Fluid mechanics

Use momentum and energy balances to predict pipe losses, boundary-layer drag, work in a pump and the flow limits of a compressible nozzle.

Read the subject in depth

Each volume develops its arguments in sequence, with worked problems and links to experiments.

- ### Momentum, motion and transport in fluids

  Derive a flow from its conservation laws and boundary conditions. Quantify how length scales and material properties decide which approximation can predict a measurement.

Advanced study across the subject

Each sequence connects the course to detailed chapters in neighbouring subjects. The complete course PDF includes their derivations, examples and exercises.

1. ### Energy, entropy and mechanical response

   Close a steady-flow energy balance, then determine which pressure or temperature change reveals an irreversible process. Connect a film force to a moving bearing.

2. ### Solving transport while conserving mass

   Follow control-volume reasoning onto a grid. Compare diffusion, interface fluxes and advective remapping. Check how a travelling perturbation moves at different resolutions.

How the course fits together

1. **Chapters 1–4**

   First specify how fluid moves and how pressure supports a stationary column. Apply momentum and mechanical-energy balances to a control volume. The force on a bend and the head lost in a pipe require different terms.

2. **Chapters 5–8**

   Resolve viscous shear in a duct, then use similarity to identify equivalent flows. Vorticity tracks local rotation. At a wall, balancing inertia with viscous diffusion predicts the growth of a thin boundary layer.

3. **Chapters 9–12**

   Integrate pressure and shear over a surface to obtain drag or lift. Couple pressure losses across networks and rotating machines, then include density changes to study compressible flow.

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Chapter 1 · Velocity fields, flow rates and conservation of mass

Read local acceleration from a velocity field and use an integral mass balance to connect inlet, outlet and storage.

FROM FIRST PRINCIPLES

Derive the continuity equation from conservation of mass

Choose a fixed region of space containing fluid of density $\rho(\mathbf{x}, t)$ and velocity $\mathbf{u}(\mathbf{x}, t)$. Mass crosses a small boundary element dA with outward normal $\mathbf{n}$ at the signed rate $\rho\mathbf{u} \cdot \mathbf{n}\, dA$. A positive flux removes mass. With no sources of mass inside, the rate of change of the enclosed mass must balance the net outflow.

$$\frac{d}{dt}\int_V \rho\, dV + \oint_{\partial V} \rho\mathbf{u} \cdot \mathbf{n}\, dA = 0.$$

For a stationary control volume the time derivative can be taken inside the integral. The divergence theorem changes the boundary integral to a volume integral. Since the resulting balance holds for every sufficiently small volume in a smooth flow, its integrand must vanish.

$$\partial_t \rho + \nabla \cdot (\rho\mathbf{u}) = 0,$$

$$\frac{D\rho}{Dt} + \rho\, \nabla \cdot \mathbf{u} = 0.$$

Here $D/Dt = \partial_t + \mathbf{u} \cdot \nabla$ follows a moving parcel. Constant parcel density therefore gives

$$\nabla \cdot \mathbf{u} = 0.$$

In a steady single-inlet, single-outlet incompressible duct, integration reduces this statement to $A_1 \bar{u}_1 = A_2 \bar{u}_2$.

Background · Material acceleration

$$D\mathbf{u}/Dt = \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}$$

A fluid parcel can accelerate as it follows a steady but spatially varying velocity. A probe fixed in space measures only the local time history.

DERIVATION AND APPLICATIONS

Distinguish a parcel trajectory from an instantaneous streamline

Dye introduced into an unsteady flow need not follow a line drawn tangent to one velocity snapshot. To calculate the difference, consider a spatially uniform planar velocity field with horizontal component $u = U > 0$ and vertical component $v = at$, where a is constant. Such a field is divergence-free because neither component varies with position. Neglect diffusion and assume a neutrally buoyant tracer. A parcel released at the origin at time t_0 obeys the two ordinary differential equations

$$\dot{x} = U,$$
$$\dot{y} = at,$$
$$x = U(t - t_0),$$
$$y = \tfrac{1}{2}a(t^2 - t_0^2).$$

Eliminate the elapsed time to find the parcel path. The first term reflects vertical velocity at release and the quadratic term accumulates the subsequent acceleration. In contrast, an instantaneous streamline uses the velocity direction at one frozen time t_s for all its points.

$$y_{\text{path}}(x) = \frac{at_0}{U}x + \frac{a}{2U^2}x^2,$$
$$\left.\frac{dy}{dx}\right|_{\text{stream}} = \frac{at_s}{U}.$$

Continuous injection at the origin generates a third observable. At a fixed observing time t_s, different coloured parcels have different release times $t_0 = t_s - x/U$. Substitution into the position solution gives the locus of all visible dye at that instant, called the streakline.

$$y_{\text{streak}}(x; t_s) = \frac{at_s}{U}x - \frac{a}{2U^2}x^2.$$

The streakline is tangent to the instantaneous velocity at the injector but curves away from that direction farther downstream. In the steady limit $a = 0$ all three constructions produce horizontal lines.

WORKED EXAMPLE

Compare a pulse of dye with a velocity image

Take $U = 0.400$ m/s and $a = 0.100$ m/s². A parcel released at $t_0 = 0$ reaches $x = 0.800$ m at 2.00 s and has risen $\frac{1}{2}(0.100)(2.00)^2 = 0.200$ m. The streamline through the injector at 2.00 s has slope 0.500, reaching 0.400 m at the same horizontal coordinate. It represents the current velocity field, which changed while the pulse was travelling.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Recover acceleration from a continuous dye trace

In another experiment $U = 0.500$ m/s. At $t_s = 3.00$ s, continuously injected dye at $x = 0.500$ m lies 0.250 m above the source. That parcel is 1.00 s old, so its release time was 2.00 s. From $y = a(3^2 - 2^2)/2$ one finds $a = 0.100$ m/s². The local velocity slope at observation is then 0.600, independently measurable from a short-exposure particle image.

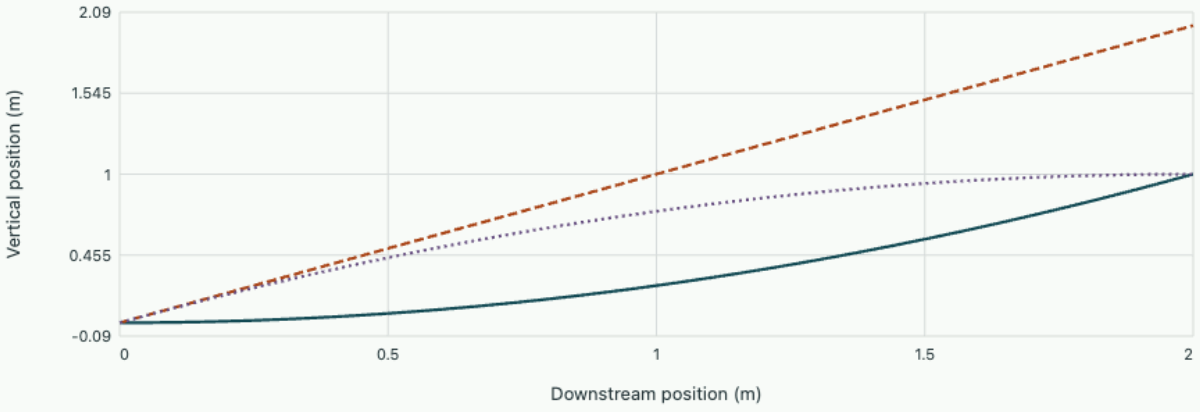
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Constant horizontal speed (m/s)	1
Vertical acceleration scale (m/s²)	0.5
Observation time (s)	2



The three curves share the source but encode different measurements. At observation time 2 s the vertical velocity field is spatially uniform at 1.000 m/s.

Look at a point or follow a parcel

At a contraction in an otherwise steady pipe, a parcel of water speeds up. A probe fixed at the throat reports a constant speed throughout the experiment. Both observations are compatible because a field

can vary along the path even when it has no explicit time dependence. Eulerian measurements describe velocity at a place. Following a marked parcel gives its changing position and velocity.

Write the velocity field as $\mathbf{u}(\mathbf{x}, t)$. During dt the parcel advances through $d\mathbf{x} = \mathbf{u}\,dt$. The chain rule along that trajectory gives

$$\mathbf{a} = \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}.$$

The first term is local acceleration and the second is convective acceleration. In steady one-dimensional flow $a_x = u\,du/dx$. A streamline is tangent to the instantaneous velocity. In a steady flow pathlines and streamlines coincide, whereas in an unsteady field the curve traced by a parcel can differ from a single snapshot's streamlines.

Background · Can a sequence of photographs identify the acceleration of one parcel?

A fixed camera that records velocity at one pixel measures an Eulerian history. A particle may enter a different pixel long before the next frame. For the planar velocity field $\mathbf{u} = (ax, -ay)$ with constant a, all fixed pixels record an unchanging field. A parcel released on the positive x axis instead obeys $x(t) = x_0 e^{at}$.

$$\frac{D\mathbf{u}}{Dt} = \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = (a^2 x, a^2 y).$$

At $a = 2\ \mathrm{s}^{-1}$ and $x = 0.10$ m on the axis, the downstream acceleration is 0.40 m/s². The parcel moves into faster neighbouring fluid. Estimating acceleration from an image pair therefore requires either tracking the same particle or combining the local time difference with spatial velocity gradients.

Count the mass crossing a surface

Enclose part of a fluid system in a fixed control volume. A small surface element with outward normal $\mathbf{n}$ passes volume $(\mathbf{u} \cdot \mathbf{n})dA\,dt$ in time dt. Multiply by density and add over the whole boundary. Normal inflow is negative, reducing the net outward flux. Conservation of mass is

$$\frac{d}{dt}\int_{\mathrm{CV}} \rho\, dV + \int_{\mathrm{CS}} \rho\, \mathbf{u} \cdot \mathbf{n}\, dA = 0.$$

In steady operation the mass flow rate entering all inlets equals that leaving all outlets. A one-inlet, one-outlet reduction with uniform cross-sectional density gives

$$\rho_1 A_1 \bar{u}_1 = \rho_2 A_2 \bar{u}_2.$$

The mean velocity is the area integral of the normal component divided by area. A centreline speed is seldom that mean. For variable density the volume rate can change along a duct while mass rate remains

constant.

Accelerate water in a contraction

Water of density 1000 kg/m³ enters a straight circular nozzle 60 mm in diameter with a mean speed of 1.50 m/s. The outlet is 30 mm in diameter. First use the inlet area to find the volume and mass passing per second.

$$Q = \bar{u}_1 \frac{\pi D_1^2}{4} = 4.24 \times 10^{-3}\ \mathrm{m^3/s},$$
$$\dot{m} = \rho Q = 4.24\ \mathrm{kg/s}.$$

For steady incompressible flow the same volume per second must pass through an outlet of one quarter the inlet area. The mean speed increases accordingly.

$$\bar{u}_2 = \bar{u}_1 \left(\frac{D_1}{D_2} \right)^2 = 1.50 \left(\frac{60}{30} \right)^2\ \mathrm{m/s} = 6.00\ \mathrm{m/s}.$$

These speeds determine the change in streamwise momentum flux if both profiles are nearly uniform. The pressure and wall forces responsible for that change require a momentum balance.

Suppose along a centreline segment of length 0.30 m the speed is approximately linear from 1.50 to 6.0 m/s. Its gradient is 15 s⁻¹. At a local speed of 3.0 m/s the convective acceleration is 45 m/s². The transit time through the segment is

$$\int dx/u = \ln(6/1.5)/15 = 0.0924\,\mathrm{s}.$$

Dividing distance by the arithmetic mean of the two speeds gives a different answer because a particle spends a larger fraction of its time in the slower region.

Let the control volume fill

A vented tank receives 4 L/s and initially delivers 2 L/s through a controlled outlet. In a constant horizontal tank section of 0.50 m² the water level rises at $(0.004 - 0.002)/0.50 = 0.0040$ m/s. The storage term remains essential even if both inlet and outlet flow meters have steady readings. A free-surface tank can therefore act as an independent calibration: the measured volume increase over a timed interval should equal the integrated difference between inflow and outflow. If the outlet is driven by the changing hydrostatic head, its rate will cease to be constant and the balance becomes a differential equation for the depth.

Check continuity directly in a velocity field

A planar test field $u = ax,\ v = -ay$ with constant a stretches a fluid square in one direction and shortens it in the other. Since $\partial_x u + \partial_y v = a - a = 0$, its area is preserved locally. A particle released at x_0, y_0 follows $x = x_0 e^{at},\ y = y_0 e^{-at}$; the product xy remains constant. This explicit trajectory shows how an incompressible parcel can deform and accelerate even when the velocity at every fixed point is independent of time.

Count mass through a control surface that moves

A flow meter on a translating piston does not see the same volume rate as a stationary meter at the same instant. Start with a material system, whose total mass is constant. Over a short time dt, a patch of control surface of area dA moves a distance $\mathbf{v}_s \cdot \mathbf{n}\, dt$ in its outward normal direction. Fluid moves a distance $\mathbf{v} \cdot \mathbf{n}\, dt$ in that direction. The volume swept through the moving patch is the difference times dA. Converting this volume into mass gives the moving-control-volume form of continuity:

$$\frac{d}{dt}\int_{CV(t)} \rho\, dV + \int_{CS(t)} \rho(\mathbf{v} - \mathbf{v}_s)\cdot \mathbf{n}\, dA = 0.$$

The storage term differentiates the mass inside the actual changing volume. A boundary following a parcel has zero relative normal velocity there, even when the parcel and the boundary move quickly in the laboratory. On an impermeable wall this no-penetration condition applies to its normal movement; the mass balance alone imposes no constraint on tangential slip. Setting $\mathbf{v}_s = 0$ recovers the fixed-volume formula. Making the entire surface material removes the flux term and recovers conservation of the system mass.

For constant density, differentiating the geometric volume and using the outward convention gives

$$dV/dt = \int \mathbf{v}_s \cdot \mathbf{n}\, dA.$$

Substitution shows how the familiar $\int \mathbf{v}\cdot\mathbf{n}\, dA = 0$ can hold for an instantaneous surface even while its enclosed volume is growing. Relative flux and volume storage must be paired consistently.

Interpret a flow reading on a moving piston

A horizontal cylinder of area $0.010\,\mathrm{m}^2$ has an inlet in its fixed left end and a small outlet passing through a piston at its right end. The piston moves to the right at 0.040 m/s while the cylinder stays full of incompressible water. A fixed inlet meter measures $0.00070\,\mathrm{m}^3/\mathrm{s}$. The volume occupied by water grows at $0.010(0.040) = 0.00040\,\mathrm{m}^3/\mathrm{s}$. Therefore the relative outlet flux through the piston is $0.00070 - 0.00040 = 0.00030\,\mathrm{m}^3/\mathrm{s}$, or 0.30 L/s.

