only at the next perturbative order. Positive α raises frequency with amplitude; negative α lowers it within the local range where the stable weakly nonlinear model applies.

DERIVATION AND APPLICATIONS

Find the third harmonic generated by a weak cubic spring

A change in oscillation period is one signature of a nonlinear restoring force. The same nonlinearity also changes the shape of the recorded motion. Let an undamped mass obey $m\ddot{x} + kx + \alpha x^3 = 0$, with $k > 0$. Release it from rest at the measured peak displacement A. Define $\omega_0 = \sqrt{k/m}$, the small signed parameter $\epsilon = \alpha A^2/k$, and a stretched time $\tau = \Omega t$. Write $x/A = \cos\tau + \epsilon u(\tau)$ with $u(0) = u'(0) = 0$ so that A remains the physical turning point.

Allow the frequency to shift as $\Omega = \omega_0(1 + \epsilon\nu)$. Substitute the expansion into the equation and retain terms only to first order in ϵ. Primes below denote differentiation with respect to τ:

$$u'' + u = 2\nu\cos\tau - \cos^3\tau = (2\nu - 3/4)\cos\tau - \tfrac{1}{4}\cos 3\tau.$$

A term driven at the unit frequency would grow proportionally to $\tau \sin \tau$, contradicting a bounded, periodic correction. Choose $\nu = 3/8$ to remove it. The remaining forcing is at three times the fundamental. A trial $C \cos 3\tau$ has $u'' + u = -8C \cos 3\tau$. Add the required homogeneous term to satisfy the turning-point conditions:

$$u(\tau) = \frac{\cos 3\tau - \cos \tau}{32}.$$

The predicted displacement therefore contains two lines with linked amplitudes and a shifted fundamental:

$$x(t) \simeq A[(1 - \epsilon/32)\cos \Omega t + (\epsilon/32)\cos 3\Omega t], \qquad \Omega/\omega_0 \simeq 1 + 3\epsilon/8.$$

The coefficient sign gives the third harmonic relative to the phase of a release at positive displacement. Squaring an amplitude to obtain a power spectrum erases that sign. The approximation assumes $|\epsilon| \ll 1$ and excludes appreciable dissipation during the observation.

WORKED EXAMPLE

Estimate a micrometre-scale distortion

Change the inputs and investigate this derivation.

Take $A = 12.0$ mm, $k = 20.0$ N/m and $\alpha = 1.00 \times 10^{4}$ N/m³. Then $\epsilon = 0.0720$, so the third-harmonic displacement amplitude is

$$12.0(0.0720)/32 = 0.0270\,\text{mm}.$$

The fundamental frequency should increase by 2.70%. With $\omega_0 = 10.0$ rad/s, the third-harmonic line lies close to 30.8 rad/s.

SECOND APPLICATION

Test whether a measured line comes from the spring

Change the inputs and investigate this derivation.

At $A = 6.00$ mm on the same apparatus, ϵ falls by four and the third-harmonic amplitude falls by eight to 3.38 micrometres. The fractional frequency shift becomes 0.675%. A third-harmonic signal that instead halves with amplitude could arise from an already distorted input or readout. The two amplitude scalings plus an independent frequency measurement test the cubic model more strongly than observing a line at three times the frequency.

Try it · Duffing harmonic

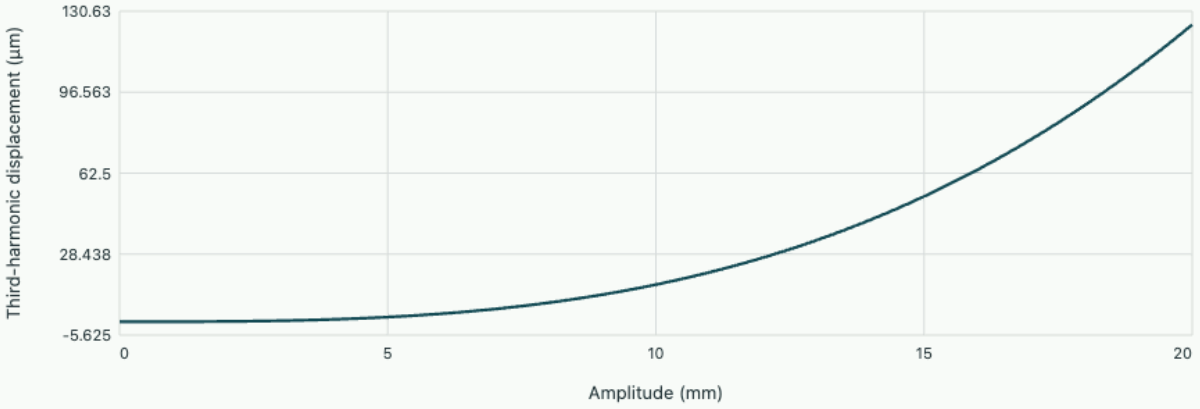
Vary the physical parameters in the derived relation. Use the units and assumptions in the readout to connect the curves to the two worked examples. A parameter study can test the forward prediction; the same relation can then be inverted to infer a parameter from a measurement.

Open this live calculation

Try the calculation

Amplitude (mm) **12**

Cubic stiffness (N/m³) **10000**



First perturbative harmonic

Linear stiffness 20 N/m. At chosen amplitude ε=0.072, third harmonic 27 µm; first frequency shift 2.7%. Requires ε≪1.

Start from a reversible force law

A spring that looks linear under small deflection can stiffen as it is displaced further. For an oscillator symmetric under $x \mapsto -x$, the restoring force expanded about equilibrium has no even powers. Keep the leading correction and write $m\ddot{x} + kx + \alpha x^3 = 0$, with $k > 0$ and α measured in N/m³. Assume the displacement is small enough that neglected fifth-order terms remain unimportant. Positive α gives a hardening spring. A negative cubic coefficient describes local softening and may need higher-order terms to represent large displacements safely within the model.

The potential is $kx^2/2 + \alpha x^4/4$. Without damping the total energy is conserved and a release at amplitude A fixes that energy. Unlike a harmonic oscillator, the quadratic curvature at the origin no longer fixes the period for every possible energy. We want a quantitative connection between the amplitude of the motion and its zero-crossing interval.

Obtain the first frequency correction

Try $x \simeq A\cos\Omega t$ and substitute it into the cubic equation. The identity $\cos^3 u = (3\cos u + \cos 3u)/4$ shows that the nonlinearity introduces a component at the original frequency and a third harmonic. Balance the fundamental coefficient to get $-m\Omega^2 + k + 3\alpha A^2/4 \simeq 0$. Expanding the square root gives

$$\Omega \simeq \omega_0 \left(1 + \frac{3\alpha A^2}{8k}\right),$$

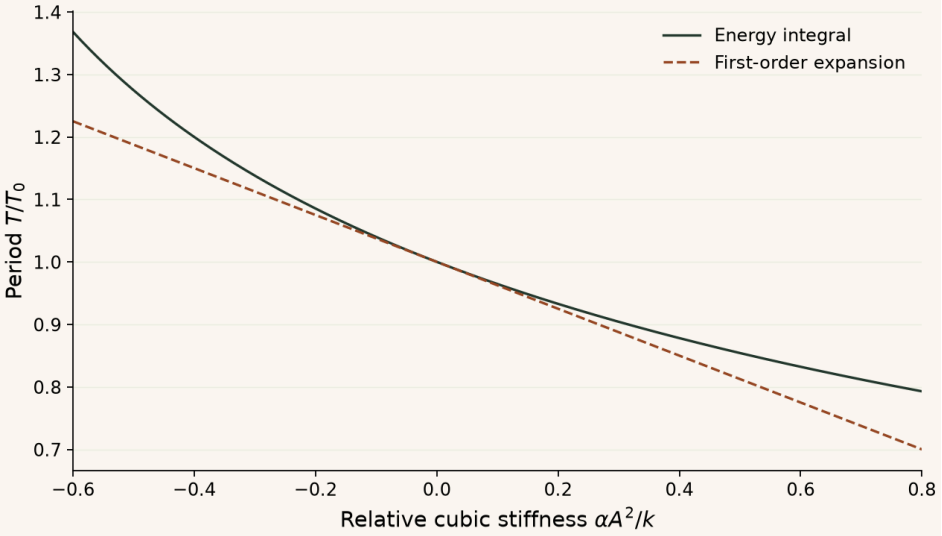
$$\omega_0 = \sqrt{k/m}.$$

The control parameter is $|\alpha| A^2/k$, the ratio of cubic to linear force at maximum displacement. The small additional third harmonic changes the waveform at first order but only feeds back into this leading frequency result at higher order. Fitting a perfect sinusoid to a high-precision trace may therefore show a residual even when the measured average period follows this formula.

Check the result using the conserved energy

The quarter-period is the time to move from equilibrium to A. Substitution of the energy at the turning point gives an integral over $dx/\sqrt{2[E - U(x)]/m}$. Let $x = A\sin u$ to remove the square-root endpoint. The remaining integrand is proportional to $\{1 + [\alpha A^2/(2k)](1 + \sin^2 u)\}^{-1/2}$. Expanding and averaging over the interval from zero to $\pi/2$ uses mean $\sin^2 u = 1/2$, yielding $T \simeq (2\pi/\omega_0)[1 - 3\alpha A^2/(8k)]$. This agrees with the reciprocal of the harmonic-balance result.

The integral approach generalises to experimentally tabulated conservative forces by integrating them first to obtain a potential. It would need modification for strong damping, where energy varies significantly between the two turning points of one cycle.



A positive cubic stiffness shortens the period. The first-order expansion is tangent to the energy-integral curve at zero and gradually loses accuracy away from that limit.

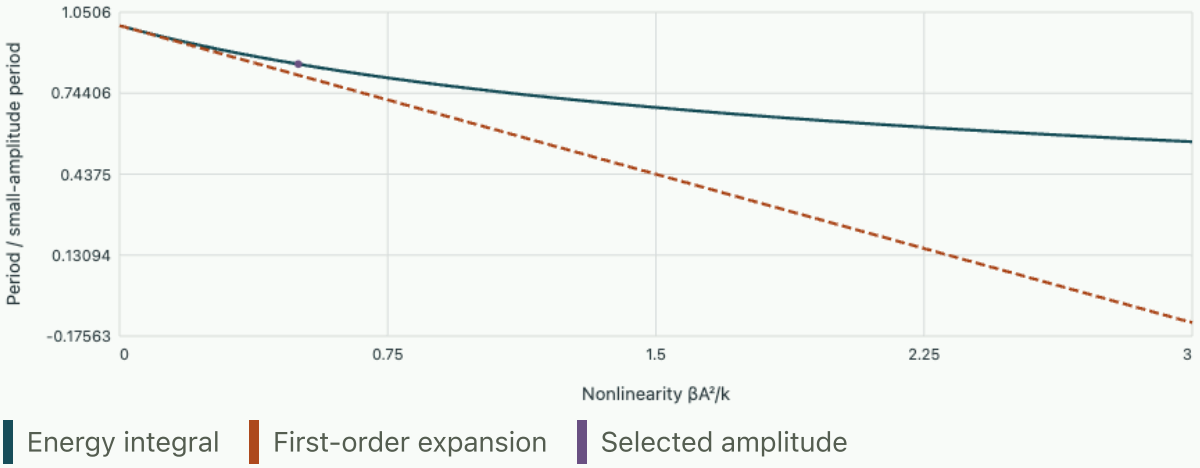
Try it · Duffing

Change the inputs and compare the recalculated curves with the preceding derivation and baseline figure. The axes give the units; the numerical readout states the model assumptions and any limiting cases. Reset the inputs to return to the reference setting.

Open this live calculation

Try the calculation

Cubic stiffness × amplitude² / linear stiffness **0.5**



Exact period ratio 0.85413. The expansion is local to βA²/k ≪ 1; it becomes negative if extrapolated far enough.

Infer stiffness from period data

For $m = 0.50$ kg, $k = 200$ N/m and $\alpha = 4.0 \times 10^4$ N/m³, $\omega_0 = 20.0$ rad/s. At $A = 0.020$ m the force ratio is 0.080, giving a leading frequency increase of 3.0% to 20.6 rad/s. At half this amplitude the predicted increase is 0.75%. Measure the period over several cycles at each amplitude, keeping the amplitude change due to loss small. A plot of frequency against the square of amplitude should be approximately straight near the origin. A nonzero intercept shift under a change of temperature could indicate a change in ordinary linear stiffness instead of a new nonlinear coefficient.

Compute the entire amplitude dependence of a pendulum

A cubic expansion is useful near equilibrium. For a pendulum it is possible to carry the energy calculation beyond every finite Taylor approximation. Take a point mass with fixed length ℓ, no friction and a release from rest at angle $0 < \theta_0 < \pi$. Its conserved energy gives

$$\dot\theta^2 = (2g/\ell)(\cos\theta - \cos\theta_0).$$

Use $\cos\theta = 1 - 2\sin^2(\theta/2)$. Defining $k = \sin(\theta_0/2)$ reduces the speed magnitude to $2\sqrt{g/\ell}\sqrt{k^2 - \sin^2(\theta/2)}$.

One quarter period is the time to travel between zero angle and the turning point. The reciprocal speed appears to diverge at the latter, but it has an integrable square-root singularity for every ordinary turning angle. To evaluate it stably, set $\sin(\theta/2) = k\sin u$. The factor $\cos u$ from differentiating cancels the same factor inside the square root. Four equivalent quarters give

$$T = 4\sqrt{\ell/g}\int_0^{\pi/2} \frac{du}{\sqrt{1 - k^2\sin^2 u}} = 4\sqrt{\ell/g}\,K(k).$$

Here K denotes the complete elliptic integral with modulus k. Expanding the integrand at small k and integrating the first correction gives

$$T/T_0 = 1 + k^2/4 + \cdots = 1 + \theta_0^2/16 + \cdots,$$

where $T_0 = 2\pi\sqrt{\ell/g}$. The positive correction is the familiar softening: increasing amplitude lengthens the period because the restoring torque grows more slowly than its linear extrapolation.

Decide when a small-angle clock loses accuracy

A pendulum of length 0.800 m has $T_0 = 1.794$ s at $g = 9.81$ m/s². At a release of 60.0°,

$$k = \sin 30^\circ = 0.500.$$

Evaluating the integral gives $K(0.500) = 1.68575$ and $T/T_0 = 2K/\pi = 1.0732$, hence $T = 1.926$ s. The weak-angle approximation gives

$$1 + (\pi/3)^2/16 = 1.0685.$$

It misses the exact period by roughly 0.008 s per cycle; a many-cycle timing record resolves the accumulated discrepancy.

As θ_0 tends to π, the integrand near $u = \pi/2$ approaches the reciprocal of a vanishing cosine. The integral diverges logarithmically, expressing the long time spent close to the unstable upper equilibrium. No finite polynomial in amplitude can reproduce that divergence. A numerical timing experiment should therefore locate regular zero crossings rather than demand a turning-point velocity of exactly zero in sampled data. Comparing the crossing intervals with the regularized integral distinguishes numerical resolution error from the genuine increase in physical period.

A way to evaluate the elliptic integral without a special-function library is to form the arithmetic–geometric mean. Begin with $a_0 = 1$ and $b_0 = \sqrt{1 - k^2}$, then replace the pair by their arithmetic and

geometric means. Both converge rapidly to a common M, and $K(k) = \pi/(2M)$. For the 60° release the first pair is 1 and 0.86603, the next pair is approximately 0.93301 and 0.93060, and the subsequent pair agrees near 0.93181. Taking the reciprocal of the common value directly gives the period ratio. This procedure supplies an accurate independent benchmark even when the small-amplitude expansion ceases to converge quickly.

Numerical quadrature and the arithmetic–geometric mean should agree at a fixed amplitude. If a period extracted from actual trajectories differs while these two evaluations agree, examine the dynamical time step, how release angle is measured, and whether the observed oscillations have suffered appreciable damping.

Read a period from the area enclosed by an orbit

Classical mechanics

A track with a V-shaped potential has straight force segments but a strongly amplitude-dependent period. Its area in phase space gives a useful way to predict both the period and a slow change of the track.

Construct the closed area

For one bound degree of freedom at energy E, position runs between two turning points. In a mechanical Hamiltonian $p^2/(2m) + U(q)$, the upper branch of the phase portrait has positive momentum and the lower branch has its negative. The directed integral around one full orbit counts the enclosed area. Define the action J using the area divided by 2π. This choice makes its conjugate phase an angle measured in radians:

$$J(E) = \frac{1}{2\pi} \oint p\, dq = \frac{1}{\pi} \int_{q_-}^{q_+} \sqrt{2m(E - U(q))}\, dq.$$

Now change the energy slightly. The integrand vanishes at each regular endpoint, so the moving-endpoint contributions to the derivative vanish. In the interior $\partial p/\partial E = m/p = 1/v$. The derivative of the upper-branch area therefore measures the time to reach the opposite turning point.

$$\frac{dJ}{dE} = \frac{T(E)}{2\pi},$$

$$\omega(E) = \frac{dE}{dJ}.$$

This argument also explains why the construction becomes delicate near a separatrix. A trajectory can spend an arbitrarily long time near an unstable equilibrium, and T can diverge even though the enclosed area stays finite.

Background · Action variable

$$J = (2\pi)^{-1} \oint p\, dq$$

This area measure is defined for a closed cycle. It can change strongly near a separatrix even under a slow parameter variation.

Carry out the integral for a constant restoring force

Use $U(q) = F|q|$ with $F > 0$. The turning amplitude is $A = E/F$. Integrate the positive branch from 0 to A and use reflection symmetry. Substitution $s = E - Fq$ gives

$$J = \frac{4\sqrt{2m}}{3\pi F} E^{3/2},$$

$$T = \frac{4\sqrt{2mE}}{F} = 4\sqrt{\frac{2mA}{F}}.$$

The answer has an independent mechanical check. From a turning point the bead accelerates uniformly with magnitude F/m until the centre. Covering a distance A from rest takes $\sqrt{2mA/F}$, a quarter period. Doubling the amplitude increases the full period by $\sqrt{2}$. The abrupt change of force direction at the centre leaves the velocity continuous.

Slowly steepen the track

A slowly adjusted parameter can preserve the orbit action to leading order when the motion remains periodic and many cycles fit inside the adjustment time. For this trap m stays constant, so constant J requires $E^{3/2}/F$ to stay fixed. Therefore $E \propto F^{2/3}$,

$$A = E/F \propto F^{-1/3},$$

$$T \propto F^{-2/3}.$$

Work done by adjusting the slope changes the energy despite the approximate conservation of action.

Start with $m = 0.25$ kg, $F = 2.0$ N and $A = 0.080$ m. The initial energy is 0.160 J and the period 0.566 s. A slow eightfold increase of F makes the energy four times larger, 0.640 J, and the amplitude half as large, 0.040 m. The period becomes 0.141 s. In contrast, switching slopes instantaneously at the centre leaves the kinetic energy and position unchanged at that instant. The same final slope would then give amplitude 0.010 m. The route by which a parameter changes matters.

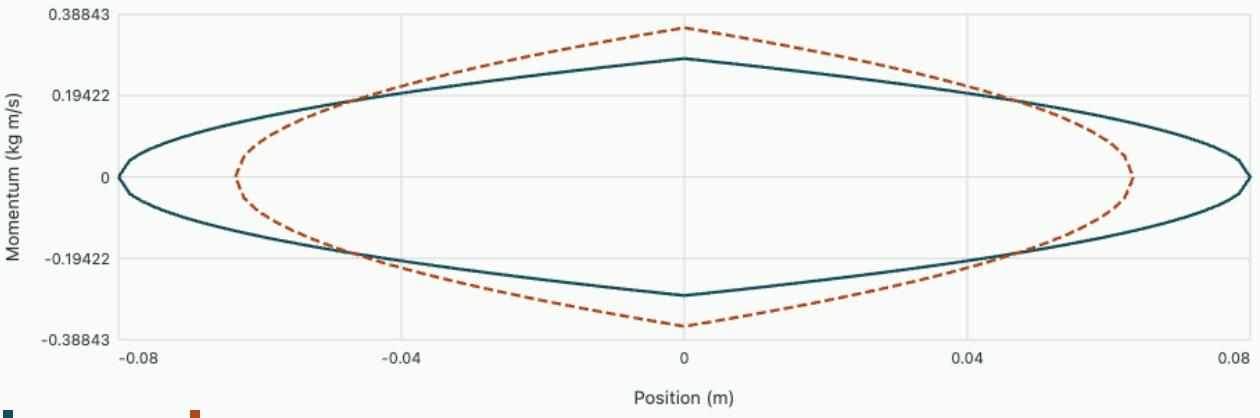
Compare closed orbits at constant action

The axes are position and momentum. Change the slope ratio. Predict how the turning positions and the central momentum move in opposite directions. Their changes balance in the enclosed area.

Open this experiment online

Try the calculation

Final to initial force ratio 2



Final amplitude 0.0635 m, energy 0.254 J and period 0.3564 s.

Tutorial on inferring a change

After a slow adjustment the period is 64% of its original value. Find the force and amplitude ratios.

▼ **Work through the solution**

From $T_f/T_i = (F_f/F_i)^{-2/3}$, the force ratio is

$$0.64^{-3/2} = 1.953125.$$

The amplitude ratio is its inverse cube root, 0.800. Squaring the period formula with these values provides an independent check.

Derive a relation between the area action and the product of energy and period in the V trap. What fraction of the total energy is kinetic on a time average over one orbit?

▼ **Work through the solution**

The explicit formulas give $J = ET/(3\pi)$, a quantity with units of action. Along the orbit $p\,dq = p\dot{q}\,dt = 2K\,dt$. Hence $2\pi J = 2T\langle K\rangle$ and $\langle K\rangle = E/3$, leaving $2E/3$ in potential energy. This is an independent check using a measurable time average.

A trap is weakened gradually until the amplitude doubles. Find the predicted changes in energy and period, and state when the slow-ramp argument is most vulnerable.

▼ **Work through the solution**

The amplitude varies as $F^{-1/3}$, so doubling it requires reducing F to one eighth. The energy then falls to one quarter and the period increases fourfold. A ramp duration adequate initially can become too short compared to the final period. Checking the endpoint period alone is insufficient if the control has sharp changes during the ramp.

Further reading · Tong, action–angle coordinates and adiabatic invariants.

Parametric resonance and Floquet growth

Classical mechanics

A pendulum can gain energy when its length varies periodically even if no additive force acts on it. A modulation near twice a natural frequency amplifies one phase of the oscillation and attenuates another. The stability test concerns the growth of a perturbation over a complete modulation period.

Linearise the timing mechanism

Consider $\ddot{x} + 2\gamma\dot{x} + \omega_0^2[1 + h\cos(\Omega t)]x = 0$. Here h is the fractional modulation of the stiffness and γ is the amplitude damping rate. If $x = \dot{x} = 0$ exactly, the solution stays zero. Small fluctuations or an initial condition provide the seed. Direct differentiation of $E = [\dot{x}^2 + \omega_0^2(1 + h\cos\Omega t)x^2]/2$ yields

$$\dot{E} = -2\gamma\dot{x}^2 - h\omega_0^2\Omega x^2\sin(\Omega t)/2.$$

Energy transfer depends on the phase of x^2, which naturally contains a component at twice the oscillation frequency.

Near $\nu = \Omega/2 \simeq \omega_0$, write $x = A(t)e^{i\nu t} + A^*(t)e^{-i\nu t}$ with slowly varying A. Multiplication by $\cos(2\nu t)$ mixes the negative-frequency term back into the positive-frequency term with coefficient $h\omega_0^2/2$. There is also a component at 3ν, rapidly oscillating in this first-resonance approximation. Retain terms proportional to $e^{i\nu t}$, neglect $\ddot{A}$ and products of small damping with $\dot{A}$. This gives

$$2i\nu\dot{A} + (\omega_0^2 - \nu^2 + 2i\gamma\nu)A + (h\omega_0^2/2)A^* = 0.$$

Find the first instability tongue

Set $\Delta = \nu - \omega_0$ and keep leading terms in the small ratios $h, \gamma/\omega_0, |\Delta|/\omega_0$. The slow equations are $\dot{A} = (-\gamma - i\Delta)A + igA^*$ and their conjugate, where $g = h\omega_0/4$. The two-by-two characteristic equation is

$$(s + \gamma)^2 = g^2 - \Delta^2.$$

The faster exponent is

$$\operatorname{Re} s_+ = -\gamma + \operatorname{Re}\sqrt{(h\omega_0/4)^2 - \Delta^2}.$$

The oscillation grows when $g^2 > \gamma^2 + \Delta^2$. Exactly on resonance this gives $h > 4\gamma/\omega_0$. With $\omega_0 = 10\,\mathrm{s}^{-1}$, $\gamma = 0.10\,\mathrm{s}^{-1}$ and $h = 0.08$, $g = 0.20\,\mathrm{s}^{-1}$, giving an on-resonance doubling time $\ln 2/0.10 = 6.93\,\mathrm{s}$. The approximate growing interval in drive angular frequency is $20 \pm 0.346\,\mathrm{s}^{-1}$. A large-amplitude real oscillator can saturate as its natural frequency changes with amplitude, a mechanism absent from this linear equation.

Compute the period map without averaging

Integrate the original linear equation over one drive period from two independent initial states $(1, 0)$ and $(0, 1)$. Their final columns form a matrix M, which maps every initial perturbation over the same period T. After j periods the map is M^j; the maximum asymptotic amplitude exponent is $\ln \max |\lambda_i(M)|/T$. Its determinant also has an exact check $e^{-2\gamma T}$ from the divergence in the $(x, \dot{x})$ plane. The calculation below integrates both columns by fourth-order Runge–Kutta. Compare the numerical Floquet exponent with the averaged prediction as the modulation depth increases. Both curves describe growth rates, including negative rates for decay.

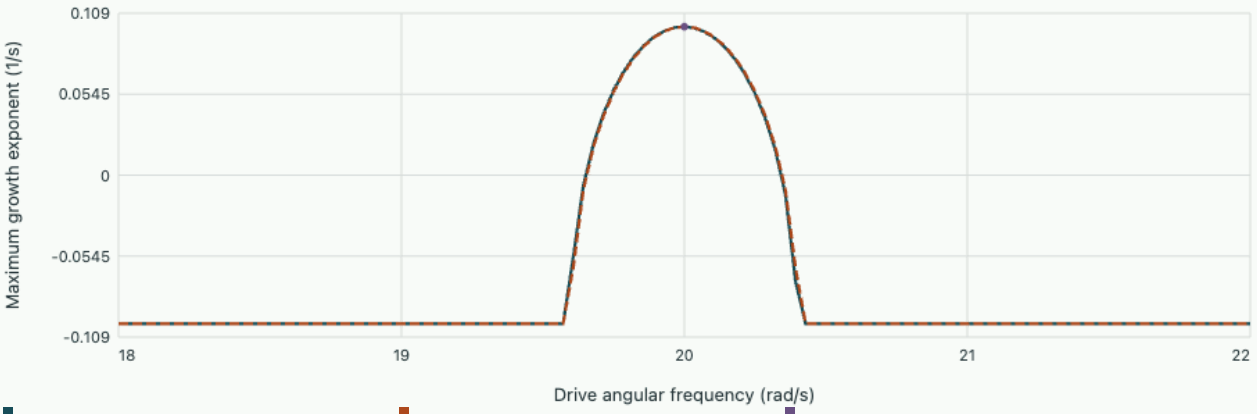
Open this experiment online

Try the calculation

Fractional stiffness modulation **0.08**

Damping γ (1/s) **0.1**

Drive frequency Ω (rad/s) **20**



At Ω = 20 rad/s, numerical exponent 0.09997 s⁻¹ and averaged exponent 0.1 s⁻¹. Amplitude doubling time 6.934 s.

Tutorial on phase-sensitive amplification

At exact resonance and zero loss, substitute $A = X + iY$ into $\dot{A} = igA^*$. Identify the two special quadrature directions.

▼ **Work through the solution**

Equating real and imaginary parts yields

$$\dot{X} = gY,\ \dot{Y} = gX.$$

Along $Y = X$ both grow at g, whereas along $Y = -X$ they decay at g. These directions reflect how timing the stiffness change can add or remove work on each cycle.

On resonance an oscillator with $\omega_0 = 40\,\mathrm{s}^{-1}$ first grows near $h = 0.03$. Estimate its small-signal damping and its free amplitude e-folding time.

▼ **Work through the solution**

From the weak-modulation threshold $\gamma = h\omega_0/4 = 0.30\,\mathrm{s}^{-1}$. Without modulation the underdamped free amplitude falls as $e^{-\gamma t}$, giving $3.33\,\mathrm{s}$. Threshold uncertainty or nonlinear frequency shifts propagate into this inference.

Further reading · Verhulst, Utrecht University, perturbation analysis of parametric resonance.

Störmer–Verlet and a discrete invariant

Numerical ordinary differential equations

Derive the leapfrog update, identify its oscillatory stability interval and quantify the difference between bounded energy and correct phase.

FROM FIRST PRINCIPLES

Derive a reversible position update from Newton's law

For a force that depends only on position, $m\ddot{q} = F(q)$, the acceleration can be evaluated at a stored position without an implicit solve. Expand the exact position one step forward and one step backward about the same time t_n. The odd derivatives reverse sign and cancel when the expansions are added:

$$q(t_n + h) + q(t_n - h) = 2q(t_n) + h^2\ddot{q}(t_n) + O(h^4).$$

Replace acceleration by force divided by mass and solve for the forward position. The resulting relation is unchanged if the labels $n + 1$ and $n - 1$ are exchanged:

$$q_{n+1} = 2q_n - q_{n-1} + \frac{h^2}{m}F(q_n).$$

Introduce the staggered momentum $p_{n-1/2} = m(q_n - q_{n-1})/h$. The same algebra becomes an impulse followed by a drift,

$$p_{n+1/2} = p_{n-1/2} + hF(q_n),$$
$$q_{n+1} = q_n + hp_{n+1/2}/m.$$

A specified momentum at $t = 0$ starts the first half step with $p_{1/2} = p_0 + hF(q_0)/2$. For smooth forces the method has second-order position accuracy. It is important to distinguish the staggered momentum from momentum at the stored position time when checking energy.

DERIVATION AND APPLICATIONS

Why a stable Verlet trajectory can arrive at the wrong time

Long-term energy behaviour is important in reversible mechanics, but phase error can dominate a timing measurement. Consider the velocity-independent linear oscillator $q'' = -\omega^2 q$. Use a uniform step h, an accurately initialized first point and the position recurrence obtained by eliminating the half-step momenta in Verlet. Write $z = h\omega$:

$$q_{n+1} - 2q_n + q_{n-1} = -z^2 q_n.$$

Assume a discrete mode $q_n = e^{in\vartheta}$. Divide by the mode value at n and combine the two exponentials. The angle advanced per step obeys

$$2\cos\vartheta - 2 = -z^2,$$
$$\sin(\vartheta/2) = z/2.$$

For $0 < z < 2$ choose the branch continuous with the exact oscillation at small h. The discrete angular frequency and its expansion are

$$\widetilde{\omega} = \frac{2}{h}\arcsin(z/2) = \omega\left[1 + \frac{z^2}{24} + O(z^4)\right].$$

The numerical oscillator runs slightly fast even though its amplitude can remain bounded indefinitely. Its phase lead grows approximately linearly with elapsed time at fixed step. Above $z = 2$ the dispersion relation permits no real phase angle, signaling a separate stability failure. The formula concerns time discretization alone; incorrect initial half-step velocities can add an initial phase offset.

WORKED EXAMPLE

Estimate a ten-period timing error

For $\omega = 5.00$ rad/s and $h = 0.100$ s, $z = 0.500$ and $\widetilde{\omega} = 20\arcsin(0.250) = 5.054$ rad/s. Ten true periods last 12.57 s. Over that duration the numerical phase gains about $(5.054 - 5.00)(12.57) \simeq 0.674$ rad. The estimated time at which the numerical oscillator completes its tenth period is $20\pi/5.054 = 12.43$ s, about 0.13 s too soon.

SECOND APPLICATION

Select a step from a phase requirement

Suppose 100 periods should accumulate less than 0.100 rad of phase error. The leading estimate gives $2\pi(100)z^2/24 < 0.100$, so $z < 0.0618$. For $\omega = 5.00$ rad/s this means $h < 0.0124$ s. A choice based only on the stability requirement $h < 0.400$ s would be far too coarse. Halving a small step should reduce the accumulated phase error by approximately four at fixed physical duration.

Position and momentum at staggered times

For unit mass and a time-independent potential $V(q)$, the equations are

$$q' = p,\; p' = -V'(q).$$

Treat the potential force over a half step, drift with the updated momentum for a full step, then use the new position for the last half kick. This yields

$$p_{n+1/2} = p_n - \tfrac{h}{2} V'(q_n),$$
$$q_{n+1} = q_n + h p_{n+1/2},$$
$$p_{n+1} = p_{n+1/2} - \tfrac{h}{2} V'(q_{n+1}).$$

Successive half kicks at the same position share a force evaluation. Reversing the sign of h exactly undoes the steps in reverse order, in exact arithmetic. Subtracting consecutive position increments gives

$$q_{n+1} - 2q_n + q_{n-1} = -h^2 V'(q_n).$$

The starting position and velocity must initialise the very first half step correctly; using the full velocity as the half-step velocity injects an initial phase error.

The oscillator has an exactly computable modified energy

For $V = \omega^2 q^2/2$ define $z = h\omega$. The position coefficient of q_n after one step is

$$a = 1 - z^2/2;$$

the momentum coefficient of q_n is $-h\omega^2(1 - z^2/4)$. The full map has equal diagonal entries a and off-diagonal entries h, $-hc$ with $c = \omega^2(1 - z^2/4)$. Since $a^2 + h^2 c = 1$, expanding squares shows $p_{n+1}^2 + c q_{n+1}^2 = p_n^2 + c q_n^2$.

When $0 < z < 2$, $c > 0$ and level sets of this invariant are bounded ellipses. The eigenvalues are $e^{\pm i\theta}$ with $\cos\theta = a$. Beyond two, one real eigenvalue has magnitude greater than one and trajectories generically grow. At exactly two the invariant loses control of position and a repeated eigenvalue can cause linear growth. An apparently bounded run starting from specially chosen data at the threshold cannot certify stability for arbitrary perturbations.

Follow a step and its timing

For $\omega = 1, h = 1, q_0 = 1, p_0 = 0$, the half kick gives -0.5, the drift gives $q_1 = 0.5$, and the final half kick gives $p_1 = -0.75$. The physical energy decreases from 0.5 to 0.40625 in this first step. The modified energy $(p^2 + 0.75q^2)/2$ equals 0.375 at both points. The discrete phase is $\theta = \pi/3$, so the computed trajectory returns after six steps. The true oscillator returns after $2\pi \approx 6.283$ seconds. A stable trajectory with a bounded energy error can therefore accumulate a large timing error over many cycles.

For small z, expanding $\theta = 2\arcsin(z/2)$ gives

$$\theta = z + z^3/24 + O(z^5).$$

At a fixed final time the accumulated phase error is therefore second order in h. A period estimate and an energy trace probe different aspects of long-time integration and should be reported separately.

Why the initial velocity test matters

A test with only the initial position nonzero may conceal the borderline instability at $h\omega = 2$. At that boundary the update is

$$q_{n+1} = -q_n + hp_n,\ p_{n+1} = -p_n.$$

For $p_0 = 0$ the positions merely alternate sign forever. With $q_0 = 0$ and $p_0 = 1$, however, explicit iteration gives

$$p_n = (-1)^n,$$

$$q_n = (-1)^{n-1} nh.$$

The position magnitude grows linearly. A stability test must account for all independent starting directions or small perturbations of the chosen one, especially when an eigenvalue of modulus one is repeated.

Prove conservation of angular momentum by individual substeps

For a particle of constant mass in a radial potential $V(\|q\|)$, the force is parallel to the position vector. Its angular momentum is $L = q \times p$. During a kick of duration τ, momentum changes by $\tau F(q)$ at fixed position, so the change in L is

$$\tau q \times F(q) = 0.$$

During a drift, position changes by $\tau p/m$ at fixed momentum, giving change $(\tau/m)p \times p = 0$. The full kick–drift–kick map therefore preserves angular momentum exactly in arithmetic without rounding, for any step length for which these evaluations are defined.

The proof uses radial force and the relationship between velocity and momentum. An external noncentral force exerts a physical torque, while a coordinate-dependent mass matrix changes the drift subflow. Under either modification the two zero cross products need rechecking. The argument also gives no bound on energy or phase. Exact conservation of angular momentum constrains orbital geometry but leaves freedom to move from the correct energy level onto a nearby one or to return to a point on the orbit at the wrong time.

Compare three diagnostics on an elliptical oscillator trajectory

Use unit mass and two-dimensional potential $V = (x^2 + y^2)/2$. Start at $q = (1, 0)$, $p = (0, 2)$, so $L = 2$ and physical energy is

$$E = (1 + 4)/2 = 2.5.$$

At step $h = 0.5$, the first half kick gives

$$p_{1/2} = (-0.25, 2).$$

The drift leads to $q_1 = (0.875, 1)$. The last half kick subtracts $0.25 q_1$, giving $p_1 = (-0.46875, 1.75)$. The resulting planar angular momentum is $0.875(1.75) - 1(-0.46875) = 2$, exactly as proved.