The outlet area is $2.0 \times 10^{-4}\,\mathrm{m}^2$ and flow exits to the right. Its mean velocity relative to the piston is 1.50 m/s, while its laboratory velocity at the moving outlet is

$$1.50 + 0.040 = 1.54\,\mathrm{m/s}.$$

Multiplying that laboratory velocity directly by the outlet area gives 0.308 L/s, too high for the outlet flow measured by an instrument carried on the piston. If the piston moved to the right at 0.070 m/s, the entire inlet rate would fill the new volume and relative outlet discharge would be zero. These conclusions assume no leakage past the piston rim and that the outlet area is part of the moving boundary; a fixed downstream collection nozzle would require its own control surface and storage terms.

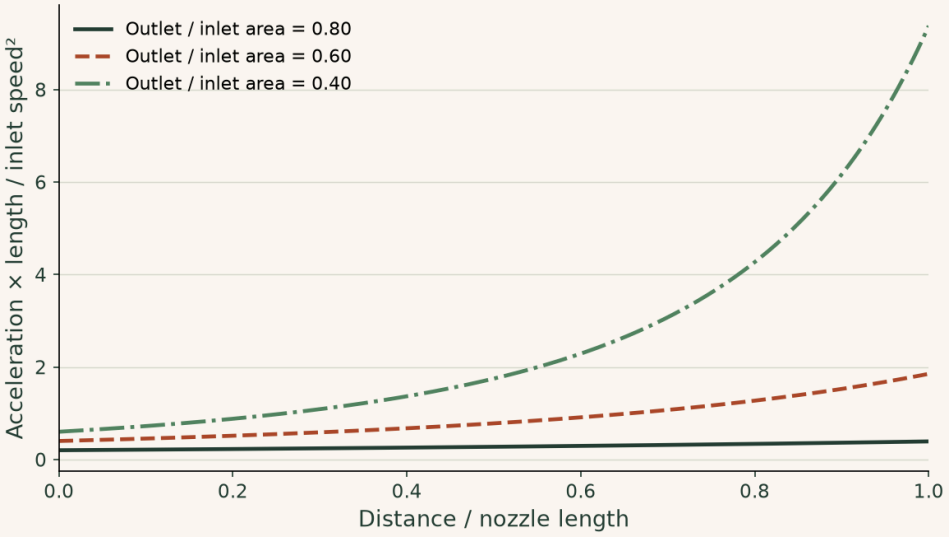
A downstream weighing tank that travels with the piston would collect 0.30 kg per second at water density 1000 kg/m³. This is the discharge relative to that apparatus. The distinction is measurable and does not depend on any pressure-loss assumption for the outlet.

Can a steady flow accelerate a fluid particle?

Let the cross-sectional area in a straight nozzle be $A(x)/A_0 = 1 - ax/L$. In an ideal steady quasi-one-dimensional incompressible flow, continuity gives

$$u = U_0/(1 - ax/L).$$

The material acceleration is $u\,du/dx$ although every stationary probe measures a constant velocity.



Reducing the area to 60% of its inlet value gives an exit acceleration of approximately 1.85 U_0^2/L. The concentrating acceleration comes from both the increasing speed and its increasing gradient.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

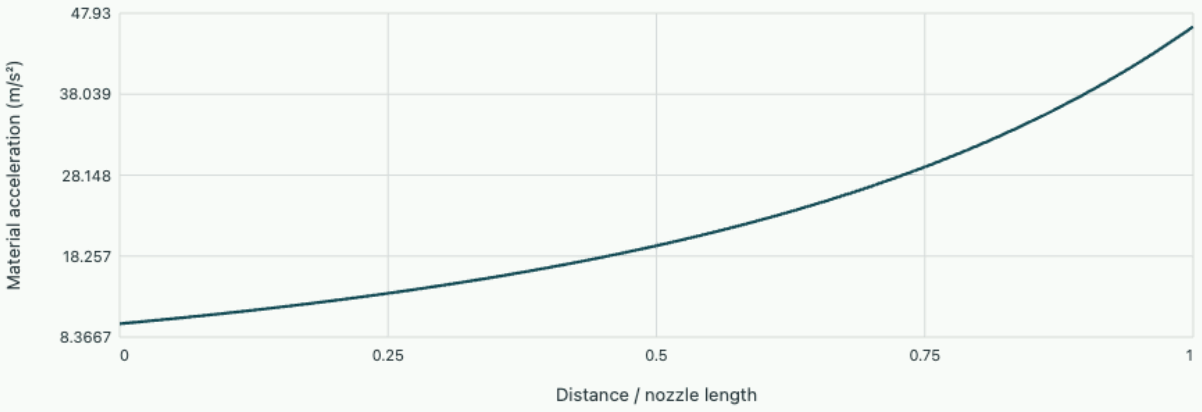
Reduce the exit area and inspect where the acceleration grows fastest. Doubling inlet velocity quadruples the acceleration.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Outlet / inlet area	**0.6**
Inlet speed (m/s)	**5**
Nozzle length (m)	**1**



Convective acceleration

Exit velocity 8.333 m/s; exit acceleration 46.296 m/s². The flow is steady at each fixed probe.

Chapter 2 · Hydrostatic pressure and forces on submerged surfaces

Derive pressure from a body-force balance and locate the resultant force on a vertical gate.

FROM FIRST PRINCIPLES

Determine gate force by integrating hydrostatic pressure

Consider a liquid at rest, with constant density ρ under uniform gravity g. Take depth h positive downwards from its free surface. A small horizontal fluid slice of area A and thickness dh has downward weight $\rho g A\,dh$. Equilibrium requires the increase in bottom pressure force to equal that weight.

$$[p(h+dh)-p(h)]A=\rho g A\,dh,$$
$$p(h)-p(0)=\rho g h.$$

For a vertical plane gate with the same atmospheric pressure on its dry side, only this gauge pressure contributes to the net force. Its resultant follows by integrating $\rho g h$ over the wetted area. Its line of action must also reproduce the pressure moment about the surface level.

$$F=\rho g\int_A h\,dA=\rho g A\bar{h},$$
$$h_R F=\rho g\int_A h^2\,dA.$$

Writing $\int h^2 dA = I_G + A\bar{h}^2$ gives

$$h_R=\bar{h}+I_G/(A\bar{h}),$$

where I_G is the second area moment about the horizontal centroidal axis. The greater pressure at depth places the resultant below the centroid.

DERIVATION AND APPLICATIONS

Find the horizontal and vertical forces on a curved gate

A curved gate generally feels both a horizontal and a vertical force from hydrostatic pressure. Consider a circular quarter-arc of radius R and width b perpendicular to the page. Place the arc centre at the free surface and choose x rightwards, z vertically downwards. The arc is $x = R\cos\theta$, $z = R\sin\theta$ for $0 \le \theta \le \pi/2$. Water occupies the circular sector on the centre side of the gate. Uniform atmospheric pressure on the other side cancels, and the liquid is stationary at constant density ρ.

$$p = \rho g R \sin\theta,$$
$$dA = b R \, d\theta,$$
$$d\mathbf{F} = p \, dA \, (\cos\theta \, \mathbf{e}_x + \sin\theta \, \mathbf{e}_z).$$

Integrating the horizontal component involves $\sin\theta\cos\theta$. The resulting force equals the pressure force on the vertical projection of the arc, of height R and centroid depth $R/2$.

$$F_x = \rho g b R^2 \int_0^{\pi/2} \sin\theta\cos\theta \, d\theta = \frac{1}{2}\rho g b R^2.$$

The vertical component instead integrates $\sin^2\theta$. It equals the weight of the imaginary water above the arc up to the free surface, which in this geometry is precisely the water sector itself.

$$F_z = \rho g b R^2 \int_0^{\pi/2} \sin^2\theta \, d\theta = \frac{\pi}{4}\rho g b R^2.$$

Every elemental force lies along an arc radius, so the resultant line passes through the circle centre. Its magnitude follows from vector addition and its direction makes angle $\arctan(\pi/2)$ below horizontal. The sign of the vertical reaction changes if the fluid bears on the opposite side.

WORKED EXAMPLE

Size the reaction on a laboratory quadrant

With $R = 0.400$ m, $b = 0.600$ m and $\rho = 1000$ kg/m³, the common factor is

$$\rho g b R^2 = 942\,\mathrm{N}.$$

The horizontal and downward forces are 471 N and 740 N respectively. The resultant is

$$\sqrt{471^2 + 740^2} = 877\,\mathrm{N}.$$

A bearing at the centre can react that force with no hydrostatic torque about its own axis.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Infer liquid density from the vertical component

A separate quarter gate has radius 0.300 m and width 0.500 m. After subtracting gate weight, its measured downward fluid force is 295 N. Inversion gives $\rho = 4F_z/(\pi g b R^2) = 851$ kg/m³. The expected horizontal reaction magnitude is

$$2F_z/\pi = 188\,\mathrm{N}.$$

This ratio tests the geometry without any independent density calibration.

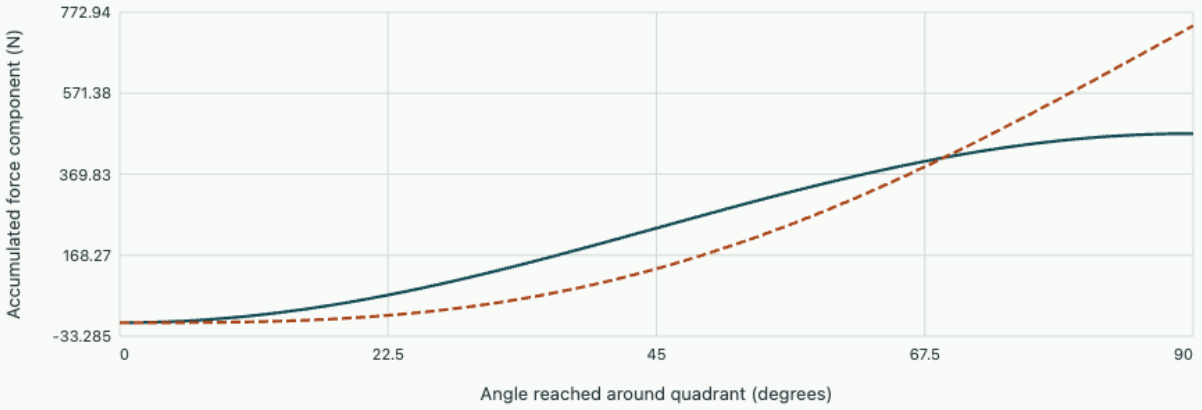
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Quadrant radius (m)	0.4
Gate width (m)	0.6
Fluid density (kg/m³)	1000



Full quadrant: Fx=470.880 N; Fz=739.657 N. Pressure vanishes at the upper free-surface edge; the vector direction varies along the gate.

Locate the force on an inclined circular viewing window

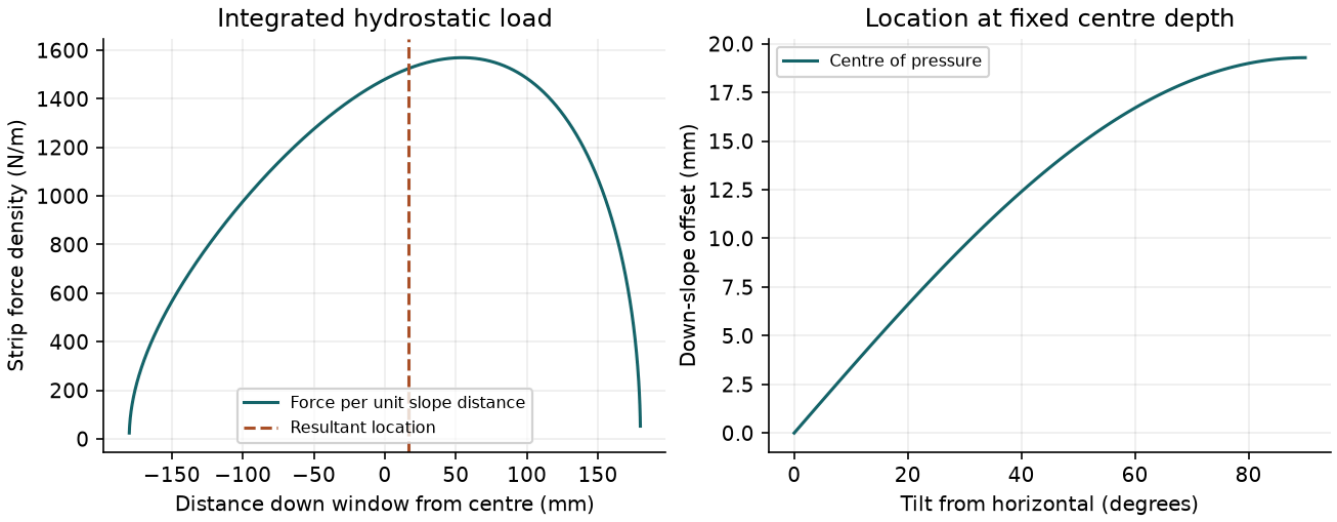
A circular window is installed below the water level in a laboratory tank. Knowing its total normal load is enough for one force balance. Sizing its hinge reaction also requires the moment of that load. The pressure increases with vertical depth, while the force elements must be integrated across an inclined surface. Distinguish distance down the window from depth below the surface before constructing the moment.

Let the window have radius a, centre depth h_c and inclination θ above the horizontal. Define s along its steepest downslope direction, measured from the disc centre. A point at s has vertical depth $h_c + s\sin\theta$. With an atmospheric gas on the dry side, the differential normal force is $\rho g(h_c + s\sin\theta)\,dA$. Since the first area moment $\int s\,dA$ vanishes by symmetry, the resultant is simply $\rho g h_c A$. Thus two windows of equal area at the same centre depth carry the same normal force even if their inclinations differ, provided both remain completely submerged.

To get the location of the force, take moments around the horizontal diameter through the window centre. The constant contribution to pressure gives zero moment about this axis, while the part proportional to s contributes $\rho g \sin\theta \int s^2 dA$. A circular disc has $A = \pi a^2$ and diametral second moment $I = \pi a^4/4$, which gives

$$s_{\mathrm{cp}} = \frac{\rho g \sin\theta\, I}{\rho g h_c A} = \frac{a^2 \sin\theta}{4 h_c}.$$

For $a = 0.18$ m, $h_c = 0.42$ m, $\theta = 60^\circ$ and $\rho = 998$ kg/m³, the force is 418.55 N and acts 16.70 mm down the window from its centre. Its vertical depth is only $16.70 \sin 60^\circ = 14.46$ mm below the centre depth. A load strip of thickness ds has width $2\sqrt{a^2 - s^2}$. The notebook integrates the strip forces and their first moments independently to check the circular formula.



The deepest point has the greatest local pressure. It carries very little integrated strip force because the circular chord becomes short at the rim.

For a hinge along the upper tangent to the disc, the resultant lever arm is $a + s_{\mathrm{cp}}$ measured down the window. Multiplying the default force by $0.180 + 0.01670$ m gives approximately 82.3 N m. Treating the force as though it acted at the centre would predict only 75.3 N m. The difference of 7.0 N m equals the moment of the pressure gradient about the centre diameter. The hinge axis must lie in the plane of the window for this simple lever arm to apply. Changing to a vertical hinge introduces a different area moment and, for this symmetric disc, a different distribution of opening torque.