Calculate energy separately. The squared position is $0.875^2 + 1 = 1.765625$, and the squared momentum is

$$0.46875^2 + 1.75^2 = 3.2822265625.$$

Hence the physical energy after this step is 2.52392578125, about 0.96 percent above the initial value. The exact continuous position at this time is $(\cos 0.5, 2 \sin 0.5) \approx (0.87758, 0.95885)$, so the position error is concentrated in the second coordinate. Angular momentum, physical energy and the individual trajectory coordinates expose three distinct properties of the same update.

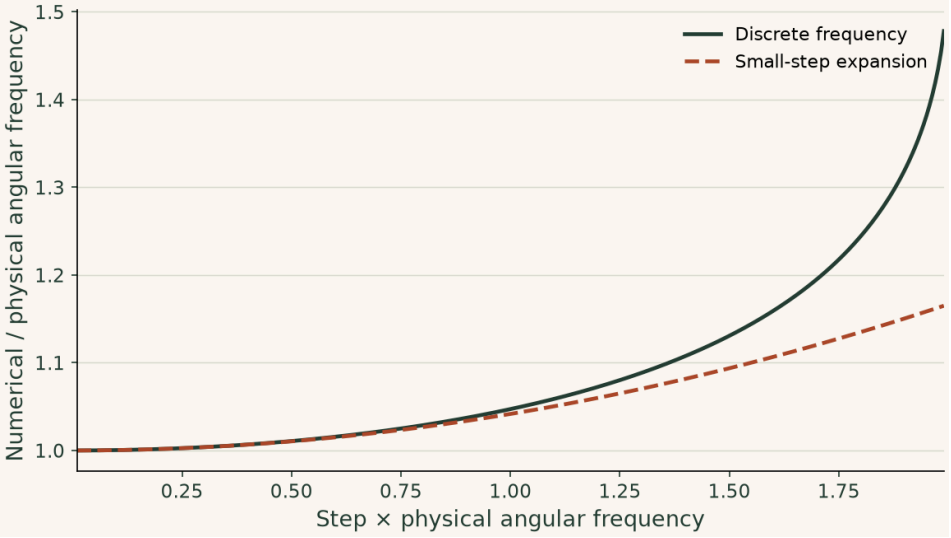
Because the harmonic force is separable by coordinate, the modified invariant from the scalar oscillator adds across the two directions: $[\|p\|^2 + (1 - h^2/4)\|q\|^2]/2$ is constant. Substitution gives 2.46875 at both steps, independently checking the arithmetic. For a general radial potential this particular quadratic modified invariant no longer follows, while the angular momentum proof remains valid. A conservation claim should identify the symmetry responsible rather than assuming every observed invariant survives a change of potential.

If there were a small constant transverse force, the physical change in angular momentum would be the time integral of $q \times F_{\text{ext}}$. A better diagnostic would subtract the accumulated applied torque from the measured change in L, rather than demand that L remain fixed. This distinction also guards against a subtle implementation error: zeroing a real torque to improve a conservation plot would change the physical problem and could hide a significant trajectory bias.

Measuring the signed torque makes its direction as well as its magnitude available for this balance check.

Estimate the phase error of a stable Verlet orbit

Eliminate momentum from velocity Verlet for the harmonic oscillator. A discrete sinusoid obeys $\cos\theta = 1 - (h\omega)^2/2$. Its angular frequency is θ/h; divide by ω before comparing steps.



At hω = 1 the numerical frequency exceeds the true one by about 4.72%. The scheme can maintain a bounded orbit yet acquire a substantial phase error after many cycles. The chart stops below the stability boundary at two.

▶ **Read values from the plot**

Change parameters · Estimate the phase error of a stable Verlet orbit

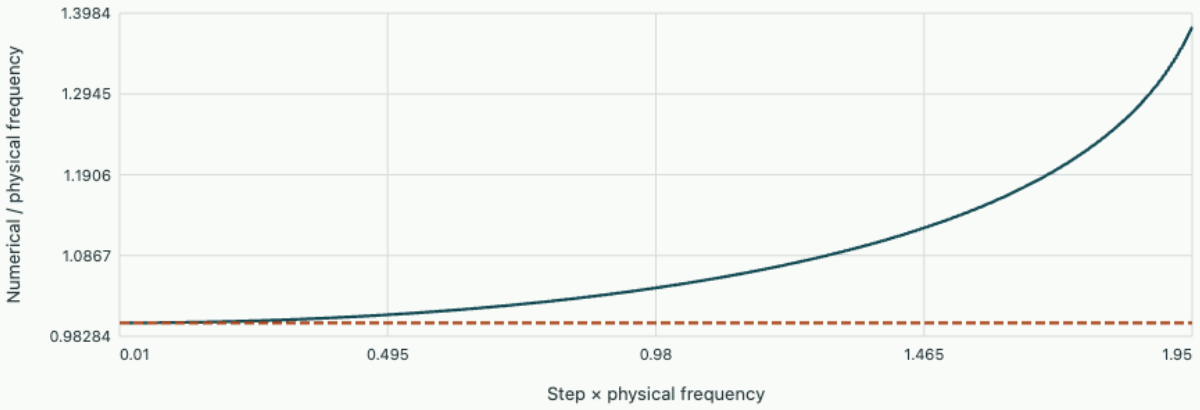
Change one input at a time and compare the recalculated curves with the argument above. The browser evaluates this small model locally; the accompanying numerical values state its assumptions and reference checks. Reset restores the reference inputs, and the data can be downloaded as CSV.

Open this calculation with adjustable inputs

Try the calculation

Selected hω **1**

Physical cycles **10**



Selected frequency ratio 1.0472; phase lead after 10 exact cycles is 2.9655 rad. The stability boundary is hω = 2.

Implicit midpoint, reversibility and the phase of an oscillator

Numerical ordinary differential equations

Imagine integrating a mechanical trajectory forward, saving only its end state and then running the same method backward. Recovering the start is a stronger demand than obtaining a plausibly small one-step error. Even exact recovery does not guarantee that an oscillation peaked at the correct time.

Build a method from a centred difference

Integrate $y' = f(t, y)$ over an interval of length h. Midpoint quadrature approximates the integral by h times its integrand at $t_n + h/2$. The arithmetic mean of the unknown endpoint states approximates the exact midpoint state up to order h^2 on a smooth trajectory. Using this mean as the derivative's state argument gives

$$y_{n+1} - y_n = h f(t_n + h/2, (y_n + y_{n+1})/2).$$

Errors from midpoint quadrature and from evaluating f at the approximate midpoint both contribute $O(h^3)$ locally, giving global order two on a fixed interval under the usual regularity and stability conditions. In general the unknown appears inside f, so an implicit equation must be solved. If f is Lipschitz with constant L_f, direct fixed-point iteration contracts when $|h|L_f/2 < 1$. Newton iteration can converge over a wider range. Neither a failed nonlinear solve nor loose tolerances inherit the exact algebraic properties of the method.

Exchange y_n, y_{n+1} and replace h by $-h$. The left side changes sign, the arithmetic midpoint is identical and the midpoint time is identical if the reverse step starts at t_{n+1}. Thus, on the same locally unique solution branch,

$$\Phi_{-h} = \Phi_h^{-1}.$$

A round trip using variable steps must repeat the accepted steps in reverse order. Independently choosing a new adaptive mesh on the way back need not retrace the states.

Solve the step for a harmonic oscillator

Write $q' = \omega p,\ p' = -\omega q$, with scaled momentum p in the same units as q, and $z = q + ip$. The ODE becomes

$$z' = -i\omega z.$$

Solving the linear midpoint equation gives the scalar multiplier

$$M(u) = \frac{1 - iu/2}{1 + iu/2} = \exp[-2i \arctan(u/2)],$$

$$u = \omega h.$$

Its numerator and denominator are complex conjugates. Hence it preserves $q^2 + p^2$, proportional to the oscillator energy, exactly in real arithmetic. For any radius or step size $M(-u)M(u) = 1$. The rotation per step is

$$\theta = 2\arctan(u/2) = u - u^3/12 + O(u^5).$$

The numerical phase lags by $u - \theta$ per step. After N steps the accumulated lag is $N(u - \theta)$, increasing linearly in elapsed time even while the energy remains exactly constant.

Compare explicit midpoint Runge–Kutta. A half Euler predictor leads to

$$M_{\mathrm{RK2}} = 1 - iu - u^2/2.$$

It also has order two, but $|M_{\mathrm{RK2}}|^2 = 1 + u^4/4$. The amplitude grows slowly at every nonzero step. A forward and reverse pair multiplies the starting amplitude by $1 + u^4/4$. In a conservative problem these two second-order procedures can produce visibly different long records.

Inspect the return and the lag separately

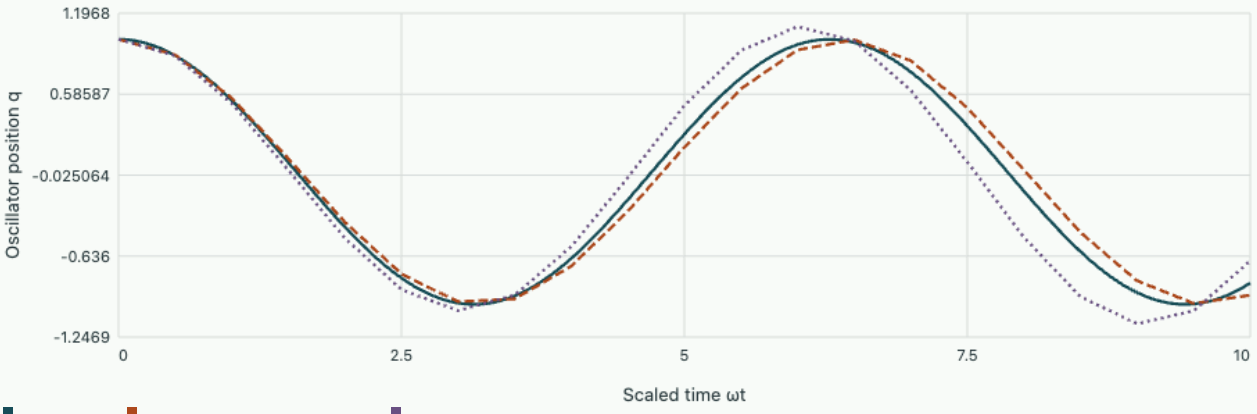
With $\omega = 1$, $h = 0.5$, $N = 20$, exact accumulation is 10 radians. Midpoint gives $40\arctan(0.25) = 9.7991$ radians, a lag of 0.2009 radians. The explicit round trip multiplies the radius by $(1 + 0.5^4/4)^{20} \simeq 1.364$. The implicit round trip returns the original radius and phase to roundoff. Run both updates below. Compare results at equal elapsed time by halving h and doubling N. The phase error should drop by about a factor of four. The graph uses computed steps for the methods and a more finely sampled exact reference.

Open this experiment online

Try the calculation

Time step h (ω = 1) **0.5**

Number of forward steps **20**



Implicit midpoint phase lag 0.2009 rad. Forward-backward distance from the start is 8.951e-16 for implicit midpoint and 0.3635 for explicit midpoint.

Tutorial on designing an accuracy criterion

A solver preserves the harmonic energy exactly over a simulation of 100 time units with $\omega = 1$. A measurement also requires phase error below 0.01 radians. Use the leading midpoint error to estimate an admissible step. How would you verify it?

▼ **Work through the solution**

The accumulated phase error is approximately $T\omega^3 h^2/12$. Solving $100h^2/12 < 0.01$ gives $h < 0.0346$. Verify at a chosen integer step count using $T - 2N\arctan[T/(2N)]$, or by convergence against a refined trajectory that measures the phase continuously. Energy alone has zero error for all these steps, so it cannot certify the phase target.

Further reading · Hairer's TUM geometric numerical integration course, symmetric methods.

Locating a grazing event between time steps

Numerical ordinary differential equations

Derive a sensitivity formula for event time, then measure how a harmless error in a trajectory can create or remove two threshold crossings.

A time is an implicitly defined output

Suppose an integrator advances $y' = f(t, y)$ and an event occurs at $g(t, y(t)) = 0$. Along the solution the derivative of the event function is $\dot{g} = g_t + \nabla_y g \cdot f$. A perturbation δg in the computed event function shifts a simple event time by $\delta t \simeq -\delta g / \dot{g}$. This estimate separates trajectory accuracy from event-time accuracy. At a tangency the denominator vanishes and the formula cannot be used.

Background · Event transversality

$$\dot{g} = \partial_t g + \nabla g \cdot f$$

For a simple event this directional derivative is nonzero. Near grazing, a small state error can create a much larger error in the event time.

Resolve two crossings analytically

Take $g(t) = (t - t_0)^2 - d$, a local model for a turning point relative to a sensor threshold. For $d > 0$ the events are $t_0 \pm \sqrt{d}$. Their separation is $2\sqrt{d}$, and their slopes are $\pm 2\sqrt{d}$. For $d = 0$ there is only a tangent touch. If $d < 0$, the sensor is never triggered. An additive constant perturbation ϵ changes the effective parameter from d to $d - \epsilon$.

At $t_0 = 1$, $d = 0.01$ and $\epsilon = 0.001$, the first event moves from 0.9 to about 0.90513. The linear prediction is

$$0.001 / 0.2 = 0.005.$$

At $d = 0.0004$ the same perturbation removes both crossings entirely. Reporting more digits from a polished scalar rootfinder would not recover them.

What two endpoint samples fail to tell you

With $d = 0.0025$, a step from 0.9 to 1.1 has equal positive endpoint values of 0.0075 but crosses the threshold twice. Dense interpolation followed by an extremum check can detect this pair if the interpolant resolves the turning point. A sign-change test at accepted endpoints alone cannot. An interpolant is itself an approximation, so event feasibility should be tested against an error scale at the extremum.

Open this experiment online

Try the calculation

Threshold offset d **0.01**

Trajectory bias ε **0.001**



Reference: 0.9 s and 1.1 s. Perturbed: 0.90513 s and 1.0949 s.

Tutorial on event tolerances

For a simple crossing with slope magnitude 0.04 units/s, what error in the event function is consistent with a 1 ms time tolerance to first order?

▼ Worked solution

Multiply the slope by the desired time uncertainty. The event function should be accurate to about 4×10^{-5} units near the crossing.

A contact model needs the first event, even if it only touches the threshold and returns. How should the experiment be reported near grazing?

▼ Worked solution

Report the minimum event-function value and an estimated error on it, in addition to any time estimates. If zero lies within that uncertainty one cannot distinguish no crossing, tangency and a nearby pair from these data alone.

Virtual work and the forces hidden by a constraint

Analytical mechanics from constraints to invariants

A bead follows a wire, a pendulum remains the same length, and a rolling wheel obeys a relation between translation and rotation. Their reduced equations describe the motion but can conceal the largest forces in the apparatus. Determine which forces disappear under a virtual displacement, recover their values, and decide when the assumed motion ceases to be physically admissible.

Hold the clock fixed when defining a virtual displacement

A system of particles has Cartesian positions $\mathbf{r}_i$ and applied forces $\mathbf{F}_i$. A holonomic constraint is an equation $g_\alpha(\mathbf{r}_1, \dots, \mathbf{r}_N, t) = 0$. At a fixed instant an admissible infinitesimal change of configuration obeys $\sum_i \nabla_i g_\alpha \cdot \delta\mathbf{r}_i = 0$. The physical velocities obey a different relation, $\sum_i \nabla_i g_\alpha \cdot \dot{\mathbf{r}}_i + \partial_t g_\alpha = 0$. A virtual displacement compares geometrically allowed configurations at the same time. It has no prescribed duration and makes no claim that the system's momentum actually points along it.

The distinction becomes measurable for a bead on a moving horizontal table. Every allowed virtual displacement has zero vertical component, so the table's upward normal force gives zero virtual work. An upward-moving table can nevertheless raise the bead and perform positive ordinary work. Ordinary power involves the actual velocity, which includes the table motion. A careless statement that constraint forces do no work confuses two different operations. For a fixed ideal frictionless surface the real velocity is tangent as well, and vanishing virtual work then does imply vanishing ordinary power.

Let $\mathbf{R}_i$ be the reaction forces and use $m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i + \mathbf{R}_i$. Multiply by an admissible $\delta\mathbf{r}_i$ and sum over particles. We call the constraints ideal when $\sum_i \mathbf{R}_i \cdot \delta\mathbf{r}_i = 0$ for every allowed virtual displacement. Eliminating those reactions gives d'Alembert's principle,

$$\sum_i (\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i) \cdot \delta\mathbf{r}_i = 0.$$

This is a dynamical equation expressed through a static-looking virtual-work test. It applies instantaneously along the motion. It does not assert that the total force vanishes or that the acceleration is an independent virtual displacement. One advantage is that the detailed construction of a rigid support need not be known to determine the admissible tangential acceleration; its force can be calculated afterwards when designing the support.

Pass from virtual work to equations for coordinates

Choose independent coordinates q_a with $\mathbf{r}_i = \mathbf{r}_i(q, t)$ and independent virtual increments δq_a. The velocity is $\sum_a \partial_{q_a}\mathbf{r}_i\,\dot{q}_a + \partial_t\mathbf{r}_i$. With kinetic energy $T = \sum_i m_i\dot{\mathbf{r}}_i^2/2$, differentiation at fixed q, t gives

$$\partial_{\dot{q}_a} T = \sum_i m_i\dot{\mathbf{r}}_i \cdot \partial_{q_a}\mathbf{r}_i.$$

Differentiate in time. The product rule produces an acceleration term and a second term involving $d(\partial_{q_a}\mathbf{r}_i)/dt$. The latter equals $\partial_{q_a}\dot{\mathbf{r}}_i$, with the generalised velocities held fixed, and supplies $\partial_{q_a} T$.

Hence

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q_a}-\frac{\partial T}{\partial q_a}=\sum_i m_i\ddot{\mathbf r}_i\cdot\frac{\partial \mathbf r_i}{\partial q_a}.$$

The projected applied force is

$$Q_a=\sum_i \mathbf F_i\cdot\partial_{q_a}\mathbf r_i.$$

Independence of the coordinate variations in d'Alembert's principle gives one equation $d(\partial_{\dot q_a}T)/dt-\partial_{q_a}T=Q_a$ per coordinate. When $\mathbf F_i$ comes from a potential $V(q,t)$, the conservative contribution is $-\partial_{q_a}V$. Other contributions such as a measured resisting torque remain explicit. A coordinate expressed as an angle has a generalised force in joules per radian; a coordinate expressed as a length has one in newtons. Checking $Q_a\,dq_a$ as work avoids mistaking a generalised force for a Cartesian component.

For a simple pendulum of length l and mass m, choose θ measured from downward vertical. Then $T=ml^2\dot\theta^2/2$ and $V=mgl(1-\cos\theta)$. The equation $ml^2\ddot\theta=-mgl\sin\theta$ follows at once. The string tension vanished under the angular displacement. It has not been shown to vanish physically. Projecting Newton's equation onto the direction from the bead toward the pivot gives

$$T_{\text{string}}=mg\cos\theta+ml\dot\theta^2.$$

This second observable is sensitive to speed even when the reduced angular acceleration depends only on angle.

Recover multipliers before choosing coordinates

Cartesian coordinates keep reaction directions explicit. Put all coordinates into a vector x, let M be the mass matrix and let G contain the rows ∇g_α. The tangent virtual displacements are precisely those satisfying $G\delta x=0$. A reaction doing zero virtual work must be a linear combination of the normal covectors, $R=G^T\lambda$, when the constraints are independent. Newton's equation becomes

$$M\ddot x=F+G^T\lambda.$$

Differentiate the constraints twice along the trajectory: $G\ddot x+\gamma=0$, where γ includes the known velocity, curvature and explicit time terms. Substituting for $\ddot x$ determines reactions from a much smaller system,

$$(GM^{-1}G^T)\lambda=-\gamma-GM^{-1}F.$$

For any vector c, $c^TGM^{-1}G^Tc=(G^Tc)^TM^{-1}(G^Tc)$ is positive if G has independent rows and particle masses are positive. Dependent constraint equations make the individual multipliers ambiguous. For instance writing both $g=0$ and $2g=0$ declares the same geometrical restriction twice. Only the

combined reaction is then determined. The magnitude of a raw multiplier also depends on the units and normalisation of g. Reporting the force $G^T\lambda$ makes comparisons between alternative constraint descriptions meaningful.

Use $g = (x^2 + y^2 - l^2)/2 = 0$ for a planar pendulum, with y upward, so the gradient is the radius vector $\mathbf{r}$. Newton's equation reads $m\ddot{\mathbf{r}} = -mg\hat{y} + \lambda\mathbf{r}$, where the symbol g in the gravitational term denotes the acceleration magnitude. The acceleration constraint $\mathbf{r}\cdot\ddot{\mathbf{r}} = -v^2$ gives

$$\lambda = (mgy - mv^2)/l^2.$$

The inward tension is $-\lambda l = mv^2/l - mgy/l$. Since $y = -l\cos\theta$, this is the radial-force result already obtained. The multiplier in this convention has units of force per length. Using the radial-distance constraint instead would make its magnitude a force.

Background · What if the support can push but cannot pull?

Replacing a surface by an equality constraint assumes the object stays in contact. A particle sliding along the inside of a smooth vertical circular track illustrates the additional physical test. At the top, gravity and the inward normal reaction together must supply the centripetal acceleration.

$$N = \frac{mv^2}{R} - mg, \qquad N \geq 0.$$

At a top speed below $\sqrt{gR}$ the equality-constrained solution requires negative N. An ordinary track cannot exert that pull. Its normal force then falls to zero and the particle must follow free flight until it meets the track again. A bead threaded on a wire could receive the oppositely directed force and would follow a different trajectory. Both models have the same geometrical path during contact.

Recognise an inequality at a contact

A bead threaded on a stiff loop can experience reaction in either direction, whereas a string can pull only. A surface without adhesion can push only. Derive the constrained motion, calculate its reaction, and compare the required sign with the physical mechanism. When a calculated tension turns negative, replace the constrained trajectory by free flight until a new contact event. Continuing to integrate the pendulum equation would retain a mathematically valid circle and a physically unavailable pushing string.

A mass starts at the bottom of a frictionless vertical loop with speed v_b. Energy conservation gives $v_{\text{top}}^2 = v_b^2 - 4gl$, since the top is $2l$ higher. A string just remaining taut at the top requires $T_{\text{top}} = mv_{\text{top}}^2/l - mg \geq 0$, giving $v_b^2 \geq 5gl$. The energy-only criterion for reaching the elevation is $v_b^2 \geq 4gl$; the interval between these two values illustrates why a trajectory test must include reactions. With $l = 0.80\,\text{m}$, the threshold bottom speed is $6.26\,\text{m/s}$. At this threshold the bottom tension is $6mg$, six times the weight, even though the tension is about to vanish at the top.

Recover the forces on a wedge that is free to slide

Place a wedge of mass M on a smooth horizontal floor and a block of mass m on its smooth upper face. Let the slope descend to the right at angle α. Choose X as the wedge's horizontal position and s as distance down the face. The block's coordinates apart from fixed offsets are $x = X + s\cos\alpha,\ y = -s\sin\alpha$. Its relative velocity points down the slope but its absolute acceleration need not. The kinetic energy contains a mixed term $m\dot{X}\dot{s}\cos\alpha$. Dropping that term is equivalent to holding the wedge stationary while allowing it to recoil, two incompatible mechanical models.

The reduced equations are $(M+m)\ddot{X} + m\ddot{s}\cos\alpha = 0$ and $\ddot{s} + \ddot{X}\cos\alpha = g\sin\alpha$. The first is conservation of total horizontal momentum, since the block–wedge normal force is internal and the floor is frictionless. Elimination gives

$$\ddot{s} = (M+m)g\sin\alpha/(M + m\sin^2\alpha),$$

$$\ddot{X} = -mg\sin\alpha\cos\alpha/(M + m\sin^2\alpha).$$

The acceleration down the moving face exceeds $g\sin\alpha$ because the wedge moves left as the block slides right. Nevertheless the absolute vertical acceleration of the block remains less than or equal to free fall because the upper face pushes upward.

To find the block's normal force, project its absolute acceleration onto the outward unit normal $(\sin\alpha, \cos\alpha)$. The part proportional to $\ddot{s}$ cancels, leaving $\ddot{X}\sin\alpha$. Newton's equation along that direction gives

$$N = m(g\cos\alpha + \ddot{X}\sin\alpha) = mMg\cos\alpha/(M + m\sin^2\alpha).$$

With $M = 3.0\,\mathrm{kg}$, $m = 1.0\,\mathrm{kg}$, $\alpha = 30^\circ$, one finds $\ddot{s} = 6.04\,\mathrm{m/s^2}$,

$$\ddot{X} = -1.31\,\mathrm{m/s^2},$$

$$N = 7.84\,\mathrm{N}.$$

The normal based on a stationary wedge would be $8.50\,\mathrm{N}$. A force sensor under the moving face could distinguish the two models even when a short record of relative slope displacement cannot resolve the difference.

Which reaction does no virtual work here? The normal force on the block has virtual work $N\hat{n}\cdot(\delta X\hat{x} + \delta s\hat{t}) = N\sin\alpha\,\delta X$. The opposite normal force on the wedge gives $-N\sin\alpha\,\delta X$, so the sum cancels. Considering only the block and asserting zero work from the normal for arbitrary δX would be incorrect. At the real velocities the same reaction transfers energy between the bodies even though its net internal power on their pair is zero. The floor's normal reaction balances the vertical momentum change of the whole system. During the slide it is $(M+m)g - m\ddot{s}\sin\alpha$, less than total weight while the block accelerates downwards.

Take the limits as an additional test. At fixed m, α, increasing M to infinity returns the familiar fixed-wedge results. At a nearly horizontal face, downslope acceleration and wedge recoil vanish, restoring the full block weight as normal force. At a vertical face the normal vanishes and the block falls. These limits hold because the contact has no friction and can be maintained with a nonnegative normal. A moving

wedge of independently prescribed acceleration would replace the horizontal momentum equation by a control input, changing both the relative acceleration and the support force.

Distinguish rolling constraints from purely geometric restrictions

A cylinder of radius R, mass m and axial inertia I rolls without slip down a fixed rough incline. Choose coordinate s down the incline and spin angle ϕ in the corresponding sense. The constraint $\dot{s} = R\dot{\phi}$ is integrable in this one-dimensional motion, so it can be inserted as $s - R\phi = \text{constant}$. The resulting kinetic energy is $(m + I/R^2)\dot{s}^2/2$. The reduced equation gives

$$\ddot{s} = g\sin\alpha/[1 + I/(mR^2)].$$

Reconstruct the static friction from translation, $f = gm\sin\alpha - m\ddot{s}$, directed up the slope, and compare it with the available maximum $\mu_s mg\cos\alpha$.

For a homogeneous solid cylinder $I = mR^2/2$, the speed-up is $2g\sin\alpha/3$ and friction magnitude $mg\sin\alpha/3$. A ring of the same mass and radius has $I = mR^2$, half the gravitational downslope acceleration and half the downslope weight component as friction. At 30° the minimum static friction coefficients are respectively $\tan 30^\circ/3 = 0.192$ and $\tan 30^\circ/2 = 0.289$. Two objects on the same surface can therefore diverge qualitatively because the ring slips while the cylinder still rolls. Treating an assumed no-slip relation as exact without testing that inequality can produce an unphysical result despite apparently consistent energy conservation.

The friction on a rigid body rolling without slip on a fixed surface has zero power because its instantaneous point of contact is stationary. The force still changes both linear and angular momentum. On a moving conveyor the contact velocity is no longer zero in the laboratory; friction can supply mechanical energy even while the relative contact velocity vanishes. For a wheel steering on a plane the restrictions $\dot{x} = R\dot{\phi}\cos\psi,\ \dot{y} = R\dot{\phi}\sin\psi$ depend on heading ψ. In general they cannot be integrated into two configuration equations relating x, y, ϕ, ψ globally. d'Alembert's fixed-time allowed variations still project out ideal rolling reactions, but replacing those velocity constraints by invented holonomic coordinates gives the wrong dynamics. The distinction can be seen geometrically by rolling and steering through a closed sequence of steering operations that leaves a net displacement.

Find loads on two supports of a suspended bar

A rigid horizontal bar of mass m and length $2a$ is supported at its ends by two vertical inextensible light cords. At one instant both cords are vertical and the bar is horizontal. An extra downward load of magnitude F is applied at horizontal coordinate s relative to the centre. Ask first for a static experiment. The two vertical reaction forces T_L, T_R obey force balance $T_L + T_R = mg + F$ and moment balance $a(T_R - T_L) = sF$. This gives

$$T_R = (mg + F + sF/a)/2,$$
$$T_L = (mg + F - sF/a)/2.$$

The lever arms encode geometry that the vertical balance alone cannot provide.

The same result can be obtained from virtual variations if the reactions are kept as multipliers until the end. For small vertical translation y of the centre and small tilt θ, the endpoint vertical displacements are $\delta y - a\delta\theta$ on the left and $\delta y + a\delta\theta$ on the right. The load point displacement is $\delta y + s\delta\theta$. Before eliminating the endpoint constraints, project all applied and reaction forces against arbitrary independent $\delta y, \delta\theta$; vanishing of both coefficients gives the preceding balances. After imposing both endpoint constraints directly, these two variations vanish and the reduced model for vertical motion contains no equation from which the support loads can be read. Removing a degree of freedom can remove precisely the observable of interest unless reactions are reconstructed.

For $m = 2.0\,\mathrm{kg}$, $a = 0.40\,\mathrm{m}$, $F = 12\,\mathrm{N}$ and $s = 0.25\,\mathrm{m}$, the total load is $31.62\,\mathrm{N}$ and the load difference $7.50\,\mathrm{N}$. Hence $T_R = 19.56\,\mathrm{N}$, $T_L = 12.06\,\mathrm{N}$. Exchanging the load from right to left reverses the difference while preserving the sum. This symmetry is a calibrating manoeuvre for two load cells with different offsets: subtract readings from mirrored configurations to isolate the lever-arm response, and compare their sum to the known total weight and applied force.

Now imagine three vertical supports at $-a, 0, +a$. Their three tensions enter only two independent balances. Perfectly rigid constraints determine no unique allocation to the centre support. The ambiguity is physical; the stiffnesses, preloads and small height differences of real supports decide the result. With three identical unstrained linear springs and a centred load F, each spring extends equally if the bar stays level, so each carries one third of the total weight. If the middle spring is twice as stiff and all have the same unloaded height, it carries one half and each outer spring one quarter. Both answers satisfy exactly the same force and moment balances. Reporting a unique value from a singular ideal multiplier system would manufacture an unmeasured constitutive assumption.

Test an acceleration constraint against drift of the position itself

Solving $G\ddot{x} + \gamma = 0$ guarantees the correct instantaneous normal acceleration at a state lying on the constraint manifold with tangent velocity. It does not magically repair earlier position or velocity errors in approximate time integration. For a particle constrained to a circle, monitor $g = (r^2 - l^2)/2$, $\dot{g} = \mathbf{r} \cdot \mathbf{v}$, and the computed force. A small tangential displacement can give tiny instantaneous force error while the radius gradually drifts. Integrating an acceleration-level constraint twice without a position or velocity correction can cause substantial mismatch over many revolutions.

A simple projection illustrates what each correction does. Normalise an approximate position vector to length l, then remove the radial component of velocity by $\mathbf{v} \leftarrow \mathbf{v} - \hat{r}(\hat{r} \cdot \mathbf{v})$. The corrected point and tangent satisfy the constraints, but the velocity correction removes kinetic energy $m(\hat{r} \cdot \mathbf{v})^2/2$, and the position change can alter potential energy. Projection is therefore a numerical operation whose physical invariant errors must be measured. A structure-preserving constrained integrator builds the position and momentum conditions into the step and often gives better long-time behaviour. Compare convergence of reaction, constraint residual and energy separately instead of treating a rounded radius as proof of a correct force.

A pendulum on a free cart tests the complete mass matrix

Mount a pendulum pivot on a cart of mass M moving horizontally on a smooth track. The bob has mass m, length l and downward angle θ. Choose the cart position X and use bob coordinates $X + l\sin\theta, -l\cos\theta$. The kinetic energy is $(M+m)\dot{X}^2/2 + ml\dot{X}\dot{\theta}\cos\theta + ml^2\dot{\theta}^2/2$. Two different off-diagonal mass terms would contradict its origin as a quadratic form; the same factor $ml\cos\theta$ multiplies the companion acceleration in each equation. This is a useful check on a manually assembled multibody model.

The equations are

$$(M+m)\ddot{X} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = 0,$$

$$l\ddot{\theta} + \ddot{X}\cos\theta + g\sin\theta = 0.$$

In the small-angle regime the horizontal centre of mass moves uniformly. Choosing zero total momentum gives

$$\dot{X} \simeq -ml\dot{\theta}/(M+m).$$

Substitute its derivative into the angular equation to obtain

$$\ddot{\theta} + [(M+m)g/(Ml)]\theta = 0.$$

Releasing the support raises the oscillation frequency because the effective angular inertia falls while the gravitational stiffness stays the same. The result is a statement about the relative pendulum angle, measured from an inertial vertical, with an unforced horizontally moving support.

For $M = 0.60\,\mathrm{kg}$, $m = 0.20\,\mathrm{kg}$ and $l = 0.50\,\mathrm{m}$, the small-angle period is

$$2\pi\sqrt{Ml/[(M+m)g]} = 1.229\,\mathrm{s}.$$

Fixing the cart with a clamp gives $1.419\,\mathrm{s}$ instead. At zero total horizontal momentum the cart displacement from its equilibrium is approximately $-0.125\,\theta$ metres, so a $0.10\,\mathrm{rad}$ angle corresponds to $12.5\,\mathrm{mm}$ of recoil. Simultaneous position measurements on cart and bob should reconstruct a horizontal centre of mass that is stationary to within tracking uncertainty. A missing cart inertia or a wrongly filled cross term can alter both period and recoil; checking just one number may not identify which error occurred.

The reaction on the moving bob can again be recovered. Its radial acceleration along the outward string direction is $\ddot{X}\sin\theta - l\dot{\theta}^2$. Thus the tension is $T = m[g\cos\theta - \ddot{X}\sin\theta + l\dot{\theta}^2]$. The acceleration of the support appears explicitly. Reusing the fixed-pivot formula at all angles would omit a term that vanishes accidentally at the bottom. To test the sign, observe a rightward tilted pendulum whose pivot accelerates right: the support's acceleration has an outward projection along the tilted string, reducing the inward tension needed for the same relative motion. Substituting a directly measured cart acceleration provides an independent load-cell prediction.

Follow a force eliminated by a constraint

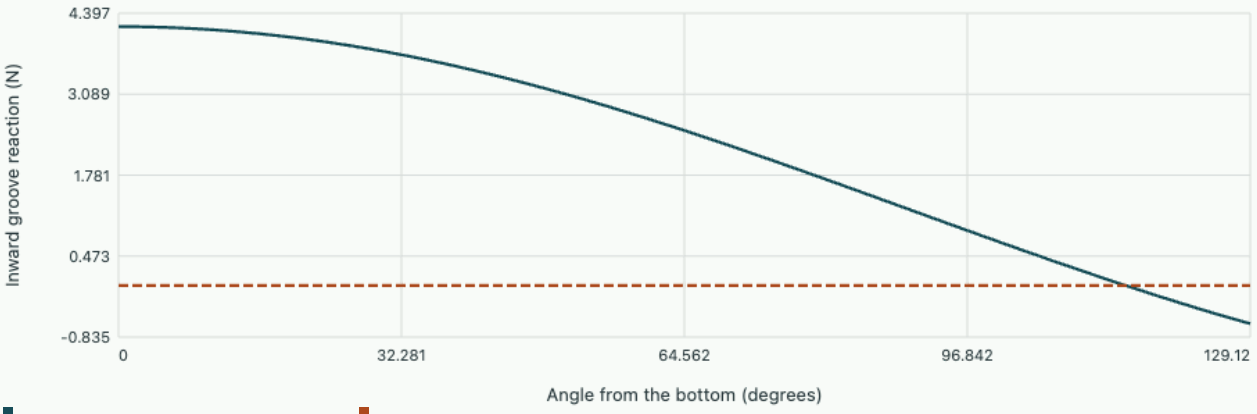
Predict when a two-sided circular groove must reverse its normal force. Check the prediction against the exact energy and radial balances.

Open this experiment online

Try the calculation

Bottom speed (m/s) **4**

Loop radius (m) **0.5**



Required radial reaction Loss of one-sided support

Reachable angle 129.12°. Inward groove reaction there -0.619 N. One-sided loop threshold v_0 = 4.9523 m/s.

Decide which force a sensor actually reports

Reaction forces can be reported as tension along a member, normal traction at a contact, total support load on a floor, or a single Cartesian component read by a load cell. A rigid-body idealisation alone does not specify a sensor's orientation or the weight of its housing. For a fixed pendulum, for example, the string carries tension T, while a vertical force channel at its pivot reads the vertical string contribution $T\cos\theta$. A horizontal channel reads $T\sin\theta$. The force on the pivot is opposite the force exerted on the bob. Selecting the wrong sign or leaving out the fixture's own weight can create a discrepancy even when the dynamical calculation is correct.