What changes when the window is lowered until its centre depth doubles at the same tilt?

▼ Work through the solution

The normal force doubles and the downslope offset halves, to about 8.35 mm. Their product, the moment about the horizontal diameter through the disc centre, stays fixed because it depends only on the pressure gradient and the geometry. Rotate a fully submerged window to horizontal in the notebook and the offset goes to zero. If a window crosses the free surface, the wetted area and its centroid change; continuing to integrate the full disc would introduce fictitious negative gauge loads on dry strips.

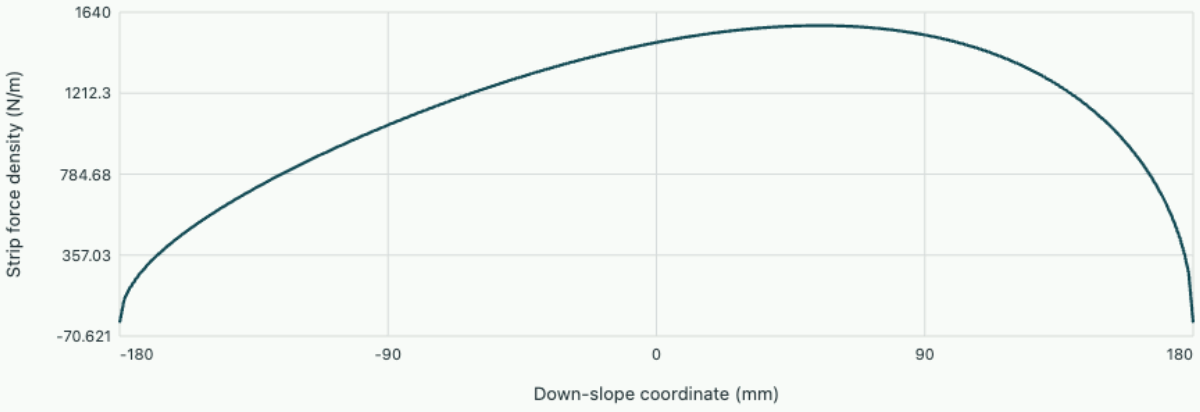
Interactive calculation · Window force

Vary the notebook inputs and compare both plotted observables. The same equations generate the browser curves and the notebook reference; fixed material, geometry and model assumptions are stated in the readout.

Open this live calculation

Try the calculation

Circular window radius (m)	**0.18**
Centre depth (m)	**0.42**
Tilt from horizontal (degrees)	**60**



Hydrostatic strip load

Resultant 418.55 N; down-slope offset 16.70 mm. Every available setting keeps the circular window fully submerged.

Download the editable notebook for this example

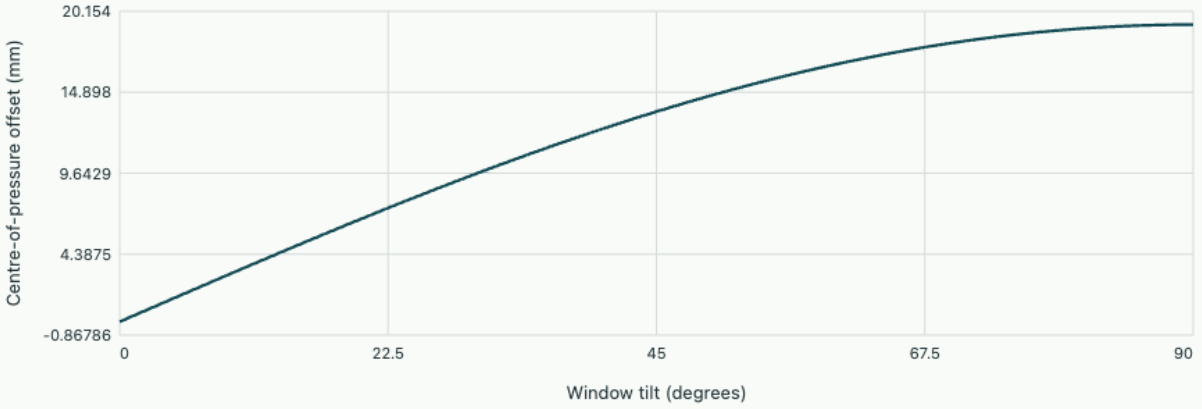
Interactive calculation · Window offset

Vary the notebook inputs and compare both plotted observables. The same equations generate the browser curves and the notebook reference; fixed material, geometry and model assumptions are stated in the readout.

Open this live calculation

Try the calculation

Circular window radius (m)	0.18
Centre depth (m)	0.42
Tilt from horizontal (degrees)	60



Resultant offset

Resultant 418.55 N; down-slope offset 16.70 mm. Every available setting keeps the circular window fully submerged.

Download the editable notebook for this example

Open editable notebook Download .ipynb

Balance a stationary fluid element

A diver experiences pressure from all directions, while a sluice gate feels a net force mainly from one side. Pressure is a scalar state variable. The force on a surface element is directed opposite its outward normal and has magnitude $p\,dA$. The net force on a real gate also depends on what presses on its back face. For a fluid at rest there is no viscous shear. Take a thin horizontal slab of area A and thickness dz, with z upward. Its vertical force balance is $p(z)A - p(z+dz)A - \rho g A\,dz = 0$, or $dp/dz = -\rho g$.

For constant density this integrates to $p = p_s + \rho g h$ at depth h below a free surface of pressure p_s. A vented water surface has atmospheric absolute pressure. Gauge pressure subtracts that same

atmospheric pressure at every point. It is convenient for net forces when the other face is exposed to the same atmosphere. A sealed gas space with appreciable excess pressure adds a uniform load component to the hydrostatic triangle.

Background · Which way does a liquid surface tilt in an accelerating tank?

Consider a tank translating horizontally at uniform acceleration a to the right. After sloshing has subsided, the liquid is stationary relative to the tank. In that frame each parcel has an effective body acceleration $-a\mathbf{e}_x - g\mathbf{e}_z$. Along a free surface at uniform atmospheric pressure the pressure gradient has no tangential component.

$$dp = -\rho a\,dx - \rho g\,dz = 0,$$

$$\frac{dz}{dx} = -\frac{a}{g}.$$

The free surface falls to the right. A tank 0.80 m long accelerating at 1.2 m/s² exhibits a height difference of about 98 mm between the ends, provided neither end spills or dries out. This argument uses the uniform translating frame; a tank that is still sloshing has additional relative accelerations.

Use two separate integrals

For a flat vertical gate, add horizontal pressure forces to get

$$F = \int_A p_g\,dA,$$

where p_g is the pressure difference across it. To find the force's depth h_R, match its moment:

$$Fh_R = \int_A h p_g\,dA.$$

With a vented free surface on one side and air on the other, this gives

$$F = \rho g A \bar{h},$$

$$h_R = \int_A h^2 dA/(A\bar{h}).$$

Decomposing the second moment into centroidal and parallel-axis pieces gives

$$h_R = \bar{h} + \frac{I_G}{A\bar{h}}.$$

Here I_G is the area's second moment about its horizontal centroidal axis. Its SI unit is m⁴. The second term is positive: lower parts of a vertical gate carry more force per unit area, moving the resultant below the area centroid.

Calculate hinge moment before actuator force

A 1.20 m wide rectangular gate extends from the water surface to depth 1.80 m. Its horizontal hinge is at the surface. At the bottom the gauge pressure is $1000(9.81)(1.80) = 17.7$ kPa. The triangular pressure profile integrates to $F = \rho g b H^2/2 = 19.1$ kN. The resultant lies at $2H/3 = 1.20$ m beneath the hinge, producing a moment of 22.9 kN·m. A horizontal actuator at the bottom must pull with magnitude $22.9/1.80 = 12.7$ kN to balance this water moment. This excludes gate weight and friction at the hinge.

If the whole gate is submerged another metre with both ends still separated from the same air pressure on the back, its pressure diagram gains a uniform rectangle. That additional resultant acts at the gate centroid. It increases hinge moment, while the centre of pressure of the total load moves closer to the centroid measured relative to the gate. Simply adding one metre to the original lever arm would confuse a depth measured from the surface with a moment arm measured from the hinge.

Include stratification and buoyancy

In layered immiscible liquids, integrate ρg through each layer. Pressure remains continuous across a flat interface when capillary effects are negligible, while its depth gradient changes. For a closed submerged object the resultant pressure force is upward with magnitude equal to the displaced fluid's weight. To see this, replace the body by the fluid that would occupy its volume. In static equilibrium surface pressure must balance that fluid's weight; the same surface pressure acts on the original object. Buoyancy concerns the net force on a closed surface. It cannot replace a hinge-moment calculation on an open gate.

Read a two-fluid differential manometer

Two water taps at the same elevation connect to opposite legs of a manometer containing a heavier immiscible liquid of density 2500 kg/m³. If the heavy-fluid interfaces differ in height by 40 mm, following a hydrostatic path gives a pressure difference $(2500 - 1000)g(0.040) = 589\,\mathrm{Pa}$. The tap above the lower heavy-fluid interface has the larger pressure. The water columns matter even though both taps sit at the same elevation.

Derive a restoring moment for a floating body

Equality of weight and buoyancy establishes vertical equilibrium of a floating body, but does not determine whether a small roll is restored. Consider a long prismatic pontoon with vertical sides floating at rest in a liquid of uniform density. Let the displaced volume be ∇, its centroid the centre of buoyancy B, and the centre of gravity G on the upright symmetry axis. For a small heel angle θ, a strip of

waterplane area dA at transverse coordinate y gains a submerged slice of volume approximately $y\theta\,dA$ on one side and loses one on the opposite side.

The first moment of these changed wedges shifts the centre of buoyancy sideways by $\Delta y_B = \theta \int y^2 dA/\nabla = \theta I_{\mathrm{wp}}/\nabla$. Thus the line of buoyancy in the tilted configuration intersects the original symmetry axis at a point M, the transverse metacentre, located $BM = I_{\mathrm{wp}}/\nabla$ above B to first order. The initial righting lever is $GM \sin\theta \simeq GM\,\theta$, and the corresponding restoring moment magnitude is $WGM\theta$ when $GM > 0$.

The waterplane second moment here is taken about the longitudinal heel axis; it is not the structural bending second moment of the vessel wall. A wide waterplane can increase stability without changing the weight because its outer strips contribute strongly to I_{wp}. A centre of gravity above the centre of buoyancy can still be stable if it remains below the metacentre.

Measure a pontoon’s centre of gravity by shifting a weight

A rectangular pontoon 4.0 m long and 2.0 m wide floats with 0.40 m draft in freshwater. Its displaced volume is $4(2)(0.4) = 3.20\,\mathrm{m}^3$, so the total supported mass is 3200 kg. The centre of buoyancy lies 0.20 m above the keel. Its waterplane moment is $4(2^3)/12 = 2.667\,\mathrm{m}^4$, giving $BM = 2.667/3.20 = 0.833\,\mathrm{m}$ and $KM = 1.033\,\mathrm{m}$ above the keel.

Shift a pre-existing 100 kg test weight transversely by 0.40 m without changing its height. A settled heel of 2.0° balances the imposed heeling moment against buoyancy. For small heel the more accurate inclination-test expression is

$$GM = md/(M_{\mathrm{tot}} \tan\theta) = 100(0.40)/[3200 \tan 2^\circ] = 0.358\,\mathrm{m}.$$

The inferred unshifted vertical centre of gravity is

$$KG = 1.033 - 0.358 = 0.675\,\mathrm{m}.$$

Raising a 400 kg installed component by 0.80 m would raise the combined centre by $400(0.80)/3200 = 0.100$ m and reduce initial GM to 0.258 m. A slack liquid surface inside the pontoon or partial emergence of its deck edge changes the small-heel model; either can invalidate an extrapolation of the measured initial restoring moment to large roll angles.

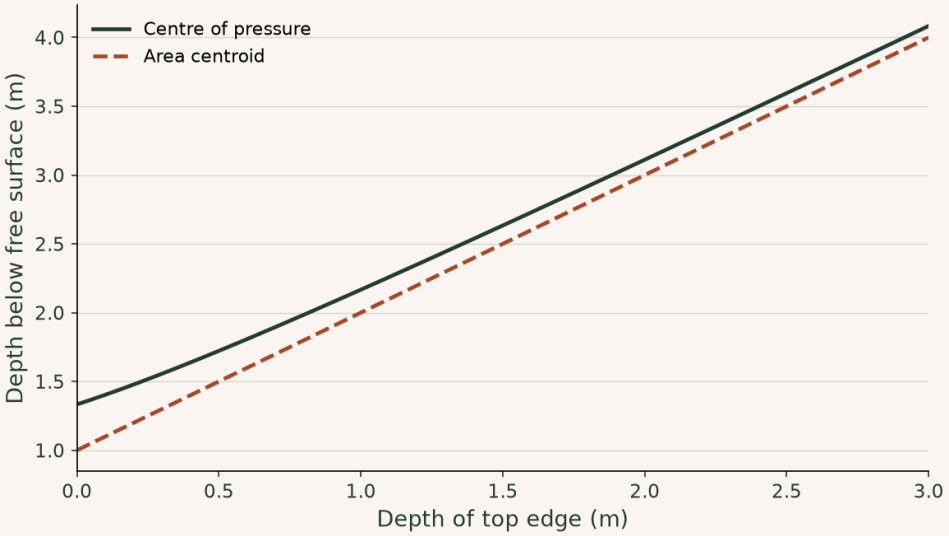
For the original centre of gravity the initial rotational stiffness is

$$WGM = 3200(9.81)(0.358) = 11.24\,\mathrm{kN\,m/rad}.$$

A loading estimate from a small known heeling moment can therefore serve as a separate check on the shifted-weight experiment. This stiffness is hydrostatic; rolling inertia and hydrodynamic damping are still required to predict a free-roll period or decay.

Place the hinge on a fully submerged vertical gate

A rectangular vertical gate is 2 m tall and of constant width. Integrate hydrostatic gauge pressure over its area. The centre of pressure is $h_c + I_c/(Ah_c)$, so the offset below the centroid shrinks as the gate is submerged more deeply.



With the top at the free surface, pressure acts 1.333 m below it rather than at the 1 m centroid. When the top is 3 m down, the offset is just 0.0833 m. Absolute pressure on both sides would cancel in this gauge-pressure loading case.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

Change gate height and compare the centroid with the centre of pressure. Width changes resultant force without changing its depth.