Suppose an ideal load cell with its sensitive axis vertical is tared with the fixture installed but the $0.40\,\mathrm{kg}$ bob removed. When the bob is held at 60° by a separate purely tangential actuator and is otherwise stationary, the radial balance gives

$$T = mg\cos 60^\circ = 1.96\,\mathrm{N}.$$

The pivot's vertical reading attributable to the string is $0.981\,\mathrm{N}$. The tangential actuator supplies the remaining vertical support as well as a horizontal component. Once the actuator releases,

instantaneously the same tension and reading hold because radial speed is still zero. At the bottom of the subsequent swing the load jumps to the dynamical value obtained earlier, even though no upper fixture mass has changed.

A student hangs the bob at the same 60° angle using a separate horizontal cord instead of a tangential actuator. Should the pivot cell read the preceding $0.981\,\mathrm{N}$?

▼ Work through the solution

The horizontal cord contributes no vertical support. Static balance now requires $T\cos 60^\circ = mg$, giving $T = 7.85\,\mathrm{N}$ and vertical pivot contribution $3.92\,\mathrm{N}$. The horizontal holding force has a nonzero radial projection and changes the radial-force balance. After the horizontal cord is removed the tension immediately changes to $mg\cos 60^\circ$ because no momentum impulse is required for a finite jump in force. A schematic that shows only the bob angle is insufficient to infer the support load without saying how the angle is maintained.

In the free cart experiment bearing drag is weak but measurable. Can an observed frequency shift distinguish it from a wrongly assumed cart mass?

▼ Work through the solution

A viscous drag changes the horizontal momentum balance and allows the formerly stationary centre of mass to move as momentum leaks to the track. A mass error in a drag-free model changes the relative period and recoil ratio without supplying an actual external impulse. Track the complete centre of mass using independently weighed masses, compare its momentum change with the integrated bearing force, and measure the decay envelope. The frequency alone is a single scalar and can often be fitted by several combinations of missing mass, spring compliance and drag.

Locate contact loss on a smooth convex surface

A small block starts sliding from near rest just to one side of the top of a smooth fixed sphere of radius R. Measure angle θ from upward vertical. Energy gives $v^2 = 2gR(1-\cos\theta)$ during contact, while the inward radial equation reads $mg\cos\theta - N = mv^2/R$. Thus $N = mg(3\cos\theta - 2)$. Contact first vanishes at $\cos\theta = 2/3$, or 48.2°. For a sphere of radius $0.50\,\mathrm{m}$, this occurs after a vertical drop $R/3 = 0.167\,\mathrm{m}$ and at speed $1.81\,\mathrm{m/s}$. A block balanced exactly at the top at exact rest is an unstable equilibrium in the ideal model; a small initial displacement or speed initiates the measured trajectory.

Once the block leaves, the same spherical coordinates can still be used to describe position, but $r = R$ is no longer an equation of motion. Continue with the measured tangential velocity and free gravitational acceleration. Enforcing the sphere condition beyond release would make the computed normal negative and predict a surface that pulls on the block. The zero normal, the geometric release angle and the subsequent ballistic arc provide three independent observables for the same force reconstruction.

Tutorial on measuring reactions

A $0.40\,\mathrm{kg}$ pendulum of length $0.75\,\mathrm{m}$ is released from rest at 60° from downward vertical. Predict the load-cell reading at the pivot when it passes the bottom. What part does the reduced equation alone omit?

▼ **Work through the solution**

The height change is

$$l(1-\cos 60^\circ) = 0.375\,\mathrm{m}.$$

Energy conservation gives

$$v^2 = 2g(0.375) = 7.36\,\mathrm{m}^2/\mathrm{s}^2.$$

At the bottom the tension is $mg + mv^2/l = 7.85\,\mathrm{N}$, twice the weight. The tangential acceleration there vanishes because gravity points along the string, but the radial acceleration remains nonzero. A sensor measuring the string force sees this centripetal contribution directly.

A bead lies on a smooth moving plane with upwards acceleration a_z. Which acceleration first makes a passive contact impossible if the bead initially shares the plane's motion?

▼ **Work through the solution**

Vertical Newton balance requires $N - mg = ma_z$, or $N = m(g + a_z)$. At $a_z = -g$ the reaction vanishes. A more negative plane acceleration would require the plane to pull the bead down, so contact is lost and the bead follows free fall until the separation closes. For upward acceleration the normal exceeds weight. In all these cases a virtual displacement tangent to the plane produces zero virtual normal work.

Two code implementations use $(x^2 + y^2 - l^2)/2$ and $x^2 + y^2 - l^2$ as constraints for the same state. Their multipliers differ by a factor two. How can the physical result be tested?

▼ **Work through the solution**

The second gradient is twice the first at the same point. Its multiplier must be half the first multiplier for $G^T\lambda$ to coincide. Compare the reaction vector in newtons, its dot product with an independently chosen tangent direction, and the normal acceleration required by the geometry. A comparison of raw multiplier values alone would flag a difference that is entirely due to normalisation.

Compare a constrained trajectory with the motion and force calculations above.

David Tong develops virtual work and constrained coordinates in his classical dynamics course.

Euler–Poinsot motion, torque-free nutation and the intermediate axis

Analytical mechanics from constraints to invariants

A freely rotating body has fixed angular momentum in space, yet the components recorded by three body-mounted gyroscopes can vary periodically or linger near an unstable rotation. Solve the nonsymmetric top from its invariants, distinguish its body-component period from its spatial orientation, and extract a stability time scale from a measured inertia tensor.

Derive the body equations from a spatial conservation law

Choose principal axes fixed in a rigid body and label its moments $I_1 < I_2 < I_3$. In the co-rotating basis the angular-momentum components are

$$L_i = I_i \omega_i .$$

For any vector expressed in that basis, its inertial derivative is its component derivative plus $\boldsymbol{\omega} \times \mathbf{L}$. With no external torque the inertial derivative is zero. Taking components gives

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\omega_2 \omega_3 ,$$

$$I_2 \dot{\omega}_2 = (I_3 - I_1)\omega_3 \omega_1 ,$$

$$I_3 \dot{\omega}_3 = (I_1 - I_2)\omega_1 \omega_2 .$$

Multiply these equations by the corresponding ω_i and sum: the coefficients of $\omega_1\omega_2\omega_3$ cancel, giving constant $E = \sum I_i \omega_i^2 / 2$. Multiplying instead by $I_i \omega_i$ gives constant $L^2 = \sum I_i^2 \omega_i^2$. In body angular-momentum space the motion lies on the intersection of a sphere and an energy ellipsoid. At fixed L, the possible energies range from $L^2/(2I_3)$ to $L^2/(2I_1)$. The intermediate value $L^2/(2I_2)$ divides two families of closed curves and contains the unstable principal-axis orientations.

In Poinsot's construction the inertia ellipsoid in angular-velocity space rolls on an invariable plane normal to the fixed spatial angular momentum. The ellipsoid corresponding to a given energy is

$$\sum I_i \omega_i^2 = 2E .$$

Its normal at the instantaneous angular velocity is proportional to $\mathbf{L}$; the tangent plane obeys $\mathbf{L} \cdot \mathbf{x} = 2E$. In space that plane therefore has fixed distance $2E/L$ from the origin. Its instantaneous point in the body along $\boldsymbol{\omega}$ has zero instantaneous rotational velocity because the vector cross product of angular velocity with itself is zero. These facts underlie the rolling construction. A photographed body axis traces a different curve from the moving contact point or from a body-frame gyro trace.

Test all three principal rotations locally

For rotation $(0, 0, \Omega)$, linearise the first two Euler equations in transverse perturbations. Eliminating one component gives

$$\ddot{\omega}_1 = -\Omega^2[(I_3 - I_2)(I_3 - I_1)/(I_1 I_2)]\omega_1.$$

Small transverse deviations oscillate. Performing the analogous process about I_1 also gives a restoring sign. About $(0, \Omega, 0)$ the sign reverses and the growing exponent is $\lambda = |\Omega|\sqrt{(I_2 - I_1)(I_3 - I_2)/(I_1 I_3)}$. The unstable rotation is still an exact solution when both perturbations vanish identically. Instability describes the evolution of arbitrarily small disturbances, not a spontaneous violation of the torque-free equations.

For moments proportional to $1, 2, 3$ and intermediate-axis rotation $\Omega = 6.0\,\mathrm{rad/s}$, $\lambda = 3.46\,\mathrm{s}^{-1}$ and the linear e-folding time is $0.289\,\mathrm{s}$. A perturbation with a growing-mode component 10^{-3} of the spin reaches roughly 10^{-1} in $\ln(100)/\lambda = 1.33\,\mathrm{s}$. Its nonlinear fate remains bounded by the momentum sphere and energy ellipsoid. An exponential growth fit carried past the first large reorientation would therefore misrepresent the global motion. Initial perturbations projected entirely on the decaying linear eigenvector can delay the first visible flip.

Solve the low-energy family with elliptic functions

For $L^2/(2I_3) < E < L^2/(2I_2)$ the trajectory surrounds the largest-inertia axis. Start at a crossing with $\omega_2 = 0, \omega_1 = A > 0, \omega_3 = C > 0$. Solving the two invariants gives

$$A^2 = (2EI_3 - L^2)/[I_1(I_3 - I_1)],$$
$$C^2 = (L^2 - 2EI_1)/[I_3(I_3 - I_1)].$$

When ω_1 reaches zero the other transverse component has maximum $B^2 = (2EI_3 - L^2)/[I_2(I_3 - I_2)]$. Use $s = \omega_2/B$. Eliminating the other squares with the invariants gives

$$\omega_1^2 = A^2(1 - s^2),$$
$$\omega_3^2 = C^2(1 - k_e^2 s^2).$$

Euler's second equation now reduces to

$$\dot{s}^2 = \nu^2(1 - s^2)(1 - k_e^2 s^2),$$
$$\nu^2 = \frac{(I_3 - I_2)(L^2 - 2EI_1)}{I_1 I_2 I_3},$$
$$k_e^2 = \frac{(I_2 - I_1)(2EI_3 - L^2)}{(I_3 - I_2)(L^2 - 2EI_1)}.$$

Separating variables defines $u = \nu t = \int_0^s dw/\sqrt{(1 - w^2)(1 - k_e^2 w^2)}$. Its inverse is Jacobi's $\mathrm{sn}(u, k_e)$. The continuously signed square roots are cn and dn, giving $(\omega_1, \omega_2, \omega_3) = (A\,\mathrm{cn}, B\,\mathrm{sn}, C\,\mathrm{dn})$. This construction defines the special functions from the mechanical quadrature instead of assuming them as a table of answers. At $k_e = 0$ the inverse is a sine and the stabilising motion is harmonic. As $k_e \to 1$, the integral to $s = 1$ diverges logarithmically and the period grows without bound.

The body-frame period is

$$T_b = 4K(k_e)/\nu,$$

with K the same integral evaluated at one. Some software defines elliptic functions in terms of parameter $m = k_e^2$ rather than modulus k_e. Supplying one to an implementation expecting the other produces a plausible curve with the wrong period. Check the derivative at the zero crossing $\dot{\omega}_2(0) = B\nu$ and the time of its first maximum K/ν before fitting a long trace. The third component dn repeats after half the full period, so a single principal-axis gyro can report a dominant frequency twice that of the full body vector.

Reconstruct the observed nutation angle

Orient the inertial vertical along $\mathbf{L}$. The angle between body axis three and this fixed direction obeys $\cos\theta = L_3/L = I_3\omega_3/L$. For $I_1, I_2, I_3 = (1, 2, 3)\,\mathrm{kg\,m^2}$ and initial $\boldsymbol{\omega} = (0.50, 0, 1.00)\,\mathrm{rad/s}$, the invariants give $L^2 = 9.25$, $2E = 3.25$ in SI, $A = B = 0.50\,\mathrm{rad/s}$, $C = 1.00\,\mathrm{rad/s}$, $\nu = 1.00\,\mathrm{s^{-1}}$, and $k_e^2 = 1/12$. Thus the nutation angle changes from $\arccos(3/\sqrt{9.25}) \approx 9.46^\circ$ to $\arccos[3\sqrt{11/12}/\sqrt{9.25}] \approx 19.2^\circ$. A torque is unnecessary for this change in the angle of a body axis relative to fixed $\mathbf{L}$.

The principal-axis components alone determine body orientation only when supplemented with an initial attitude and the kinematic equation. For the Euler-angle convention with polar axis $\mathbf{L}$, the azimuthal rate of body axis three is $\dot{\phi} = L(L_1^2/I_1 + L_2^2/I_2)/(L_1^2 + L_2^2)$ away from the polar singularity. It follows by differentiating the body-axis unit vector in space and projecting onto its azimuthal direction. A complete period of the body components therefore leaves a finite spatial rotation about fixed $\mathbf{L}$. Return of the three gyro readings does not imply that a painted mark has returned to its starting position in the laboratory.

Solve the high-energy family and join both sides at the separatrix

For $L^2/(2I_2) < E < L^2/(2I_1)$ the closed curves on the angular-momentum sphere surround axis one. The component ω_1 now keeps its sign while ω_3 changes sign through a cycle. As in the first family, choose a phase with $\omega_2 = 0$ and both other components positive. The two invariants give the same initial $A^2 = (2EI_3 - L^2)/[I_1(I_3 - I_1)]$ and $C^2 = (L^2 - 2EI_1)/[I_3(I_3 - I_1)]$. Now the maximum of ω_2 occurs when $\omega_3 = 0$, so

$$B^2 = (L^2 - 2EI_1)/[I_2(I_2 - I_1)].$$

Which component reaches zero is the physical branch choice that determines the arrangement of elliptic functions.

Let $s = \omega_2/B$. The invariant equations give

$$\omega_3^2 = C^2(1 - s^2),$$
$$\omega_1^2 = A^2(1 - k_h^2 s^2).$$

Substitution in the second Euler equation yields the same elliptic quadrature but with $\nu_h^2 = (I_2 - I_1)(2EI_3 - L^2)/(I_1 I_2 I_3)$ and $k_h^2 = (I_3 - I_2)(L^2 - 2EI_1)/[(I_2 - I_1)(2EI_3 - L^2)]$. All components are given by $(A\,\mathrm{dn}, B\,\mathrm{sn}, C\,\mathrm{cn})(\nu_h t, k_h)$, where the notation means each function is evaluated at the same time and modulus. On this energy side $0 < k_h < 1$. The low-energy modulus formula evaluated here would exceed one, signalling that the original choice of component scaling and physical extrema must be changed.

For the same inertias $1, 2, 3$, choose initial $(1, 0, 0.20)\,\mathrm{rad/s}$. Then $L^2 = 1.36,\ 2E = 1.12$, and the high-energy inequality holds since $2E > L^2/I_2 = 0.68$. The formulas give

$$A = 1,\ B = \sqrt{0.12} = 0.3464,\ C = 0.20,\ \nu_h = \sqrt{1/3} = 0.5774\,\mathrm{s}^{-1},\ k_h^2 = 0.12.$$

The minimum ω_1 is $A\sqrt{1 - k_h^2} = 0.9381\,\mathrm{rad/s}$, so that principal gyro never reverses even though the third one does. A body-marking aligned with axis three can undergo a conspicuous reversal while the dominant axis-one spin changes only modestly. These component predictions are more specific than an energy or angular-speed trace.

At $2E = L^2/I_2$, both moduli approach one. The definitions give

$$\mathrm{sn}(u, 1) = \tanh u,$$
$$\mathrm{cn}(u, 1) = \mathrm{dn}(u, 1) = \mathrm{sech}\, u.$$

One separatrix solution therefore has $\omega_2 = B\tanh(\nu t)$ with $\omega_1 = A\,\mathrm{sech}(\nu t)$ and $\omega_3 = C\,\mathrm{sech}(\nu t)$, up to signs and time shifts consistent with Euler's equations. It connects opposite steady orientations along axis two as t goes from negative to positive infinity. Its asymptotic approach is exponential, with ν equal to the small-disturbance growth exponent at the intermediate-axis equilibrium after inserting $B = L/I_2$. A nearby periodic orbit spends much of its time near the unstable spin and flips more rapidly between those intervals.

Reconstruct spatial attitude without a singular azimuth

Let $R(t)$ map body components into a laboratory basis. For body angular velocity $\boldsymbol{\omega}_b$, its kinematic equation is

$$\dot{R} = R[\boldsymbol{\omega}_b]_\times,$$

where the skew matrix acts by a cross product. Starting from measured $R(0)$, propagate by multiplying by the rotation appropriate to each short time interval. Exponentiating the skew matrix for an interval with constant estimated body angular velocity preserves orthogonality and determinant exactly. Applying

a naive forward-Euler matrix addition $R + hR[\omega]_\times$ generally stretches the basis, even if the angular velocity components came from an exact elliptic solution.

For a constant rotation about axis three at one radian per second, a single Euler step of duration $0.10\,\mathrm{s}$ takes a transverse unit vector to one of length $\sqrt{1 + 0.10^2} = 1.00499$. After one hundred such steps the transverse length would grow by $(1.01)^{50} \approx 1.64$. A quaternion or matrix exponential avoids that artificial stretch. Normalising each axis separately is insufficient if the axes lose orthogonality; check $R^T R$ against identity and $\det R$ against one. Also compute $R(I_1\omega_1, I_2\omega_2, I_3\omega_3)^T$ at every time. It should give the same spatial vector in a torque-free run.

A measured attitude time series allows a different inference from fitting Euler angles separately. Use relative rotations between successive images to estimate angular velocity in the correct body or spatial frame. A camera can momentarily lose sensitivity to yaw when a marked axis points towards it; simultaneous orthogonal views or multiple non-collinear marks avoid that degeneracy. The gyro and attitude measurements together distinguish a true physical precession from an Euler-angle wrap by 2π. Synchronise their clocks before comparing the phase of a nutation extremum, since even a small time delay produces a plausible-looking mismatch near a fast flip.

Infer inertia ratios from a free-rotation record

A calibrated triad records $\omega_i(t)$ in axes thought to be principal. Euler's equations predict that $\dot\omega_1/(\omega_2\omega_3)$ is the constant $(I_2 - I_3)/I_1$ wherever the denominator is appreciable, and similarly for cyclic indices. Instead of dividing noisy derivatives near zeros, integrate each component over a time interval. For example $\omega_1(t_2) - \omega_1(t_1) = a_1 \int_{t_1}^{t_2} \omega_2\omega_3\, dt$. Fit a_1 over several intervals and repeat for a_2, a_3. Select intervals with different angular configurations so that products do not all average nearly to zero. The ratios depend on inertia shape; multiplying every moment by a common constant leaves a torque-free angular-velocity record unchanged.

For a record giving $a_1 = -1.0$, $a_2 = 1.0$ and $a_3 = -1/3$, solve the homogeneous equations $I_2 - I_3 = -I_1$, $I_3 - I_1 = I_2$, $I_1 - I_2 = -I_3/3$. They give $I_1 : I_2 : I_3 = 1 : 2 : 3$. An independent angular-momentum or torque calibration is needed to set the absolute scale. A known short applied torque impulse that changes one principal angular velocity by a measured amount can do this, provided the other gyroscopic contributions over that impulse are small or accounted for. A fit that estimates all absolute inertias from angular velocity alone can have an apparently excellent residual and still be structurally underdetermined.

If the sensors are not along principal axes the inertia matrix contains off-diagonal entries. The coordinate-free body equation is

$$I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\boldsymbol{\omega}) = 0.$$

Each measured time sample gives equations linear in the constant entries of the symmetric inertia tensor. With sufficiently rich motion one may estimate these entries up to scale, then diagonalise the fitted tensor to recover principal directions. Pure rotation along one axis gives insufficient excitation to reveal three distinct directions. Comparing a family of initial spins with a single tensor provides a stronger validity test than fitting a different set of effective principal inertias to each record.

Relate a body-frame period to a measured geometric phase

After a closed polhode in body angular-momentum space the laboratory orientation differs by a rotation about the fixed spatial momentum. That phase can be separated into a term accumulated at a dynamical rate and a term depending on the oriented solid angle of the body momentum path. For a fixed L, the angular-velocity component along the momentum is $\boldsymbol{\omega}\cdot\hat{L} = 2E/L$, since $\mathbf{L}\cdot\boldsymbol{\omega} = 2E$. Over one body period it contributes $2ET_b/L$. Parallel transport of a transverse reference direction along the unit momentum curve contributes minus the oriented solid angle enclosed, up to whole turns and the chosen orientation of the sphere. The resulting net attitude phase is $2ET_b/L - \Omega_{\mathrm{solid}} \pmod{2\pi}$.

The geometric part can be visualised by carrying a local tangent basis along the closed curve. Its two basis vectors are constrained to stay perpendicular to the moving unit momentum and are transported with no imposed twist around that direction. A return to the initial unit momentum need not return that basis to its starting orientation. For a latitude circle on the unit sphere the enclosed cap area is $2\pi(1-\cos\theta)$; shrinking the path to a point makes the geometric phase vanish modulo 2π. This is why comparing a painted mark after a nutation only to an integrated scalar spin speed can give a discrepancy even with exact torque-free motion.

In practice the total phase can be measured without choosing a cap convention. Fit the attitude at two times separated by the full gyro period, form $R(t+T_b)R(t)^{T}$, and extract the relative rotation around the known fixed spatial momentum. Independently integrate $2E/L$ over that interval and compare the difference modulo 2π with the signed spherical area traced by normalised body momentum. A factor-of-two error in the identified period, common if only ω_3 is recorded in the low-energy family, generally leaves the relative attitude rotation axis different from $\mathbf{L}$. The axis check can reveal the error before assigning a numerical geometric phase.

Identify which inertias can exist in an actual rigid body

Three positive principal moments inserted into Euler's equations define a mathematical top, but the moments of a physical nonnegative mass distribution also satisfy triangle inequalities. In principal coordinates $I_1 = \int (y^2+z^2)\,dm$,

$$I_2 = \int (z^2+x^2)\,dm,$$

$$I_3 = \int (x^2+y^2)\,dm.$$

Therefore $I_1 + I_2 - I_3 = 2\int z^2\,dm \geq 0$, and likewise by cyclic permutation. Equality means all mass is confined to the corresponding coordinate plane. The convenient example $1, 2, 3$ lies on that boundary and can be realised by an ideal planar lamina. A fitted ratio $1, 2, 3.5$ would imply negative mean squared distance along one direction and should prompt a calibration or modelling check.

A uniform thin rectangular plate of mass M and side lengths b along body axis one and a along body axis two has $I_1 = Ma^2/12,\ I_2 = Mb^2/12,\ I_3 = M(a^2+b^2)/12$. Choosing $b = \sqrt{2}\,a$ gives exactly $1:2:3$. For $M = 0.60\,\mathrm{kg}$, $a = 0.20\,\mathrm{m}$, $b = 0.283\,\mathrm{m}$, the moments are approximately $0.0020, 0.0040, 0.0060\,\mathrm{kg\,m^2}$. The component equations for these moments give the same time

histories as the scaled numerical example for identical initial angular velocities, while the actual angular momentum and energy are smaller by a factor 500. This gives a physical specimen for checking principal-axis stability without requiring a massive rotor.

Adding a small mass at a known location changes both inertia and possibly the centre of mass. If a symmetric pair of equal small masses is attached at $\pm d\hat{e}_3$, their combined mass m_a adds $m_a d^2$ to I_1 and I_2, nothing to I_3, and leaves the centre fixed. For the plate above and $m_a = 0.10\,\mathrm{kg}, d = 0.10\,\mathrm{m}$, the new moments are $0.0030, 0.0050, 0.0060\,\mathrm{kg\,m^2}$. The intermediate-axis growth exponent at fixed Ω changes from $|\Omega|/\sqrt{3}$ to $|\Omega|\sqrt{(0.005-0.003)(0.006-0.005)/(0.003\cdot 0.006)} = |\Omega|/3$. A reversible mass-placement experiment tests the inertia dependence of instability, not merely the existence of a visually dramatic flip.

Test intermediate-axis stability with the full body equations

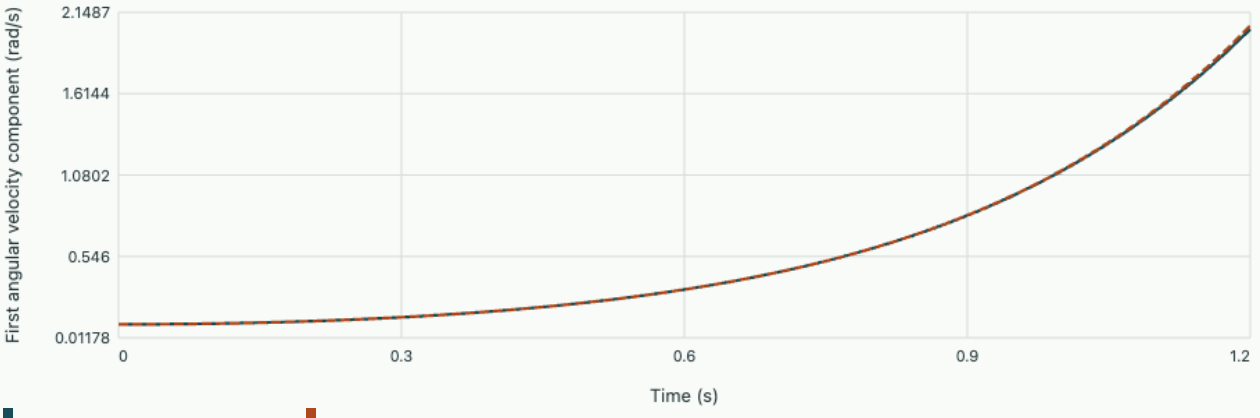
Change the initial tilt and spin. Compare the linear perturbation and the complete torque-free motion while monitoring energy and angular momentum in the OneSimulation output.

Open this experiment online

Try the calculation

Nominal middle-axis speed (rad/s) **12**

Initial transverse speed (rad/s) **0.1**



Initial linear growth rate 3.0984 s⁻¹. With zero initial transverse slope the predicted first doubling time is 0.42505 s.

Download the OneSimulation model

In your OneSimulation environment, run the model with these inputs.

```
python spectra-and-orbits.py rotor --spin 12 --seed 0.1
```

Estimate distance from a separatrix using dwell time

Near the intermediate-axis energy, elliptic modulus approaches one on either side. Writing $d = 1 - k_e^2$ on the low-energy family gives $K \sim \ln(4/\sqrt{d})$. Thus $T_b \sim 4\ln(4/\sqrt{d})/\nu$ for a slowly varying scale ν. Changing d by a factor of one hundred increases the expected body period by approximately $4\ln 10/\nu$. A tiny difference in the initial inertia-weighted energy can produce a readily observed increase in the time between reorientation events while the maximum angular speed scarcely changes. The logarithm follows from passages near a hyperbolic equilibrium, where the signed unstable perturbation grows exponentially.

If $\nu = 3.0\,\mathrm{s}^{-1}$ and $d = 10^{-4}$, the leading estimate is

$$T_b = (4/3)\ln 400 \approx 7.99\,\mathrm{s}.$$

Reducing d to 10^{-6} gives $T_b \approx 11.06\,\mathrm{s}$. Inverting this relation makes estimates of d exponentially sensitive to a period offset. An uncertain start or stop event of a few tenths of a second, or a weak external torque during a long dwell, can affect an inferred separatrix energy considerably. Report the uncertainty of $E - L^2/(2I_2)$ as well as the fitted period when claiming close approach to the unstable family.

One experimental check is to use both the crossing speed of $\omega_2 = 0$ and the time between successive such crossings. At a low-energy upward crossing the slope has magnitude $B\nu$; it remains finite near the separatrix even while the period diverges. A simple global slowdown from increased bearing drag tends to modify the crossing speed as well as the dwell. A correctly prepared near-separatrix torque-free motion has relatively rapid crossings separated by long intervals close to intermediate-axis rotation. The distinction is visible in the time domain and does not depend on estimating a derivative at the slowest, noisiest part of the record.

A fit places a nominally torque-free experiment slightly above the separatrix when E, L are computed from gyros, but below it when inferred from the oscillating component pattern. What should be checked first?

▼ Work through the solution

Recalculate both invariants from the same time stamps, calibrated orientation and inertia axes. The energy classification subtracts comparable quantities and can be sensitive to cross-axis bias. The low-energy family keeps its third component's sign, whereas the high-energy family keeps its first component's sign with the stated ordering. Verify that a full body period was actually recorded and that sign changes were resolvable. Persistent disagreement after uncertainty and sampling checks can indicate a slow torque, changing mass distribution or a misspecified principal-axis alignment.

Quantify a small external torque before fitting a free-top period

Suppose the laboratory angular momentum has magnitude $3.0\,\mathrm{kg\,m^2/s}$ and an approximately constant transverse torque $0.010\,\mathrm{N\,m}$ acts for a $6.0\,\mathrm{s}$ body period. The accumulated transverse impulse is

$0.060\,\mathrm{kg\,m^2/s}$, enough to tilt the spatial angular-momentum direction by roughly $0.020\,\mathrm{rad}$, or 1.15°. Its magnitude changes only to second order if the torque remains perpendicular to the initial momentum. A check using only L^2 could therefore show a small relative error while the invariable plane has moved perceptibly.

Use two marked inertial reference directions to track the spatial momentum vector and plot its Cartesian increments against the integrated measured torque. When gravity acts about a support whose line passes through the mass centre, its torque vanishes ideally, but a small offset can supply just such a transverse disturbance. Reversing the initial spin while retaining the support geometry helps distinguish a fixed parasitic torque from gyro bias. If the spatial direction changes appreciably during one long near-separatrix dwell, fitting a torque-free elliptic period to the complete signal becomes inaccurate even if the early, fast gyro crossings agree with the free model.

Tutorial on a torque-free body's time scales

What period should be measured for the transverse gyro in the worked example? Use the small-parameter expansion $K = (\pi/2)[1 + m/4 + 9m^2/64 + \cdots]$ with $m = k_e^2$.

▼ **Work through the solution**

Substitution of $m = 1/12$ gives $K \approx 1.605$ and $T_b \approx 6.42\,\mathrm{s}$. The third-axis component repeats after about $3.21\,\mathrm{s}$, while the second component first reaches its positive maximum at $1.61\,\mathrm{s}$. Checking all three features tests the phase relation among the gyros and helps identify a sensor axis swap.

For a body with $I_1 = I_2 < I_3$, how do Euler's equations simplify and which feature of the asymmetric example disappears?

▼ **Work through the solution**

The third body component remains exactly constant, and the first two rotate sinusoidally at body-frame frequency $(I_3 - I_1)|\omega_3|/I_1$. The polar angle to fixed angular momentum is constant. A body axis can still precess in space, but the varying torque-free polar nutation caused by distinct transverse inertias disappears. Taking this limit in the low-energy solution gives $k_e \to 0$ consistently.

A gyro experiment reports slow amplitude decay along with fixed spatial angular momentum. Can that arise in a strictly rigid, torque-free model conserving kinetic energy?

▼ **Work through the solution**

The two rigid-body invariants fix the allowed intersection curve, so a persistent drift between such curves requires an effect beyond that ideal model or a changing calibration. Internal dissipation in a deformable body can decrease kinetic energy while conserving total spatial angular momentum, favouring rotation about the largest-inertia axis. An external torque could instead change angular momentum. Measuring both invariants helps distinguish the two explanations.

Compare principal-axis stability before fitting an asymmetric-top gyro trace.

David Tong derives the Euler equations, Euler angles and free-top motion.

The heavy symmetric top, precession branches and the sleeping state

Analytical mechanics from constraints to invariants

Gravity supplies a torque to a top pivoted away from its centre of mass. Its axis may precess steadily, nutate between two turning angles, or trace loops. Derive all three from the same angular kinetic energy and determine when an upright spinning top can remain stable.

Project gravity onto a body with one fixed point

Let $I_1 = I_2 = I$ and I_3 be moments about the pivot, with the centre of mass a distance l along the symmetry axis. Use Euler angles θ, ϕ, ψ, where θ measures the tilt from upward laboratory vertical, ϕ measures the azimuth of the symmetry axis, and ψ spins the body about that axis relative to the nodal line. The centre-of-mass height is $l\cos\theta$, so gravity contributes $V = Mgl\cos\theta$. In the body the spin component of angular velocity is $\omega_3 = \dot\psi + \dot\phi\cos\theta$. The squared transverse component is $\dot\theta^2 + \dot\phi^2\sin^2\theta$. Consequently

$$L = \tfrac{1}{2}I(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \tfrac{1}{2}I_3(\dot\psi + \dot\phi\cos\theta)^2 - Mgl\cos\theta.$$

The angles ϕ and ψ are absent themselves. Their momenta $L_z = I\dot\phi\sin^2\theta + L_3\cos\theta$ and $L_3 = I_3\omega_3$ are constants. Gravity torque about the pivot is perpendicular to the vertical and to the body symmetry axis, consistent with these two conservation laws. The entire vector $\mathbf{L}$ need not be constant. Distinguish the axial spin ω_3 measured by a body gyro from the Euler rate $\dot\psi$. A precessing top can have appreciable $\dot\phi\cos\theta$; using the painted-mark rate in a spin estimate without specifying the reference can bias a precession prediction.

Eliminate cyclic velocities to find the nutation potential

Solve $\dot\phi = (L_z - L_3\cos\theta)/(I\sin^2\theta)$ and $\dot\psi = L_3/I_3 - \dot\phi\cos\theta$. The energy can now be written $E = L_3^2/(2I_3) + I\dot\theta^2/2 + V_{\text{eff}}(\theta)$, with

$$V_{\text{eff}}(\theta) = Mgl\cos\theta + \frac{(L_z - L_3\cos\theta)^2}{2I\sin^2\theta}.$$

The second term is not an additional gravitational interaction. It is the transverse rotational energy required by the two fixed angular momenta at each tilt. A root of $E - L_3^2/(2I_3) - V_{\text{eff}} = 0$ is a polar turning angle; a simple root is generally reached and reversed in finite time. A minimum gives stable regular precession under sufficiently small nutations at fixed conserved momenta. The apparent divergence at a pole disappears if the numerator vanishes in the same limit. The possibility of the upright sleeping state therefore requires

$$L_z = L_3.$$

Putting $u = \cos\theta$ gives a polynomial first integral. With $E' = E - L_3^2/(2I_3)$, one obtains

$$I^2\dot{u}^2 = 2I(E' - Mglu)(1-u^2) - (L_z - L_3 u)^2 = P_3(u).$$

The accessible range lies within $[-1,1]$ where $P_3 \geq 0$. The physical turning points can be found without assigning an Euler azimuth at a pole. A nutation period is $2I\int_{u_{\min}}^{u_{\max}} du/\sqrt{P_3(u)}$. Computing the azimuthal advance over the same cycle requires averaging $\dot{\phi}(u)$ with the same time weight. An arithmetic average of the two turning-point precession rates misses the longer dwell times near the endpoints.

Find slow and fast regular precession

For constant tilt and constant $\dot{\phi} = \Omega$, the θ equation at a nonzero tilt gives

$$I\Omega^2\cos\theta - L_3\Omega + Mgl = 0.$$

This relation follows directly by differentiating the Lagrangian with respect to θ, using $\partial_\theta[\dot{\psi} + \dot{\phi}\cos\theta] = -\dot{\phi}\sin\theta$. It is quadratic in precession speed, so the often-used $\Omega \simeq Mgl/L_3$ is only one limit. For a top above the horizontal plane and $L_3 > 0$, two positive precession branches exist if $L_3^2 \geq 4IMgl\cos\theta$. Their sum is $L_3/(I\cos\theta)$, and their product is $Mgl/(I\cos\theta)$.

Take $I = 0.020\,\mathrm{kg\,m^2}$, $I_3 = 0.010\,\mathrm{kg\,m^2}$, $Mgl = 0.200\,\mathrm{N\,m}$,

$$L_3 = 0.200\,\mathrm{kg\,m^2/s},$$

$$\theta = 60^\circ.$$

The quadratic is $0.010\Omega^2 - 0.200\Omega + 0.200 = 0$, with roots 1.056 and $18.94\,\mathrm{rad/s}$. Both have the same material spin $\omega_3 = 20\,\mathrm{rad/s}$, but their vertical angular momenta are different. The slow branch has $L_z = 0.020(1.056)(0.75) + 0.200(0.5) = 0.1158\,\mathrm{kg\,m^2/s}$; the fast branch has about $0.384\,\mathrm{kg\,m^2/s}$. Preparing a tilt and a spin alone therefore does not select a unique regular precession. The initial angular velocity of the tilted axis also matters.

Distinguish looping, cusped and monotone azimuth traces

During nutation the sign of $\dot{\phi}$ is the sign of $L_z - L_3 u$. If $u_* = L_z/L_3$ lies strictly between the turning points, the azimuth reverses along each nutation and the axis can trace loops on the unit sphere. If u_* equals a turning point, both $\dot{\theta}$ and $\dot{\phi}$ can vanish together there, producing a cusp in the ideal trace. If the numerator keeps one sign the azimuth advances monotonically even as tilt oscillates. Seeing an apparent loop in a two-dimensional camera projection is insufficient by itself; recover the three-dimensional orientation or check the signed azimuthal rate.

One classic release starts with a tilted axis that initially has no angular velocity except spin about its own direction. Then $L_z = L_3\cos\theta_0$, $\dot{\phi} = 0$, $\dot{\theta} = 0$. The gravitational torque starts tilting the axis down; on returning to that initial turning point the projected angular speed again momentarily vanishes. By contrast, giving the top the correct nonzero precession velocity at release puts it on a regular precession

branch with no initial nutation. Identical tilt and spin measurements can thus yield dramatically different trajectories if the release mechanism imparts a different transverse angular velocity.

Derive the upright stability threshold

At $L_z = L_3$, reduce the ratio $(1-\cos\theta)^2/\sin^2\theta$ to $(1-\cos\theta)/(1+\cos\theta)$. Near zero tilt the effective potential is $Mgl + [L_3^2/(8I) - Mgl/2]\theta^2 + O(\theta^4)$. A positive coefficient gives an energy minimum and requires $L_3^2 > 4IMgl$. For the preceding I, Mgl this critical angular momentum is $0.1265\,\mathrm{kg\,m^2/s}$, or an axial spin speed $12.65\,\mathrm{rad/s}$ for the chosen I_3. A high spin stabilises the upper position by coupling tilt to conservation of angular momentum. The purely gravitational potential by itself has a maximum there.

The fixed-$L_z = L_3$ quadratic energy curvature describes a polar disturbance within that momentum class. Near the axis ϕ, ψ individually become coordinate singularities. To analyse arbitrary small perturbations, use horizontal components of the unit axis and linearise their coupled gyroscopic equations. The two circular precession frequencies at the pole solve $I\Omega^2 - L_3\Omega + Mgl = 0$, giving the same discriminant. Exactly at threshold the linear frequencies coalesce, and nonlinear terms or dissipation can dominate a finite disturbance. Distinguishing a stable ideal top from an experimentally long-lived slowly slowing top requires monitoring spin over the entire observation interval.

Measure the nutation frequency of a regular precession branch

Take any regular precession tilted away from the poles, with prescribed constants L_z, L_3. A small perturbation of the polar angle gives $I\ddot{\eta} + V_{\mathrm{eff}}''(\theta_0)\eta = 0$, so the nutation frequency at fixed momenta follows from the second derivative of the effective potential. Differentiating while holding the instantaneous $\dot{\phi}$ fixed would give a different result, because a displaced top with the same momenta must change its precession rate. This distinction matters when preparing a perturbation: a small tap that adds a transverse angular impulse can change the constants as well as the tilt.

A convenient explicit formula follows by writing $F(\theta) = L_z - L_3\cos\theta$,

$$w(\theta) = F/(I\sin^2\theta).$$

Then $V_{\mathrm{eff}}' = \sin\theta[-Mgl + L_3 w - Iw^2\cos\theta]$. At a regular solution the bracket vanishes. Differentiate w as $w' = (L_3/I - 2w\cos\theta)/\sin\theta$, substitute into the derivative of the bracket, and obtain

$$\omega_{\mathrm{nut}}^2 = \frac{V_{\mathrm{eff}}''(\theta_0)}{I} = \left(\frac{L_3}{I} - 2\Omega\cos\theta_0\right)^2 + \Omega^2\sin^2\theta_0.$$

For the 60° example,

$$L_3/I = 10.0\,\mathrm{s^{-1}}.$$

On the slow branch $\Omega = 1.056\,\mathrm{s}^{-1}$, this gives $\omega_{\mathrm{nut}} \approx 8.99\,\mathrm{s}^{-1}$, or a nutation period of $0.699\,\mathrm{s}$. On the fast branch $\Omega = 18.94\,\mathrm{s}^{-1}$, it gives approximately $18.7\,\mathrm{s}^{-1}$, or $0.336\,\mathrm{s}$. Both tilted regular solutions have positive fixed-momentum curvature in this example. Comparing just the precession rates would miss the different small oscillations around their two reduced equilibria.

Suppose a small angular displacement conserves L_z, L_3 on the slow branch. The correlated rate shift is $w'\delta\theta$. At the example point $w' = (10 - 1.056)/\sin 60^\circ = 10.33\,\mathrm{s}^{-1}/\mathrm{rad}$. A nutation of $0.020\,\mathrm{rad}$ therefore modulates the precession rate by about $0.207\,\mathrm{rad/s}$, a substantial fraction of its slow steady value. A low-resolution video that measures azimuth per frame can show a conspicuous modulation even when the nutation angle itself is visually small. Filtering only one angle and fitting it with constant angular speed would hide this dynamical correlation.

Recover a nutation trajectory from energy and two turn angles

Suppose a measured top has $I = 0.020\,\mathrm{kg\,m^2}$, $Mgl = 0.20\,\mathrm{J}$, $L_3 = 0.20\,\mathrm{kg\,m^2/s}$, and is released without precession or nutation at $\cos\theta_0 = 0.50$. Its constants are $L_z = 0.10\,\mathrm{kg\,m^2/s}$ and $E' = 0.10\,\mathrm{J}$. The polar polynomial becomes

$$P_3(u) = 0.04(0.10 - 0.20u)(1 - u^2) - (0.10 - 0.20u)^2.$$

Factoring out $0.10 - 0.20u$ leaves $0.04(1 - u^2) - (0.10 - 0.20u)$. The other roots satisfy $0.04u^2 - 0.20u + 0.06 = 0$, giving $u \simeq 0.3206$ and $u \simeq 4.68$. Only the first lies within the physical interval. Thus the tilt runs from 60° to about 71.3°, and the unphysical cubic root must be discarded.

At the initial $u = 0.50$, the azimuthal numerator vanishes and the top momentarily has no precession. At the lower turning $u \simeq 0.3206$, the precession rate is $(0.10 - 0.20u)/[0.020(1 - u^2)] \approx 2.0\,\mathrm{rad/s}$. These two measurements together discriminate the cusped release from regular slow precession. A detector reporting only a mean precession rate near one radian per second could confuse them. Timing the first turning angle after release offers an independent check of Mgl/I, while a direct gyro constrains L_3/I_3.

To evaluate the nutation quadrature numerically without endpoint difficulties, substitute $u = u_{\min} + (u_{\max} - u_{\min})\sin^2\alpha$. As α approaches either endpoint, du cancels the simple square-root singularity from P_3. Integrate once with numerator one for time and once with the extra factor $\dot\phi(u)$ for azimuth. Repeat after halving the quadrature spacing. An exactly repeated root signals a limiting regular solution or a separatrix; the simple-turning substitution alone cannot regularise an actual logarithmically divergent period. In the cusped trajectory above the angular trace is continuous at the polar turning, even though the tangent to that trace can vanish.

Use horizontal tilt coordinates at the sleeping pole

Near the upright orientation the separate Euler angles ϕ, ψ are ill conditioned. Instead let the symmetry-axis unit vector be $\mathbf{n} = (x, y, 1) + O(x^2 + y^2)$ in an inertial frame with z upward. Here x, y are small dimensionless tilts, not translations. The instantaneous transverse angular velocity is

$\mathbf{n} \times \dot{\mathbf{n}} \approx (-\dot{y}, \dot{x}, 0)$, because $(\mathbf{n} \times \dot{\mathbf{n}}) \times \mathbf{n} = \dot{\mathbf{n}}$. The laboratory angular momentum to first order is $I\mathbf{n} \times \dot{\mathbf{n}} + L_3 \mathbf{n}$. Its derivative equals the gravitational torque $Mgl\,\hat{z} \times \mathbf{n}$. Equating the two horizontal components yields

$$I\ddot{x} + L_3 \dot{y} - Mgl\,x = 0,$$
$$I\ddot{y} - L_3 \dot{x} - Mgl\,y = 0.$$

Let $w = x + iy$. These combine into $I\ddot{w} - iL_3\dot{w} - Mgl\,w = 0$. Substituting $w \propto e^{i\Omega t}$ returns $I\Omega^2 - L_3\Omega + Mgl = 0$ without ever assigning azimuth to the exact pole. For spin strictly above threshold, a general small tilt is a superposition of two circular motions of the two real frequencies. Their vector sum traces a wobble. For spin below threshold, the two frequencies form a complex conjugate pair and one mode grows exponentially. A perfectly vertical initial state can remain exactly vertical in an ideal simulation on either side, so stability requires a nonzero test perturbation.

Take a below-threshold spin $L_3 = 0.100\,\mathrm{kg\,m^2/s}$ with the earlier $I = 0.020$ and $Mgl = 0.200$. The real precession rate is $L_3/(2I) = 2.50\,\mathrm{s^{-1}}$ and the exponential growth rate is

$$\sqrt{4IMgl - L_3^2}/(2I) = 1.94\,\mathrm{s^{-1}}.$$

If the growing-mode component starts at $1\,\mathrm{mrad}$, a $20\,\mathrm{mrad}$ excursion takes about $\ln 20/1.94 = 1.55\,\mathrm{s}$ within the linear model. Below threshold the top need not collapse immediately after release. A short video can show several degrees of apparently orderly precession before growing tilt becomes unmistakable.

Just above threshold the mode frequencies are nearly degenerate. Decomposing noisy initial conditions into two circular amplitudes then involves division by their small frequency separation. A finite tilt impulse can produce a large beat envelope even though both ideal modes remain bounded. At exactly the repeated root the general linear solution contains $te^{i\Omega t}$ as well as $e^{i\Omega t}$; neutral-looking roots alone do not ensure boundedness. This distinction matters when inferring a critical spin from finite observation windows. Report the initial perturbation and the tilt threshold used to define loss of stability.

Infer the centre-of-mass offset from controlled precession

If regular precession can be prepared at a known tilt, the quadratic balance gives

$$Mgl = L_3\Omega - I\Omega^2 \cos\theta.$$

Neglecting the second term is customary only on a sufficiently slow branch. With the slow 60° case the neglected term is about $0.0112\,\mathrm{N\,m}$, or 5.6% of the true gravitational torque scale. Inferring the centre-of-mass offset by using $l = L_3\Omega/(Mg)$ would overestimate l by this fraction. On the fast branch the two terms nearly cancel and a small error in I or Ω can be magnified substantially in the inferred Mgl.

For a top of measured mass $0.50\,\mathrm{kg}$ and $Mgl = 0.200\,\mathrm{N\,m}$, the offset is $0.0408\,\mathrm{m}$. An uncertainty of $0.002\,\mathrm{N\,m}$ in gravitational torque corresponds to about $0.41\,\mathrm{mm}$ before mass and local gravity uncertainty. Repeat at several known tilts while measuring body spin and laboratory precession

individually. A consistent l across slow and fast branches checks the kinetic-energy model, whereas a gradual decrease of apparent L_3 during a single run suggests pivot drag or aerodynamic torque. A static suspension measurement of the centre of mass is a useful independent reference when estimating inertias from dynamic data.

Verify the torque law with a horizontal gyroscope

An instructive limiting experiment supports one end of a spinning wheel axle at a pivot, leaving its symmetry axis nearly horizontal. Choose the centre of mass along the free end of the axle and define the wheel's axial angular momentum by the right-hand rule. The gravitational torque is $\boldsymbol{\tau} = Mgl\,\hat{z} \times \mathbf{n}$. If $\mathbf{L}$ is approximately $L_3\mathbf{n}$, a regular precession at laboratory angular speed Ω gives

$$\dot{\mathbf{L}} \approx L_3 \Omega\, \hat{z} \times \mathbf{n}.$$

Equating the torque gives

$$\Omega = Mgl / L_3.$$

At exactly horizontal tilt this agrees with the finite root of the full quadratic; slightly away from horizontal the small transverse angular momentum affects the quantitative result.

For a wheel with $I_3 = 0.012\,\mathrm{kg\,m^2}$, axial spin $40\,\mathrm{rad/s}$, mass $0.40\,\mathrm{kg}$ and pivot offset $0.060\,\mathrm{m}$,

$$L_3 = 0.480\,\mathrm{kg\,m^2/s},$$
$$Mgl = 0.235\,\mathrm{N\,m}.$$

Horizontal precession is $0.490\,\mathrm{rad/s}$, one revolution in $12.8\,\mathrm{s}$. Reversing the spin reverses the precession direction. Moving the pivot to the other side of the centre also reverses the gravitational torque for a fixed axis convention. Testing each reversal separately helps distinguish the true gyroscopic response from a static torque due to a tilted support or a twisting suspension lead.

The phrase that gravity causes the wheel to stay up hides a vertical support force and a changing horizontal angular momentum. The pivot supplies most or all of the vertical force required to support the wheel's weight. With steady horizontal precession, the centre of mass also accelerates horizontally toward the precession axis with magnitude $l\Omega^2$. Thus the pivot must supply a horizontal radial reaction $Ml\Omega^2$. For the numerical wheel this is only $0.0058\,\mathrm{N}$, much smaller than the $3.92\,\mathrm{N}$ vertical support. A torque signal can be easy to see through slow precession while the corresponding horizontal reaction is weak. A load cell measuring only vertical force would not directly measure the gravitational torque acting about the pivot.

Spin friction alters the interpretation of a long run. Suppose the measured axial spin falls from 40 to $35\,\mathrm{rad/s}$ over one minute and the axis remains close to horizontal regular precession. The predicted precession speed rises from 0.490 to $0.561\,\mathrm{rad/s}$, even though mechanical energy is being dissipated. This is not negative damping. Less axial angular momentum must change direction under essentially the same gravitational torque, so its direction turns more rapidly. If the spin-down time becomes comparable

with a nutation period, an instantaneous regular-precession approximation is no longer sufficient; include dissipative torques in the full equations and measure both tilt and angular momentum.

Identify the precession branches

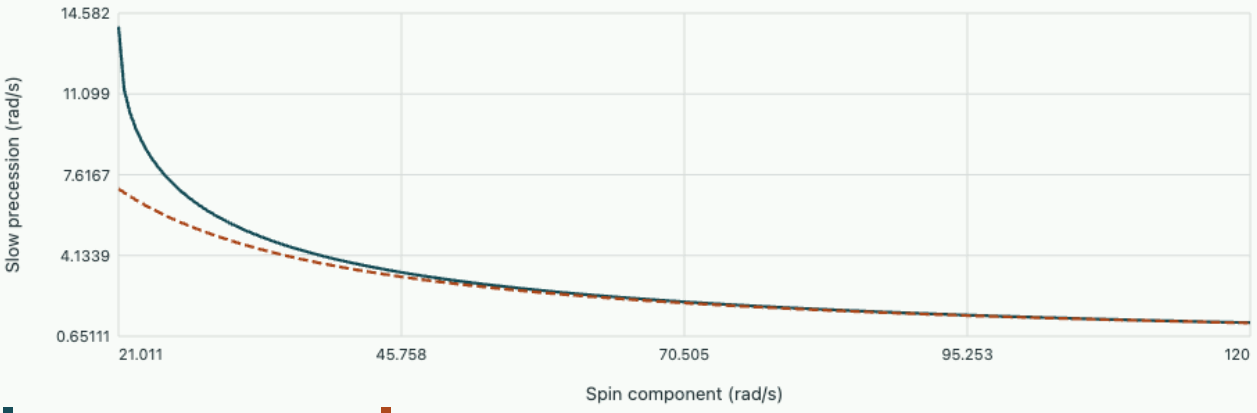
Change the angular momentum of a supported top before identifying its steady rotation rates. Check which approximation singles out the slower branch.

Open this experiment online

Try the calculation

Tilt from vertical (°) **60**

Moment arm (mm) **60**



For this tilt the two regular-precession branches first exist at spin 21.0 rad/s. At 100 rad/s the exact slow rate is 1.488 rad/s.

Extract polar turning points from uncertain orientation data

A camera supplies a unit direction $\mathbf{n}(t)$, while body and pivot calibrations supply I, I_3, Mgl. To infer L_3, either measure the gyro component along $\mathbf{n}$ or combine two independent orientation marks so that spin about the axis is observable. A single unmarked axis direction cannot reveal that spin. Vertical angular momentum follows from $L_z = I(\mathbf{n} \times \dot{\mathbf{n}})_z + L_3 n_z$. This expression remains regular at a vertical pose and avoids differentiating an azimuth divided by a small $\sin\theta$. Estimating $\dot{\mathbf{n}}$ still requires timing and finite-difference checks, especially near a fast turn.

The transverse kinetic energy is $I|\dot{\mathbf{n}}|^2/2$, so $E' = I|\dot{\mathbf{n}}|^2/2 + Mgl\, n_z$. These vector formulas allow each time sample to supply independent estimates of the invariants. Compute their median or uncertainty-weighted values over a torque-free-about-vertical segment, then locate the physical roots of $P_3(u)$. Reject roots outside $[-1, 1]$ before converting to angles, and propagate measured parameter ranges into intervals for the roots. Near a repeated root a small coefficient error can split the pair substantially or

apparently remove it. A nearly steady precession should therefore be assessed using the dynamic measurement uncertainty, not classified solely by the discriminant of a fitted cubic.

A direction-only camera sees apparently steady 60° precession at $1.0\,\mathrm{rad/s}$. Can it infer the axial spin without measuring a second body mark?

▼ **Work through the solution**

In a precisely known ideal regular state the torque balance gives $L_3 = (I\Omega^2\cos\theta + Mgl)/\Omega$, so one can infer spin from a dynamical model and calibrated inertias. Using $I = 0.020$, $Mgl = 0.200$ gives

$$L_3 = 0.210\,\mathrm{kg\,m^2/s}.$$

That is an indirect inference, sensitive to gravity calibration, pivot torque, and the assumption of exactly regular precession. The orientation of one axis alone supplies no purely kinematic measurement of rotation about itself. A second mark or gyro offers an independent check rather than fitting the same torque law twice.

A numerical implementation rescales $\mathbf{n}$ to unit length at every step and observes bounded plotted angles. Does this establish correct top dynamics?

▼ **Work through the solution**

Unit length is only the configuration constraint. Reconstruct L_z, L_3, E', compare polar turning points with the cubic, and repeat at smaller time step. The vector can stay exactly on the sphere while its azimuth and nutation phase drift, or while a spurious torque transfers axial angular momentum. A support torque and gravitational work budget, with a clear reference for the potential, test complementary aspects that normalisation alone cannot enforce.

Check the fixed-pivot inertia before drawing an instability boundary

A laboratory data sheet may quote the transverse inertia about the centre of mass, whereas the Lagrangian here needs the inertia about the pivot. With a pivot on the symmetry axis a distance l from the centre, the parallel-axis relation gives $I = I_{\mathrm{cm}} + Ml^2$ for each transverse direction, while I_3 remains unchanged. For $M = 0.50\,\mathrm{kg}$, $l = 0.080\,\mathrm{m}$, and $I_{\mathrm{cm}} = 0.010\,\mathrm{kg\,m^2}$, the correct value is $0.0132\,\mathrm{kg\,m^2}$. Neglecting the shift underpredicts the critical spin by a factor $\sqrt{0.010/0.0132} = 0.870$.

Moving the pivot changes both the gravitational torque scale Mgl and the transverse inertia. Doubling l does not merely multiply the stability threshold by $\sqrt{2}$ if Ml^2 is appreciable. Compute the new I and then the product that enters $L_{3,\mathrm{crit}}^2 = 4IMgl$. A two-pivot experiment with one unchanged rotor provides a direct test of the parallel-axis correction and checks why a table of freely rotating principal moments cannot be inserted unaltered into a supported top calculation.

Tutorial on a heavy top

With the same constants, impose a tilt of 90°. Does the regular precession formula become singular when $\cos\theta = 0$?

▼ **Work through the solution**

The unsimplified quadratic becomes $-L_3\Omega + Mgl = 0$, so the finite regular precession is

$$\Omega = Mgl/L_3 = 1.0\,\mathrm{rad/s}.$$

A quadratic formula written with $2I\cos\theta$ in the denominator masks this regular solution. The other branch becomes unbounded as the horizontal configuration is approached from the side with positive cosine at fixed spin. Keep the original equation when taking a degenerate limit.

A top undergoes nutation with u between 0.30 and 0.70, with $L_z = 0.11$ and $L_3 = 0.20$ in consistent angular-momentum units. Does the azimuth reverse?

▼ **Work through the solution**

The numerator of $\dot\phi$ vanishes at $u = 0.55$, inside the allowed interval. For $u < 0.55$ precession is positive in this convention; for $u > 0.55$ it is negative. A camera recording the full axis should therefore find an azimuthal reversal twice per nutation period. The polar velocity need not vanish at those two reversals, so these instants are distinct from the polar turning points.

A frictionless rigid-top simulation reports exact E, L_z, L_3 but returns a nutation frequency differing by ten percent from an independent measurement. Name two checks that use more than the conserved values.

▼ **Work through the solution**

Compute the turning points from the cubic and compare the measured times between them against its quadrature, then reconstruct the azimuthal phase over the same cycle. Check whether the measured spin was ω_3 or a painted-mark Euler rate and whether the inertias include the parallel-axis contribution about the pivot. A numerical method can conserve constants while producing a phase error, and an inconsistent initial state can preserve different constants perfectly.

Compare a free principal-axis rotation with a pivoted top subject to gravity.

David Tong's rigid-body chapter includes the heavy symmetric top.

Canonical perturbations, small denominators and resonant capture

Analytical mechanics from constraints to invariants

A weak periodic force can produce a bounded ripple on most trajectories and a large change on a narrow band of initial conditions. Derive the averaging transformation away from resonance, construct the slow resonant dynamics where that transformation fails, and estimate the physically observable width and time scale of a resonance.

Separate the unperturbed clock from a weak correction

Take an integrable Hamiltonian $H(I,\theta,t)=H_0(I)+\epsilon H_1(I,\theta,t)$ in action–angle variables. With $\epsilon=0$, actions are constants and angles advance at $\boldsymbol{\omega}(I)=\partial_I H_0$. A perturbation with Fourier component $h_{\mathbf{n}}(I)\cos(\mathbf{n}\cdot\theta-\Omega t)$ drives action changes proportional to its sine. If the phase sweeps quickly through all angles, positive and negative contributions largely cancel. If $\mathbf{n}\cdot\boldsymbol{\omega}-\Omega$ is small, the perturbation acts with a coherent sign over a long time. We want a transformation that makes both statements quantitative.

Use a near-identity type-two generating function $F_2(\theta,J,t)=\theta\cdot J+\epsilon S_1(\theta,J,t)$. Old actions are $I=J+\epsilon\partial_\theta S_1$, and new angles are $\psi=\theta+\epsilon\partial_J S_1$. Expand $H_0(I)=H_0(J)+\epsilon\boldsymbol{\omega}(J)\cdot\partial_\theta S_1+O(\epsilon^2)$. At first order the new Hamiltonian is $K=H_0(J)+\epsilon[H_1+\partial_t S_1+\boldsymbol{\omega}\cdot\partial_\theta S_1]$. Replacing old by new angles inside that bracket produces only higher-order errors. To leave the averaged part $\langle H_1\rangle$, solve the homological equation

$$(\partial_t+\boldsymbol{\omega}\cdot\partial_\theta)S_1=-\big(H_1-\langle H_1\rangle\big).$$

For an oscillating cosine with detuning $\Delta=\mathbf{n}\cdot\boldsymbol{\omega}-\Omega$, choose $S_1=-h_{\mathbf{n}}\sin(\mathbf{n}\cdot\theta-\Omega t)/\Delta$. The transformation is useful while the resulting action displacement $\epsilon|\partial_\theta S_1|$ is small on the relevant action scale and the derivatives entering the angle transformation remain controlled. Small ϵ alone is insufficient if Δ is even smaller. A singular denominator says that this choice to remove the harmonic is unsuitable, not that the underlying mechanical force becomes infinite.

Obtain a nonlinear frequency shift from the angle average

A free oscillator with mass m, small-amplitude frequency ω_0, and quartic correction $\beta q^4/4$ has unperturbed coordinates $q=\sqrt{2I/(m\omega_0)}\sin\theta$. Substituting gives the correction $\beta I^2\sin^4\theta/(m^2\omega_0^2)$. Since $\langle\sin^4\theta\rangle=3/8$, the first-order averaged Hamiltonian is $K=\omega_0 J+3\beta J^2/(8m^2\omega_0^2)$. Its derivative gives

$$\omega=\omega_0+3\beta J/(4m^2\omega_0^2).$$

Expressing $J\simeq m\omega_0 A^2/2$ in terms of leading oscillation amplitude yields the frequency shift $3\beta A^2/(8m\omega_0)$. Oscillatory harmonics at $2\theta,4\theta$ change the waveform but do not contribute directly to this averaged first-order clock.

For $m = 0.20\,\mathrm{kg}$, $\omega_0 = 10.0\,\mathrm{s}^{-1}$, $\beta = 80\,\mathrm{N/m^3}$, and $A = 0.10\,\mathrm{m}$, the predicted shift is $0.150\,\mathrm{s}^{-1}$, about 1.5%. The nonlinear-to-linear force ratio at maximum displacement is $\beta A^2/(m\omega_0^2) = 0.040$, consistent with the expansion. At twice the amplitude the shift quadruples and the expansion parameter grows to 0.16. Measuring period over several amplitudes and plotting the frequency against A^2 tests both the slope and the onset of curvature from higher orders. A positive β stiffens the spring; a negative quartic term is only a local approximation unless higher powers restore global confinement.

Solve the slow motion in the resonant frame

Consider a rotor $H = I^2/(2\mathcal{I}) - \epsilon V\cos(\theta - \Omega t)$, where $\mathcal{I}$ is moment of inertia and $\epsilon V > 0$ has energy units. Choose $F_2 = (\theta - \Omega t)P$, so $P = I$,

$$\phi = \theta - \Omega t,$$

$$K = P^2/(2\mathcal{I}) - \Omega P - \epsilon V\cos\phi.$$

Completing the square gives $K = (P - P_*)^2/(2\mathcal{I}) - \epsilon V\cos\phi + \text{constant}$ with $P_* = \mathcal{I}\Omega$. This is an exact pendulum in the rotating frame. It retains the resonant cosine instead of attempting to average it away. Phase locking means libration of ϕ around zero; outside its separatrix the relative phase circulates.

The stable fixed point has $P = P_*, \phi = 0$; its small libration frequency is

$$\omega_{\mathrm{lib}} = \sqrt{\epsilon V/\mathcal{I}}.$$

The saddle at $\phi = \pi$ has energy $+\epsilon V$ relative to the same constant. At $\phi = 0$ that energy is reached for $|P - P_*| = 2\sqrt{\mathcal{I}\epsilon V}$, the resonance half-width in action. Its complete width is twice that number. For $\mathcal{I} = 0.050\,\mathrm{kg\,m^2}$, $\Omega = 8.0\,\mathrm{s}^{-1}$ and $\epsilon V = 0.0020\,\mathrm{J}$, the central action is $0.40\,\mathrm{kg\,m^2/s}$; the half-width is $0.020\,\mathrm{kg\,m^2/s}$ and small libration period $31.4\,\mathrm{s}$. The rotor makes many laboratory turns while its slow relative phase completes one libration.

An initial action inside the quoted half-width is necessary but not sufficient for trapping at every phase. At $\phi = \pi$, any nonzero $P - P_*$ is above the saddle energy. Compute the actual rotating-frame energy of the initial state and compare with the saddle. For $\phi = 0, P = 0.415$, the kinetic detuning energy is $0.00225\,\mathrm{J}$; subtracting $0.0020\,\mathrm{J}$ gives $0.00025\,\mathrm{J}$, below the saddle. With the same action at $\phi = \pi$ the value is $0.00425\,\mathrm{J}$, above the saddle. Selecting trajectories by mean frequency alone can therefore mix trapped and circulating populations.

Retain a resonant harmonic in a general integrable system

For one action with $H_0'(I_*) = \Omega$, use the rotating frame and expand $H_0(I) - \Omega I$ about I_*. The linear term vanishes, leaving $H_0''(I_*)(I - I_*)^2/2$. Retain the slowly varying resonant cosine and average the rapidly varying harmonics. When $H_0'' \neq 0$, this gives the same pendulum normal form with an effective inertia and possibly a reversed sign of the stable phase. The action half-width scales as $2\sqrt{\epsilon|V/H_0''|}$. Thus a weak perturbation can influence a band of width proportional to $\sqrt{\epsilon}$, parametrically larger than a typical nonresonant action ripple proportional to ϵ.

For an exactly harmonic oscillator $H_0 = \omega_0 I$ the curvature vanishes. One cannot replace H_0'' with zero in the width formula. An exactly tuned additive drive in a linear oscillator instead produces secular growth until damping, detuning, nonlinearity or a finite drive interval intervenes. In multiple actions, one first transforms to the resonant phase $\mathbf{n} \cdot \theta - \Omega t$ and complementary angles, preserving the canonical one-form. Other near-resonances can prevent isolation of a single pendulum. Apparent chaos inferred from a short irregular time trace alone needs additional tests, including resolved phase portraits, convergence under time refinement and identification of competing resonances.

Calculate a phase-locking threshold for a drive that sweeps in frequency

The exact rotating rotor also allows a tractable time-dependent drive frequency. Write the pattern angle as $\alpha(t)$, with $\Omega(t) = \dot{\alpha}$. In its frame $K = P^2/(2\mathcal{I}) - \Omega(t) P - \epsilon V \cos\phi$. Hamilton's equations give

$$\dot{\phi} = P/\mathcal{I} - \Omega(t),$$

$$\dot{P} = -\epsilon V \sin\phi.$$

Define the detuning momentum $p = P - \mathcal{I}\Omega(t)$. Even when $\dot{\phi} = 0$ at one instant, phase locking throughout a ramp requires $\dot{p} = 0$ and hence $\sin\phi_0 = -\mathcal{I}\dot{\Omega}/(\epsilon V)$. The restoring torque must be sufficient to accelerate the rotor along with the changing pattern.

For constant $\dot{\Omega}$ and $|\mathcal{I}\dot{\Omega}| < \epsilon V$, a stable locked phase with $\cos\phi_0 > 0$ exists. Perturbing the phase by η gives $\ddot{\eta} + (\epsilon V/\mathcal{I})\cos\phi_0\,\eta = 0$, so the libration softens as the required torque approaches its maximum. At threshold the stable and unstable stationary phases meet. Above it no fixed phase can provide the torque needed by the ramp. The mere existence of a stable locked solution still does not imply that an arbitrary initial phase and momentum will be captured in it. In a conservative system initial phase-space location and separatrix crossing are essential.

With the previous $\mathcal{I} = 0.050\,\mathrm{kg\,m^2}$ and $\epsilon V = 0.0020\,\mathrm{J}$, the largest trackable constant acceleration in this model is $0.040\,\mathrm{rad/s^2}$. At half that value the stable phase lags by $\pi/6$, and the small relative-phase frequency is

$$0.20\sqrt{\cos(\pi/6)} = 0.186\,\mathrm{s^{-1}}.$$

Increasing the drive frequency by $1.0\,\mathrm{rad/s}$ at this rate takes $50\,\mathrm{s}$, only about one and a half libration periods. Adiabatic capture estimates that additionally require many libration periods may therefore be unconvincing even though the elementary torque threshold is satisfied. A measured lag versus sweep rate assesses the static torque condition, while a full phase trajectory tests dynamic capture.

The same equations admit a tilted effective washboard potential in the accelerating frame, $U_{\mathrm{rel}} = -\epsilon V\cos\phi + \mathcal{I}\dot{\Omega}\,\phi$, as long as $\dot{\Omega}$ is constant. Its potential difference over a whole 2π turn is nonzero, so unwrap the relative phase before using its energy. Mapping every phase back to $[-\pi, \pi)$ hides this difference and produces artificial jumps in the inferred energy. In the original variables the

physical motor does work. There is no contradiction between a conserved local accelerating-frame energy for a constant ramp and the changing laboratory kinetic energy of a locked rotor.

Derive secular energy exchange in a pair of weakly coupled oscillators

Take two equal masses m with individual natural frequency ω_0 and potential coupling $k_c(q_1 - q_2)^2/2$. The normal coordinates $Q_\pm = (q_1 \pm q_2)/\sqrt{2}$ have frequencies $\omega_+ = \omega_0$,

$$\omega_- = \sqrt{\omega_0^2 + 2k_c/m}.$$

Release mass one at displacement A with both velocities and q_2 initially zero. Superposition gives

$$q_1 = A[\cos\omega_+ t + \cos\omega_- t]/2,$$
$$q_2 = A[\cos\omega_+ t - \cos\omega_- t]/2.$$

The slowly varying difference phase transfers oscillation amplitude between the masses over a time much longer than their rapid periods.

For $\omega_0 = 10\,\mathrm{s}^{-1}$ and $k_c/m = 1.0\,\mathrm{s}^{-2}$, the exact separation is

$$\sqrt{102} - 10 = 0.0995\,\mathrm{s}^{-1}.$$

The first near-complete exchange occurs at $t = \pi/|\omega_- - \omega_+| \approx 31.6\,\mathrm{s}$. Over a single fast period the displacements can look almost like uncoupled oscillations, despite large later energy redistribution. Perturbing two isolated oscillators while trying to average the coupling over both independent fast angles would discard a term depending on their slow difference phase. Retaining the resonant combination predicts the exchange; diagonalising the linear model supplies a reference solution for that approximation.

Detune the uncoupled springs so their natural frequencies have a separation large compared with $k_c/(m\omega_0)$. The coupled normal modes then align more closely with the original oscillators, and the energy exchange cannot reach its former depth. In a nonlinear problem the individual frequencies also depend on action, so the energy transfer can detune itself. To interpret an observed beating envelope, measure both normal frequencies and relative phases. An apparent loss of amplitude in oscillator one can be energy stored in oscillator two, distinguishable from damping through recovery at later times and a balance of the total Hamiltonian.

Estimate the accuracy of averaging over a finite observation time

For a nonresonant perturbation with initially slow action change, one can integrate the first-order drive along the unperturbed trajectory over the actual observation window instead of assuming an infinite-time average. For $H_1 = A\cos(\theta - \Omega t)$, $\dot{I} = A\sin(\Delta t + \phi_0)$ to leading order with $\Delta = \omega_0 - \Omega$. Thus $\Delta I(T) = A[\cos\phi_0 - \cos(\Delta T + \phi_0)]/\Delta$. For $\Delta \neq 0$ its absolute value remains at most $2|A/\Delta|$, and

its largest possible value over initial phase is $2|A\sin(\Delta T/2)/\Delta|$. For $\Delta \to 0$ the phase-dependent result tends to $AT\sin\phi_0$. The finite-time expression is regular right at resonance and shows exactly when a secular term emerges.

For $A = 0.001\,\mathrm{J}$, $T = 10\,\mathrm{s}$ and $\Delta = 0.20\,\mathrm{s}^{-1}$, the maximum-amplitude change in action over initial phase is $0.00841\,\mathrm{J\,s}$. At zero detuning the leading largest change is $0.0100\,\mathrm{J\,s}$, only modestly larger because this observation window resolves a frequency band of order $1/T$. For $\Delta = 2\pi/T$ the change after exactly T vanishes, although the action swung during the interval. A final-minus-initial measurement alone can mistake a completed nonresonant excursion for an entirely unperturbed trajectory. Recording at an additional end time tests that cancellation.

At leading order a force of finite duration produces this sine-over-detuning envelope even in a linear system. It is an observation-time bandwidth, distinct from the dynamical resonance width proportional to $\sqrt{\epsilon}$ in the nonlinear rotor. Repeat the experiment with doubled drive duration at fixed amplitude. Features that narrow inversely with the duration are evidence of finite-time spectral selection. A persistent bounded island in a stroboscopic phase portrait, with the predicted amplitude-dependent width and libration frequency, is evidence of the phase-locking dynamics. If drive amplitude is large enough to change the natural frequency appreciably over the interval, the unperturbed phase-integration formula ceases to describe the trajectory.

Construct a stroboscopic section with the right clock

For a time-periodic Hamiltonian sample (I, θ) every period of the external drive at a fixed drive phase. This creates a two-dimensional area-preserving map for one periodically forced degree of freedom. A phase-locked solution can become a fixed point of this map, and librating trajectories around it can form an island. Sampling at each crossing of a moving coordinate threshold instead can produce a different return map and a different phase weight. A return plot can look irregular simply because the drive phase was not held fixed at the samples.

Consider a rigid rotor commanded at $\Omega = 8\,\mathrm{s}^{-1}$. Its drive period is

$$2\pi/8 = 0.7854\,\mathrm{s}.$$

An acquisition system sampling nominally every $0.7900\,\mathrm{s}$ accumulates approximately $8(0.7900 - 0.7854) = 0.0368\,\mathrm{rad}$ of phase error per sample. After one hundred samples the sampling phase has moved about $3.68\,\mathrm{rad}$, enough to smear a narrow island across a substantial fraction of the plot. Interpolating trajectories to timestamps from a synchronised drive encoder controls that effect. Keep unwrapped relative phase as a separate channel to count actual slips.