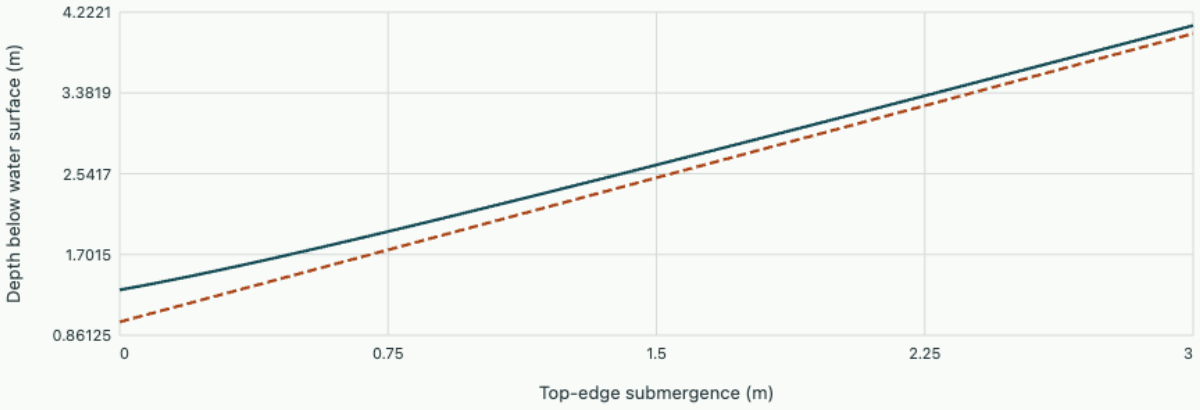
The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Gate height (m) 2

Gate width (m) 1



Centre of pressure Area centroid

With the top at the surface the resultant force is 19.620 kN at depth 1.333 m. Width changes force but not pressure-centre depth.

Study the full derivation

Gravity, capillarity and instability at a moving interface · Momentum, motion and transport in fluids

A flat interface can carry waves restored by gravity and surface tension. If its two liquids or gases move at different speeds, a band of those waves can grow. Derive the ideal threshold, identify the most vulnerable wavelength at onset and distinguish growth of an instability from advection of an otherwise stable disturbance.

Problem sheet 1 · Mass balances and hydrostatic loading

Determine the correct area, depth and system boundary before evaluating the force or volume flow.

1. Audit a branching pipeline

A 100 mm pipe supplies water at mean speed 1.00 m/s. One 60 mm branch carries 2.50 L/s; the remaining water leaves through a 50 mm branch. (a) Find the inlet flow and both branch speeds. (b) If discharge from the smaller branch is actually 4.00 L/s, infer an accumulation or leak rate. (c) What observation distinguishes temporary storage from a persistent leak?

▼ Close the volume budget

The inlet carries

$$0.25\pi(0.10)^2(1) = 7.85\,\mathrm{L/s}.$$

The nominal branch speeds are

$$0.00250/[\pi(0.06)^2/4] = 0.884\,\mathrm{m/s},$$
$$0.005354/[\pi(0.05)^2/4] = 2.73\,\mathrm{m/s}.$$

At the measured discharge the unaccounted rate is

$$7.85 - 2.50 - 4.00 = 1.35\,\mathrm{L/s}.$$

A rising storage level or an expanding system volume could temporarily account for it. At constant inventory a persistent imbalance points to another flow path or meter offset.

---

2. Locate pressure with two liquids

A vented tank has 0.40 m of oil of density 800 kg/m³ over 0.80 m of water. (a) Find gauge pressure at the interface and at the base. (b) Find the net horizontal force on a 0.50 m wide vertical wall segment extending only over the water depth. (c) Why is a single triangular pressure diagram across the water segment insufficient?

▼ Add the uniform and triangular loads

Interface pressure is $800(9.81)(0.40) = 3.14$ kPa. At the base add $1000(9.81)(0.80) = 7.85$ kPa to obtain 10.99 kPa. Assuming atmospheric pressure on the back, the average pressure on the water segment is

$$(3.14 + 10.99)/2 = 7.06\,\mathrm{kPa}.$$

On 0.40 m² this gives 2.83 kN. Oil adds the uniform 3.14 kPa across the water segment. Using only the water triangle would omit 1.26 kN.

---

3. Move a gate hinge

A vertical gate is 1.0 m wide, extending from 0.50 to 1.50 m below a vented water surface. There is air behind it. (a) Find resultant force and its depth. (b) Find the water moment about a hinge at its lower edge. (c) Compare with a force applied at the centroid.

▼ **Match the force and its moment**

The centroid depth is 1.0 m and the area is 1.0 m², giving 9.81 kN. The centroidal second moment is $1(1)^3/12 = 0.0833$ m⁴. The resultant acts at 1.0833 m depth, with lower-hinge arm $1.50 - 1.0833 = 0.4167$ m, giving 4.09 kN·m. Using the centroid would give 4.91 kN·m, too large because the greater pressure near the lower edge shifts the resultant towards that hinge.

Explore the governing equations after solving the sheet: Velocity fields, flow rates and conservation of mass; Hydrostatic pressure and forces on submerged surfaces. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Notebook practice · Locate the force on an inclined circular viewing window

Chapter 3 · Control-volume momentum and forces on fittings

Use the velocity vector, gauge pressure and a consistent force boundary to find the support load on a pipe bend.

FROM FIRST PRINCIPLES

Relate a support reaction to momentum carried by a stream

Newton's second law applies to a fixed set of fluid particles: external force equals the time derivative of their momentum. In a pipe bend it is easier to observe a stationary control volume that particles continually enter and leave. The rate of change for the material particles consists of storage in that volume plus the net rate at which momentum crosses its boundary.

$$\sum \mathbf{F} = \frac{\partial}{\partial t}\int_V \rho \mathbf{u}\, dV + \oint_{\partial V} \rho \mathbf{u}(\mathbf{u}\cdot\mathbf{n})\, dA.$$

The dot product determines whether mass crosses inwards or outwards, while the other velocity factor retains the direction of the momentum carried. For steady uniform inlet and outlet sections, the flux integral reduces to

$$\mathbf{F}_{\text{wall}} + \mathbf{F}_{\text{pressure}} + \mathbf{W} = \dot{m}(\mathbf{V}_2 - \mathbf{V}_1).$$

Pressure pushes inward normal to each open section, so its contribution must be included before solving for the wall force. The force of the fluid on the bend is opposite to the wall's force on the fluid. Equal inlet and outlet speed can still give a substantial support load when their directions differ.

DERIVATION AND APPLICATIONS

Derive the irreversible head loss across a sudden expansion

A sudden enlargement slows a liquid and partly restores static pressure. Consider steady incompressible flow from area A_1 into area $A_2 > A_1$ in a horizontal pipe. Choose the downstream section after mixing has produced an approximately uniform velocity again. Approximate both station profiles as uniform, neglect wall shear over the short control volume and take the mean pressure on the annular shoulder to be the upstream pressure p_1. Mass conservation makes the two mean speeds V_1, V_2 proportional to inverse area.

$$Q = A_1 V_1 = A_2 V_2,$$
$$p_1 A_2 - p_2 A_2 = \rho Q (V_2 - V_1).$$

The inlet and shoulder together contribute $p_1 A_2$ to the axial pressure force. Rearranging the momentum balance gives a positive downstream pressure rise, despite the separated mixing region.

$$p_2 - p_1 = \rho V_2 (V_1 - V_2).$$

Insert that pressure rise into the mechanical energy balance. The difference between kinetic energy released and pressure recovered is the specific energy dissipated in mixing, expressed below as a head of flowing liquid.

$$h_L = \frac{V_1^2 - V_2^2}{2g} - \frac{p_2 - p_1}{\rho g} = \frac{(V_1 - V_2)^2}{2g}.$$

This result tends to zero quadratically as the area change vanishes. A gradual diffuser would require a different shoulder-pressure and separation model. Pressure recovery can thus coexist with positive mechanical-energy loss and its measurement alone does not provide the loss.

WORKED EXAMPLE

Audit an abrupt pipe enlargement

Water moving at 3.00 m/s enters a pipe section with three times the area. Its final speed is 1.00 m/s. Taking $\rho = 1000$ kg/m³, the pressure rises by $1000(1.00)(2.00) = 2.00$ kPa. The kinetic pressure falls by 4.00 kPa; the missing 2.00 kPa is a head loss of 0.204 m. For 6.00 L/s, the rate of mechanical energy dissipation is $Q\rho g h_L = 12.0$ W.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Recover the enlargement ratio from a head test

Another rig measures $V_1 = 2.50$ m/s and $h_L = 0.0800$ m. Taking the positive speed drop yields

$$V_2 = 2.50 - \sqrt{2(9.81)(0.0800)} = 1.25\,\mathrm{m/s}.$$

The inferred area ratio is $A_2/A_1 = V_1/V_2 = 2.00$ to the stated precision. The expected water pressure rise is about 1.56 kPa, a separately measurable check on the shoulder-pressure assumption.

Interactive calculation · Change the inputs of this example

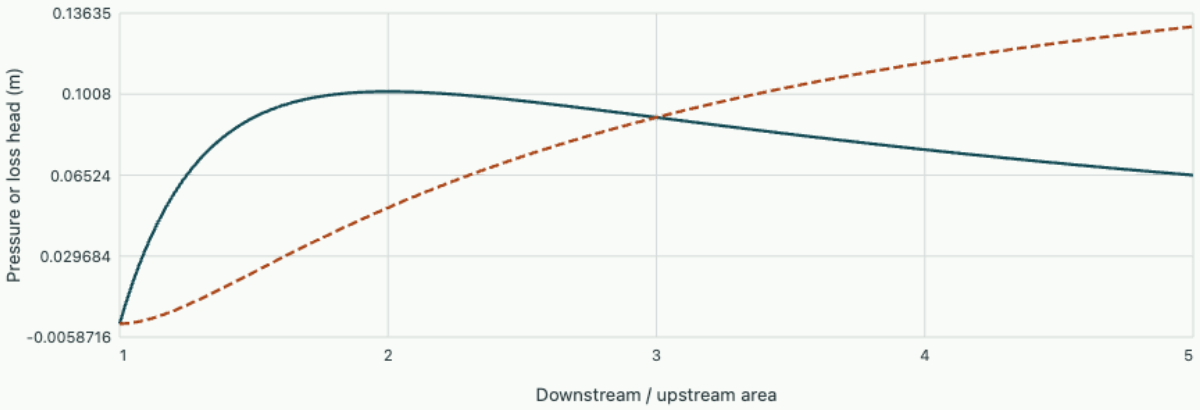
Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Upstream speed (m/s) **2**

Selected downstream/upstream area **2**



At selected area ratio 2, downstream speed=1.000 m/s and loss=0.051 m. The pressure recovery is smaller than the ideal deceleration estimate.

Momentum crosses the boundary with the mass

A pipe bend can exert a substantial force on its supports even when its inlet and outlet speeds agree. Momentum has direction. Turning a stream requires force and gives the fitting an equal and opposite load. Enclose the fluid inside the fitting in a fixed control volume. The external forces include wall traction, pressure on the control surfaces and body forces such as gravity.

A surface element transports mass per second $\rho(\mathbf{u}\cdot\mathbf{n})dA$. Multiply by the local velocity to obtain momentum per second. Newton's second law applied to the control volume reads

$$\sum \mathbf{F}_{\text{ext}} = \frac{d}{dt}\int_{\text{CV}} \rho\mathbf{u}\, dV + \int_{\text{CS}} \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\, dA.$$

For a steady fitting with nearly uniform inlet and outlet velocities this reduces to a sum of outgoing $\dot{m}\mathbf{u}$ minus incoming $\dot{m}\mathbf{u}$. Use one fixed set of axes even as pipe directions change. Pressure always pushes into the control volume, whereas the flux carries the direction of the local velocity, including its sign on the chosen axis.

Include the end pressures explicitly

Take a horizontal bend turning water from the positive x direction to the positive y direction. The inlet and outlet areas are each 0.010 m² and $Q = 0.040\,\mathrm{m^3/s}$. Their speeds are both 4.0 m/s and $\dot{m} = 40$ kg/s. The net outward momentum flux vector is $(-160, +160)$ N. Let the gauge pressures be 90 kPa at inlet and 80 kPa at outlet. Pressure pushes +900 N in x through the inlet and −800 N in y through the outlet.

Let $\mathbf{W}$ be the fitting's wall force on the enclosed fluid, with external atmospheric effects accounted for consistently using gauge pressures. Then $\mathbf{W} + (900, -800) = (-160, 160)$ N, giving $\mathbf{W} = (-1060, 960)$ N. The fluid's net gauge load on the fitting is the opposite, $(1060, -960)$ N, of magnitude 1.43 kN. Supports plus connected pipework must provide the opposite force for a stationary fitting, in addition to carrying the weight of hardware and fluid. The strongest contribution here stems from the pressure terms, despite the conspicuous turn in velocity.

Recover momentum from a nonuniform profile

A flow meter usually measures Q, not $\int u^2 dA$. At a plane normal to a unidirectional profile define $\beta = \int u^2 dA / (A\bar{u}^2)$. Then the axial momentum rate is $\beta \rho Q \bar{u}$. Uniform flow gives $\beta = 1$. For a developed laminar circular pipe $u = 2\bar{u}[1 - (r/R)^2]$; inserting $dA = 2\pi r\, dr$ in the integral gives $\beta = 4/3$. The largest velocities carry a disproportionate share of momentum. If one bend cross-section admits strong swirl or backflow, resolve vector fluxes instead of applying a single profile coefficient.

Recognise an open-stream boundary

A free jet at atmospheric pressure strikes a fixed flat plate normal to its axis and turns into a thin sheet that leaves sideways. Neglecting rebound and friction far from the impact, the outgoing axial momentum flux is zero. For water flowing at 10 m/s through a 20 mm diameter nozzle, $\dot{m} = 3.14$ kg/s, so the axial force on the plate is 31.4 N. Both the incoming and outgoing free-stream control surfaces have zero gauge pressure. The plate force then follows entirely from the change of momentum. A nozzle bolted to a pressurised pipe would require different pressure forces even if it produced exactly the same free jet.

A jet catches a moving vane

Consider a single flat vane moving steadily in the jet direction at speed u. In the vane frame the arriving volume rate is $A(V - u)$ and the incoming streamwise relative velocity is $V - u$. If the jet spreads tangentially across the face, the resulting normal force is $\rho A(V - u)^2$. For water, $A = 2.0\,\mathrm{cm^2}$, $V = 10$

m/s and $u = 4$ m/s, this gives 7.2 N. Its mechanical power on the moving vane is $Fu = 28.8\,\mathrm{W}$. Using the stationary volume rate would count fluid parcels that the moving vane never catches.

Derive the two depths of a hydraulic jump

A hydraulic jump converts shallow fast open-channel flow into deeper slow flow over a short distance. Turbulent mixing dissipates mechanical energy, so applying an inviscid energy balance between the two uniform sections would miss its defining loss. Instead choose a control volume bounded by sections before and after the roller, the horizontal bed and the free surface. Approximate the velocity as uniform and the pressure distribution as hydrostatic at the two end sections. Atmospheric pressure cancels, and over the short reach assume the integral of bed friction is small compared with the pressure and momentum forces.