A numerical area test on the stroboscopic map should use canonical action and angle or an equivalent canonical coordinate pair. A scatter plot of polar angle against angular velocity for a state-dependent inertia can distort area. Changes of pixel density, trajectory dwell time and resolution near the separatrix further affect visual appearance. To identify a resonance quantitatively, measure the action extent at the stable phase, the local libration period and the saddle crossing energy on independently prepared trajectories. Agreement among those three observations is less easily mimicked by a sampling artefact than the existence of a visually closed loop alone.

Recover a frequency shift by removing a secular term

Canonical averaging yields a frequency correction for the quartic oscillator. An independent derivation makes clear why a small difference in frequency can become a large trajectory error at long times. Write the equation as $\ddot{q} + \omega_0^2 q + (\beta/m) q^3 = 0$. Put $q = A x$, $\tau = \omega t$, $\lambda = \beta A^2/(m\omega_0^2)$ and seek $\omega = \omega_0(1 + c\lambda + \cdots)$,

$$x = x_0 + \lambda x_1 + \cdots .$$

Leading initial conditions $x(0) = 1, x'(0) = 0$ give

$$x_0 = \cos\tau .$$

At first order one finds

$$x_1'' + x_1 = 2c\cos\tau - \cos^3\tau .$$

Using $\cos^3\tau = (3\cos\tau + \cos 3\tau)/4$, the right-hand side contains a term $(2c - 3/4)\cos\tau$ resonant with the leading motion. Choosing $c = 3/8$ removes the forcing that would otherwise generate a term proportional to $\tau\sin\tau$. The frequency then agrees with the action-angle calculation. The remaining particular solution is $\cos 3\tau/32$. Including the homogeneous term required by $x_1(0) = x_1'(0) = 0$ gives

$$x_1 = (\cos 3\tau - \cos\tau)/32 .$$

This separates the leading frequency shift from an actual third-harmonic distortion of the orbit. Fitting only frequency and overlooking the small third harmonic cannot generally reproduce a precise waveform.

In the earlier example $\lambda = 0.040$, so the displacement amplitude at the third harmonic is $A\lambda/32 = 0.125\,\mathrm{mm}$ at $A = 0.10\,\mathrm{m}$. The frequency shift is about $0.15\,\mathrm{s}^{-1}$. After twenty seconds the uncorrected harmonic prediction is nearly three radians out of phase, far larger than the instantaneous third-harmonic displacement. A small-amplitude correction to the right-hand side can therefore cause an order-one error in the long-time phase. Measuring both the third harmonic and frequency gives two independent checks of the same inferred quartic stiffness. A drive electronics harmonic can mimic the former without producing the amplitude-dependent free-oscillation frequency.

Inspect growth on a parametrically driven orbit

Move the drive away from resonance and examine stroboscopic growth. Determine whether changing phase, amplitude or forcing frequency is responsible.

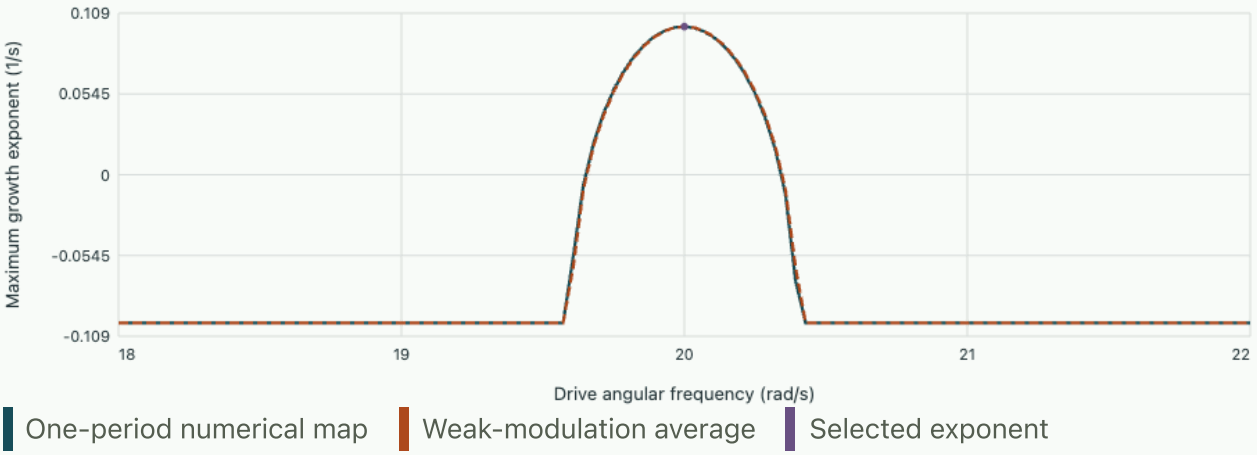
Open this experiment online

Try the calculation

Fractional stiffness modulation **0.08**

Damping γ (1/s) **0.1**

Drive frequency Ω (rad/s) **20**



At Ω = 20 rad/s, numerical exponent 0.09997 s⁻¹ and averaged exponent 0.1 s⁻¹. Amplitude doubling time 6.934 s.

Test overlap of nearby resonances without declaring chaos from one curve

Take a rotor driven by two weak periodic patterns with distinct angular speeds Ω_1, Ω_2. For an isolated pattern j, the earlier calculation gives a resonant centre $P_j = \mathcal{I}\Omega_j$ and half-width $\Delta P_j = 2\sqrt{\mathcal{I}\epsilon V_j}$. Comparing the sum of estimated half-widths with the separation $|P_2 - P_1|$ is a practical indication of when the isolated-island approximation may fail. It does not by itself prove every trajectory between the resonances is chaotic. The relative phases, other harmonics and the actual time dependence determine the global structure.

For the same inertia $0.050\,\mathrm{kg\,m^2}$, choose $\Omega_1 = 8.0\,\mathrm{s^{-1}}$, $\Omega_2 = 8.6\,\mathrm{s^{-1}}$, and forcing energies 0.0020 and $0.00050\,\mathrm{J}$. The centres differ by $0.030\,\mathrm{kg\,m^2/s}$; the individual half-widths are 0.020 and $0.010\,\mathrm{kg\,m^2/s}$. Their nominal intervals just touch. Halving both forcing energies shrinks both widths by $1/\sqrt{2}$, leaving a gap. Doubling both makes the estimated intervals overlap. The speed difference corresponds to a beat period $2\pi/0.6 = 10.47\,\mathrm{s}$, another clock that must be resolved when recording nearby motion.

Transforming into the first rotating frame makes the first perturbation stationary and leaves the second with time dependence proportional to $\cos[\phi - (\Omega_2 - \Omega_1)t]$. A stroboscopic map sampled at the beat period is therefore more informative than a map with unsynchronised lab sampling. Prepare nearby initial conditions, compare their separation over multiple beat periods, and repeat with a refined symplectic time step and a modestly varied observation interval. Short-term stretching near an ordinary hyperbolic fixed point is possible even in a fully integrable pendulum. Evidence for widespread chaotic transport requires more than the growth of one separation during one saddle passage.

A student finds an apparent threshold by plotting whether the momentum ever crosses the midpoint between the two resonant centres. What additional evidence distinguishes irregular transport from a single large regular libration?

▼ **Work through the solution**

A regular libration may cross that midpoint repeatedly inside one island, especially near its separatrix. Plot return pairs at a fixed beat phase, check for closed invariant curves or broader area filling, and track unwrapped relative phase to see whether it slips irregularly between the two patterns. Vary initial conditions and compare the statistical trend with the isolated-width estimate. Confirm that energy balance in the driven system includes explicit work by both patterns; a drift of the autonomous single-pattern energy is expected once the second drive is present.

Use phase reversal to isolate the resonant action transfer

For an initially weak sinusoidal drive, prepare two otherwise identical trajectories with initial relative phases differing by π. Their leading changes of action during a short resonant pulse have opposite signs, since $\dot{I}$ is proportional to $\sin\phi_0$. Half the difference isolates that phase-sensitive term, whereas half the sum can reveal phase-independent heating or higher-order effects. With drive coefficient $A = 0.001\,\mathrm{J}$, duration $2.0\,\mathrm{s}$ and initial phases $\pm\pi/2$, short-pulse first-order predictions differ by $0.0040\,\mathrm{J\,s}$. The test requires that detuning and nonlinear phase drift over the pulse remain small.

Perform the comparison at several durations. A linear growth of the early phase-sensitive difference is compatible with coherent transfer. Subsequent saturation, reversal or a slow oscillation can reflect the resonant pendulum dynamics once the action shift changes the phase. Friction can also saturate a response, but it produces mechanical energy loss and typically modifies the two phase preparations in different ways from conservative trapping. Measuring work from the applied torque and the full phase history adds the independent information needed to separate those explanations. A final amplitude alone gives neither the sign of the action transfer nor its phase dependence.

Tutorial on weak forcing

Far from resonance let a one-mode perturbation be $A\cos(\theta - \Omega t)$ with $A = 0.001\,\mathrm{J}$ and detuning $0.50\,\mathrm{s^{-1}}$. What first-order ripple scale should appear in the old action?

▼ **Work through the solution**

The generating function has amplitude $A/|\Delta| = 0.002\,\mathrm{J\,s}$. For harmonic number one its angle derivative has the same amplitude, so the old action oscillates about the averaged action on that scale. At detuning $0.01\,\mathrm{s^{-1}}$ the nominal ripple would be fifty times larger. Check its size against the action scale and the change of detuning across the ripple before trusting that continuation.

In the rotor example the forcing energy is multiplied by four. What happens to the resonant action, action width and small-amplitude libration period?

▼ **Work through the solution**

The central action is set by $H_0' = \Omega$ and stays $0.40\,\mathrm{kg\,m^2/s}$. The half-width doubles to $0.040\,\mathrm{kg\,m^2/s}$. The libration frequency doubles, so the period halves to $15.7\,\mathrm{s}$. These predictions apply to the exact sinusoidal rotor model; other harmonics of a physical drive may also change as its amplitude increases.

A slowly swept drive seems to carry a phase-locked rotor beyond its original resonance band. Is this inconsistent with the static separatrix estimate?

▼ **Work through the solution**

A varying $\Omega(t)$ moves the resonance centre $\mathcal{I}\Omega(t)$. In coordinates measured from that moving centre the equation for detuning momentum has an additional torque-like term $-\mathcal{I}\dot{\Omega}$. Locking can persist if the available restoring torque can track the ramp and the initial motion stays inside the evolving island. The static width describes a fixed-frequency section. Comparing a captured trajectory to its initial centre after the sweep tests the wrong phase portrait.

Compare a resonance driven by a varying stiffness with a rotating additive potential.

David Tong's Hamiltonian course develops canonical variables and adiabatic invariants.

Dynamic coupling and measured response

Combine structural elasticity, a transducer and an estimator. Compare a split resonance, a free decay and a periodically loaded shaft.

1. Elastic curves and virtual work
2. Torsional stiffness of shafts and closed sections
3. Creep, relaxation and what a holding test measures
4. A moving-coil galvanometer as an electrical and mechanical system
5. Eddy-current braking, magnetic penetration and the heat left behind
6. Coupled optical and mechanical resonances
7. Estimate a cavity linewidth from a finite, biased record
8. Measure gravity with a pendulum of finite amplitude
9. Measure inertia and gravity with two suspension axes

Elastic curves and virtual work

Engineering mechanics

Predict a specified displacement from the bending-moment field and distinguish a stiffness limit from a stress limit.

FROM FIRST PRINCIPLES

Why cantilever stiffness scales as the inverse cube of span

Take a straight slender beam clamped horizontally, with a downward end force P. The empirical inputs are small-strain linear elasticity and the beam assumption that plane sections remain normal to the deflected centreline. Denote its centroidal bending rigidity by EI. With constant section and modulus the stress moment of the stretched and compressed fibres is proportional to curvature. With downward deflection positive, equilibrium of a cut at x from the clamp supplies

$$M(x) = P(L - x),$$
$$w''(x) = M(x)/(EI).$$

Curvature is the rate at which slope changes. Integrating it once gives the slope, and integrating slope gives deflection. Combine both integrations into a weighted curvature integral: curvature occurring near the clamp rotates a longer remaining segment than curvature near the tip. The clamp fixes initial slope and displacement to zero, hence

$$w(L) = \int_0^L (L - x) w''(x)\, dx = \frac{P}{EI} \int_0^L (L - x)^2 dx = \frac{PL^3}{3EI}.$$

This is a displacement prediction under small slopes, not just a stress limit. Its input $M = EIw''$ is established from individual fibre strains and the neutral-axis condition in the detailed derivation. Short beams can also require shear deflection.

Background · Second moment of area

$$I = \int_A y^2\, dA$$

Measure the transverse coordinate from the stated bending axis. This section property has units of length to the fourth power and enters the flexural rigidity.

DERIVATION AND APPLICATIONS

From the extension of a fibre to cantilever curvature and displacement

Why does section depth enter a bending calculation through a second moment of area? Take a straight prismatic cantilever with small slopes, uniform Young's modulus E, linear elastic response and slender dimensions. Euler–Bernoulli kinematics assumes that cross-sections remain plane and normal to the deflected centreline. Measure x from the clamp and transverse deflection w downward. Measure cross-section coordinate y downward from an initially arbitrary reference line. Sections separated by dx rotate relative to one another by $d\theta$; a fibre at y shortens relative to the reference line by $y\,d\theta$.

$$\epsilon_x(y) = \epsilon_0 - y\,d\theta/dx \simeq \epsilon_0 - y w''(x),$$
$$\sigma_x = E\epsilon_x.$$

With no imposed axial resultant, the integral of this normal stress must vanish. Writing $\bar{y} = A^{-1}\int_A y\,dA$, we obtain the strain of the reference line, and thereby identify the zero-strain fibre:

$$0 = N = \int_A E(\epsilon_0 - y w'')\,dA,$$
$$\epsilon_0 = \bar{y} w'',$$
$$\sigma_x = -E(y - \bar{y})w''.$$

Define bending moment positive for this downward curvature. Each fibre contributes its force times its lever arm about the neutral axis. Integrating introduces the centroidal second moment I:

$$M = -\int_A (y - \bar{y})\sigma_x\,dA = EIw'',$$
$$I = \int_A (y - \bar{y})^2\,dA.$$

A downward tip load P gives $M(x) = P(L - x)$ by equilibrium of the portion beyond the cut. For constant EI, integrate twice and enforce the actual clamp conditions $w(0) = w'(0) = 0$:

$$EIw' = P(Lx - x^2/2),$$
$$EIw = P(Lx^2/2 - x^3/6),$$
$$w(L) = PL^3/(3EI).$$

The slope and displacement constants follow from compatibility at the support. The zero moment at the free tip follows independently from the load on the right-hand portion. Plane sections can also remain plane under pure bending of a short specimen, but shear deformation near loads and supports can matter in short cantilevers.

WORKED EXAMPLE

Calculate a tip reading and maximum fibre strain

A rectangular steel strip has $L = 0.200$ m, $b = 20.0$ mm, $h = 8.00$ mm and $E = 200$ GPa. Depth h lies along the deflection direction, so $I = bh^3/12 = 8.53 \times 10^{-10}$ m⁴. A 10.0 N tip load produces $w(L) = 0.156$ mm. The clamp moment is 2.00 N m; its curvature 0.0117 m⁻¹ gives outer-fibre strain magnitude $(0.00400)(0.0117) = 46.9$ microstrain. The top is tensile and the bottom compressive for the stated load.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Vary the span to distinguish two sources of compliance

On the same strip double the free span to 0.400 m. Bending predicts eight times the displacement, or 1.25 mm; outer-fibre strain doubles to 93.8 microstrain. A nearly span-independent displacement offset would instead suggest motion of the clamp fixture, and a shear-displacement component at fixed load and section scales only linearly with span. The different scalings let two displacement readings test the assumptions behind a bending-only interpretation.

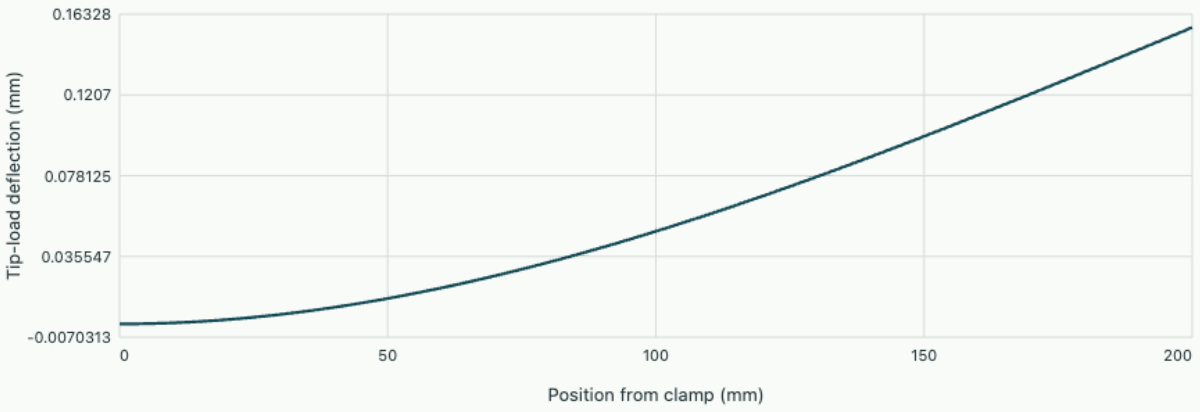
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Length (mm)	200
Section depth (mm)	8
Tip force (N)	10



Integrated curvature

Tip reading 0.1563 mm; clamp outer-fibre strain 46.87 microstrain. Width20 mm and E200 GPa.

Test a beam reduction against a solid in OneSimulation

A 120 mm cantilever of width 18 mm and modulus 70 GPa carries 30 N uniformly distributed over its end face. Make a prediction before using the slider. If the height changes from 12 to 24 mm, which changes more, the section area or its bending stiffness?

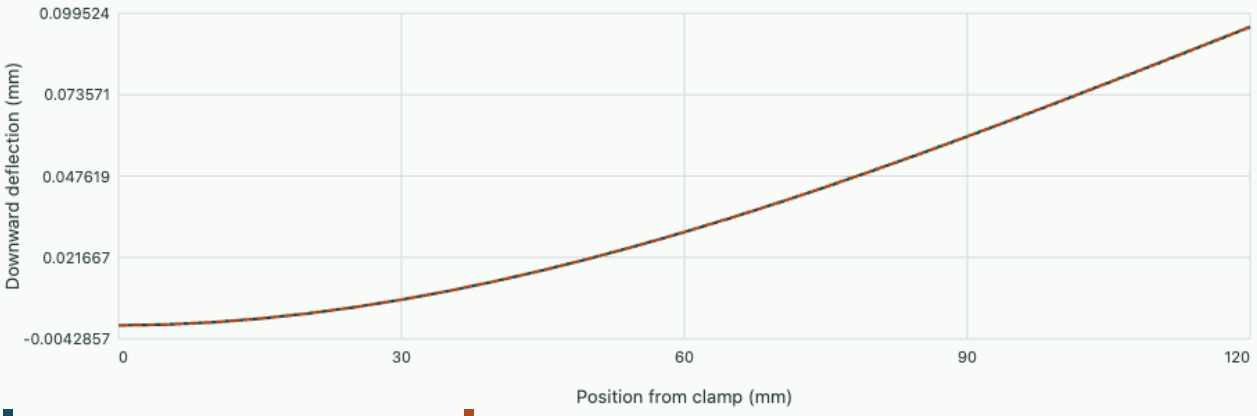
The end moment about a section a distance x from the clamp has magnitude $F(L-x)$. The small-slope beam model first converts it into curvature $w'' = F(L-x)/(EI)$, then integrates twice with $w(0) = w'(0) = 0$. The browser performs both integrations on the selected grid by trapezoidal sums. The first integral is exact for linear curvature. The second integrates a quadratic slope and retains a grid error. Its reference is $w(x) = Fx^2(3L-x)/(6EI)$.

Open this experiment online

Try the calculation

Section height (mm) 12

Axial cells 24



Integrated curvature on the grid Euler–Bernoulli solution

Tip deflection 0.095197 mm on this 1D grid, 0.095238 mm analytically. The OneSimulation model solves the same load on a 3D solid.

Download the OneSimulation model

In your OneSimulation environment, run the model with these inputs.

```
python cantilever.py --height 12 --cells 24
```

Download the model for the same dimensions and change the parameters shown with the graph. OneSimulation discretises the three-dimensional solid with quadratic hexahedral elements and applies distributed end traction. Its CSV compares centreline deflection with the analytical curve. First fix height and refine axially. Then increase height at a converged mesh. The first experiment addresses numerical error within a model, the second probes the slender-beam assumptions. Under those assumptions doubling height reduces deflection eightfold because $I = bh^3/12$.

When a beam is strong enough but too flexible

A tool rail may stay below its elastic limit while its cutting head moves outside a position tolerance. The maximum bending stress depends on the largest moment at one section. Deflection accumulates the effect of curvature over a length. A displacement sensor near an unloaded end can therefore move even if the nearby bending moment is zero. To predict it, identify both the point and direction of the required displacement.

Consider a straight, slender Euler–Bernoulli beam. Cross-sections remain plane and perpendicular to the deformed centreline. Take w positive upward and M positive in sagging. For small slopes the signed

curvature is w'', with $\kappa = M/(EI)$. Integrate once to find the slope and once more to find the displacement. Each constant of integration has physical meaning. A clamp fixes displacement and rotation. A simple support fixes its vertical displacement while permitting rotation. At an ordinary join without a hinge, both displacement and slope are continuous.

Work through a tip reading

Clamp a beam at $x = 0$. Apply a downward point load P at $x = a$ on a beam of total length $L > a$. The signed moment in $0 < x < a$ is $-P(a - x)$. For $x > a$ the free segment has no external force and no moment. Taking constant stiffness and using the clamp conditions gives

$$EIw'(x) = -P(ax - x^2/2),$$

$$EIw(x) = -P(ax^2/2 - x^3/6), \quad 0 \le x \le a.$$

At the load point the displacement is $-Pa^3/(3EI)$ and the slope is $-Pa^2/(2EI)$. The free extension beyond it is straight but tilted. Use those values as initial conditions for the unloaded segment:

$$w(L) = w(a) + (L - a)w'(a) = -Pa^2(3L - a)/(6EI).$$

This expression correctly vanishes when the force is applied at the clamp and reduces to the standard end-load expression when $a = L$.

Calculate both position and slope

Let $EI = 8400\,\mathrm{N\,m^2}$, $L = 1.20$ m, $a = 0.80$ m and $P = 100$ N. Under the load the downward displacement is 2.03 mm and the rotation magnitude is

$$100(0.80)^2/[2(8400)] = 0.00381\,\mathrm{rad}.$$

The 0.40 m overhang contributes another 1.52 mm at the tip, for a total downward reading of 3.56 mm. The slope is small compared with a radian, consistent with the linear curvature assumption. A gauge at the load and one at the tip would give different stiffness estimates if their positions were omitted.

Obtain one displacement directly

There is another way to compute a single sensor reading. Introduce a fictitious force Q at the sensor in its desired positive direction. The moment becomes $M_P + Qm$, where m is the moment per unit sensor force. The bending energy is

$$U = \int (M_P + Qm)^2/(2EI)\,dx.$$

Differentiating with respect to Q gives the displacement conjugate to that force. Set $Q = 0$ afterwards to recover the original beam.

$$\delta = \int_0^L \frac{M_P(x) m(x)}{E(x) I(x)}\, dx.$$

For a downward unit force at the free tip, $m = -(L - x)$. Only the portion up to a contributes because $M_P = 0$ further along. The integral gives $Pa^2(3L - a)/(6EI)$, the positive downward reading obtained above. This form is especially useful for stepped sections: split the integral where either stiffness or loading changes. Its positive self-flexibility for a load and a sensor pointing the same way is a useful sign check.

Exchange the load and the gauge

The unit-load formula supplies a test that can be made on the same beam without knowing its exact stiffness distribution. Put the same 100 N force at the tip and move the displacement sensor to $x = 0.80$ m. The new integral has exactly the previous pair of moment fields with their roles exchanged. Since their product is unchanged, the new reading is also 3.56 mm downwards. This Maxwell–Betti reciprocity applies to these work-conjugate forces and displacements in a linear elastic structure with unchanged restraints. A slipping clamp, a support that lifts off or motion large enough to change the load geometry can cause discrepancies between reciprocal tests. Swapping load and sensor positions is therefore also a practical check of the assumed support behaviour.

Quantify the deformation omitted by a bending-only model

The straight-normal assumption of Euler–Bernoulli theory omits transverse shear distortion. For a slender beam this is often small, but for a deep beam or a material with a small effective shear modulus it can dominate a displacement measurement. In a simple Timoshenko description let $\kappa_s G A$ be the effective shear stiffness. The dimensionless shear correction factor κ_s accounts for the nonuniform cross-sectional shear field when a single transverse shear resultant V is used. For the illustration use $\kappa_s = 5/6$ for a homogeneous rectangular section.

For an end-loaded cantilever the bending moment magnitude is $P(L - x)$ and the shear force magnitude is P throughout the span. Add the elastic shear energy to the bending energy. Differentiating with respect to the end force yields the total deflection under that force:

$$U = \int_0^L \left[\frac{P^2 (L - x)^2}{2EI} + \frac{P^2}{2\kappa_s G A} \right] dx,$$

$$\delta = \frac{PL^3}{3EI} + \frac{PL}{\kappa_s G A}.$$

The shear contribution grows linearly with span while the bending contribution grows cubically. Their ratio is $3EI/(\kappa_s G A L^2)$. Writing $I/A = h^2/12$ for a rectangle gives $[E/(4\kappa_s G)](h/L)^2$. This form identifies both the geometric slenderness and material stiffness ratio required for a sensible model

choice. A fixed rule based on span-to-depth alone can be poor for a sandwich section with a compliant core.

Interpret a short cantilever stiffness test

An aluminium fixture projects 120 mm from a rigid clamp. Its cross-section is 20 mm wide and 40 mm deep in the bending direction. Take $E = 70{,}000\,\mathrm{MPa}$, $G = 26{,}000\,\mathrm{MPa}$, and an end load of 500 N. The area is 800 mm² and the bending second moment is

$$20(40)^3/12 = 106667\,\mathrm{mm}^4.$$

In consistent N–mm units the bending deflection is

$$500(120)^3/[3(70000)(106667)] = 0.0386\,\mathrm{mm}.$$

The shear deflection is

$$500(120)/[(5/6)(26000)(800)] = 0.00346\,\mathrm{mm}.$$

A dial indicator would measure about 0.0420 mm under these ideal assumptions.

Fitting a bending-only formula to this reading suggests an apparent modulus of about $70(0.0386/0.0420) = 64.2\,\mathrm{GPa}$. The discrepancy arises from omitted shear deformation despite a perfectly elastic material. Repeat the test at twice the overhang while keeping the force unchanged: bending deflection rises by eight, shear deflection by two. The fractional shear error falls to one quarter of its original value. Measuring several spans can therefore separate an incorrect modulus from a shear correction. Clamp compliance is another candidate in a real experiment. A clamp rotation produces a term with a different span dependence, and a gauge mounted close to the fixture can reveal that motion independently.

For equal applied loads at spans L and $2L$, express measured compliances as $C_1 = B + S$ and $C_2 = 8B + 2S$, where B and S are the original bending and shear contributions per unit load. Then $B = (C_2 - 2C_1)/6$ and $S = C_1 - B$. A negative inferred S signals measurement uncertainty or an omitted boundary deformation, since the assumed elastic shear energy must be positive. This inversion makes the two-span proposal a quantitative model test.

Taking three spans instead of two permits a residual check, because the third measured displacement is then predicted without adjusting either fitted contribution.

Torsional stiffness of shafts and closed sections

Engineering mechanics

Derive torque–twist relations and explain the large difference in stiffness between an intact tube and an open strip.

FROM FIRST PRINCIPLES

Derive the torque–twist relation from local shear deformation

Consider a straight circular shaft in small elastic torsion. In Saint-Venant torsion of a circular section, each cross-section can rotate as a rigid plane relative to the next. Let its rotation be $\phi(x)$ and take two sections separated by dx. At radius r their relative tangential displacement is $r\,d\phi$. Dividing by the axial spacing gives engineering shear strain and, through the measured shear modulus G, shear stress:

$$\gamma_{x\theta} = r\,d\phi/dx,$$
$$\tau_{x\theta} = Gr\,d\phi/dx.$$

Each patch of area dA carries tangential force $\tau\,dA$ with moment arm r. Sum these moments over the section and integrate the resulting twist rate along the shaft:

$$T = \int_A r\tau\,dA = GJ\phi',$$
$$J = \int_A r^2\,dA,$$
$$\Delta\phi = \int \frac{T(x)}{G(x)J(x)}dx.$$

For a homogeneous solid circular shaft $J = \pi R^4/2$, so doubling radius strongly increases torsional stiffness. Noncircular sections generally warp and their torsion constant need not equal this polar area moment. The circular-shaft kinematics should not be used unchanged for an open thin-walled section.

DERIVATION AND APPLICATIONS

How two bonded shaft materials share one applied torque

Two shafts in series carry the same torque; a bonded core and sleeve instead undergo the same twist per length. Consider a circular core of radius a and shear modulus G_c, bonded to a concentric sleeve with outer radius b and modulus G_s. Use a long prismatic region away from load introduction, small elastic deformations, perfect bond and no interfacial slip. Let $\phi(x)$ describe section rotation. Two cross-sections separated by dx acquire a relative tangential motion $r\,d\phi$ at radius r:

$$\gamma_{x\theta} = r\phi',$$
$$\tau_i(r) = G_i r\phi'.$$

Compatibility makes ϕ' common. The shear stress may change at the material boundary because the moduli differ. The torque contribution from a thin annulus is its area $2\pi r\,dr$ times shear stress times radius. Integrating in each material gives

$$T_c = G_c J_c \phi',$$
$$T_s = G_s J_s \phi',$$
$$J_c = \frac{\pi a^4}{2},$$
$$J_s = \frac{\pi(b^4 - a^4)}{2}.$$

The applied torque T is their sum. Solving first for the common twist and then for the load split yields

$$\phi' = \frac{T}{G_c J_c + G_s J_s},$$
$$\frac{T_c}{T} = \frac{G_c J_c}{G_c J_c + G_s J_s}.$$

The denominator has units of N m². Setting both shear moduli equal recovers the homogeneous solid shaft independently of the arbitrary division at a. If the sleeve is unbonded and torque is applied only to the core, equal twist is no longer a valid compatibility condition.

WORKED EXAMPLE

Predict the twist of a reinforced shaft

Take $a = 10.0$ mm, $b = 15.0$ mm, $G_c = 80.0$ GPa and $G_s = 30.0$ GPa. The rigidities are 1257 and 1914 N m². Under $T = 60.0$ N m the twist rate is 0.0189 rad/m; over 0.500 m the end rotates 0.00946 rad. The core carries 23.8 N m and the sleeve 36.2 N m. Their stresses at a are 15.1 and 5.68 MPa respectively despite equal shear strain there.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Infer a sleeve modulus from a rotation test

A second shaft with the same dimensions and core rotates through 0.0100 rad over 0.500 m under 50.0 N m. The measured total rigidity is $TL/\phi = 2500$ N m². Subtracting the known core leaves 1243 N m². Since $J_s = 6.38 \times 10^{-8}$ m⁴, $G_s = 19.5$ GPa. The test would yield an apparent modulus if compliance of the grips were included in the measured rotation.

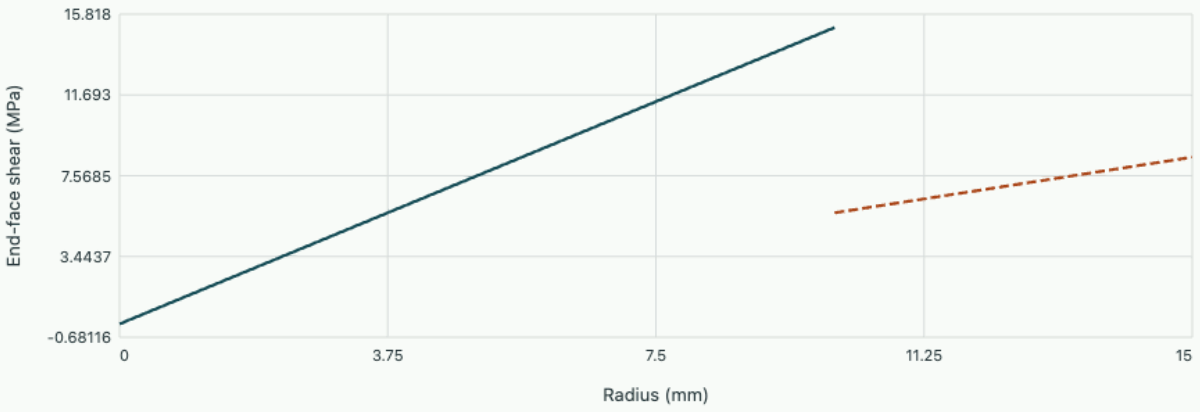
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Core radius (mm)	10
Outer radius (mm)	15
Torque (N m)	60



Twist rate 0.01892 rad/m; core torque 23.777 N m; sleeve 36.223 N m. Compatibility sets equal twist, not equal interface stress.

How much torque does the sleeve actually carry?

A steel core and an aluminium sleeve extend the full length of a circular shaft. They are bonded along their cylindrical interface, and the end fittings transmit the resultant torque to the complete section. A student dividing the torque in proportion to cross-sectional area obtains a plausible answer that misses both lever arm and material stiffness. We can reconstruct the division from the displacement at the interface before writing any torque balance.

Let a denote the core radius and b the outside radius. A cross section rotating by angle $\varphi(x)$ moves a point at radius r tangentially by $r\varphi$. Over a short axial interval the resulting engineering shear strain is $\gamma_{x\theta} = r\,d\varphi/dx$. Bonding requires a common twist per length φ' for the two materials. The shear on a cross-sectional end face is $Gr\varphi'$, so it can jump at $r = a$ even though the strain there is continuous. It is

the traction on a radial-normal interface that must equilibrate on opposite sides of the bond. Confusing it with the traction on the shaft end face would wrongly force the plotted shear stresses to coincide.

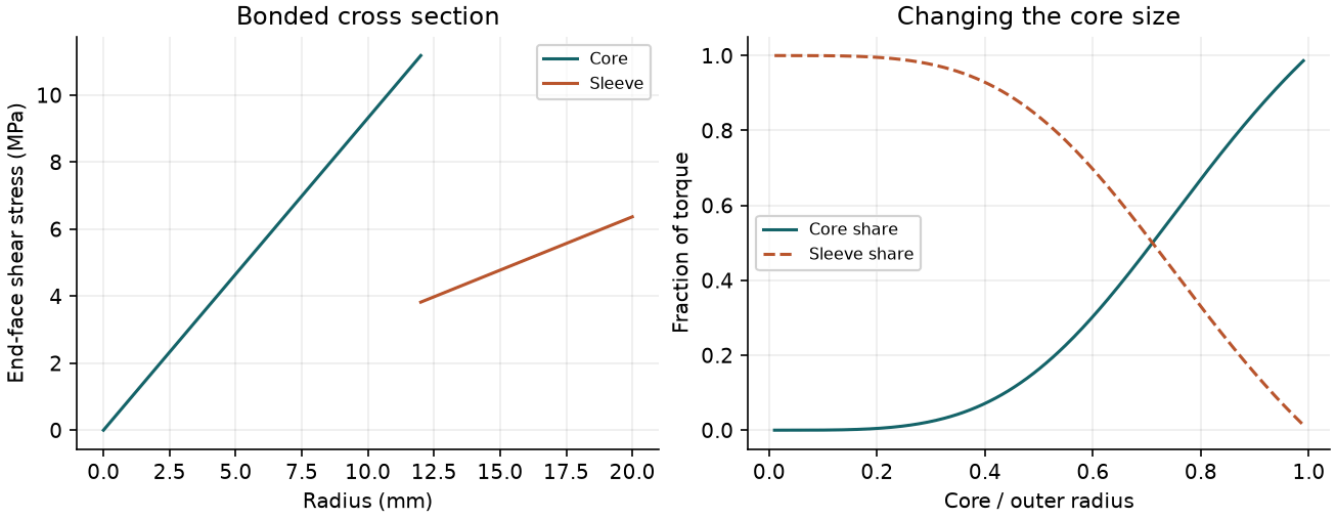
The torque on an annular element is the shear times its area times its lever arm. Integration of $r(Gr\varphi')(2\pi r\,dr)$ gives

$$T = [G_c J_c + G_s J_s]\varphi',$$

$$J_c = \pi a^4/2,$$

$$J_s = \pi(b^4 - a^4)/2.$$

Use $a = 12$ mm, $b = 20$ mm, $G_c = 79$ GPa and $G_s = 27$ GPa. Under 100 N m the core carries 30.35 N m and the sleeve 69.65 N m. A 600 mm length rotates by 0.405 degrees. Immediately inside the interface the end-face shear is 11.18 MPa. On the other side of that same radius the lower modulus reduces it to 3.82 MPa; continuing outward increases it again. An accurate shaft twist cannot by itself identify the local peak stress unless the section and material distribution are known.