In a rectangular channel per unit width let the volume rate be q, depths y_1, y_2 and speeds $v_i = q/y_i$. The horizontal pressure resultant per width is $\rho g y_i^2/2$. Equating pressure force difference to momentum-flux change gives

$$\frac{y_1^2}{2} + \frac{q^2}{g y_1} = \frac{y_2^2}{2} + \frac{q^2}{g y_2}.$$

One solution is the unchanged depth. Factoring out $y_2 - y_1$ for the nontrivial solution gives

$$q^2 = g y_1 y_2 (y_1 + y_2)/2.$$

With upstream Froude number $Fr_1 = v_1/\sqrt{g y_1}$, the downstream-to-upstream depth ratio becomes $[\sqrt{1 + 8 Fr_1^2} - 1]/2$. The deeper downstream branch occurs when $Fr_1 > 1$. As $Fr_1 \to 1$, the ratio tends to unity and the finite jump disappears.

Calculate the energy dissipated in a channel transition

A horizontal test channel carries water at $y_1 = 0.20\,\mathrm{m}$ and $v_1 = 4.0\,\mathrm{m/s}$. Its unit-width discharge is $q = 0.80\,\mathrm{m^2/s}$ and Froude number is

$$4/\sqrt{9.81(0.20)} = 2.86.$$

The depth ratio is about 3.57, giving $y_2 = 0.714\,\mathrm{m}$ and $v_2 = 0.80/0.714 = 1.12\,\mathrm{m/s}$. The downstream Froude number is about 0.42. These are conjugate depths for the specified momentum balance; the tailwater has to support the downstream depth for a stationary jump to sit at the chosen location.

Using specific mechanical head $E = y + v^2/(2g)$, the inlet and outlet give 1.015 and 0.778 m respectively. The loss is approximately 0.237 m of water, or $1000(9.81)(0.80)(0.237) = 1.86\,\mathrm{kW}$ dissipated per metre of channel width. Substitution of the momentum relation into the energy difference gives

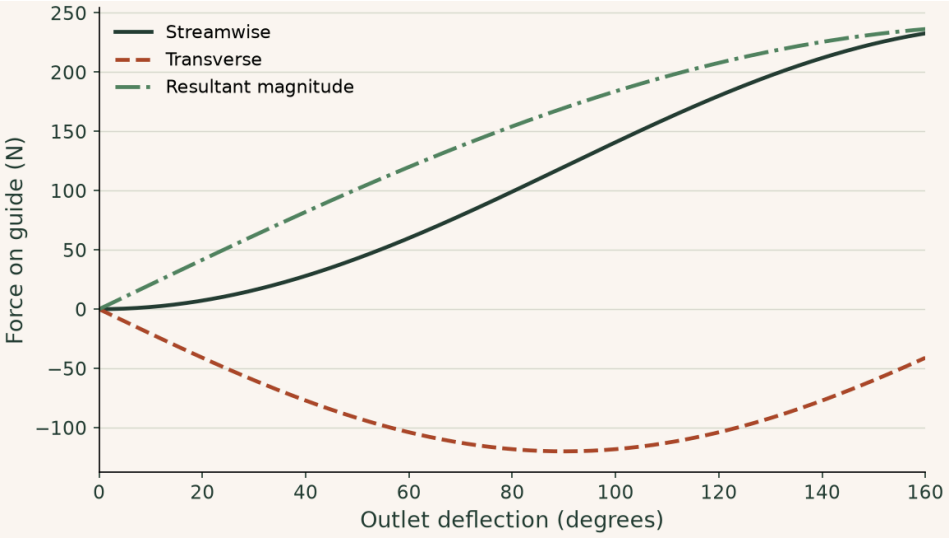
$$\Delta E = (y_2 - y_1)^3 / (4 y_1 y_2),$$

which is positive for the physical jump and approaches zero rapidly for a weak jump. This is a useful cross-check on a measured depth pair. If measurements are taken within the roller, pressure may be nonhydrostatic and velocity markedly nonuniform; using those sections in the simple end-state formula can produce an apparent energy or momentum imbalance.

Another discrimination comes from comparing the two depths with an upstream energy curve. At $q = 0.80$ m²/s, a downstream depth giving exactly the upstream energy would have to satisfy $y + 0.03262/y^2 = 1.015$. The deeper root is about 0.981 m, appreciably deeper than the momentum-predicted 0.714 m. Pairing those equal-energy depths would require an additional horizontal force or a different channel geometry to satisfy momentum. A smooth reversible change over a channel hump can conserve specific head while changing velocity and depth; a jump on a flat bed instead conserves the appropriate momentum function while dissipating head. Laboratory measurements of both levels are sufficient to distinguish the two idealisations.

Resolve the momentum force on a turning guide

A steady free water jet of mass flow 6 kg/s enters a smooth stationary guide at 20 m/s and leaves with the same speed at the indicated angle. Inlet and exit are at ambient pressure. Neglect weight for this short control volume. Reaction on the guide is the negative of the fluid momentum change.



A 90° turn gives +120 N streamwise and −120 N transverse, with resultant about 170 N. A 180° reversal would give 240 N streamwise. Using just the change in speed would predict zero for every turn.

▸ **Read values from the plot**

Interactive calculation · change the inputs of this example

Compare a right-angle turn with a complete reversal. The transverse force requires vector momentum, not just speed magnitude.

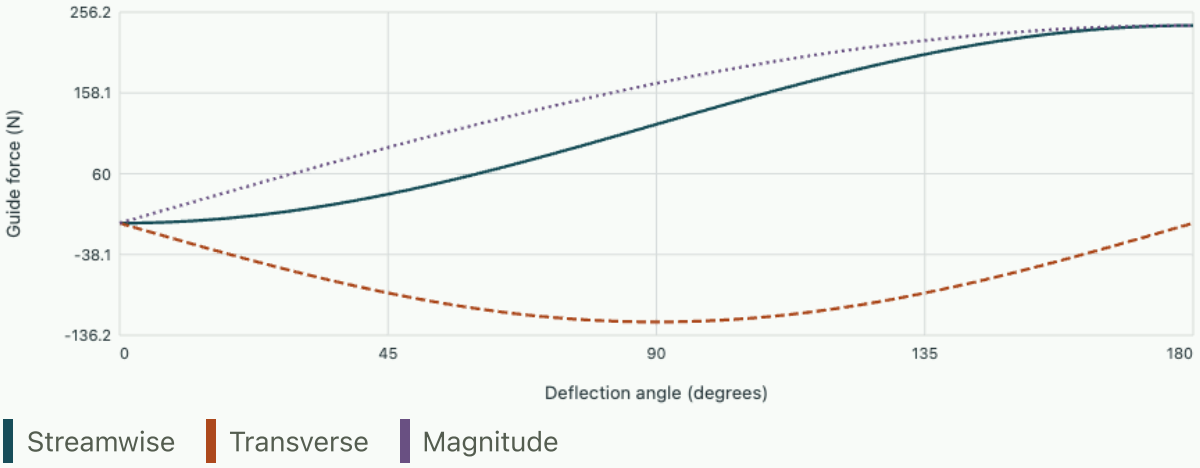
The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Mass flow (kg/s) **6**

Jet speed (m/s) **20**



A 90° turn gives components 120.000 and -120.000 N. Both ports are at ambient pressure.

Chapter 4 · Bernoulli head, losses and pump duty

Build a mechanical-energy balance between two stations and distinguish a recoverable pressure change from irreversible loss.

FROM FIRST PRINCIPLES

Derive mechanical head along a streamline

Take steady inviscid flow of constant-density liquid under gravity. Along a streamline use distance s, speed V and elevation z positive upwards. Tangential acceleration of a parcel is $V\,dV/ds$ because the parcel moves from one position to the next. Pressure gradient and gravity provide the tangential force per unit volume. Dividing their balance by density gives

$$V\frac{dV}{ds} = -\frac{1}{\rho}\frac{dp}{ds} - g\frac{dz}{ds}.$$

Each term is a derivative along the path. Integrating between two positions shows that their sum, written as energy per unit mass, is conserved under the stated assumptions.

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant along a streamline.}$$

Dividing by g converts all terms into lengths called heads. In a pipe system one measures added pump head H_p, removed turbine head H_t and irreversible loss h_L to form a section-to-section engineering balance. For uniform profiles this reads

$$H_1 + H_p = H_2 + H_t + h_L,$$
$$H_j = p_j/(\rho g) + V_j^2/(2g) + z_j.$$

This distinguishes a reversible pressure drop associated with acceleration from a loss of total mechanical head.

Background · Head loss

$$h_L = \Delta p_{\text{loss}}/(\rho g)$$

This is an irreversible loss expressed as a length. A static pressure change can also include reversible changes of speed or elevation.

DERIVATION AND APPLICATIONS

Derive an ideal Venturi flow reading from two pressure taps

A narrowing can measure flow by reversibly trading pressure for speed. Take a horizontal Venturi meter carrying steady, incompressible liquid from a developed upstream section of area A_1 into a throat of area A_2. For an ideal first model assume uniform velocity at the two taps, negligible viscous loss between them, and no pump or turbine between stations. Pressure taps register static pressure. Define $\beta = d_2/d_1$ for geometrically circular sections and use continuity to eliminate upstream speed.

$$V_1 = \frac{A_2}{A_1} V_2 = \beta^2 V_2,$$

$$p_1 - p_2 = \frac{\rho}{2}(V_2^2 - V_1^2).$$

The squared velocity is the source of the fourth power of diameter ratio. Substituting the continuity expression and multiplying throat speed by throat area gives the ideal meter law.

$$Q_{\mathrm{id}} = A_2 \sqrt{\frac{2(p_1 - p_2)}{\rho(1 - \beta^4)}}.$$

In an installed meter a calibrated discharge coefficient C_d relates actual flow to this model. At fixed geometry, density and coefficient, logarithmic differentiation produces a useful uncertainty relation for pressure-transducer readings.

$$Q = C_d Q_{\mathrm{id}},$$

$$\frac{dQ}{Q} = \frac{1}{2}\frac{d(\Delta p)}{\Delta p},$$

$$\Delta p = p_1 - p_2.$$

The pressure difference between upstream tap and throat is larger than the permanent loss measured after the diffuser. Estimating pump power from the meter differential would therefore include recoverable pressure. At very small flow, a fixed pressure offset has a particularly large relative effect.

WORKED EXAMPLE

Translate a pressure reading into discharge

A water meter has $d_1 = 80.0$ mm, $d_2 = 40.0$ mm, $\rho = 1000$ kg/m³ and $C_d = 0.980$. A 3.00 kPa differential corresponds to ideal throat speed $\sqrt{6/0.9375} = 2.53$ m/s. Actual discharge is

$$0.980\pi(0.0200)^2(2.53) = 3.12\,\mathrm{L/s}.$$

A 30 Pa differential error represents one percent of the measured drop, giving approximately 0.5 percent flow error.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Calibrate a second meter against a collection tank

A meter with $\beta = 0.600$ and $A_2 = 600$ mm² gives 2.00 kPa differential for a liquid of density 800 kg/m³. Its ideal flow is

$$0.000600\sqrt{4000/[800(1-0.6^4)]} = 1.44\,\mathrm{L/s}.$$

Collecting 14.0 L in 10.0 s indicates 1.40 L/s, hence $C_d = 0.974$ after using unrounded intermediate values. Repeating at other flow rates tests whether one constant coefficient is appropriate.

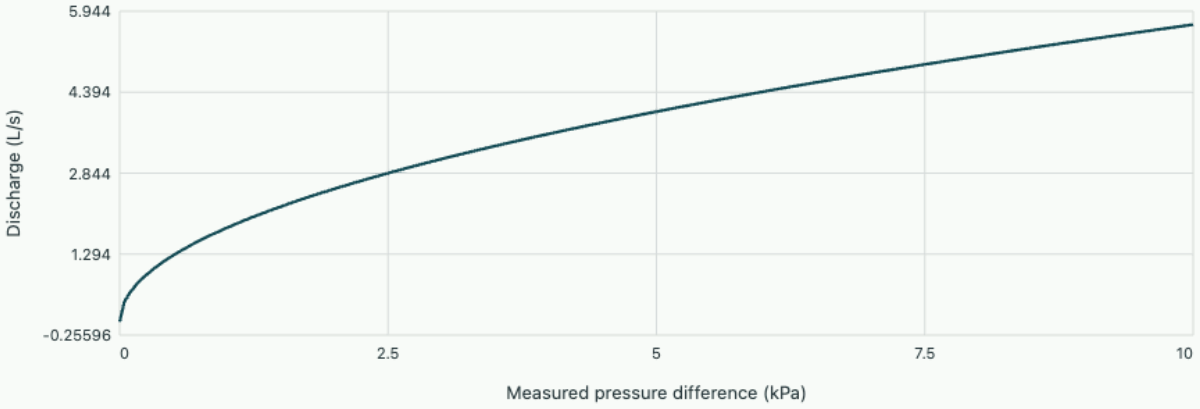
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Throat/upstream diameter β	**0.5**
Discharge coefficient	**0.98**
Fluid density (kg/m³)	**1000**



Calibrated differential meter

Fixed throat radius20 mm. At3 kPa, Q=3.115 L/s. A small fractional pressure error produces half that fractional flow error.

Integrate the acceleration along a streamline

A nozzle can lower static pressure while increasing speed. A valve can also lower pressure, yet recover little useful speed downstream. To separate the mechanisms, start from a steady, inviscid streamline in constant-density flow. Project Newton's equation along its tangent s, with gravity acting downward. The acceleration in that direction is $u\,du/ds$, giving $\rho u\,du = -dp - \rho g\,dz$. Integration between points on that streamline gives $p/\rho + u^2/2 + gz =$ constant.

The three terms have units J/kg. Dividing by g measures the same energies as equivalent lengths of fluid column, called pressure head, velocity head and elevation head. A rise in speed at constant elevation can be paid for by a drop in static pressure even in lossless flow. Bringing such a stream to rest along a short,

nearly lossless streamline would convert velocity head back into stagnation pressure. An irreversible loss reduces that recoverable total.

Add distributed losses and machinery

For a steady incompressible stream between stations with approximately uniform profiles, a pump supplies head H_p and a turbine extracts head H_t. Viscous losses h_L appear as mechanical energy converted into internal energy. With endpoints numbered in the flow direction the energy balance is

$$\frac{p_1}{\rho g} + \frac{\bar{u}_1^2}{2g} + z_1 + H_p = \frac{p_2}{\rho g} + \frac{\bar{u}_2^2}{2g} + z_2 + H_t + h_L.$$

For a noticeably nonuniform profile, multiply each velocity-head term by the kinetic-energy correction $\alpha = \int_A u^3 dA / (A\bar{u}^3)$. It differs from the momentum correction because energy flux weights the cube of velocity. Long straight pipes are commonly modelled with the Darcy loss $f_D (L/D)\bar{u}^2/(2g)$ and fittings with $K\bar{u}^2/(2g)$. State which pipe diameter supplies the reference velocity for each coefficient.

Lift water between open tanks

Two large vented reservoirs differ in elevation by 8.0 m. Water flows at $Q = 0.010\,\mathrm{m^3/s}$ through 40 m of 80 mm diameter pipe. Use a supplied Darcy friction factor 0.025 and total fitting coefficient 3.0, including entrance and exit losses. Both free surfaces have atmospheric pressure and negligible velocity. The pipe area is 5.03×10^{-3} m², giving mean speed 1.99 m/s and velocity head 0.202 m. Friction and fittings dissipate $[0.025(40/0.080) + 3.0]0.202 = 3.13$ m. The pump therefore supplies 11.13 m head.

The hydraulic power is $\rho g Q H_p = 1.09$ kW for $\rho = 1000$ kg/m³. A pump with efficiency 0.70 at this duty requires 1.56 kW of shaft power. Elevation work alone would predict 0.785 kW hydraulic, and electrical input would also require the motor and drive efficiencies. These differences reflect which boundary is used for the energy balance.

Find the intersection with the pump characteristic

If coefficients are treated as fixed, the system head here has the form $H_{\mathrm{sys}} = 8 + BQ^2$. Doubling the flow quadruples the losses, not the elevation rise. A real pump produces a head that depends on flow rate and rotational speed. Its steady operating point is where available head equals required head. Partially closing a valve increases B; it does not directly prescribe a particular flow. Readjusting pump speed changes its characteristic instead. In a real design recheck friction factor and efficiency at the new rate. At low Reynolds number, where laminar friction factor varies inversely with speed, the distributed loss becomes linear rather than quadratic in flow.

Check the absolute pressure on a pump inlet

A pump inlet sits 3.0 m above an open water reservoir. At a particular duty the suction-line speed is 2.0 m/s and the loss between reservoir and inlet is 0.60 m. Atmospheric pressure is 101 kPa. Applying the mechanical energy balance upstream of the impeller gives absolute inlet pressure $101000 - 1000(9.81)(3.0 + 0.60) - 1000(2.0)^2/2 = 63.7\,\mathrm{kPa}$. If the supplied vapour pressure is 2.3 kPa, this tapping remains above it. Pressure may fall further inside the impeller. Evaluating cavitation margin requires the pump's suction requirement at the actual duty and an absolute pressure reference; a negative gauge pressure at the tapping alone does not establish cavitation.

Locate the critical depth from a constrained energy minimum

In open-channel flow the atmosphere fixes the free-surface pressure, while the depth and speed can both adjust. Consider a rectangular channel of constant width with slowly changing bed elevation z_b. Define the discharge per unit width $q = vy$ and use the bed as the local elevation datum. With nearly hydrostatic pressure and approximately uniform section velocity, the mechanical head relative to the bed is $E(y) = y + q^2/(2gy^2)$. This specific energy is finite at a given depth but diverges both for a very shallow stream at fixed discharge and for a very deep stream.

At fixed q, differentiate with respect to y: $dE/dy = 1 - q^2/(gy^3)$. The stationary depth is $y_c = (q^2/g)^{1/3}$, its Froude number is one, and the second derivative $3q^2/(gy^4)$ is positive. The minimum value is

$$E_c = 3y_c/2.$$

For any specific energy above this minimum two positive depths are possible. One is deep and slow and the other shallow and fast. One must know the upstream flow regime and the physical control to choose a branch.

Across a smooth low hump with negligible loss the total head $z_b + E$ remains constant. Raising the bed reduces the specific energy available at the crest. On the deep branch depth decreases towards critical, whereas on the shallow branch depth increases towards it. These opposite trends follow from the sign of dE/dy. Neither branch can pass below E_c at the prescribed discharge.

Predict when a raised bed changes the upstream water level

Water enters a wide rectangular flume at $y = 0.80\,\mathrm{m}$ and $v = 1.0\,\mathrm{m/s}$, giving $q = 0.80\,\mathrm{m^2/s}$ and upstream specific energy $E_1 = 0.8510\,\mathrm{m}$. The critical depth is $(0.64/9.81)^{1/3} = 0.4025\,\mathrm{m}$, so $E_c = 0.604\,\mathrm{m}$. The largest ideal bed rise that leaves the upstream state unchanged is

$$0.851 - 0.604 = 0.247\,\mathrm{m}.$$

For a smaller 0.15 m bed rise the crest must have $E = 0.7010\,\mathrm{m}$. Solving $y + 0.03262/y^2 = 0.7010$ on the deep branch gives approximately 0.614 m, which can be checked by substitution. Speed increases to about 1.30 m/s while the stream remains subcritical.

Raise the crest instead by 0.30 m. The unchanged upstream level would provide only 0.551 m specific energy, less than the minimum. If discharge stays fixed, the upstream surface must rise until at least $E_1 = 0.30 + 0.604 = 0.904\,\mathrm{m}$ is available. The subcritical solution upstream is then approximately 0.860 m depth, a rise of about 60 mm. The physical limitation is the minimum section energy for the prescribed flow, not a loss of mass. Adding appreciable crest losses would require an even larger upstream head to sustain the same discharge.

A depth gauge right at a critical crest is relatively insensitive as an energy measurement: the first derivative dE/dy vanishes there. Conversely, inferring depth from a noisy measured head becomes sensitive near the minimum because the inverse slope becomes large. Measurements upstream and downstream help identify which of the two nearby depth branches is physically present.

Problem sheet 2 · Forces, flow work and losses

Use momentum to predict a support load and mechanical energy to predict pressures and pump power.

1. Calculate the load on a contraction

A horizontal water nozzle carries 0.020 m³/s. Its inlet area is 0.010 m², outlet area 0.0020 m², inlet gauge pressure 60 kPa and outlet atmospheric. Assume uniform profiles. (a) Find the momentum change per second. (b) Find the gauge force of the fluid on the nozzle. (c) Infer the head loss from the specified data.

▼ **Separate force and energy checks**

The inlet and exit speeds are 2.0 and 10 m/s. Net downstream momentum flux is

$$20(10-2)=160\,\mathrm{N}.$$

Inlet pressure supplies 600 N, so the nozzle wall exerts −440 N on the fluid and the fluid exerts +440 N on the nozzle. The available pressure head is

$$60\,000/(1000g)=6.12\,\mathrm{m}.$$

The velocity head increases by $(100-4)/(2g)=4.89$ m, leaving a positive 1.22 m loss. A lossless calculation would yield a different inlet pressure for that flow.

---

2. Select pump duty for two reservoirs

A pump delivers 6.0 L/s to a vented reservoir 12 m above its source. The losses at this rate are 3.0 m. (a) Find hydraulic and shaft power at 75% efficiency. (b) If loss varies as Q^2, repeat at 9.0 L/s. (c) Why is multiplying the first power by the flow ratio inadequate?

▼ **Add static lift and dissipative head**

The original head is 15.0 m, so hydraulic power is $1000(9.81)(0.006)(15)=883$ W and shaft power 1.18 kW. At 9 L/s the loss grows to $3(1.5)^2=6.75$ m, giving 18.75 m total head. Hydraulic and shaft powers become 1.66 and 2.21 kW if efficiency remains unchanged. Flow enters directly in power and indirectly through the extra head needed to overcome losses.

---

3. Turn a free jet through 120 degrees

A 5.0 kg/s water jet travels horizontally at 8.0 m/s and leaves a fixed smooth guide at the same speed, turned 120° counterclockwise. (a) Find inlet and outlet velocity components. (b) Obtain the force of water on the guide. (c) Determine the guide’s mechanical power and reconcile it with its force.

▼ Balance vector momentum

With the incoming jet along $+x$, the outlet velocity is $(-4.0, 6.93)$ m/s. The guide's force on the water is $5[(-4, 6.93) - (8, 0)] = (-60.0, 34.6)$ N when gravity is negligible and both surfaces have atmospheric pressure. Water acts on the guide with $(60.0, -34.6)$ N. The stationary guide has zero velocity and its mechanical power is zero. It changes momentum direction without changing the jet's speed or kinetic-energy rate.

Explore the governing equations after solving the sheet: Control-volume momentum and forces on fittings; Bernoulli head, losses and pump duty. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Chapter 5 · Viscous stresses, duct resistance and wall-driven flow

Derive a developed velocity profile from momentum balance and distinguish a moving-wall contribution from a pressure-driven contribution.

FROM FIRST PRINCIPLES

Derive the parabolic velocity profile between stationary plates

Consider steady fully developed flow between wide parallel plates at $y = 0$ and $y = H$. Assume incompressible Newtonian fluid, constant viscosity μ, no slip, no variation across the channel width and no streamwise body force. Let $G = -dp/dx > 0$ drive flow in the positive x direction. Since the axial velocity depends only on y, the convective acceleration is zero. A thin fluid layer balances pressure force against the change of shear stress across its thickness.

$$0 = -\frac{dp}{dx} + \frac{d\tau_{xy}}{dy},$$

$$\tau_{xy} = \mu \frac{du}{dy},$$

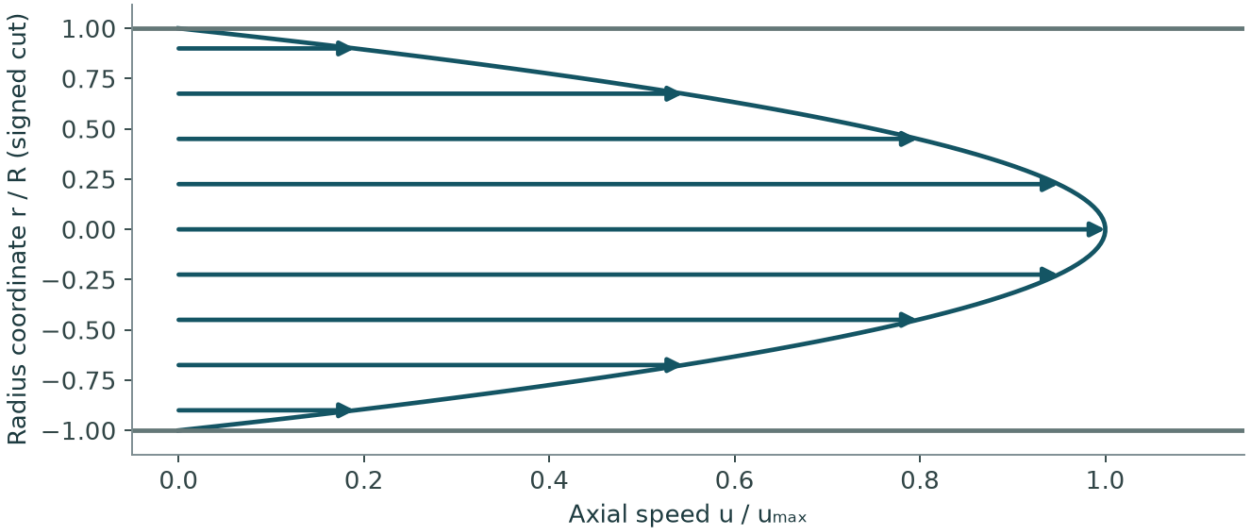
$$\mu u'' = -G.$$

Integrate twice and apply zero velocity at both walls. The resulting velocity is largest halfway between the walls, where the shear stress vanishes. Integrating the profile gives discharge per unit width.

$$u(y) = \frac{G}{2\mu} y(H - y),$$

$$\frac{Q}{b} = \int_0^H u(y)\, dy = \frac{G H^3}{12\mu}.$$

The cube of the gap arises because widening the channel increases both conducting area and the speed at a given pressure gradient. A moving wall adds a linear shear-driven profile through the boundary condition because the governing equation is linear.



Why the mean velocity is half the centreline velocity

Consider steady, fully developed laminar flow of a Newtonian liquid through a straight circular pipe with no slip. The pressure force balances viscous shear on each cylindrical fluid core. Its area grows as r² and its shear surface grows as r, so the needed wall-parallel shear grows linearly in r. Integrating once gives a parabola. The cross-sectional mean uses the annular weight 2πr dr and equals half the centreline value for this parabolic profile.

DERIVATION AND APPLICATIONS

Quantify the effect of wall slip on laminar pipe resistance

A pressure-flow calibration of a small tube can reveal a change in the assumed wall condition. Consider steady, fully developed, axisymmetric flow of a Newtonian liquid of viscosity μ in a round tube of radius R. Take the pressure gradient magnitude $G = -dp/dz > 0$ and neglect body forces along the tube. Balance pressure force against viscous drag on a coaxial fluid cylinder of radius r. Positive axial speed decreases approaching the wall.

$$G\pi r^2 = -\mu \frac{du}{dr}\, 2\pi r,$$

$$\frac{du}{dr} = -\frac{Gr}{2\mu}.$$

Allow a Navier slip length $b \geq 0$ at the stationary wall: the wall speed extrapolates to zero a distance b beyond it in a local linear approximation. This gives

$$u(R) = -b\, u'(R).$$

Integrating the shear rate from the wall into the flow adds a uniform velocity offset to the usual parabola.

$$u(r) = \frac{G}{4\mu}\left(R^2 - r^2 + 2bR\right).$$

Integrate the velocity over annuli to turn a local change of boundary condition into a measurable discharge. The contribution from the offset is proportional to the full area because the velocity addition is constant over the cross-section.

$$Q = 2\pi \int_0^R u(r) r \, dr = \frac{\pi G R^4}{8\mu}\left(1 + \frac{4b}{R}\right).$$

The shear stress needed to balance a fixed pressure gradient is unchanged. Effective slip lowers the required pressure gradient for a fixed flow. When b/R is small the fractional change is proportional to b/R, making small tubes more sensitive at a given slip length.

WORKED EXAMPLE

Calculate the extra discharge of a coated tube

Take $R = 100$ µm, $b = 5.00$ µm, $\mu = 1.00$ mPa s and $G = 10.0$ kPa/m. The no-slip flow is $\pi(10^4)(10^{-4})^4/(8 \times 10^{-3}) = 0.393$ µL/s. Slip increases it by $4b/R = 0.200$, yielding 0.471 µL/s. The predicted wall speed $GbR/(2\mu) = 2.50$ mm/s could be examined by a near-wall tracer.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Infer a slip length from pressure reduction

A different tube of radius 40.0 µm requires only 80.0 percent of the reference no-slip pressure drop to sustain the same flow. Equal flow implies $1 + 4b/R = 1/0.800 = 1.25$, giving $b = 2.50$ µm. A radius calibration error can mimic this signal because the no-slip flow varies as R^4. Independent bore measurements are therefore especially informative in this inverse experiment.