The radial stress is local, while each torque share is an integral weighted by the lever arm. A thin outer ring may contribute disproportionately to the total rigidity.

In the notebook, integrate the stress directly over the core and sleeve and compare the total with the imposed torque. Evaluate the two integrals separately so a numerical trapezoid does not smear the material jump. Then make the two moduli equal: the end-face shear curve becomes straight across the entire section, while the fraction of torque in the core equals $(a/b)^4$. This limit offers an independent check against a uniform solid shaft.

A separate way to check the torque split uses stored energy. In a unit length of shaft the strain energy density is $\tau\gamma/2$. Integrating it over each annulus gives $GJ(\varphi')^2/2$ for that material. Because the twist is common, the energy fractions equal the torque fractions. For the example the total strain energy is $T\varphi/2$, about 0.35 J over 600 mm. If the two materials were joined end to end, the applied torque would be common and the angles of twist would add. The continuous cylindrical bond is the feature that selects the parallel arrangement used here.

At what radius fraction will the default materials divide torque exactly in half?

▼ **Work through the solution**

Set $G_c a^4 = G_s(b^4 - a^4)$. Rearranging gives

$$a/b = [G_s/(G_c + G_s)]^{1/4} = 0.7106.$$

At $b = 20$ mm this corresponds to a core radius of about 14.2 mm. Run the notebook with the revised radius and read both shares. This calculation assumes a bonded circular shaft away from the ends; it cannot predict the local loading of a short end fitting.

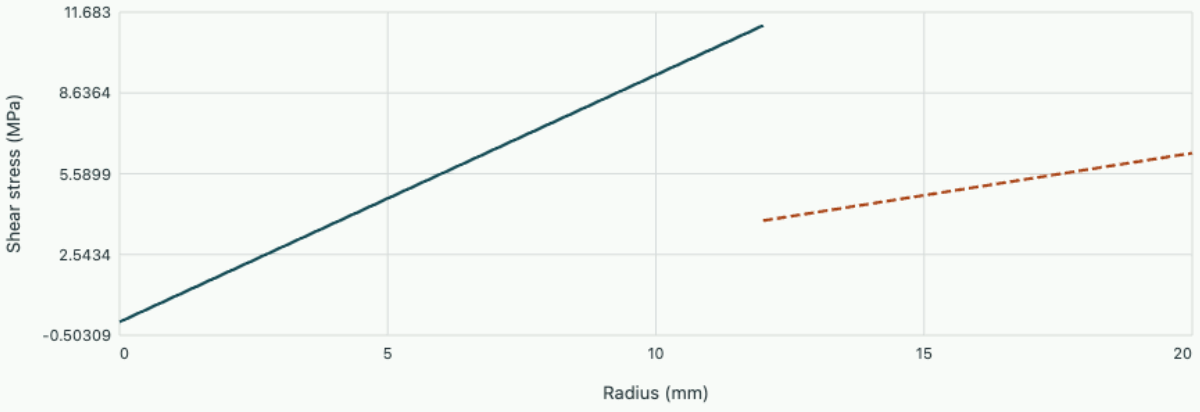
Interactive calculation · Bonded torsion

Vary the notebook inputs and compare both plotted observables. The same equations generate the browser curves and the notebook reference; fixed material, geometry and model assumptions are stated in the readout.

Open this live calculation

Try the calculation

Core radius (mm)	12
Core shear modulus (GPa)	79
Sleeve shear modulus (GPa)	27
Applied torque (N m)	100



At length0.60 m the rotation is 0.405°. Core torque 30.346 N m. Shear strain is continuous while stress jumps when the moduli differ.

Download the editable notebook for this example

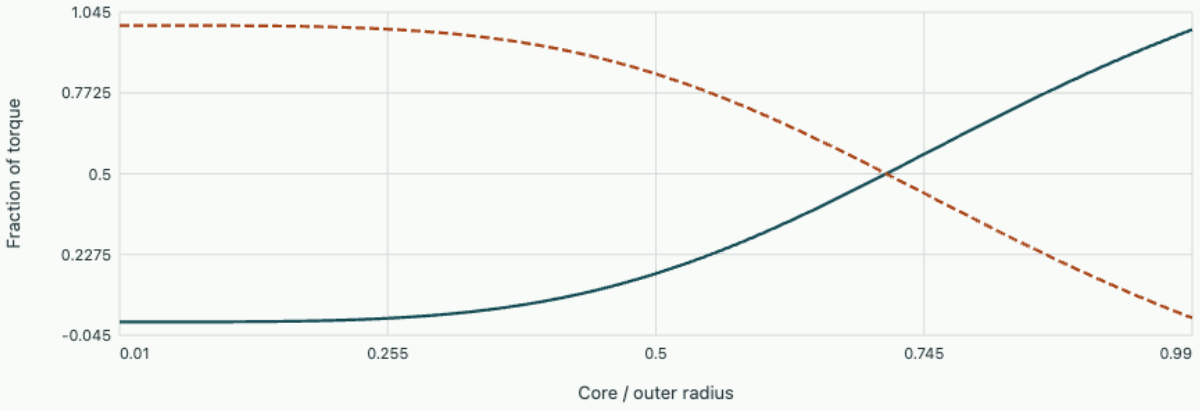
Interactive calculation · Bonded torque share

Vary the notebook inputs and compare both plotted observables. The same equations generate the browser curves and the notebook reference; fixed material, geometry and model assumptions are stated in the readout.

Open this live calculation

Try the calculation

Core radius (mm)	12
Core shear modulus (GPa)	79
Sleeve shear modulus (GPa)	27
Applied torque (N m)	100



At length0.60 m the rotation is 0.405°. Core torque 30.346 N m. Shear strain is continuous while stress jumps when the moduli differ.

Download the editable notebook for this example

Open editable notebook Download .ipynb

Rotate two adjacent sections

A torque meter measures twist between two stations on a shaft. To infer torque, first ask how material between the stations deforms. A straight circular shaft in uniform, small-deformation torsion has cross-sections that rotate as rigid discs. Two discs separated by dx rotate relative to one another by $d\theta$. A fibre at radius r moves sideways by $r\,d\theta$, giving shear strain $\gamma = r\,d\theta/dx$. With elastic shear modulus G, the shear stress increases linearly from the centre to the surface.

Each area element transmits a torque $r\tau\,dA$. Integrating produces the polar second moment of area J, whose dimensions are length to the fourth power. With constant torque and cross-section, integration along the shaft then converts twist per length into total relative angle.

$$T = \int_A r\tau\,dA = G\theta' \int_A r^2 dA,$$

$$J = \frac{\pi}{2}(r_o^4 - r_i^4),$$

$$\theta = \frac{TL}{GJ}.$$

Follow a torque through a stepped shaft

A 0.75 m aluminium tube of outer radius 15 mm and inner radius 12 mm transmits 45 N m. For $G = 26\,\mathrm{GPa}$, $J = 4.695 \times 10^{-8}\,\mathrm{m}^4$. Its twist is

$$45(0.75)/(26\times10^{9}J) = 0.02765\,\mathrm{rad} = 1.58^\circ.$$

Surface shear is

$$45(0.015)/J = 14.4\,\mathrm{MPa}.$$

The inside surface still carries 80% of this stress. Drilling a central hole removes the least highly stressed material first.

Twist can also identify an unknown joint compliance. If an instrument measures 2.00° across this tube plus its end couplings, and all parts carry the same torque, the couplings contribute about 0.42°. Their combined torsional compliance is approximately $0.42\pi/[180(45)] = 1.6\times10^{-4}\,\mathrm{rad/(N\,m)}$. Such a comparison assumes that the gauges do not include a second shaft segment and that the torque is transmitted without slip.

Carry torque around a single closed cell

For a long, thin-walled closed tube with a single cell, use the shear flow $q = \tau t$, the force per unit length of wall. Longitudinal equilibrium makes this flow approximately constant around the perimeter if no intermediate longitudinal force is introduced. A length ds carries tangential force $q\,ds$. Its moment about an arbitrary origin is q times twice the triangular area swept by the radius vector. Around the median wall contour this gives

$$T = 2A_m q.$$

For a rectangular cell with median dimensions 80 by 40 mm, a torque of 320 N m gives

$$q = 320000/[2(80)(40)] = 50\,\mathrm{N/mm}.$$

A 2 mm wall carries 25 MPa while a 4 mm wall carries 12.5 MPa. A slender longitudinal slit interrupts the closed path. The constant circulating flow solution then ceases to apply. A seam that reconnects the two lips must transmit the longitudinal shear flow if the component is to recover its closed-section behaviour.

Distinguish two section properties

For a noncircular closed tube, the polar area moment and torsion constant generally differ. Compatibility around a single thin-walled cell gives

$$\theta' = T \oint ds/(Gt)/(4A_m^2).$$

Making only a short wall segment very thin substantially increases this integral. A wall segment can therefore control angle of twist even when equilibrium fixes the same shear flow everywhere around the cell.

Determine torque sharing between two closed cells

A single closed cell supports one constant circulating shear flow. In an aircraft-style box or a partitioned machine boom, adjacent cells share a web. Torque balance alone cannot tell how much flow each cell carries. The missing equation is twist compatibility: because the cells are bonded along the common web, their twist per unit length must agree. Assume a long, thin-walled prismatic box of one material carrying Saint-Venant torsion with negligible distortion. Use median-line areas and length-to-thickness ratios throughout.

Assign positive clockwise circulations q_1, q_2 to both cells. They oppose each other on the common web, where physical flow becomes $q_1 - q_2$ measured in the first cell's direction. Let S_1, S_2 denote $\int ds/t$ over the exterior walls of each cell and S_w that quantity along their common web. The torque and the two contour compatibility relations become

$$T = 2A_1 q_1 + 2A_2 q_2,$$
$$2A_1 G\theta' = (S_1 + S_w)q_1 - S_w q_2,$$
$$2A_2 G\theta' = (S_2 + S_w)q_2 - S_w q_1.$$

These equations express the same shear-strain compatibility that leads to $J_t = 4A_m^2 / \oint ds/t$ for one cell. If both cells and exterior wall thicknesses are mirror images, symmetry requires equal circulations and zero common-web flow under pure torque. The common web may still carry transverse shear from a separate load case. A missing web flow under pure torque is therefore a property of that loading and symmetry.

Compute the stress in an internal dividing web

Take adjacent median rectangles 40 mm high, 60 and 30 mm wide, with every wall 2 mm thick. Then

$$A_1 = 2400,\ A_2 = 1200\,\mathrm{mm}^2.$$

Their three exterior walls give $S_1 = (60+40+60)/2 = 80$ and $S_2 = (30+40+30)/2 = 50$, while $S_w = 20$. Equal twist requires $(100q_1 - 20q_2)/2400 = (70q_2 - 20q_1)/1200$, giving $140q_1 = 160q_2$. Thus $q_2 = 0.875q_1$, larger than one might expect from the smaller area alone.

Applying $T = 345\,\mathrm{N\,m} = 345000\,\mathrm{N\,mm}$ gives $345000 = 4800q_1 + 2400(0.875)q_1$, so $q_1 = 50.0$ and $q_2 = 43.75\,\mathrm{N/mm}$. The dividing web carries only 6.25 N/mm, or 3.13 MPa. The primary exterior walls carry 25.0 and 21.9 MPa. Using $G = 27000\,\mathrm{MPa}$, either contour gives $\theta' = (100\times 50 - 20\times 43.75)/(2\times 2400\times 27000) = 3.18\times 10^{-5}\,\mathrm{rad/mm}$, about 1.82° per metre. Checking the other contour independently catches a reversed sign on the shared wall. If a weld along that wall fails, equal twist and the closed paths that led to these flows can no longer be assumed.

Individual cell torque contributions are informative when interpreting a section cut. The larger cell contributes $2(2400)(50) = 240\,\mathrm{N\,m}$, and the smaller contributes $2(1200)(43.75) = 105\,\mathrm{N\,m}$. Their 345 N m sum is independent of the coordinate origin. These are bookkeeping circulations rather than two torques applied independently to separate tubes. Cutting through the common web changes the compatibility and available flow paths, so using the original partition after a cut is not justified. Surface strain measurements on the upper exterior walls and the shared web would distinguish the predicted 50, 43.75 and 6.25 N/mm flows more effectively than measuring a single outer-wall strain alone.

A comparison with a thin unpartitioned box is possible without changing outer dimensions. For the outer median rectangle 90 by 40 mm, $A_m = 3600\,\mathrm{mm}^2$ and $\oint ds/t = 2(90+40)/2 = 130$. Its single-cell torsion constant is

$$4A_m^2/130 = 398769\,\mathrm{mm}^4.$$

The partitioned solution implies

$$J_t = T/(G\theta') = 345000/[27000(3.183\times 10^{-5})] \simeq 401500\,\mathrm{mm}^4.$$

Under this specific pure torque the additional web therefore raises torsional stiffness by less than one percent. It experiences little shear because the neighbouring circulations are similar. Its contribution to transverse load sharing, panel buckling resistance or local attachment stiffness can nevertheless be important and needs separate calculations.

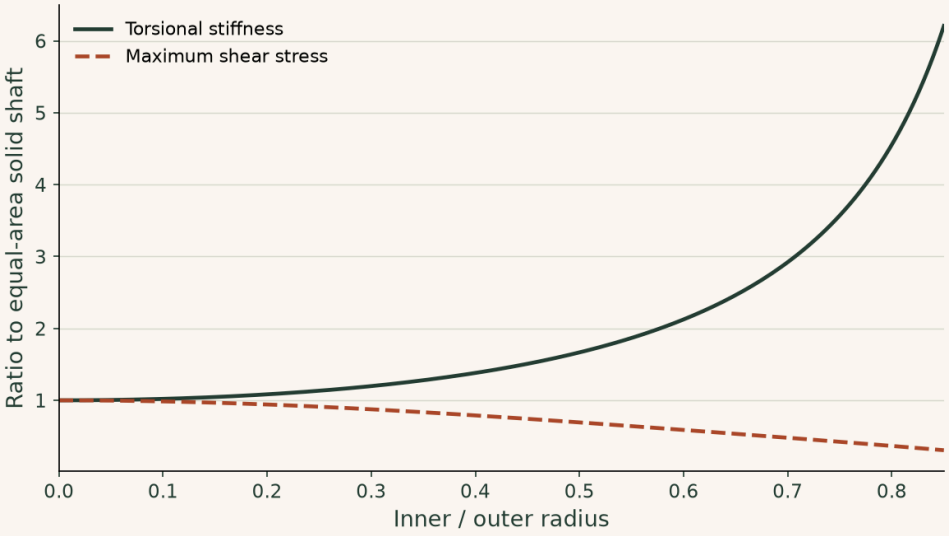
Move the web to the centre, producing identical cells of median width 45 mm. Pure torque then gives equal circulations, zero shear flow on the common web and precisely the single outer-cell twist in this idealisation. This limiting case is a demanding model check. A numerical section calculation that predicts double the torsional stiffness simply because there are now two cells has likely treated the common web as two independent exterior walls.

Compare a tube and a shaft at equal material area

Hold length, material, torque and cross-sectional area fixed while hollowing a circular shaft and increasing its outer radius. For radius ratio r, area conservation gives

$$R/R_s = 1/\sqrt{1-r^2}.$$

Substitute in $J = \pi R^4(1-r^4)/2$ and $\tau_{\max} = TR/J$.



At inner/outer radius 0.6 the torsional stiffness is 2.125 times the solid-shaft value, while peak shear is about 0.588 times as large. This comparison requires 25% greater outer radius; under a fixed envelope it would answer a different design question.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

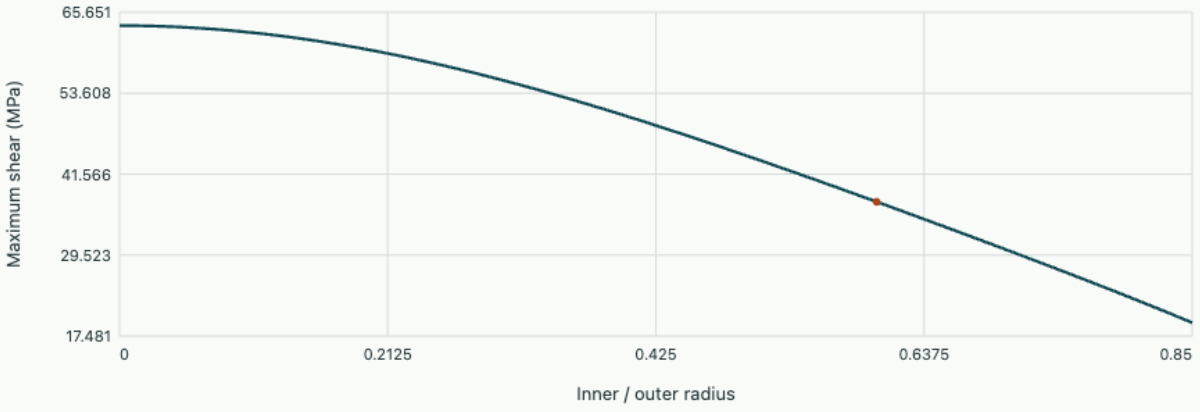
Hollow the shaft while keeping material area fixed. Watch both the required outer envelope and the peak shear change.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Equivalent solid radius (mm)	10
Torque (N m)	100
Hollow radius ratio	0.6



Equal material area | Selected tube

Selected outer radius 12.500 mm; stiffness / solid 2.125; peak shear 37.448 MPa. The outer envelope increases.

Creep, relaxation and what a holding test measures

Engineering mechanics

Two polymer specimens can show the same initial force and quite different readings ten minutes later. Change the quantity held fixed and the same material reveals a different time scale.

Connect a spring to a dashpot and keep track of the variables

An ideal spring obeys $\sigma = E\epsilon_s$. A linear dashpot obeys $\sigma = \eta\dot{\epsilon}_d$, where the viscosity η has units Pa s. These elements describe recoverable storage and dissipative flow. Start with a Maxwell element by putting them in series. Both carry the same stress and their strains add, so differentiating total strain gives

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}.$$

A fixed strain sets the left side to zero. Stored spring deformation slowly transfers to the dashpot while total extension stays fixed, producing a decaying stress. Under a constant nonzero stress the dashpot instead accumulates strain indefinitely. This second prediction makes the fluid-like long-time behaviour of the Maxwell element easy to test.

Add an equilibrium load path

Place a spring of modulus E_0 parallel to the Maxwell element and call the Maxwell stress s. Both branches now have the same total strain. The measured stress is $E_0\epsilon + s$. Replace s by $\sigma - E_0\epsilon$ in the series equation and gather the derivatives:

$$\dot{\sigma} + \frac{E_1}{\eta}\sigma = (E_0 + E_1)\dot{\epsilon} + \frac{E_0 E_1}{\eta}\epsilon.$$

In a sudden small strain step of magnitude ϵ_*, a finite dashpot cannot change strain instantaneously, so both springs respond at first. Holding the extension then allows the Maxwell stress to relax to zero. Conversely, a sudden stress step loads both springs at first but ends with the equilibrium spring carrying the complete applied stress. Solving the constant-coefficient equations with these initial and final conditions gives

$$\sigma(t) = \epsilon_*[E_0 + E_1 e^{-t/\tau_r}],$$

$$\tau_r = \eta/E_1,$$

$$\epsilon(t) = \frac{\sigma_*}{E_0} + \left[\frac{\sigma_*}{E_0 + E_1} - \frac{\sigma_*}{E_0}\right] e^{-t/\tau_c},$$

$$\tau_c = \frac{\eta(E_0 + E_1)}{E_0 E_1}.$$

The two exponential times differ because under load control the equilibrium spring gradually takes a different share of the force. At $t = 0$ the effective modulus is $E_0 + E_1$ in either test.

Worked example · identify parameters from one record

Impose 1% strain. Stress starts at 0.90 MPa and eventually settles at 0.30 MPa. The final value gives $E_0 = 30$ MPa and the drop gives $E_1 = 60$ MPa. Suppose the excess above the plateau falls to $1/e$ of its initial value after 40 s. The viscosity is then $60(40) = 2400$ MPa s. After 80 s the stress prediction is

$$0.30 + 0.60 e^{-2} = 0.3812 \,\mathrm{MPa}.$$

Subtracting the plateau before fitting a logarithm is essential: the logarithm of the total stress has a changing slope.

Worked example · make a prediction the fit did not use

In an independent test impose a stress of 0.45 MPa. Strain jumps to $0.45/90 = 0.005$ and approaches $0.45/30 = 0.015$. The predicted creep time is

$$40(90/30) = 120 \,\mathrm{s}.$$

After one minute the prediction is

$$0.015 - 0.010 e^{-0.5} = 0.00893.$$

A model that used the 40 s relaxation time for creep would yield 0.01277, readily distinguishable if the load has settled. Both experiments should use the same temperature because molecular mobility, and therefore the viscosity, can be highly temperature dependent.

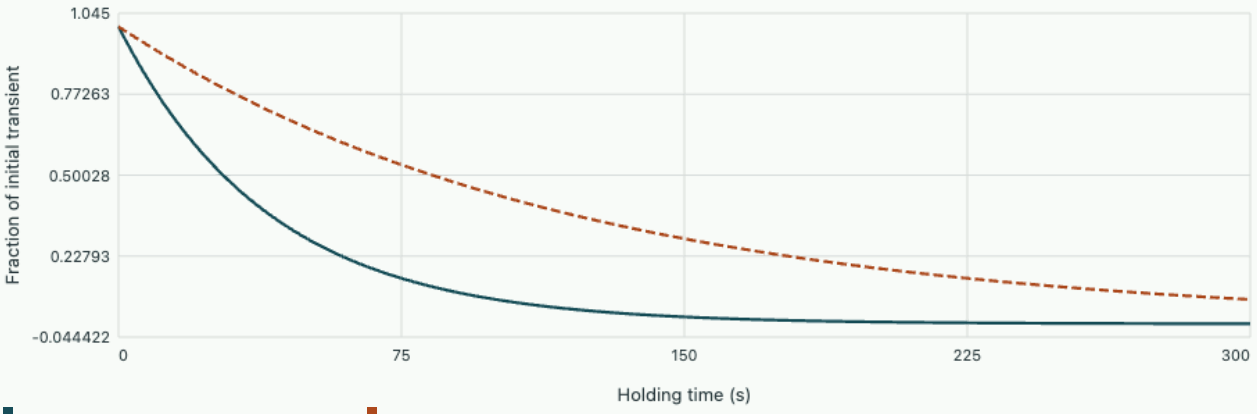
Double viscosity while leaving both spring moduli fixed. Predict which curve reaches its original one-minute value after two minutes. Next change the parallel spring alone and use the normalized curves to distinguish the two characteristic times.

Open this experiment online

Try the calculation

Viscosity (MPa s) **2400**

Parallel spring modulus (MPa) **30**



The relaxation time is 40.0 s and the creep time is 120.0 s. At 1% fixed strain the stress after 80 s is 0.381 MPa.

Tutorial · interrupt a relaxation experiment

Release the prescribed 1% strain instantaneously back to zero after 40 s. Calculate the stress immediately before and after release. What should happen over a further 40 s while zero extension is held?

▼ **Track the spring jumps**

Before release the stress is

$$0.30 + 0.60/e = 0.5207\,\mathrm{MPa}.$$

At the abrupt negative strain step the dashpot strain cannot jump. Both springs together change stress by $-0.01(90) = -0.90$ MPa, leaving −0.3793 MPa. At zero total strain the equilibrium spring carries zero and the remaining compressive Maxwell stress relaxes with the original 40 s time. It reaches −0.1395 MPa one relaxation time later.

Further reading · A. F. Bower, Applied Mechanics of Solids, §3.6, small-strain viscoelasticity

A moving-coil galvanometer as an electrical and mechanical system

Electromechanics, optics and measurement

A scale division records an angle, yet the quantity marked on the dial is electric current. A spring, a magnetic field and the resistance of the external measuring circuit connect the two. Derive the calibration, predict how a pointer approaches its reading, and find which experiment measures the charge of a short pulse.

Convert the force on two straight sides into a torque

A light rectangular coil has N turns, active length l, width w, and area per turn $A = lw$. It pivots about an axis parallel to its long sides. Shaped pole pieces confine an approximately radial magnetic field of magnitude B to an air gap. Over the working angular range the magnetic field is perpendicular to the active wire and lies approximately in the plane of the coil. An elementary length carrying conventional current i experiences $d\mathbf{F} = i\,d\mathbf{l} \times \mathbf{B}$. The two long sides carry current in opposite directions and experience forces of equal magnitude $NilB$. Acting with lever arms $w/2$, the forces make a couple whose torque is $NiAB$. Their net force is approximately zero. The short ends supply negligible torque about the pivot under the ideal geometry.

Write $K = NAB$, and choose the positive sense of current to give positive torque. Its units are $\mathrm{N\,m/A}$. In a uniform Cartesian field an ordinary rigid loop instead gives torque $NiAB\sin\alpha$, where α is the angle between its normal and the field. The radial gap maintains the factor equivalent to $\sin\alpha$ close to unity as the pointer turns. This is a design choice that produces a nearly linear scale. If a measured calibration curve flattens at a large angle, field nonuniformity and a nonlinear torsion spring are candidate causes that can be tested separately.

For a soft torsion spring with stiffness κ, the return torque is $-\kappa\theta$ at small deflections from the zero-current setting. At static equilibrium $\theta = Ki/\kappa$. With $N = 200$, $A = 1.00\,\mathrm{cm^2}$, $B = 0.250\,\mathrm{T}$ and $\kappa = 8.0 \times 10^{-4}\,\mathrm{N\,m/rad}$, the torque constant is $0.0050\,\mathrm{N\,m/A}$ and the current sensitivity is $6.25\,\mathrm{rad/A}$. A $5.0\,\mathrm{mA}$ current turns the pointer through $0.03125\,\mathrm{rad}$, about 1.79°. A mirror mounted on the same axle deflects a reflected laser through twice this angle. At a screen distance of $0.50\,\mathrm{m}$ the spot shifts $0.50\tan(2\theta) = 31.3\,\mathrm{mm}$, making the small mechanical rotation easy to resolve.

Recover the reciprocal voltage from work

Now rotate the coil through a small positive angle while it carries positive current. The electromagnetic torque does mechanical work $Ki\,d\theta$. The voltage associated with the motion must therefore draw electrical work $ie_{\mathrm{back}}\,dt$ of equal magnitude when the motor torque assists that motion. This gives the back-emf relation $e_{\mathrm{back}} = K\dot\theta$, measured as a positive drop in the motor sign convention. Its coefficient has units $\mathrm{V\,s/rad}$, numerically the same SI coefficient as the torque constant. One can also derive it from the motional emf on the two active sides. Each moves at speed $(w/2)\dot\theta$, crosses field B, and generates a voltage $Bl(w/2)\dot\theta$ along its wire. The two contributions add over every turn.

Let J denote the rotational inertia, b the mechanical viscous damping coefficient, L the coil inductance and R the sum of winding plus external series resistance seen by a voltage source $v(t)$. The

electromechanical equations are

$$J\ddot{\theta} + b\dot{\theta} + \kappa\theta = Ki,$$

$$L\dot{i} + Ri + K\dot{\theta} = v(t).$$

Multiplying the first equation by $\dot{\theta}$ and the second by i eliminates the coupling power $Ki\dot{\theta}$. The result says that source power vi equals the rate of increase of $J\dot{\theta}^2/2 + \kappa\theta^2/2 + Li^2/2$, plus $b\dot{\theta}^2 + Ri^2$. This identity fixes the relative signs of the torque and back emf. If a sign mistake turns a passive closed coil into a self-accelerating pointer, the same mistake produces a residual coupling term in this power balance. Mechanical friction and winding heat represent distinct sinks even though both can damp the observed displacement.

Predict overshoot and find a critical external resistance

If the electrical relaxation time L/R is short compared with the oscillation time $1/\sqrt{\kappa/J}$, set $i \simeq (v - K\dot{\theta})/R$. Substituting into the mechanical equation increases its effective damping to $b_{\mathrm{eff}} = b + K^2/R$. Set $v = 0$ when discussing free relaxation with the electrical terminals connected through a resistor. A moving shorted coil generates a current that opposes its motion. Opening the circuit instead suppresses that current and leaves the mechanical damping. An ideal current source holding specified current through all pointer motions also changes the back-emf response because its output voltage adjusts to enforce that current.

The undamped angular frequency is $\omega_0 = \sqrt{\kappa/J}$ and the damping ratio is

$$\zeta = b_{\mathrm{eff}}/(2\sqrt{\kappa J}).$$

Take $J = 2.0 \times 10^{-6}\,\mathrm{kg\,m^2}$, $b = 1.6 \times 10^{-5}\,\mathrm{N\,m\,s/rad}$, and the preceding K, κ. Then $\omega_0 = 20.0\,\mathrm{rad/s}$, $2\sqrt{\kappa J} = 8.0 \times 10^{-5}\,\mathrm{N\,m\,s/rad}$, and an open circuit gives $\zeta = 0.20$. With $R = 1.00\,\Omega$ the electric contribution to damping is $2.50 \times 10^{-5}\,\mathrm{N\,m\,s/rad}$, giving $\zeta = 0.5125$. For a voltage step the ideal fractional overshoot in the reduced model is

$$\exp[-\pi\zeta/\sqrt{1-\zeta^2}] = 0.153.$$

The first peak occurs after $\pi/(\omega_0\sqrt{1-\zeta^2}) = 0.183\,\mathrm{s}$.

Critical damping occurs when $b + K^2/R = 2\sqrt{\kappa J}$, which requires a total resistance of about $0.391\,\Omega$. A winding resistance of $0.300\,\Omega$ therefore leaves $0.091\,\Omega$ for external source and leads. If the winding alone exceeds the target resistance, a simple passive external series resistor cannot reach this damping in the reduced model. A finite L must also be checked: with $L = 5.0\,\mathrm{mH}$ and $R = 1.0\,\Omega$, $\omega_0 L/R = 0.10$, so the approximation provides a useful starting estimate. Reducing R indefinitely eventually invalidates the elimination of current. The full pair of equations then predicts the damping and frequency dependence.

Distinguish a charge impulse from a slow current reading

Suppose a brief imposed current pulse ends long before the pointer has turned appreciably. Integrating the torque equation over that interval gives

$$J\Delta\dot{\theta} \simeq K\int i\,dt = Kq.$$

The initial signal is now angular velocity proportional to total transmitted charge. For weak subsequent damping, the first excursion is approximately $Kq/(J\omega_0)$. A dose of $20\,\mu\mathrm{C}$ gives $2.50\,\mathrm{mrad}$ with the instrument above, corresponding to about $2.50\,\mathrm{mm}$ on the mirror scale. Two current pulses of identical charge but different shapes give nearly the same reading if both are short on the mechanical time scale. Their peak currents can be entirely different.

For appreciable damping, use the impulse solution $\theta(t) = Kq\,e^{-\zeta\omega_0 t}\sin(\omega_d t)/(J\omega_d)$, with $\omega_d = \omega_0\sqrt{1-\zeta^2}$ for an underdamped pointer. Its maximum obeys $\tan(\omega_d t_p) = \omega_d/(\zeta\omega_0)$ in the first quarter cycle. Substitution corrects the ballistic calibration instead of treating the damped peak as an undamped amplitude. The state of the external circuit after the pulse matters because it determines ζ during the throw. A calibration pulse should leave the measuring circuit connected in the same way as the unknown pulse.

Measure the torque constant without assuming the magnetic field

The preceding design calculation multiplies turn count, active area and a nominal air-gap induction. An independent calibration can measure the same constant without knowing the field distribution along every wire. Move the pointer through a small controlled angle while measuring its open-circuit voltage with a high-impedance instrument. In the same working interval the voltage is $K(\theta)\dot{\theta}$ up to a polarity choice. Integrating the voltage over the sweep gives $\int_{\theta_1}^{\theta_2} K(\theta)\,d\theta$. If K is constant this is simply $K\Delta\theta$, independent of sweep speed. It is often easier to integrate a recorded voltage pulse accurately than to differentiate the angle trace to obtain instantaneous speed.

For a sweep through $0.0800\,\mathrm{rad}$, suppose the measured voltage integral is $0.401\,\mathrm{mV\,s}$. The generator constant is $5.01\times10^{-3}\,\mathrm{V\,s/rad}$. Independently add a known small torque to the spring and measure its static angle. A torque $32.0\,\mu\mathrm{N\,m}$ rotates the pointer by $0.0400\,\mathrm{rad}$, giving $\kappa = 8.00\times10^{-4}\,\mathrm{N\,m/rad}$. Combining the two measurements predicts a current sensitivity of $6.27\,\mathrm{rad/A}$. Splitting the sweep into smaller equal angular intervals tests whether the measured K depends on angle. Sweeping backwards should reverse the integrated voltage. A voltage offset instead contributes its offset times duration, so repeating at two sweep speeds helps separate it.

A dynamic cross-check uses free decay under two load conditions. Measure the exponential decay rate α_{open} with a high-impedance voltage probe and α_R with a known total resistance in the closed circuit. In the short-electrical-time limit, subtraction of the two damping laws yields

$$K^2 = 2JR(\alpha_R - \alpha_{\mathrm{open}}).$$

With $J = 2.0 \times 10^{-6}\,\mathrm{kg\,m^2}$, $R = 1.00\,\Omega$, and decay rates 4.0 and $10.25\,\mathrm{s^{-1}}$, this gives

$$K = 0.0050\,\mathrm{N\,m/A}.$$

The method needs two identifiable decay rates and independent inertia. If the loaded response is overdamped, the envelope of two oscillation peaks is no longer available; fit the full two-real-pole response to the measured displacement.

Keep the third dynamical variable in a frequency-response experiment

For a sinusoidal voltage at angular frequency ω the exact linear transduction law follows by eliminating the complex current after transforming both differential equations. The mechanical impedance in angular-displacement form is $\kappa - J\omega^2 + i\omega b$; the winding impedance is $R + i\omega L$. Using phasors with time dependence $e^{i\omega t}$ gives

$$\frac{\Theta}{V} = \frac{K}{(R + i\omega L)(\kappa - J\omega^2 + i\omega b) + i\omega K^2}.$$

At zero frequency the gain is $K/(R\kappa)$ and the phase tends to zero, as the static calibration requires. At a sufficiently high frequency both the electrical and inertial impedances grow. For $L > 0$ the magnitude approaches $K/(LJ\omega^3)$. The reduced model with the current eliminated algebraically falls only as $1/\omega^2$. The change of the asymptotic slope provides an experimental way to distinguish finite inductance from an incorrectly adjusted viscous damping constant. Beyond the working band of a real instrument, elastic modes of the pointer or parasitic capacitance can enter before that asymptote is resolved.

At the nominal undamped frequency $20\,\mathrm{rad/s}$, with $R = 1.0\,\Omega$ and $L = 5.0\,\mathrm{mH}$, the stiffness and inertia terms cancel. The denominator becomes $-\omega^2 L b + i\omega(Rb + K^2)$, or approximately $-0.000032 + i\,0.000820$ in consistent SI units. Its magnitude gives about $6.09\,\mathrm{rad/V}$. A sinusoid of $1.0\,\mathrm{mV}$ peak produces about $6.09\,\mathrm{mrad}$ peak, whereas the static response to $1.0\,\mathrm{mV}$ is $6.25\,\mathrm{mrad}$. The phases differ radically despite their similar amplitudes: near this frequency the pointer lags by about 92.2°. A gain-only calibration at a single frequency would miss the phase lag.

Change the electrical load and compare two step models

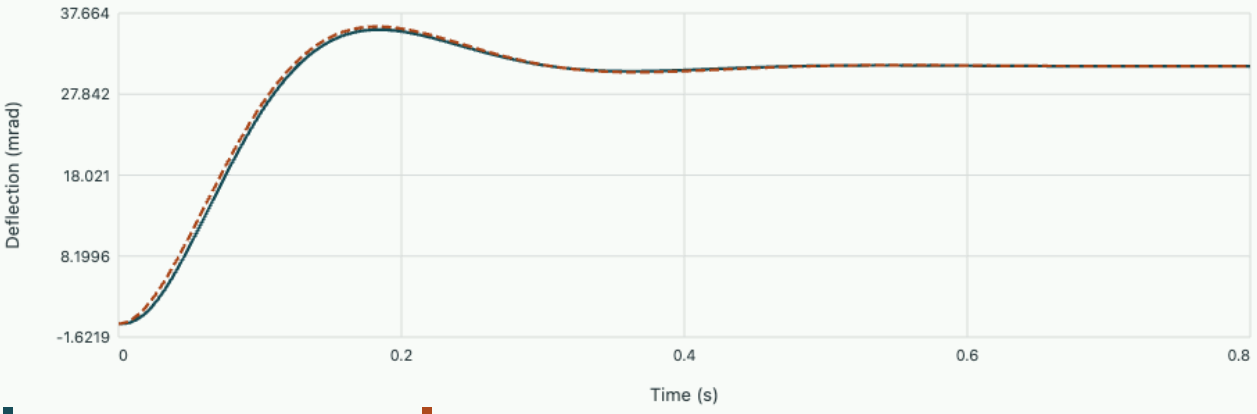
In the live graph the source voltage is chosen anew at each resistance to give a final current of $5.0\,\mathrm{mA}$ after the pointer settles. That convention keeps the final angle common to all curves and isolates the transient response of the load. The full model retains $i, \theta, \dot\theta$; the second curve sets $L\dot i = 0$. Predict first which trace should start flatter. With finite inductance the current initially vanishes and increases at a finite rate. The initial torque and angular acceleration therefore vanish, while the reduced model applies a finite torque immediately. Increasing the inductance makes that distinction easier to see, especially when the electrical response time becomes comparable to the period of mechanical oscillation.

Open the live experiment

Try the calculation

Total resistance (Ω) 1

Winding inductance (mH) 5



The reduced damping ratio is 0.512. The largest deflection in the first 0.80 s is 35.67 mrad. The common static limit is 31.25 mrad.

How could a student change the static range of the instrument without changing its suspension or its torsional stiffness?

▼ **Work through the solution**

For a dc ammeter, connect a calibrated shunt resistance in parallel with the coil, allowing only a known fraction of total current into it. With a coil winding resistance $0.300\,\Omega$ and a $0.0333\,\Omega$ shunt, the coil carries about ten percent of a steady terminal current and the indicated full-scale terminal current increases about tenfold. The total circuit seen by a motional voltage during a pointer oscillation depends on the shunt and source impedance, so the static range multiplier does not by itself give the new damping. For a voltmeter, a series resistance instead sets the current produced by an applied terminal voltage. In both cases record the resistance network whenever reporting a dynamic response.

Estimate the back action on the signal being measured

A galvanometer becomes part of the circuit it observes. Suppose a source with open-circuit voltage $10.0\,\mathrm{mV}$ and internal resistance $2.00\,\Omega$ drives a coil of resistance $0.300\,\Omega$. After settling, the current is $4.35\,\mathrm{mA}$ and the pointer turns by $27.2\,\mathrm{mrad}$. The undisturbed short-circuit current of the source would have been $5.00\,\mathrm{mA}$. Interpreting the indicated value as the current that would flow with a perfect zero-resistance meter underestimates that quantity by about thirteen percent. Measuring the open-circuit voltage instead requires a sufficiently large input resistance, another electrical boundary condition.

During pointer motion the disturbance includes the back emf as well as winding resistance. If the same coil is briefly turning at $0.40\,\mathrm{rad/s}$ it develops a $2.0\,\mathrm{mV}$ back emf. Neglecting inductance for that estimate, the resulting current is

$$(10.0 - 2.0)\,\mathrm{mV} / 2.30\,\Omega = 3.48\,\mathrm{mA}.$$

The deflection therefore contains information about both the source and the pointer's earlier motion. A static calibration alone cannot recover a rapid source waveform. A known source resistance and the two coupled dynamic equations provide a way to estimate the waveform from synchronised angle and current measurements.

Tutorial on calibration and transient identification

A current source produces a static mirror shift of $12.5\,\mathrm{mm}$ on a screen $0.50\,\mathrm{m}$ away. Estimate the current and identify a reversible test for an angular zero offset.

▼ **Work through the solution**

Use $\theta = \tfrac{1}{2}\arctan(0.0125/0.50) = 0.01250\,\mathrm{rad}$ and $i = \kappa\theta/K = 2.00\,\mathrm{mA}$. Repeat with reversed current of equal magnitude. Half the difference of the two angles cancels a fixed angular zero offset and isolates the response odd in current. A changing temperature during reversal can alter the spring or winding resistance, so alternate the order and record the zero afterwards if the required uncertainty is small.

For an open-circuit free decay the ratio of two successive same-side positive peak angles is 3.60. Their time separation is $0.321\,\mathrm{s}$. Recover the mechanical parameters that can be inferred if J is known.

▼ **Work through the solution**

The logarithmic decrement is $\ln 3.60 = 1.281$. The decay rate is $\alpha = 1.281/0.321 = 3.99\,\mathrm{s^{-1}}$, and the damped angular frequency is

$$2\pi/0.321 = 19.57\,\mathrm{rad/s}.$$

Then $\omega_0 = \sqrt{\omega_d^2 + \alpha^2} = 19.97\,\mathrm{rad/s}$,

$$b = 2J\alpha = 1.60 \times 10^{-5}\,\mathrm{N\,m\,s/rad},$$

$$\kappa = J\omega_0^2 = 7.98 \times 10^{-4}\,\mathrm{N\,m/rad}.$$

This identification assumes viscous damping and a sufficiently high impedance to make the electrical damping negligible.

Explore how motion generates voltage in the induction chapter.

Eddy-current braking, magnetic penetration and the heat left behind

Electromechanics, optics and measurement

A conducting loop moves above a row of magnetic poles. The drag first increases with speed, but a rapid crossing need not generate proportionally greater braking. Trace the work done against the Lorentz force into circuit energy, heat and a measurable temperature change. Then decide which features of a solid plate the single-loop model can capture.

Generate a voltage from a spatially periodic field

Let a rectangular loop of height h along y and width w along x translate to the right above a stationary magnetic pattern. At the loop let the normal induction be $B_z(x) = B_0 \cos(kx)$, with $k = 2\pi/\lambda$. Its two active sides are centred symmetrically about the coordinate x_c. Assume the field varies negligibly along y over this height. Integrating through the loop gives

$$\Phi = h \int_{x_c - w/2}^{x_c + w/2} B_0 \cos(kx)\, dx = [2hB_0 \sin(kw/2)/k] \cos(kx_c).$$

A uniform translation $x_c = ut$ therefore produces an external emf with peak amplitude $E_{\mathrm{pk}} = Ku$, where $K = 2hB_0 |\sin(kw/2)|$.

This spatial averaging explains an initially surprising design feature. A loop spanning precisely one full spatial period always links zero net flux from the sinusoidal field and therefore generates no emf in this ideal geometry. A loop of half a period has maximum coupling at a specified height. If w is much smaller than the period,

$$\sin(kw/2) \simeq kw/2,$$

$$K \simeq hwB_0 k.$$

This is exactly the small-loop result from using its area times the local gradient of induction. Reversing velocity reverses the emf at a fixed position. At a fixed positive speed the temporal frequency is $f = u/\lambda$, so the same conductor experiences different electrical frequencies at different mechanical speeds.

Include finite inductance and derive the drag curve

Model the current path by a constant resistance R and self-inductance L. Its steady periodic current solves $L\dot{i} + Ri = \mathcal{E}(t)$. In phasor notation its magnitude is $E_{\mathrm{pk}}/\sqrt{R^2 + (\omega L)^2}$, lagging the emf by $\arctan(\omega L/R)$, where $\omega = ku$. The current develops field energy $Li^2/2$ and dissipates Ri^2. Multiplying the circuit equation by i and averaging over a period eliminates the derivative of stored energy. The average mechanical power delivered against the drag is therefore the mean Joule heating $P = E_{\mathrm{pk}}^2 R/[2(R^2 + \omega^2 L^2)]$. Dividing by the speed gives the drag magnitude

$$F(u) = \frac{K^2 u}{2R[1 + (u/u_c)^2]},$$

$$u_c = \frac{R}{kL}.$$

At low speed the drag is proportional to velocity. At the crossover $u = u_c$ it reaches $K^2 u_c/(4R)$, and at much larger speeds it falls approximately as $1/u$ within the assumed lumped model. The incoming mechanical power then tends to a constant rather than growing without bound. Why is there a maximum? Increasing speed raises the emf, but it also raises the reversal frequency. An inductive current increasingly lags the driving emf, and its component in phase with the emf does the net dissipative work. A measured turnover therefore tests both an electrical time scale and the spatial scale of the magnet pattern.

Use $h = 50\,\mathrm{mm}$, $w = 25\,\mathrm{mm}$, $\lambda = 0.100\,\mathrm{m}$, $B_0 = 0.20\,\mathrm{T}$, $R = 0.050\,\Omega$ and $L = 0.150\,\mathrm{mH}$. Then $K = 0.01414\,\mathrm{T\,m}$, $u_c = 5.31\,\mathrm{m/s}$, and maximum mean drag is $5.31\,\mathrm{mN}$. At $u = 2.0\,\mathrm{m/s}$ the field alternates at $20\,\mathrm{Hz}$, giving an emf of $28.3\,\mathrm{mV}$ peak. The current is $0.529\,\mathrm{A}$ peak and its phase lag 20.7°. The resistor receives approximately $7.00\,\mathrm{mW}$, and the motor maintaining translation supplies a mean force of $3.50\,\mathrm{mN}$. Using peak current in I^2R without its factor one half would overstate the mechanical force by a factor two.

Determine whether the current penetrates a real plate

A continuous metal plate contains many possible current paths, unlike a preformed loop. Their resistances, inductances and mutual couplings depend on the plate boundary, magnet geometry and frequency. For a good homogeneous conductor the magnetic diffusion equation gives a skin depth $\delta = \sqrt{2/(\mu\sigma\omega)}$ under sinusoidal excitation. At a depth δ a field entering a locally planar half-space has lost a factor e in amplitude and acquired a phase delay of one radian. The frequency to use is the frequency experienced by the moving material; the fact that the magnets are stationary in the laboratory does not make this frequency zero.

For copper with $\sigma = 5.8 \times 10^7\,\mathrm{S/m}$ and permeability close to μ_0, the skin depth is about $14.8\,\mathrm{mm}$ at $20\,\mathrm{Hz}$ and $1.48\,\mathrm{mm}$ at $2.0\,\mathrm{kHz}$. A $2.0\,\mathrm{mm}$ plate therefore sees little through-thickness attenuation in the first experiment but may need a diffusion calculation in the second. The plate's lateral current pattern can still matter when the through-thickness current density is nearly uniform. Slots interrupt wide current loops and usually reduce drag even when the mass and the bulk conductivity change little. Comparing a slotted and an unslotted plate directly tests the importance of current closure.

If a plate experiences sharply separated poles, its applied field in the moving frame contains a fundamental and higher harmonics. Harmonic n has a skin depth smaller by $\sqrt{n}$, so checking only the fundamental can miss surface concentration associated with narrow features. A Hall scan along the track supplies the actual spatial field spectrum. Multiplying each spatial wavenumber by the speed converts it into a temporal angular frequency for a constant-speed pass. Once the speed changes significantly over one pole spacing, a time-domain calculation using the measured position trace is preferable to this fixed-frequency conversion.

Close a mechanical and thermal energy budget

Take a rotating brake as a second experiment. A flywheel of known inertia $J_{\mathrm{rot}} = 0.020\,\mathrm{kg\,m^2}$ slows from 120 to $80\,\mathrm{rad/s}$ while no drive supplies power. The released rotational kinetic energy is

$$J_{\mathrm{rot}}(120^2 - 80^2)/2 = 80.0\,\mathrm{J}.$$

A separate run with the magnet removed measures bearing and air losses. Suppose the corresponding speed-dependent losses integrated over the magnetic braking trajectory total $8.0\,\mathrm{J}$. Then the electromagnetic channel receives $72.0\,\mathrm{J}$. Using the reference run's total energy over an arbitrary equal time could give a different answer, since the two speed histories affect the parasitic losses.

A calorimeter enclosing the conducting rotor has effective heat capacity $C = 60\,\mathrm{J/K}$ and a thermal link of conductance $G = 0.12\,\mathrm{W/K}$ to an environment at constant temperature T_a. Its balance is

$$C\,dT/dt = P_{\mathrm{deposited}} - G(T - T_a).$$

During a $20\,\mathrm{s}$ run, suppose the time-averaged rise above ambient is $0.50\,\mathrm{K}$. Then $1.20\,\mathrm{J}$ flowed through the thermal link. If all electromagnetic heat entered the calorimeter, its final rise from an ambient start is

$$(72.0 - 1.20)/60 = 1.18\,\mathrm{K}.$$

A measured rise near $0.80\,\mathrm{K}$ would leave an energy discrepancy of approximately $23\,\mathrm{J}$. Check conduction into a shaft, heat stored outside the instrumented volume, and the estimate of parasitic torque before adjusting the electromagnetic theory.

Find the spatial features that dominate a measured braking signal

A single Fourier component of field already produces an informative turnover. In a tracked experiment the field often has a measurable third harmonic. Let $B(x) = B_1 \cos kx + B_3 \cos(3kx + \varphi)$, with constant translation speed. Flux integration across the finite loop multiplies each harmonic by its own width factor. The corresponding emf peak is $2h|B_n|u|\sin(nkw/2)|$, and its temporal angular frequency is nku. The linear RL equation treats the two drives independently. Orthogonality over a complete fundamental period makes the mean of their product vanish, so total mean heat can be found by adding the individual resistive powers. Within one period the interference terms do affect instantaneous torque and power.

Suppose $B_3/B_1 = 0.20$ and keep the preceding $w = \lambda/4$. Here $\sin(kw/2)$ and $\sin(3kw/2)$ both have magnitude $1/\sqrt{2}$. At low speed the third harmonic therefore adds about four percent to mean drag. At $u = u_c$, however, its angular frequency sees inductive impedance $3R$ while the fundamental sees R. The third-harmonic share relative to the fundamental becomes $0.20^2(1+1)/(1+9) = 0.008$, only 0.8%. A peak-to-peak field reading can change visibly without a comparable change in mean drag. Conversely, if the loop width happens to null one spatial harmonic, another smaller harmonic can supply most of the signal. A field map and the loop geometry together determine what source drives the circuit.

In a solid plate, predicting force from field and speed needs a field solution that enforces current closure. In the slowly moving conductor, $\mathbf{J} = \sigma[-\nabla\phi + \mathbf{u} \times \mathbf{B}]$ when the imposed magnetic pattern is effectively time independent in the laboratory and the induced magnetic field can be neglected. Charge conservation then gives

$$\nabla \cdot [\sigma(-\nabla\phi + \mathbf{u} \times \mathbf{B})] = 0.$$

On an electrically insulated plate edge, enforce $\mathbf{n} \cdot \mathbf{J} = 0$. Solving this potential problem accounts for charge accumulation that redirects currents at a boundary. Simply setting $\mathbf{J} = \sigma\mathbf{u} \times \mathbf{B}$ can predict current flowing straight out of an insulating edge. The force follows from integrating $\mathbf{J} \times \mathbf{B}$ through the conducting volume, and Joule heat follows from $\int J^2/\sigma \, dV$.

A useful check on such a field calculation is a finite plate translating wholly inside an exactly uniform static magnetic field. Charges can rearrange until a uniform electrostatic field cancels $\mathbf{u} \times \mathbf{B}$. An isolated plate in this steady limit then has zero transport current and zero drag. Motion alone does not guarantee braking. At an entry edge, the moving conductor spans a spatial change of induction, and one scalar potential generally cannot cancel the motional drive around every closed path. Slits, thickness variations, a conducting contact at a boundary, and the induced field at higher speeds each change this test problem in an identifiable way.

Interpret a thermal transient without assuming it is instantaneous

If the brake runs at a stable speed and equilibrium electrical conditions, a local single-temperature model sees a nearly constant power. Its step response from ambient is $\Delta T = (P/G)[1 - \exp(-Gt/C)]$. Removing the drive gives an exponential decay with the same time constant C/G. Measuring that cooling tail fixes the ratio of heat capacity to conductance but does not set either one separately. A known heater power or an independent heat-capacity measurement supplies the missing absolute scale. A slope evaluated over a long interval without correcting leakage will underestimate the input power as the sample warms.

Take a small bench calorimeter enclosing the single test loop with $C = 1.20\,\mathrm{J/K}$ and $G = 0.020\,\mathrm{W/K}$. The $7.00\,\mathrm{mW}$ low-speed run gives a $0.350\,\mathrm{K}$ eventual rise and a $60\,\mathrm{s}$ time constant. After $30\,\mathrm{s}$ the rise is about $0.138\,\mathrm{K}$, whereas a thermally isolated calculation would give $0.175\,\mathrm{K}$. Its stored heat $0.166\,\mathrm{J}$ is less than the $0.210\,\mathrm{J}$ mechanical work over that interval; the thermal link accounts for the remainder. Loading the conductor thermally with an additional copper holder increases the response time and changes the initial slope without changing the final rise if conductance to the bath remains the same.

Compare predictions at a chosen travel speed

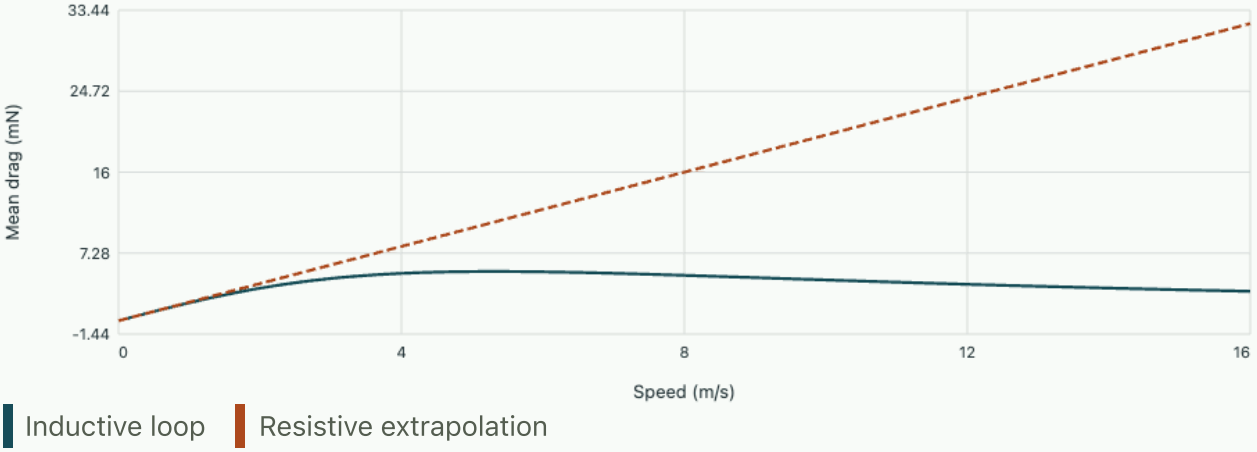
The live graph displays the force predicted by the finite-inductance loop alongside the resistive low-frequency extrapolation. Choose a speed, calculate the apparent excitation frequency, and compare the returned skin depth to the actual conductor thickness before interpreting the prediction as a plate measurement. Try choosing an inductance twice as large. The location of maximum force halves in speed, but the low-speed slope at fixed geometry and resistance is unchanged. A real plate whose high-

speed drag does not follow the single-loop curve can still agree with the model on its low-speed slope. Plotting the measured residuals against speed can reveal the point where extra current paths or through-thickness diffusion become important.

Open the live experiment

Try the calculation

Travel speed (m/s)	2
Loop inductance (mH)	0.15
Peak field (T)	0.2



At 2.00 m/s the drag is 3.50 mN and the mean heat is 7.00 mW. The copper skin depth at 20.0 Hz is 14.78 mm. Peak drag occurs at 5.31 m/s.

A plate and holder cool from $0.240\,\mathrm{K}$ above ambient to $0.120\,\mathrm{K}$ in $42\,\mathrm{s}$ after a run. Can these two readings establish the braking power?

▼ **Work through the solution**

The half-life gives

$$C/G = 42/\ln 2 = 60.6\,\mathrm{s},$$

which agrees approximately with the small calorimeter example. It determines only a ratio. If an independent steady $10.0\,\mathrm{mW}$ heater then produces a $0.500\,\mathrm{K}$ temperature rise, $G = 0.0200\,\mathrm{W/K}$ and $C = 1.21\,\mathrm{J/K}$. A magnetic run that settles at $0.350\,\mathrm{K}$ now corresponds to $7.00\,\mathrm{mW}$, provided the thermal boundary and deposition pattern are sufficiently similar to the calibration.

Check the sign of instantaneous electromagnetic work

At a phase where the applied emf has just crossed zero, a finite-inductance loop may still carry considerable current. During part of a cycle the emf and current can have opposite signs. The product $\mathcal{E} i$ is then negative even though $R i^2$ remains positive. The difference is a negative derivative of the magnetic store $L i^2/2$. The loop briefly returns energy mechanically while its stored field decreases, and some of that release also becomes heat. A force transducer resolving separate pole passages can therefore see instantaneous variations that a mean braking-force estimate suppresses.

For the numerical $2.0\,\mathrm{m/s}$ case the peak field energy is $L I_{\mathrm{pk}}^2/2 \approx 21.0\,\mu\mathrm{J}$. The dissipated energy per full pole period is

$$7.00\,\mathrm{mW}/20\,\mathrm{Hz} = 350\,\mu\mathrm{J}.$$

These quantities serve different purposes. The first is a reversible instantaneous store; the second is net irreversible energy delivered each time the conductor moves one spatial period. Dividing the second by the spatial period of $0.100\,\mathrm{m}$ recovers the mean drag of $3.50\,\mathrm{mN}$ without using the translational speed explicitly.

Tutorial on braking measurements

At $u = 2.0\,\mathrm{m/s}$ the loop above experiences a mean drag of $3.50\,\mathrm{mN}$. Predict what halving the induction and doubling the speed does, keeping its geometry and electrical properties fixed.

▼ **Work through the solution**

Halving induction quarters K^2. The speed factor contributes $2[1+(2/5.31)^2]/[1+(4/5.31)^2] \approx 1.46$. The combined change gives a drag about 0.364 times the original, or $1.27\,\mathrm{mN}$. The mechanical power decreases less because speed has doubled. A linear-speed extrapolation would miss the inductive phase lag, already significant at $4\,\mathrm{m/s}$.

In a repeated flywheel test the brake becomes warmer with every run. Which electrical parameter change can explain a change of drag even with the magnets held fixed?

▼ **Work through the solution**

The resistance of an ordinary metal generally increases with temperature. At a fixed speed F is proportional to $R/[R^2+(\omega L)^2]$. Raising resistance reduces drag on the low-frequency resistive side and can increase it on the high-frequency inductive side. The maximum occurs at $R = \omega L$ if electrical geometry and inductance stay fixed. Measuring the temperature and resistance before each run helps distinguish this effect from a weakening field or a changed gap.

Compare a step-driven diffusion profile with sinusoidal skin penetration.

Coupled optical and mechanical resonances

Electromechanics, optics and measurement

A vibrating mirror can imprint its position on an optical field, while the stored light pushes back on the mirror. An apparent pair of peaks can indicate two hybrid modes, interference at a readout port or two unrelated resonances. Starting from energy in a moving cavity, derive the coupled response and learn which measurements establish a coupling strength and its uncertainty.

From a moving boundary to a force per photon

Take a nearly empty Fabry–Pérot cavity of optical length L with one moving end. A longitudinal resonance has $\omega_c = m\pi c/L$ for integer m, neglecting frequency-dependent mirror phase. For a small extension x, $\delta\omega_c = (\partial\omega_c/\partial x)x$ with signed slope $G_x = -\omega_c/L$. At fixed photon number n the optical energy is $n\hbar\omega_c(x)$. Its derivative gives the radiation-pressure force on the coordinate. For this mirror $F = -n\hbar G_x > 0$ pushes outward. The derivative must be at fixed photon number. Differentiating a driven steady-state photon number as though it were a closed-system energy would incorrectly include work exchanged with the pump.

For a 10 mm cavity resonant near 1064 nm the frequency slope magnitude in ordinary frequency is about 28.2 MHz per nanometre. A static shift of 10 pm therefore changes the optical resonance by approximately 282 kHz. One photon exerts a mean force $\hbar\omega_c/L \simeq 1.87 \times 10^{-17}$ N. An equivalent classical derivation counts a momentum change twice per normal reflection for each relevant round trip. The two descriptions yield the same mean force. The mirror displacement is the properly normalised mode coordinate; in a flexural device the optically sampled position and the total physical mass are generally different from centre-of-mass displacement and mass.

Introduce a single mechanical eigenmode with effective mass m_{eff}, angular frequency Ω_m and displacement scale $x_{\mathrm{zpf}} = \sqrt{\hbar/(2m_{\mathrm{eff}}\Omega_m)}$. The single-photon coupling magnitude is $g_0 = |G_x|x_{\mathrm{zpf}}$. This normalisation is convenient even when the observed vibrations are entirely classical. For $m_{\mathrm{eff}} = 10$ ng and $\Omega_m/(2\pi) = 1.0$ MHz, $x_{\mathrm{zpf}} = 9.16 \times 10^{-16}$ m and $g_0/(2\pi) = 25.8$ Hz for the stated optical geometry. The small displacement scale explains why a coherent pump is normally used to make the interaction measurable.

How a laser couples frequencies that are otherwise far apart

An optical carrier oscillates at hundreds of terahertz, whereas the mirror may vibrate at only megahertz. Direct comparison of these two bare frequencies would never suggest a resonant exchange. A strong coherent pump supplies a phase reference and converts a mechanical displacement into optical sidebands. Move into a frame rotating at the pump frequency and write the cavity amplitude as its large steady part plus a small perturbation. The leading frequency-shift interaction is proportional to photon number times displacement. Expanding the intensity around the large pump keeps terms linear in both the probe field and displacement. The resulting coupling is $G = g_0\sqrt{n_c}$, where n_c is mean intracavity pump photon number.

Tune the pump about one mechanical frequency below the optical cavity resonance. In the resolved-sideband and weak-fluctuation limit a near-resonant process moves one excitation between the

mechanical mode and an upper optical sideband. The rapidly rotating opposite-sideband terms average out. Now the two coupled frequencies are an optical sideband offset and the mechanical frequency, measured relative to a common rotating reference. The large carrier frequency has already been accounted for by the pump. The pump phase can be absorbed into the mode definitions, so G may be chosen positive for this isolated pair.

If instead the pump is placed near the upper sideband, the surviving interaction creates or annihilates excitations in pairs. The passive exchange equations below cease to apply and strong pumping can produce amplification or instability. A bare two-oscillator machine driven mechanically near a shared natural frequency often has analogous exchange equations without a carrier pump. Identical matrices can describe different energy supplies; identify the drive and rotating reference before assigning a literal photon-to-phonon energy conversion.

Derive complex poles and the effect of unequal damping

Let a, b be small energy-normalised optical and mechanical complex amplitudes. In a frame where the uncoupled optical sideband has zero frequency, let the mechanical detuning be δ. Let κ, Γ denote energy linewidths in angular frequency, so amplitude decay rates are half these values. With an optical probe s at offset ω, the linear system is

$$\begin{pmatrix} \kappa/2 - i\omega & iG \\ iG & \Gamma/2 - i(\omega - \delta) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} s \\ 0 \end{pmatrix}.$$

Zeros of the determinant give autonomous complex mode frequencies. In the lossless case the eigenvalues are $\delta/2 \pm \sqrt{G^2 + (\delta/2)^2}$. Far off resonance each branch approaches a bare mode. At zero detuning the two eigenvectors have equal optical and mechanical weights and the splitting is $2G$. The eigenvectors rotate continuously through the avoided crossing, an important discriminator from two uncoupled lines simply passing through one another.

At zero detuning with losses, completing the square in the determinant gives

$$\omega_{\pm} = -i(\kappa + \Gamma)/4 \; \pm \sqrt{G^2 - (\kappa - \Gamma)^2/16}.$$

The real parts separate once $G > |\kappa - \Gamma|/4$. The shared imaginary part above that threshold corresponds to a hybrid energy linewidth $(\kappa + \Gamma)/2$. Resolving two intensity features on a given instrument is a stronger, readout-dependent condition. The dimensionless cooperativity $C = 4G^2/(\kappa\Gamma)$ is yet another measure, comparing interaction against the product of dissipation scales. A value greater than one does not automatically prove two resolved normal-mode poles in a device with very unequal linewidths.

Use frequency numbers divided by 2π consistently from here for examples. With optical linewidth 240 kHz, mechanical linewidth 40 kHz and coupling 160 kHz, the real pole separation is

$$2\sqrt{160^2 - 50^2} = 304\,\mathrm{kHz}.$$

Both hybrid energy linewidths are 140 kHz, and cooperativity is 10.7. If coupling falls to 45 kHz, cooperativity is still about 0.844 and no real zero-detuning pole separation appears. At 50 kHz the square root reaches zero. Near that threshold even a small linewidth or detuning calibration error has a large effect on a coupling estimate inferred from a single splitting.

Compute what an optical or mechanical receiver actually sees

Invert the matrix. With $D = (\kappa/2 - i\omega)[\Gamma/2 - i(\omega - \delta)] + G^2$, the small amplitudes are $a = s[\Gamma/2 - i(\omega - \delta)]/D$ and $b = -iGs/D$. The poles are shared. Their numerators are different, so the two channels can show different apparent maxima. A port in reflection also has a directly reflected field that interferes with the leaking intracavity amplitude. Its intensity can display a narrow dip or asymmetric feature without a matching resonance-height pattern inside the cavity. A legitimate forward model must describe the measured port.

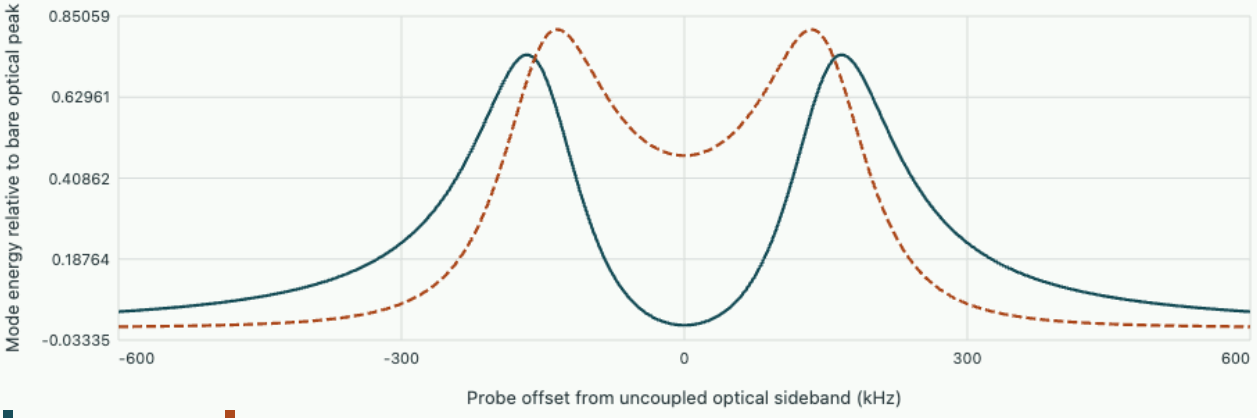
Open the live experiment

Try the calculation

Coupling G/(2π) (kHz) **160**

Optical linewidth κ/(2π) (kHz) **240**

Mechanical detuning (kHz) **0**



At zero bare detuning the real pole separation is 304.0 kHz. For the selected coupling and linewidths the cooperativity is 10.67. The mechanical linewidth is fixed at 40 kHz.

The graph plots the internal squared amplitudes relative to the bare on-resonance optical probe energy. Set the mechanical detuning to zero and reduce the coupling through 50 kHz. Compare the optical dip with the predicted real pole separation. Then set a substantial positive detuning and follow the transfer of mechanical response onto one branch. As a control set the coupling exactly to zero. The optical curve recovers the bare cavity Lorentzian and the mechanical probe-induced response vanishes. A thermally excited mechanical background, absent from this driven calculation, could remain in a real detector.

At zero detuning and zero probe offset,

$$a/a_{\mathrm{bare}} = 1/(1 + C).$$

For the default parameters the internal optical probe intensity at the centre is only $1/(1 + 10.67)^2 = 0.00735$ of the bare peak. This deep feature alone still does not establish the pole spacing. An incoherent background, a frequency-dependent detector transfer or coherent feedthrough can alter peak and dip positions. Fit complex quadratures across a detuning series when available, and verify bare linewidths by independent ringdown or far-detuned spectra.

Check how energy linewidth and amplitude ringdown are related for one isolated resonance

Watch energy exchange in time and identify the linewidth convention

In an ideal equal-linewidth system starting with optical perturbation $a(0) = a_0$, $b(0) = 0$ at zero detuning, change to symmetric and antisymmetric modes. Recombination gives

$$a(t) = a_0 e^{-\kappa t/2} \cos Gt,$$

$$b(t) = -i a_0 e^{-\kappa t/2} \sin Gt.$$

Their normalised total energy decays as $e^{-\kappa t}$, while energy moves back and forth between optical sideband and vibration. The first full transfer occurs at $t = \pi/(2G)$ if the rotating-wave approximation holds. For $G/(2\pi) = 100$ kHz, this is 2.50 microseconds. With a shared energy linewidth of 20 kHz in ordinary frequency the surviving energy fraction then is

$$e^{-2\pi(20\ \mathrm{kHz})(2.50\ \mu\mathrm{s})} = 0.730.$$

A gated excitation shorter than the exchange period but long enough to avoid unwanted modes can prepare an approximately optical initial perturbation. Independent detection of an optical revival and a delayed mechanical response tests coherent exchange more strongly than the presence of two peaks alone. Comparing times of maxima directly with one half the reciprocal measured optical linewidth misidentifies the exchange physics. Check the decay of the sum of appropriately calibrated energies and the relative phase of the exchange. Electrical or optical feedthrough can produce a prompt apparent response with no mechanical build-up.

Infer a coupling with propagated measurement errors

When the two hybrid pole frequencies have been obtained from a justified fit and zero bare detuning is established, the coupling follows from the measured real separation S:

$$G = \sqrt{S^2/4 + (\kappa - \Gamma)^2/16}.$$

All quantities here can be expressed in the same ordinary-frequency units. If $S = 300 \pm 8$ kHz, $\kappa = 240 \pm 10$ kHz and $\Gamma = 40 \pm 4$ kHz, the estimate is 158.1 kHz. Independent small errors give

$$\partial G/\partial S = S/(4G) = 0.474,$$

$$\partial G/\partial \kappa = -\partial G/\partial \Gamma = (\kappa - \Gamma)/(16G) = 0.0791.$$

Combining in quadrature gives about 3.9 kHz uncertainty. A common frequency calibration can correlate all three inputs, and a fitted detuning and splitting from one scan may also be correlated. Use their covariance rather than multiplying independent error bars blindly.

A second inference varies intracavity pump power. Since $G^2 = g_0^2 n_c$, plot the squared fitted coupling against independently calibrated n_c. A nonzero intercept could indicate an additional direct coupling, baseline fitting bias or a mode incorrectly assigned at low power. The linewidths themselves can vary with pump power because of heating or backaction, so fit or measure them across the scan. Incident optical power is not identical to stored photon number; cavity coupling, detuning and loss determine their relation.

A measured frequency shift and linewidth change reveal optical feedback

The same instrument can be operated below the resolved splitting regime and read out a dispersive shift of the mechanical response. Algebraically eliminate the optical amplitude in the two-mode equations while driving the mechanical mode weakly. Its inverse susceptibility acquires a self-energy contribution proportional to the optical susceptibility. Let x be probe frequency measured from the bare optical sideband and use the same linewidth convention as above. In the mechanical denominator the additional term is

$$\Sigma(x) = \frac{G^2}{\kappa/2 - i x} = \frac{G^2(\kappa/2 + i x)}{(\kappa/2)^2 + x^2}.$$

Its real part adds to the amplitude damping, while its imaginary part shifts the resonant condition. At very weak coupling one can evaluate it at the bare mechanical frequency $x = \delta$, because variation across the narrow mechanical peak is small. Near an avoided crossing or when the mechanical peak samples a large change in optical susceptibility, solve the coupled poles directly; simply adding a linewidth and shift then neglects the very hybridisation being tested.

In ordinary-frequency units choose $G = 10$ kHz, optical linewidth 200 kHz, mechanical linewidth 2 kHz and detuning $+100$ kHz. At that offset the optical amplitude half-width is 100 kHz, giving $\mathrm{Re}\,\Sigma = 0.50$ kHz and $\mathrm{Im}\,\Sigma = 0.50$ kHz. The full mechanical linewidth increases from 2.0 to approximately 3.0 kHz. Its centre moves upwards by about 0.50 kHz with the adopted matrix signs. Moving to minus 100 kHz reverses the shift and preserves the added damping in this symmetric model. At zero detuning the predicted shift vanishes but the added linewidth is largest.

A scan on both sides is more discriminating than measuring broadening alone. A background rise in mechanical temperature changes integrated thermal noise and may change intrinsic loss, but need not

reproduce this odd shift and even broadening pattern. Measure intrinsic linewidth far from the optical response and subtract that baseline. An unknown constant frequency offset can be removed with the two symmetric detunings, but an optical power dependence of cavity frequency can move the actual symmetry centre. Monitor the optical resonance during the scan.

The effective susceptibility describes classical feedback through the delayed optical field. It does not by itself specify the final mechanical temperature. A bath occupation or noise spectrum is also required to convert a damping change into an energy distribution. A driven response peak that broadens and lowers in amplitude is evidence for a changed susceptibility; thermal cooling additionally requires a calibrated measurement of the fluctuating mechanical displacement and its optical transduction.

Tutorial · A high cooperativity without resolved modes

A device has ordinary optical linewidth 1.00 MHz, mechanical linewidth 1.0 kHz and pump-enhanced coupling 30 kHz. Calculate cooperativity, compare the coupling with the real-splitting threshold and estimate the relative intracavity probe intensity at the central frequency. What observation would distinguish a narrow interference window from two separated poles?