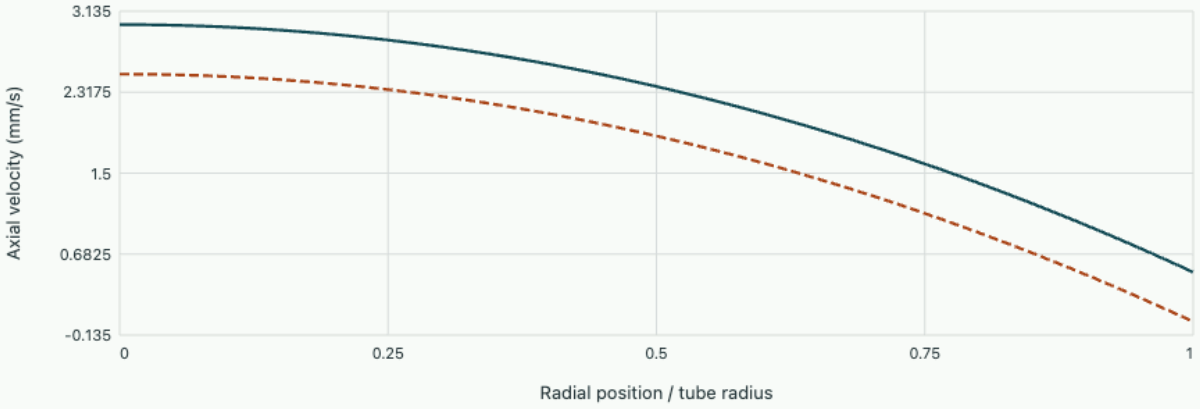
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Tube radius (µm)	100
Slip length / radius	0.1
Pressure gradient (Pa/m)	1000



Flow=0.0550 mm³/s, 1.400 times the no-slip flow. Dynamic viscosity1 mPa s; slip condition changes the wall value and the integrated conductance.

Transport momentum between neighbouring layers

Push two parallel plates past one another with liquid in the gap. Even at constant velocity the moving plate requires force. In a Newtonian fluid the tangential shear stress on a plane parallel to the motion is $\tau_{xy} = \mu\, du/dy$. Dynamic viscosity μ has units Pa·s. Divide it by density to obtain kinematic viscosity ν, an indicator of how rapidly momentum spreads through the fluid. Viscous resistance depends on a velocity gradient; uniform translation of the entire fluid would not create such stress.

Take two long horizontal plates separated by H. Far from inlet and side walls assume constant density, constant viscosity and steady fully developed laminar motion $u = u(y)$. The lower plate is fixed and the upper moves at speed U. Represent the constant driving pressure gradient by $G = -dp/dx$, positive

for pressure falling downstream. A differential slab balances the downstream pressure force with the change in shear, giving $0 = G + d\tau_{xy}/dy = G + \mu u''$.

Impose the two wall velocities

The first integration gives

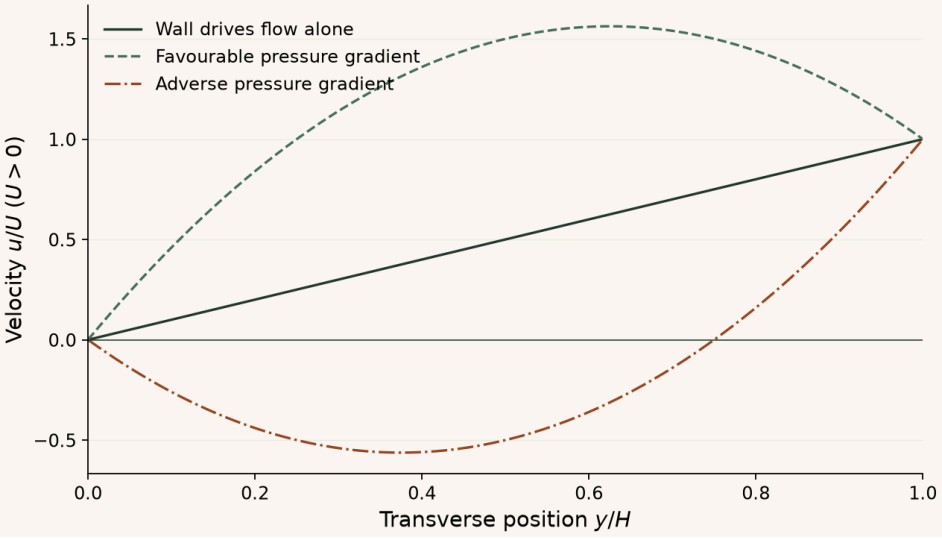
$$\mu u' = -Gy + C_1.$$

Integrating again and imposing no slip $u(0) = 0,\ u(H) = U$ determines the constants. The result is the sum of a straight wall-driven profile and a parabolic pressure-driven one:

$$u(y) = U\frac{y}{H} + \frac{G}{2\mu}y(H - y),$$

$$\frac{Q}{b} = \int_0^H u\,dy = \frac{UH}{2} + \frac{GH^3}{12\mu}.$$

Superposition here follows because the reduced equation and boundary conditions are linear in these inputs. It is not a universal property of the full nonlinear fluid equations. A negative G represents an adverse pressure rise and can produce a reverse-flow layer close to the stationary wall even when the moving wall drags fluid forward elsewhere.



The moving upper plate has the same speed in all three cases. The dimensionless pressure-gradient term multiplying (y/H)(1−y/H) is +4, 0 or −4.

Predict the pressure a moving surface can hold

An oil film has $\mu = 0.050$ Pa·s, $H = 1.0$ mm and $U = 0.20$ m/s. The wall-driven flux per unit width is $UH/2 = 1.0 \times 10^{-4}$ m²/s. An adverse pressure gradient sufficient to reduce net flux to zero must

satisfy $G = -6\mu U/H^2 = -60\,000$ Pa/m. Over 50 mm, pressure rises by 3.0 kPa. Zero net transport does not make the oil stationary.

Set $s = y/H$ and insert the zero-flux gradient. The velocity is $u/U = 3s^2 - 2s$, which changes sign at $s = 2/3$ inside the film. The velocity gradient at the upper moving wall is $4U/H = 800\,\mathrm{s}^{-1}$, so the wall supplies shear of magnitude 40 Pa. Even though the pressure-driven and wall-driven volume rates cancel, the moving surface still performs work, dissipated through the internal shear field.

Connect a local solution to a network

In a horizontal circular tube with fixed wall, axisymmetry changes the area that carries shear. Balancing forces on a cylinder of radius r gives

$$|\tau| = Gr/2.$$

Integrating the velocity gradient with no slip yields

$$u = G(R^2 - r^2)/(4\mu),$$

$$Q = \pi R^4 \Delta p/(8\mu L).$$

Thus a straight developed pipe behaves as a hydraulic resistance $8\mu L/(\pi R^4)$ in Pa·s/m³. Series resistances carry the same Q and their pressure drops add. Parallel branches share pressure drop and their flows add. The analogy requires consistent units and a laminar regime; short entrance sections and strongly accelerating junctions may add losses outside the fully developed solution.

Express pipe resistance as a friction factor

For a circular pipe of diameter $D = 2R$, the developed laminar solution gives

$$\Delta p = 32\mu L\bar{u}/D^2.$$

Define the Darcy friction factor by $\Delta p = f_D(L/D)\rho\bar{u}^2/2$. Equating these expressions produces $f_D = 64/\mathrm{Re}_D$. For 2 m of 10 mm pipe with mean speed 0.10 m/s, water viscosity 0.001 Pa s and density 1000 kg/m³, the Reynolds number is 1000 and the pressure drop is 64 Pa. A friction factor listed in the Fanning convention is one quarter of the Darcy value, so identify the convention before substituting a tabulated number into a loss calculation.

Derive the profile of a shear-thinning liquid in a pipe

A Newtonian viscosity inferred in one duct may give poor predictions for a suspension or polymer solution at another rate. In a simple power-law model the shear stress magnitude is $\tau = K|\dot{\gamma}|^n$, with consistency K in Pa sⁿ and index n. A liquid with $n < 1$ has apparent viscosity decreasing with shear rate. Assume this law is supported over the shear range being sampled, flow is steady, axisymmetric,

laminar and fully developed, and the circular pipe has no wall slip. The axial force balance is independent of rheology: at radius r, $\tau = Gr/2$ for driving gradient $G = -dp/dx > 0$.

Since the positive downstream velocity decreases towards the wall, write $-du/dr = (Gr/2K)^{1/n}$. Integrate from radius r to the stationary wall at R, then integrate the result over the circular cross-section to obtain

$$u(r) = \frac{n}{n+1}(G/2K)^{1/n}(R^{1+1/n} - r^{1+1/n}),$$

$$Q = \frac{\pi n}{3n+1}(G/2K)^{1/n}R^{3+1/n}.$$

Setting $n = 1,\ K = \mu$ gives the parabolic profile and $Q = \pi G R^4/(8\mu)$. For $n < 1$ the velocity is flatter towards the centre and the relationship between pressure gradient and flow rate is nonlinear. The ideal power law can imply unrealistic apparent viscosity near zero shear at the axis; a fitted model with finite limiting viscosities is preferable when measurements resolve those regimes.

Infer a pressure demand from wall shear

A model resin follows $n = 0.50,\ K = 0.80\,\mathrm{Pa\,s^{0.5}}$ in the operating range. A 2.0 m pipe of radius 5.0 mm experiences a pressure drop of 8.0 kPa, giving $G = 4000\,\mathrm{Pa/m}$ and wall shear $\tau_w = GR/2 = 10.0\,\mathrm{Pa}$. The wall shear rate is

$$(10/0.80)^2 = 156.25\,\mathrm{s^{-1}}.$$

Writing the mean speed in terms of this wall value gives

$$\bar{u} = nR|\dot{\gamma}_w|/(3n+1) = 0.15625\,\mathrm{m/s}.$$

The corresponding flow is $\pi(0.005)^2(0.15625) = 1.227 \times 10^{-5}\,\mathrm{m^3/s}$, about 0.736 L/min.

To double flow in this same pipe, the power-law relation requires the gradient to rise by $2^n = \sqrt{2}$. The new pressure drop is 11.3 kPa, and wall shear rate doubles to 312.5 s⁻¹. A linear resistance model extrapolated from the original measurement would predict 16 kPa. Conversely at fixed gradient a 10% radius increase multiplies rate by $1.1^{3+1/n} = 1.61$, a stronger dependence than the Newtonian fourth power. Verify the resulting shear rate still falls in the tested rheological range, and check entrance length and temperature. Temperature changes produced by viscous dissipation can otherwise mimic a shear-dependent viscosity.

Two well-spaced steady pressure tests can independently infer the unknown exponent. At fixed pipe geometry,

$$\ln Q = (1/n)\ln \Delta p + \text{constant}.$$

If pressure drop rises from 8.0 to 12.0 kPa while flow rises from 0.736 to 1.66 L/min, the log slope is $\ln(1.66/0.736)/\ln(12/8) \simeq 2.00$, giving $n \simeq 0.50$. Repeating the higher-pressure point after cooling checks whether heating caused hysteresis between runs. Using an inlet pressure transducer alone would

also include a changing downstream pressure; both end pressures or a differential gauge are needed to identify the flow law from this experiment.

The pressure taps should bracket the same developed length at every setting because entrance losses may change differently with flow.

How does a local pressure gradient accelerate a fluid particle?

A duct solution generally begins by cancelling terms in the momentum balance. To decide which terms can be cancelled, follow a small volume of fluid. Its acceleration includes the change in the velocity at its present location and the change as it moves into a different part of the velocity field. Applying the chain rule along its trajectory gives

$$D\mathbf{u}/Dt = \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}.$$

Consider the two faces normal to the first coordinate axis. Their net surface force per unit volume in direction i approaches $\partial_x \sigma_{ix}$ as the face separation tends to zero. Summing the three face pairs and adding weight gives

$$\rho Du_i/Dt = \sum_j \partial_j \sigma_{ij} + \rho g_i.$$

For an incompressible Newtonian liquid of constant viscosity μ, insert $\sigma_{ij} = -p\delta_{ij} + \mu(\partial_j u_i + \partial_i u_j)$. The cross derivative sums to $\partial_i(\nabla\cdot\mathbf{u}) = 0$. The remaining terms are the local equations

$$\rho[\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}] = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{g},$$
$$\nabla\cdot\mathbf{u} = 0.$$

Take a horizontal slice through an ideal planar stagnation region, away from confining walls. The steady field $u = ax,\ v = -ay$ satisfies continuity. Every velocity component is linear, so its Laplacian vanishes. Nevertheless the material acceleration is (a^2x, a^2y). Integrating the two horizontal gradients gives

$$p(x,y) = p_0 - \rho a^2(x^2+y^2)/2,$$

where p_0 is the stagnation-point pressure at this elevation. For water at $1000\,\mathrm{kg\,m^{-3}}$, $a = 4\,\mathrm{s^{-1}}$ and $y = 0.10\,\mathrm{m},\ x = 0$, pressure is 80 Pa below p_0. Fluid entering along positive y moves towards the origin but accelerates in positive y: it slows as pressure rises. At the stagnation point itself both velocity and acceleration vanish. Specifying p_0 fixes the pressure datum. The velocity gradients determine the spatial change.

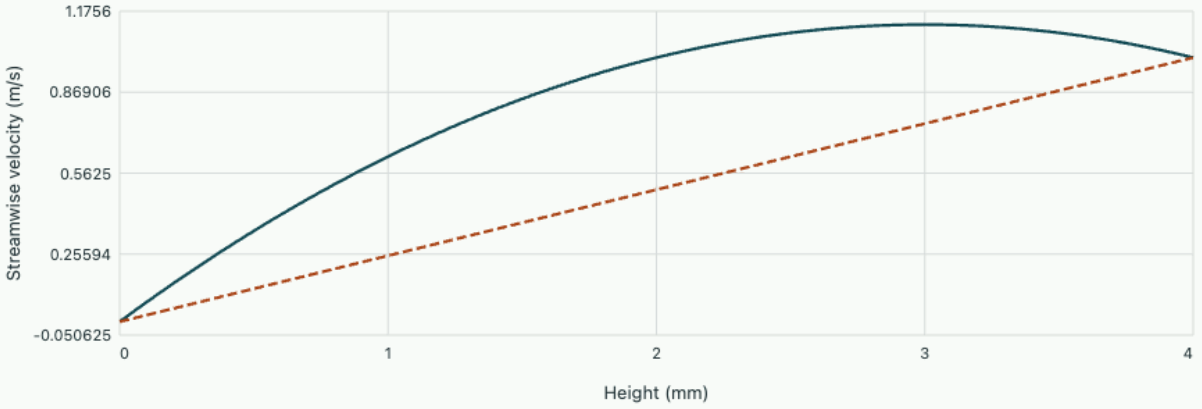
Interactive calculation · Change the inputs of this example

Change the parameters of this reference plot and compare its limiting cases. The fixed illustration remains beside the derivation; the adjustable calculation exposes the same dependence on geometry, material properties or loading.

Open this live calculation

Try the calculation

Moving-wall speed (m/s)	1
Pressure forcing / wall forcing	2
Gap (mm)	4



Mean speed 0.833 m/s. Pressure parameter equals $Gh^2/(2\mu U)$; adverse pressure can reverse flow close to the stationary wall.

Practise on sheet 3 · Which pressure difference establishes an accelerating uniform stream?.
Further reading · MIT · 1.060 Engineering Mechanics II · Lectures 35–36, Differential Analysis of Fluid Flows: The Navier–Stokes Equation.

Study the full derivation

Why shear disperses a tracer faster than molecular diffusion · Momentum, motion and transport in fluids

A short dye pulse in a straight tube first stretches into a filament and much later resembles a broadened one-dimensional cloud. Derive the long-time Taylor–Aris coefficient, predict the width of a measured pulse and identify the mixing time needed before fitting the familiar one-dimensional approximation.