▼ **Evaluate the three criteria independently**

In kHz units the cooperativity is

$$4(30)^2/(1000\cdot 1)=3.6.$$

The real-splitting threshold is $999/4=249.75$ kHz, far above the actual coupling. Yet the zero-detuning centre intensity is $1/(1+3.6)^2=0.0473$ relative to the bare optical peak. A narrow, deep structured feature is possible with no separated real pole frequencies in this model. The linewidth contrast permits mechanical response to interfere strongly within a broad optical susceptibility.

Acquire both amplitude and phase while scanning the absolute mechanical detuning and compare a coupled complex-response model with independently measured bare parameters. A pair of well-resolved poles should trace two consistent trajectories in both observables and can generate underdamped exchange in the time domain. A single optical intensity scan containing two flanks around a dip is insufficient to make that inference.

Tutorial · Diagnose an erroneous estimate of the single-photon coupling

An independent linewidth analysis gives pump-enhanced couplings 80 and 160 kHz at photon numbers 4.0×10^6 and 1.6×10^7. Estimate $g_0/(2\pi)$ and predict the coupling at 9.0×10^6 photons. A calibration revision doubles all inferred photon numbers while leaving the measured spectra unchanged. Which conclusions move?

▼ Use coherent amplitude scaling and separate calibration from spectroscopy

Divide the respective couplings in ordinary frequency by the square roots of their photon numbers. Both give 40 Hz and the intermediate prediction is 120 kHz. Doubling all inferred photon numbers lowers the inferred single-photon coupling by $1/\sqrt{2}$, to 28.3 Hz. The observed enhanced couplings, pole locations and cooperativities computed from measured linewidths stay the same. A prediction labelled by fixed physical photon number changes; a prediction at a repeatable laser setting can still be calibrated empirically.

To test the absolute calibration, compare the inferred single-photon value with the independent cavity frequency slope times the mechanical zero-point displacement. This requires an effective mass appropriate to the measured displacement and the actual cavity optical overlap. An incorrect choice of mechanical mode normalisation can mimic an error in stored photon number. Agreement across a power ratio tests the square-root scaling but cannot by itself determine the absolute optical or mechanical calibration.

Aspelmeyer, Kippenberg and Marquardt · cavity optomechanics, linearisation and strong coupling

Estimate a cavity linewidth from a finite, biased record

Numerical field laboratory

Connect the time and frequency descriptions of one damped mode, then diagnose why a weak detector background systematically biases the inferred quality factor.

Know whether the detector reads field or energy

After a drive switches off, let a single cavity mode have complex analytic amplitude $a(t) = a_0 e^{-\kappa t/2} e^{-i\omega_0 t}$ for $t \geq 0$. The energy is proportional to $|a|^2$, giving $U = U_0 e^{-\kappa t}$. Its instantaneous decay rate $-U'/U = \kappa$ implies

$$Q = \omega_0 U / P_{\text{loss}} = \omega_0 / \kappa.$$

A phase-sensitive receiver measuring field amplitude sees a time constant twice as long as the energy lifetime. Before fitting the data, ask how the measurement transducer converts the field into a reported number.

Fourier transforming the causal decaying amplitude gives an integral proportional to $1/[\kappa/2 - i(\omega - \omega_0)]$, up to the time convention. Its squared magnitude has half-power points at $|\omega - \omega_0| = \kappa/2$. Therefore the full power linewidth in hertz is $\Delta f = \kappa/(2\pi)$. A measurement of either the energy decay time $\tau_U = 1/\kappa$ or the spectral width can test the other without having to calibrate the absolute detector gain.

At $f_0 = 2.0$ GHz and $\tau_U = 0.50$ μs, we obtain $\kappa = 2.0 \times 10^6$ s^{-1}, $\Delta f \simeq 318.3$ kHz and $Q \simeq 6283$. A field receiver would report an amplitude time constant of 1.0 µs. Substituting that time directly into $Q = 2\pi f_0 \tau_U$ would double the apparent quality.

Derive the two-point fit with a detector offset

Let a power detector report $S(t) = A e^{-\kappa t} + B$, with a constant background $B > 0$. In the absence of background two samples separated by Δt give

$$\widehat{\kappa} = \log[S(t)/S(t + \Delta t)]/\Delta t.$$

With background, the logarithmic derivative is

$$-\frac{d}{dt}\log S = \kappa \frac{A e^{-\kappa t}}{A e^{-\kappa t} + B}.$$

It starts below the true rate and tends to zero in the tail. A straight line fitted to late log data can therefore look impressively precise while overestimating the lifetime. Increasing the acquisition duration can worsen this bias if the entire record is forced into a straight-line model.

In the experiment set $b = B/A = 0.04$. A two-point fit from one true lifetime to two has dimensionless rate $\log[(e^{-1} + 0.04)/(e^{-2} + 0.04)]$, approximately 0.844. Its apparent lifetime is about 18.5% high. The source, walls and coupling have not changed. Only the interpretation of detector output has

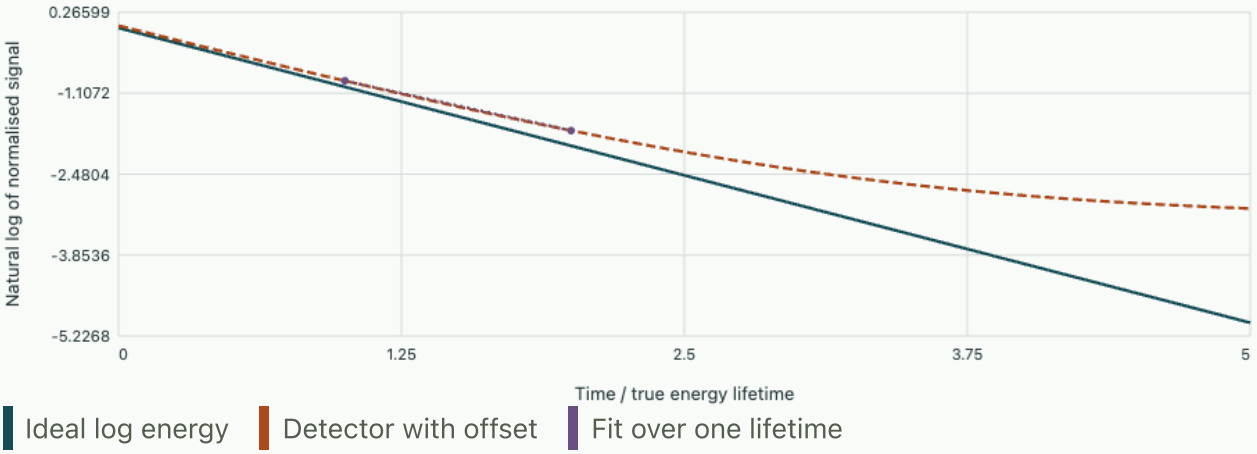
changed. Move the start of the fit towards three lifetimes and inspect the increasing gap from the ideal log line.

Open the live experiment

Try the calculation

Additive detector offset / initial energy **0.04**

Start of fitting window (lifetimes) **1**



Two-point inferred lifetime 1.1845 times the true lifetime. The same fractional bias transfers to Q for a known mode frequency.

Design a second independent inference

With three equally spaced, noiseless readings S_0, S_1, S_2 of one exponential plus a constant, subtract consecutive values before taking a logarithm. If $a = e^{-\kappa \Delta t}$, the differences are $A'(1-a)$ and $A'a(1-a)$, giving $a = (S_1 - S_2)/(S_0 - S_1)$. This cancels the constant background. At an interval of one lifetime, a first signal of unit energy and background 0.04 gives approximately $1.04, 0.407879, 0.175335$. Their ratio of differences recovers e^{-1}. In a real noise floor two nearly equal readings make this division unreliable. Use multiple windows or fit the raw signal with a measured noise model.

Finite spectral resolution creates a separate challenge. Multiplying the measured field by a rectangular window of duration T convolves its spectrum with a sinc-like response of width of order $1/T$. For the 318 kHz linewidth in the example, $T = 1$ µs gives frequency bins spaced by 1 MHz in a basic Fourier transform. Zero padding can interpolate its curve, but cannot restore the unmeasured late-time signal. A parametric fit may extract a sub-bin frequency or decay rate by using its model assumptions; its uncertainty then depends on signal-to-noise ratio and on whether the one-mode theory is adequate.

Check the single-mode model before refining its fit

If two nearby modes reach the same detector, their fields add before squaring. The result contains an interference term oscillating at their frequency difference on top of a decaying envelope. A scalar positive background cannot explain that alternating residual. Test by narrowing the excitation band, moving the pickup to a symmetry node of one mode, or examining a complex receiver's quadratures. A model with two exponentials has more degrees of freedom and can fit noise as well as signal; require consistency across independent records.

An honest linewidth report specifies the measured quantity, acquisition window, background treatment, coupling state, reference frequency and uncertainty. A fit to stored energy measures total loss. Inferring internal absorption alone additionally requires a known port coupling or a measurement that changes it separately, as in the course's cavity project.

Tutorial

A detector background is independently measured but subtracted too aggressively. What feature can appear in the logarithmic tail?

▼ **Work through the solution**

The corrected signal can approach zero and cross below it. The logarithm then becomes undefined. Even before that point its apparent local slope can become arbitrarily large. Inspect the raw signed residual and its uncertainty instead of dropping nonpositive points without accounting for selection bias.

A second experiment doubles the peak stored energy while the absolute detector offset is fixed. What should happen to the shape of the normalised log tail?

▼ **Work through the solution**

$b = B/A$ halves. Background contamination becomes important later by about $\tau_U \ln 2$. A real change in the physical lifetime need not follow that shift when the response remains linear.

Can a passive resonator's intrinsic losses be determined from a single ring-down measured at one port?

▼ **Work through the solution**

The observed decay is the sum of internal and external rates. A single accurately measured lifetime identifies their sum. A calibrated change in coupling or an independent power budget is needed to separate them.

Measure gravity with a pendulum of finite amplitude

Mechanics in the laboratory

A metre-long pendulum is an appealing clock because its small-angle period depends only on length and gravity. A precise clock can nevertheless give a precisely wrong value of gravity. Derive the full period, trace the systematic error to an identifiable approximation and design a measurement that distinguishes a length error from a release-angle error.

Define the measured length and time

Use a point bob on a massless, inextensible string attached to a fixed pivot. Let L be the distance from pivot to the bob's centre of mass, and θ the signed departure from the downward vertical. For a rigid extended bob, rotational inertia about its own centre adds to mL^2; a length measured to the bottom of the bob is neither of these dynamical lengths. Measure the elapsed time between successive crossings of a photogate in the same direction. Crossings in opposite directions give half of the period in a symmetric oscillation. A stationary turning point is a poor timing marker because the position error there produces a large time error.

At rest at the release angle θ_0, the energy is $mgL(1-\cos\theta_0)$. At a later angle the speed is $L|\dot\theta|$. Equating kinetic plus potential energy to the release value gives

$$\dot\theta^2 = (2g/L)(\cos\theta - \cos\theta_0).$$

Only a quarter cycle is needed. Inverting the angular speed and integrating from zero to θ_0 produces a square-root singularity at release. The period is finite since integrating the inverse square root over a small interval gives a square root.

Resolve the turning point before approximating

Write $k = \sin(\theta_0/2)$ and substitute $\sin(\theta/2) = k\sin u$. The vanishing square root cancels a factor $\cos u$ from the differential. One obtains the complete elliptic integral of the first kind with modulus k.

$$T = 4\sqrt{\frac{L}{g}}\,K(k),$$

$$K(k) = \int_0^{\pi/2} \frac{du}{\sqrt{1-k^2\sin^2 u}}.$$

Expanding its integrand yields

$$K(k) = (\pi/2)(1 + k^2/4 + 9k^4/64 + \cdots).$$

Since $k \simeq \theta_0/2$ at small amplitude, the first fractional period correction is $\theta_0^2/16$ with angle in radians. It is positive. At larger amplitude the tangential component of gravity rises more slowly with angle than the linear restoring law predicts. The pendulum spends longer near the turns. Passing degrees directly into the quadratic correction would give nonsense even if the timer and recorded angle were perfect.

Infer the bias in gravity

Suppose the true acceleration is $9.81\,\mathrm{m\,s^{-2}}$, the length is 1.000 m and the release angle is 30 degrees. The small-angle period is about 2.0061 s and the finite-amplitude period about 2.0410 s. Interpreting the longer measured period by the linear formula would give about $9.48\,\mathrm{m\,s^{-2}}$. The fractional gravity error is about twice the fractional period error with opposite sign because gravity is inferred from T^{-2}. Timing a hundred periods reduces random timing error but leaves this systematic shift.

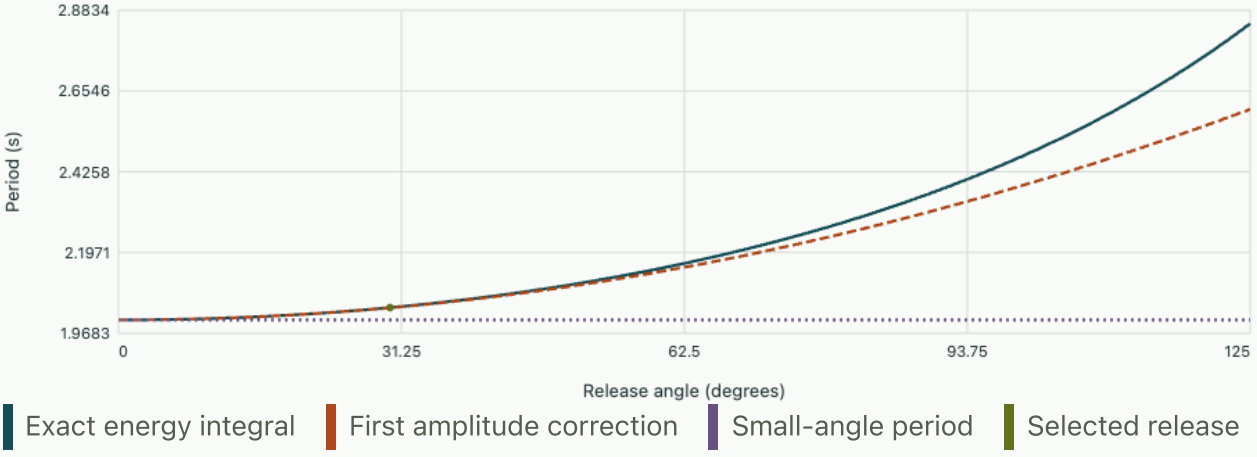
The numerical curve evaluates the complete elliptic integral. It uses the arithmetic–geometric mean identity $K(k) = \pi/[2\,\mathrm{AGM}(1, \sqrt{1-k^2})]$, so no sampled point sits on the turning point. The single-correction curve truncates the amplitude series after its first nonconstant term. Vary angle and length, comparing both the predicted period and the result a linear analysis would infer. Which change rescales both period curves together? Which changes their relative discrepancy?

Open this experiment online

Try the calculation

Release angle (degrees) **30**

Pendulum length (m) **1**



Period 2.041 s. Small-angle analysis would infer g = 9.4772 m/s².

Design a two-release check

For the same pendulum the ratio $T(\theta_a)/T(\theta_b) = K(k_a)/K(k_b)$ cancels the unknown length and local gravity. Measure at two amplitudes before deciding whether to recalibrate the length. If the ratio agrees but both absolute times differ from prediction, check the effective length, clock rate and model of the bob. If the ratio changes during a decaying swing, record initial and final amplitudes for the timed sequence. The period associated with a long sequence is an average over its slowly changing amplitude. Substituting the initial angle throughout would overcorrect the measurement.

Tutorial

At small amplitude, what release angle gives a fractional period bias of approximately one part in a thousand?

▼ **Work through the solution**

Set $\theta_0^2/16 = 0.001$. This gives 0.1265 rad, about 7.25 degrees. The gravity bias from an uncorrected measurement would be approximately two parts in a thousand. A tolerance on one quantity is not automatically the same tolerance on the other.

A photogate records successive interruptions every 1.02 seconds for the metre pendulum. A calculation gives nearly four times local gravity. Identify one possible error in the observable.

▼ **Work through the solution**

The gate may have counted both directions and returned a half period. Doubling that interval gives about 2.04 s. Because the inferred acceleration is inversely proportional to the square of the period, the missing factor of two gives a factor of four.

Why does reducing the grid spacing in an integration of the linear equation fail to recover the finite-amplitude curve?

▼ **Work through the solution**

Integration error vanishes as the step is refined, but the restoring torque in that equation is $mgL\theta$. The true torque is $mgL\sin\theta$. A converged numerical solution can therefore provide independent evidence of an inadequate physical approximation.

For the equation of motion, phase portrait and dissipation, see the MIT treatment of the simple pendulum.

Measure inertia and gravity with two suspension axes

Mechanics in the laboratory

A rigid pendulum can oscillate from either of two knife edges. If the periods agree at distinct distances from the centre of mass, the spacing of the edges determines gravity. Derive the cancellation, identify a second way equal periods can arise and find why a poorly chosen geometry magnifies timing errors.

Torque about an offset pivot

Take a rigid body of mass m with parallel suspension axes perpendicular to its plane of oscillation. Its moment of inertia about the centre of mass is $I_C = m\rho^2$; ρ is the radius of gyration. For a pivot a distance $a > 0$ from the centre, the parallel-axis theorem gives

$$I_a = m(\rho^2 + a^2).$$

Gravity exerts torque $-mga\sin\theta$. In the small-angle limit,

$$T_a^2 = \frac{4\pi^2}{g}\,\frac{\rho^2 + a^2}{a},$$

$$L_a = a + \frac{\rho^2}{a}.$$

Here L_a is the length of a point pendulum with the same period. It can lie beyond the physical centre of mass. A tap applied at this distance from a pivot gives the centre of percussion when the ideal instantaneous pivot reaction is to vanish. The same combination of mass distribution and geometry appears in an impulsive and in an oscillatory measurement because both depend on angular momentum about that pivot.

Match the periods and keep both algebraic branches

Put the second knife edge on the opposite side of the centre at distance b. Equal periods require

$$a + \rho^2/a = b + \rho^2/b.$$

Rearrange before dividing by $a - b$:

$$(a - b)\left(1 - \frac{\rho^2}{ab}\right) = 0.$$

Two equal distances always give equal periods. That branch provides no information about ρ. If the distances are distinct, equality implies

$$\rho^2 = ab.$$

Both equivalent lengths then equal $a + b$, the directly measurable edge separation. For $a = 0.200\,\mathrm{m},\ b = 0.450\,\mathrm{m},\ \rho = 0.300\,\mathrm{m}$, the equivalent length is 0.650 m and both periods at standard laboratory gravity are about 1.62 s. A small value of $a - b$ is an experimental warning sign, even if the observed equality is exceptionally clean.

Background · Can unequal release amplitudes imitate a mismatch between pivots?

A rigid pendulum about either knife edge obeys the same sine nonlinearity as a simple pendulum. To leading order in its release angle θ in radians, the measured period exceeds the small-amplitude period by the fraction $\theta^2/16$. At matched small-amplitude periods, measurements with amplitudes θ_a, θ_b therefore give

$$\frac{T_a - T_b}{T_0} \simeq \frac{\theta_a^2 - \theta_b^2}{16}.$$

Releasing at 6° about the first edge and 2° about the other creates a fractional mismatch of about 6.1×10^{-4}. For a two-second period this is 1.2 ms. Record the two amplitudes along with the timing data and reduce both periods to the same limit before moving the trim mass. Otherwise the trim can compensate a preparation error.

Extract gravity when the two measured periods differ

Multiply the first period equation by a and the second by b. Subtract to eliminate ρ^2. The exact small-angle result for unequal distances is

$$g = 4\pi^2\,\frac{a^2 - b^2}{aT_a^2 - bT_b^2}.$$

At matched periods this reduces to the simple expression in terms of the edge spacing. The general formula involves a difference in its denominator. If a and b nearly coincide, the denominator is small and two independent timing errors subtract with poor relative precision. This is a property of the inverse measurement, even if computing each individual period is perfectly well conditioned.

Once g is known, use $\rho^2 = gaT_a^2/(4\pi^2) - a^2$ to recover the body inertia per unit mass. A negative value may indicate a wrong distance measured to the centre or a timing observable that is actually a half period. Verify the result using the second pivot and recover a separately predicted period after shifting a suspension point. A single successful cancellation is weaker than reproducing an additional measurement.

Move one knife edge in the experiment

Hold $a = 0.200\,\mathrm{m}$ fixed and vary both the radius of gyration and the second distance. The graph plots the period against the possible second distance. Before inspecting intersections, predict where the

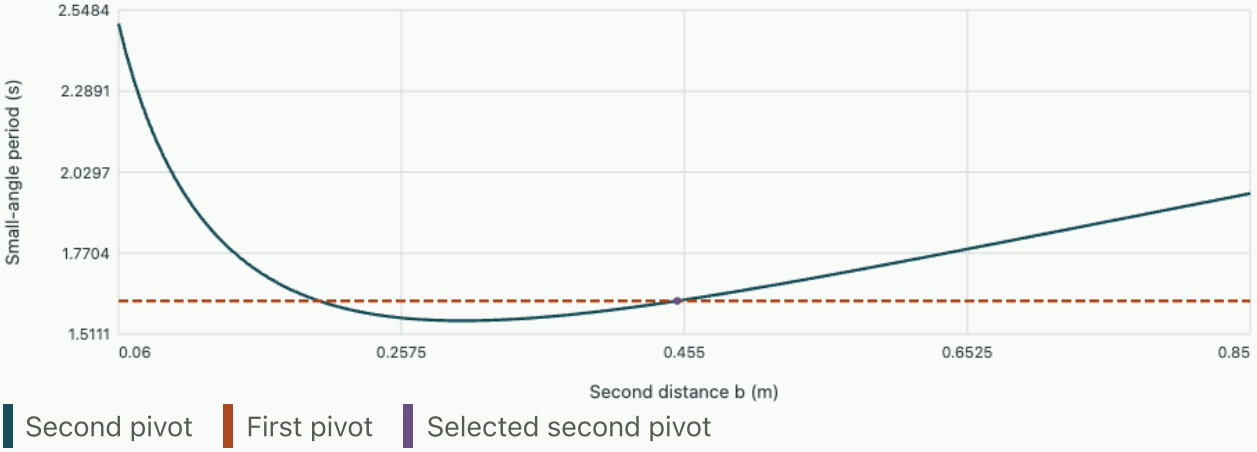
second match should lie as ρ grows. The dependence on distance has a minimum at $b = \rho$ because effective length comprises a growing and a decreasing term.

Open this experiment online

Try the calculation

Radius of gyration (m) **0.3**

Second pivot distance (m) **0.45**



First period 1.6173 s; second 1.6173 s. The other matching distance is 0.45 m.

With a finite oscillation amplitude, the physical pendulum has the same elliptic correction as a point pendulum of length L_a, provided the body is rigid and the frictionless pivot stays fixed. Matching different amplitudes can bias the cancellation. Measuring both at the same small angle removes the leading difference, and extrapolating both to zero amplitude provides a check. Rotational compliance of the knife edge and a shifting centre of mass require a different model.

Tutorial

For $a = 0.240$ m and $\rho = 0.300$ m, where should the second distinct match occur and what length should be used in $g = 4\pi^2 L/T^2$?

▼ **Work through the solution**

The second branch gives

$$b = 0.0900/0.240 = 0.375\,\mathrm{m}.$$

Because the pivots straddle the centre of mass the equivalent length is 0.615 m.

Both pivots lie on the same side of the centre, at the same distances a, b as in the previous question. What changes in the cancellation?

▼ **Work through the solution**

The periods depend on the magnitudes of the offsets, so equal periods still imply $\rho^2 = ab$ on the distinct branch. The sum $a + b$ still gives the equivalent length, but the physical ruler distance between pivots is $|a - b|$. The direct measurement shortcut relies on the chosen geometry.

Why does setting both axes exactly at the same distance make even noise-free measurements insufficient to determine both g and ρ?

▼ **Work through the solution**

The two equations are then identical. For every positive choice of inertia radius the first period determines a corresponding value of gravity. Improving timing precision cannot provide the missing independent equation.

Connect the offset-pivot calculation to the rigid-body inertia chapter.

Applied study

Move from a single oscillator to the natural frequencies and mode shapes of a machine on a flexible frame.

Can a motor support resonate at running speed?

A compact motor sits on the horizontal arm of a steel L-frame. After a disturbance it can move horizontally, bounce vertically and rock. A spring scale measures how far the mount yields under static load. To predict the frequencies of those three motions, we also need to know where the mass sits.

Start with one undamped coordinate q, measured from equilibrium. A spring with compliance c moves cP under force P, so its restoring force is $-q/c$. Newton's law gives $m\ddot{q} + q/c = 0$ and $f = 1/(2\pi\sqrt{mc})$. Gravity sets the static equilibrium. For small vibrations about it, constant gravity drops out of the linear equation.

Find the frame's vertical compliance

The frame has a clamped column of height H and a horizontal arm of length L, rigidly joined. Both have area A, bending second moment I and modulus E. Under a vertical tip force P, the arm bends with moment $P(L - x)$. The column carries axial force P and constant bending moment PL. Neglect shear deformation and store the bending and axial energies. A slowly applied load performs work $P^2 c_{zz}/2$, so

$$c_{zz} = \int_0^L \frac{(L-x)^2}{EI}\,dx + \int_0^H \frac{L^2}{EI}\,dz + \int_0^H \frac{1}{EA}\,dz = \frac{L^3/3 + L^2 H}{EI} + \frac{H}{EA}.$$

Take $H = 2.0$ m, $L = 1.5$ m, $A = 0.0050\,\mathrm{m}^2$, $I = 8.0 \times 10^{-6}\,\mathrm{m}^4$ and $E = 210$ GPa. This gives $c_{zz} = 3.35 \times 10^{-6}$ m/N. A 1 kN vertical force displaces the tip by 3.35 mm. The column-bending term alone contributes 2.68 mm. Calling the arm a cantilever with a perfectly fixed root would miss the dominant compliance.

Translation and rocking share the restoring energy

If only vertical inertia were retained, a 200 kg motor would give $f = 6.15$ Hz. In fact the column sways and the arm rotates when the motor moves vertically. In a light-frame approximation collect the tip coordinates in $\mathbf{q} = (u_x, u_z, \theta)$, measured at the machine's centre of mass. Apply a horizontal force, a vertical force and an end moment separately to obtain the three columns of the flexibility matrix $\mathbf{C}$. Its off-diagonal entries measure the movement in one coordinate caused by a load in another. For linear elastic members reciprocity makes this matrix symmetric.

The stiffness matrix is $\mathbf{K} = \mathbf{C}^{-1}$. With motor mass m and rotary inertia J, let $\mathbf{M} = \mathrm{diag}(m, m, J)$. Substituting harmonic motion into $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0$ gives

$$\mathbf{K}\boldsymbol{\phi} = \omega^2 \mathbf{M}\boldsymbol{\phi}.$$

A mode shape $\boldsymbol{\phi}$ describes a ratio of motions. The frequency $f = \omega/(2\pi)$ is independent of the scale used to draw that shape.

For $m = 200$ kg and $J = 60\,\mathrm{kg\,m^2}$, the three light-frame frequencies are 5.048, 15.449 and 44.645 Hz. If the steel frame's distributed mass is included they fall to 4.701, 13.550 and 42.41 Hz. The first light-frame mode corresponds to about 303 rpm for an excitation that repeats once per revolution. Whether it produces a large vibration also depends on damping and on the projection of the applied force onto that mode. Explore how inertia affects the rotation of a rigid body.

Transfer problem · Isolate a machine from floor vibration

A 50 kg instrument on a spring and viscous damper sits on a floor moving as $y(t)$. The mount extends by $x - y$, where x is the instrument's absolute displacement. Force balance gives

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0.$$

For sinusoidal floor motion the ratio of instrument to floor amplitude follows by replacing each derivative by $i\omega$:

$$\frac{|X|}{|Y|} = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}},$$

$$r = \omega/\omega_n,$$

$$\omega_n = \sqrt{k/m},$$

$$\zeta = \frac{c}{2m\omega_n}.$$

Choose $k = 20\,000$ N/m and $c = 200$ N·s/m. Then $\omega_n = 20$ rad/s and $\zeta = 0.10$. At 40 rad/s, $r = 2$ and the ratio is

$$\sqrt{1.16/9.16} = 0.356.$$

A 1.0 mm floor oscillation gives 0.356 mm of instrument motion. At 20 rad/s the same mount amplifies motion by $\sqrt{1.04/0.04} = 5.10$. A mount selected for stable operation at one speed can amplify vibration as a machine runs up through resonance.

Exercises

1. Infer a frequency from measured sag

A light support with one vertical coordinate deflects 4.0 mm under a machine's weight. Infer its natural frequency directly from the sag. Compare it with the forcing frequency of a once-per-revolution

imbalance at 450 rpm.

▼ **Show the estimate**

From $mg = k\delta$, $k/m = g/\delta$. Hence

$$f = (2\pi)^{-1}\sqrt{9.81/0.0040} = 7.88\,\text{Hz}.$$

The imbalance frequency is

$$450/60 = 7.50\,\text{Hz}.$$

The forcing is about 5% below the estimated frequency. Estimating a response amplitude would additionally require the forcing magnitude and damping.

---

2. Add a geometrically identical payload

For the light L-frame double both m and J while holding the support stiffness fixed. Predict the change in all three frequencies and the first critical speed. Does the same factor apply to a frame whose own mass stays fixed?

▼ **Show the modal scaling**

The entire tip mass matrix doubles. The same eigenvectors then solve the equation with each ω^2 halved. The three frequencies become 3.570, 10.924 and 31.569 Hz. The first once-per-revolution critical speed is about 214 rpm. With distributed frame mass only part of the assembled mass matrix doubles, so there is no common frequency scaling factor.

ONESIMULATION LABORATORY

Separate the effect of mass from rotary inertia

Compare recorded OneSimulation runs on this page. To test your own inputs, download the notebook and model files and run the model in an activated OneSimulation environment.

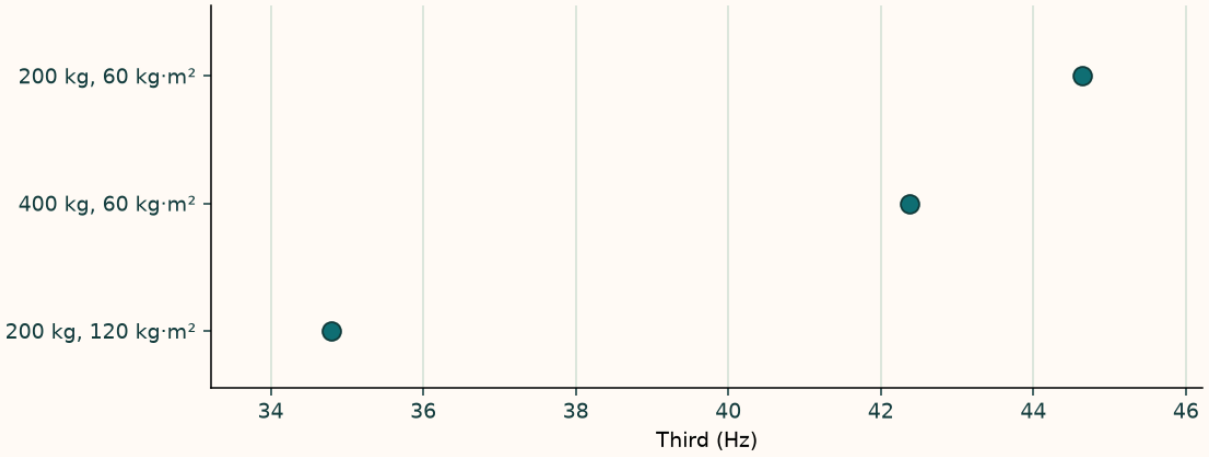
Run the frame example and read the row labelled “massless members”. Its baseline at `mass=200.0` and `rotary=60.0` is 5.048, 15.449 and 44.645 Hz. Record the light-frame and steel-frame results in separate columns.

1. Set `mass=400.0` kg and retain `rotary=60.0` kg·m². This represents adding mass close to the tip reference point. Expect light-frame frequencies of 3.666, 11.206 and 42.379 Hz. Which mode is least sensitive to the added translation inertia?

2. Reset `mass=200.0` and set `rotary=120.0`. This represents redistributing the same mass further from the rotation axis. Expect 4.798, 14.750 and 34.791 Hz. The highest mode now changes substantially. Compare these changes with the corresponding steel-frame frequencies at eight elements per member.

Natural frequencies from the frame model with negligible member mass

Tip mass and inertia	First (Hz)	Second (Hz)	Third (Hz)
200 kg, 60 kg·m²	5.048	15.449	44.645
400 kg, 60 kg·m²	3.666	11.206	42.379
200 kg, 120 kg·m²	4.798	14.750	34.791



The third frame mode falls from 44.645 to 42.379 Hz when the tip mass doubles. Doubling only the rotary inertia gives a much larger shift to 34.791 Hz.

View the frame's geometry, eigenproblem and numerical results.

Download experiment files (ZIP)

Extract the files together and open `planar_frame_point_mass_modes.ipynb` in a local OneSimulation environment.

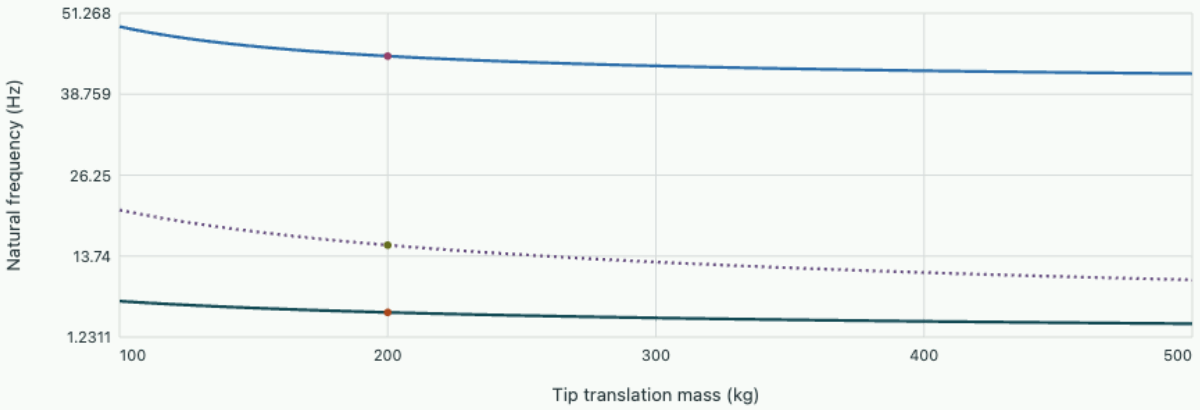
Interactive reference for the OneSimulation laboratory

Change tip mass, rotary inertia and column height. The browser solves the3×3 flexibility eigenproblem of the notebook's massless L-frame reference; distributed steel mass and mesh refinement remain available in the OneSimulation notebook.

Open the live laboratory reference

Try the calculation

Tip translation mass (kg)	200
Tip rotary inertia (kg m²)	60
Column height (m)	2



Massless-frame reference: 5.048, 15.449, 44.645 Hz. Recomputed symmetric3×3 flexibility eigenproblem includes axial compliance, E210 GPa, A5000 mm², I8×10⁶ mm⁴, arm1.5 m. The linked notebook additionally includes distributed steel mass and mesh refinement.

Problem sheet 2 · Estimate how a frame mode distributes its kinetic energy

Notation and assumptions

Read each defining equation with its assumptions and units, then follow the link to the argument that gives the equation its meaning.

Damped frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

This free-decay oscillation frequency exists in the underdamped regime. A resonance peak in a driven response uses a different frequency condition.

Derivation · Harmonic forcing, phase and dissipated power

Damping ratio

$$\zeta = c/(2\sqrt{km})$$

This compares a viscous damper with critical damping for a single linear oscillator. It cannot represent Coulomb friction at every amplitude.

Derivation · Free vibration, energy loss and ring-down data

Force impulse

$$J = \int F \, dt$$

An impulse changes linear momentum over an event. A coefficient of restitution supplies a separate relative-velocity condition for an impact.

Derivation · Impulse, angular momentum and impact

Generalised inertia

$$M(q) = 2T/\dot{q}^2$$

This scalar definition applies to one independent moving coordinate with purely quadratic kinetic energy. Changing the coordinate changes the numerical inertia.

Derivation · Coordinates, energy and equivalent inertia

Modal participation

$$Q_j = \phi_j^{\mathsf{T}} f$$

A mode is excited according to its overlap with the force. Always state how its mode vector and modal mass were normalised.

Derivation · Modal participation and absorber tuning

Rayleigh quotient

$$R(u) = u^{\mathsf{T}} K u / (u^{\mathsf{T}} M u)$$

For symmetric stiffness and positive mass this estimates squared angular frequency. A kinematically admissible displacement is required for a useful structural estimate.

Derivation · Estimating a cantilever frequency from energy

Further reading

- Use the worked recitations to compare Newton–Euler and energy-coordinate models. Derive a two-coordinate vibration model and project the forcing onto both modes. MIT · Engineering Dynamics

Compare specific topics with the university sources