Chapter 6 · Dynamic similarity, boundary layers and model tests

Choose dimensionless groups for a fluid experiment and determine which full-scale predictions a scale model can support.

FROM FIRST PRINCIPLES

Read similarity conditions from the nondimensional momentum balance

Two geometrically similar channels can behave differently when inertia, viscosity and gravity contribute in different proportions. Use incompressible Newtonian flow with density ρ, kinematic viscosity ν and constant gravitational acceleration $\mathbf{g}$. Choose length L, speed U, elapsed time L/U and pressure scale ρU^2. The underlying balance is force per unit mass equals acceleration.

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\rho^{-1}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{g}.$$

Insert $\mathbf{u} = U\mathbf{u}^*$, $\mathbf{x} = L\mathbf{x}^*$ and the other scales. Divide through by U^2/L. The viscosity and gravity terms acquire coefficients that compare their acceleration scales with inertia.

$$\partial_{t^*}\mathbf{u}^* + \mathbf{u}^*\cdot\nabla^*\mathbf{u}^* = -\nabla^* p^* + \frac{1}{Re}\nabla^{*2}\mathbf{u}^* + \frac{1}{Fr^2}\mathbf{e}_g,$$

$$Re = \frac{UL}{\nu},$$

$$Fr = \frac{U}{\sqrt{gL}}.$$

Here $\mathbf{e}_g$ points downwards. Matching the relevant coefficients and dimensionless initial and boundary conditions gives the same nondimensional problem. Choosing what to match therefore depends on which physical effects matter for the measured observable.

Background · Reynolds number

$$\mathrm{Re} = \rho U L/\mu$$

Choose speed and length scales appropriate to the geometry. Matching this ratio compares inertial and viscous terms under the stated nondimensionalisation.

DERIVATION AND APPLICATIONS

Scale a rising interface with Reynolds and Froude numbers

A free-surface model can match gravitational wave motion and still misrepresent viscous drag. Consider two geometrically similar open-channel experiments with characteristic lengths L_m, L_p, speeds U_m, U_p, gravitational accelerations g_m, g_p and kinematic viscosities ν_m, ν_p. Ignore surface tension only when the Bond number is sufficiently large in both channels. Compare the inertial acceleration U^2/L with gravity and viscous acceleration $\nu U/L^2$. Their ratios define

$$Fr = \frac{U}{\sqrt{gL}},$$

$$Re = \frac{UL}{\nu}.$$

Let $\lambda = L_m/L_p$ be the model-to-prototype length ratio. Froude similarity fixes the velocity ratio because the shape and timing of long free-surface disturbances depend on the competition with gravity. Substituting this speed into Reynolds similarity fixes the required viscosity ratio as a second independent condition.

$$\frac{U_m}{U_p} = \sqrt{\lambda \frac{g_m}{g_p}},$$

$$\frac{\nu_m}{\nu_p} = \lambda^{3/2}\sqrt{\frac{g_m}{g_p}}.$$

A travel time scales as length divided by velocity, and discharge scales as velocity times geometrically similar area. The matching free-surface kinematics therefore predict two distinct dimensional scale factors.

$$\frac{t_m}{t_p} = \sqrt{\lambda \frac{g_p}{g_m}},$$

$$\frac{Q_m}{Q_p} = \lambda^{5/2}\sqrt{\frac{g_m}{g_p}}.$$

When the same liquid and gravity are used, exact equality of both dimensionless numbers is possible only at full scale. If boundary-layer details matter in a smaller channel, the calculated Reynolds mismatch quantifies what a Froude-scaled measurement leaves unresolved.

WORKED EXAMPLE

Interpret a one-to-nine water channel

At equal gravity, $\lambda = 1/9$ gives a speed ratio of $1/3$. A model flow of 0.400 m/s corresponds to 1.20 m/s at full scale. A 2.00 s wave passage scales to 6.00 s. With the same water, $Re_m/Re_p = \lambda^{3/2} = 1/27$,

so a prototype Reynolds number of 270,000 becomes 10,000. Matching model surface-wave timing alone gives no guarantee of the same wall-shear coefficient.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Choose the viscosity for simultaneous matching

In a separate equal-gravity test, $\lambda = 1/4$ and the prototype liquid has $\nu_p = 8.00$ mm²/s. Simultaneous similarity requires $\nu_m = (1/4)^{3/2}(8.00) = 1.00$ mm²/s. A prototype discharge of 64.0 L/s maps to $64.0(1/4)^{5/2} = 2.00$ L/s. This pair of viscosity and discharge targets specifies two independently measurable conditions for commissioning the model.

Interactive calculation · Change the inputs of this example

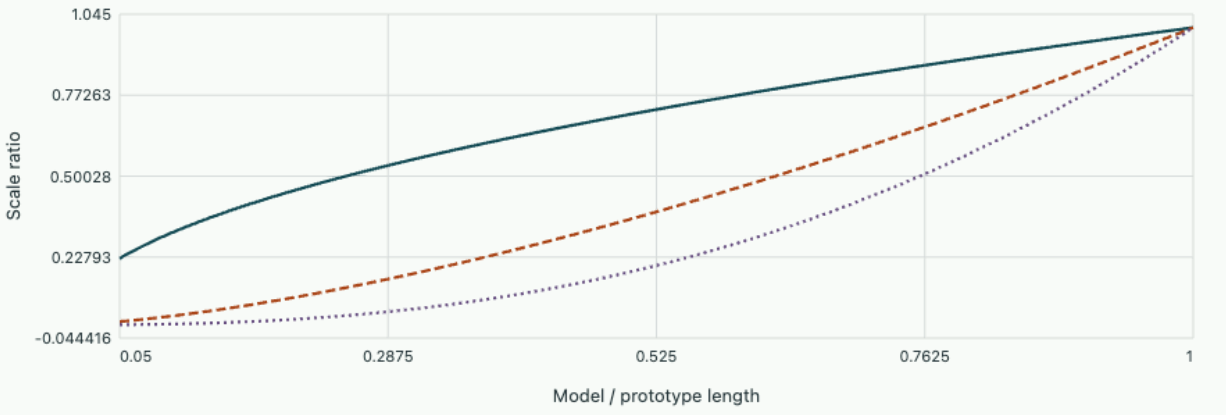
Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Model/prototype gravity **1**

Chosen length scale **0.11**



At selected length ratio 0.110: time ratio=0.332 and required viscosity ratio=0.03648. Using the same fluid usually cannot match both Fr and Re.

Start from this chapter. At fixed distance, the Reynolds number doubles when the stream speed doubles. Predict the change in the ratio of layer thickness to distance before reading the output.

OneSimulation laboratory · Measure boundary-layer growth from a velocity profile

Compare competing mechanisms

A scale model can reproduce a shape exactly and still display a different flow pattern. A fast oil stream and a slow water stream may behave similarly if the relevant dimensionless groups match. To see which groups enter, choose a typical length L, speed U, time L/U and pressure change ρU^2 in the incompressible Newtonian momentum equation. Divide by the inertial scale $\rho U^2/L$. The coefficient multiplying viscous diffusion becomes $1/Re$, where $Re = \rho U L/\mu$.

If gravity must be retained, its coefficient is $1/Fr^2$, with $Fr = U/\sqrt{gL}$. A free water surface makes this particularly important because gravity restores disturbances. For a fully submerged shape in a constant-density fluid, gravity can often be absorbed into hydrostatic pressure, leaving the Reynolds number dominant. Extra effects can introduce more groups, for example roughness divided by length or oscillation frequency times L/U. Choose groups to match the phenomenon being measured.

Derive a near-wall length scale

Consider a smooth flat plate parallel to a uniform stream with a negligible pressure gradient. In a laminar boundary layer, momentum is advected downstream on a scale U^2/x and diffuses across the shear layer on a scale $\nu U/\delta^2$. Balancing them gives $\delta \sim \sqrt{\nu x/U}$. The ratio $\delta/x \sim Re_x^{-1/2}$ may be small even though the tangential shear force from the layer matters to the machine. The similarity solution places the 99% speed level near $5x/\sqrt{Re_x}$, and the local skin-friction coefficient $\tau_w/(\rho U^2/2)$ near $0.664/\sqrt{Re_x}$.

Doubling speed at a fixed station multiplies the local Reynolds number by two. The dimensionless friction coefficient falls by $1/\sqrt{2}$, but dynamic pressure grows fourfold. The dimensional shear therefore rises by $2^{3/2}$. This distinction matters when reading plots of nondimensional test results: a lower plotted coefficient can accompany a larger actual force.

Scale a submerged component

A streamlined full-size probe of length 0.60 m moves at 2.0 m/s in water with $\nu = 1.0 \times 10^{-6}\,\mathrm{m^2/s}$. A geometrically similar third-scale model in the same water needs 6.0 m/s for equal Reynolds number 1.2×10^6. Assume matching surface finish in relative terms, sufficiently small blockage, the same flow regime and no free-surface effect. Then the dimensionless drag $C_D = F_D/(\rho U^2 A/2)$ should agree. Since the model’s reference area is one ninth and its speed squared is nine times as large, its drag is equal to the prototype drag in this example.

If only 2.0 m/s can be reached in the test tank, the model Reynolds number is one third of full size. An observed transition, separation point or drag coefficient may then shift. Document this mismatch rather than scaling the raw measured drag by area alone. Matching a single group does not fix every boundary condition, especially the upstream disturbance level that can trigger transition.

Scale a different experiment with the same geometry

For a one-ninth length model of a surface vessel, matching Froude number requires

$$U_m = U_p/3.$$

In the same water its Reynolds number relative to the prototype is

$$(1/3)(1/9) = 1/27.$$

Exact Froude and Reynolds similarity cannot both hold at that length and speed with the same fluid. Gravity-wave resistance can be compared at matched Froude number, while viscous skin friction needs a separate correction. Report both dimensionless numbers with the measured force and its reference area. That allows a reader to distinguish the physical part of the scaling argument from assumptions used for extrapolation.

Recover a dimensional frequency from a model

Matching a dimensionless frequency can transfer an unsteady effect. For an oscillation of frequency f in a stream of speed U past a length L, the Strouhal number is fL/U. A 0.10 m model in a 3 m/s stream oscillating at 6 Hz gives 0.20. If relevant geometry and governing dimensionless parameters are matched, a 0.50 m prototype at 2 m/s would oscillate at $0.20(2)/0.50 = 0.80$ Hz. Frequency alone cannot be rescaled by the geometric factor when the reference speeds differ.

Determine when surface tension prevents gravity-wave similarity

A reduced-scale flume can match gravity and inertia by matching Froude number. The air–water interface also has surface tension σ. A disturbance with characteristic length L and speed U has inertial pressure scale ρU^2, gravity scale $\rho g L$, and capillary pressure scale σ/L. The ratios produce $\mathrm{Fr}^2 = U^2/(gL)$ and $\mathrm{We} = \rho U^2 L/\sigma$. Dividing the latter by the former gives the Bond number $\mathrm{Bo} = \rho g L^2/\sigma$. Choosing model speed to match Froude number leaves Bond number determined by geometry and material properties. With the same liquid and gravity, reducing length by ten reduces Bond number by one hundred.

For a more precise test use the small-amplitude, inviscid deep-water dispersion relation $\omega^2 = gk + \sigma k^3/\rho$ for wave number $k = 2\pi/\lambda$. The gravity term arises from surface elevation, while the capillary pressure perturbation is proportional to surface curvature $k^2\eta$. The ratio of capillary to gravity restoring contributions is $\sigma k^2/(\rho g)$. Equal contributions occur at wavelength $2\pi\sqrt{\sigma/(\rho g)}$. The

appropriate length in this ratio is $1/k$; omitting the 2π when using wavelength changes the inferred correction substantially.

Check a wave-frequency prediction at two geometric scales

For water use $\sigma = 0.072\,\mathrm{N/m},\ \rho = 1000\,\mathrm{kg/m^3}$ and $g = 9.81\,\mathrm{m/s^2}$. The crossover wavelength is

$$2\pi\sqrt{0.072/9810} = 0.0170\,\mathrm{m}.$$

Consider a prototype wave of wavelength 0.40 m. Its wave number is 15.71 m⁻¹ and the capillary-to-gravity ratio is

$$0.072(15.71)^2/9810 = 0.00181.$$

Its frequency is approximately $\sqrt{9.81(15.71)(1.00181)}/(2\pi) = 1.98\,\mathrm{Hz}$, essentially the gravity prediction.

A geometrically similar one-tenth model uses wavelength 0.040 m, wave number 157.1 m⁻¹ and capillary-to-gravity ratio 0.181. Pure gravity scaling would predict frequency $\sqrt{10}(1.98) = 6.26\,\mathrm{Hz}$. Including capillarity gives $\sqrt{9.81(157.1)(1.181)}/(2\pi) = 6.79\,\mathrm{Hz}$, about 8.5% higher. A frequency discrepancy need not indicate a clock error or a viscosity error. The model has moved into a region where a restoring force previously negligible is perceptible. Deep water also requires kh large at both scales; shallow flume depth adds a separate $\tanh(kh)$ factor. Enlarging the model or choosing a different measured observable may be preferable to enforcing incompatible similarity conditions.

A useful comparison can be made by timing wave packets as well as crests. The phase speed is

$$c_p = \omega/k.$$

The group speed of a narrow wave packet follows from differentiating the dispersion relation,

$$c_g = [g + 3\sigma k^2/\rho]/(2\omega).$$

Their ratio is $[1 + 3B]/[2(1 + B)]$, where $B = \sigma k^2/(\rho g)$. In the prototype example it is 0.502, close to the deep-water gravity value of one half. In the short model it is 0.653. Thus scaling time to match one measured crest frequency still would not make packets travel relative to their crests in exactly the same way. A video showing both individual crest motion and arrival of a packet envelope provides experimental evidence for the altered mechanism.

Surface contamination is another experimental variable: an altered surface tension changes the capillary correction without changing either geometric length or gravity-wave Froude number.

Study the full derivation

Settling spheres, Stokes drag and a wall correction · Momentum, motion and transport in fluids

A falling-sphere viscometer infers viscosity from terminal speed. Derive the drag on a no-slip sphere, distinguish creeping flow from merely steady motion and use radius, vessel width and orientation checks to decide what the speed experiment actually measures.

Problem sheet 3 · Viscous transport and similarity

Derive resistance or a scaling law first, then test that its flow-regime assumptions still apply.

1. Split a laminar flow

Two long circular pipes of the same length connect common manifolds. Their radii are 2 mm and 3 mm and the fluid is the same. (a) Find the flow ratio. (b) Split a total of 6.0 mL/s. (c) If both radii are reduced by 10%, how does the required pressure drop change at fixed total flow?

▼ **Apply the common-pressure condition**

At equal pressure drop the radius-fourth law gives

$$Q_2/Q_1 = (3/2)^4 = 5.0625.$$

Thus $Q_1 = 6/(1 + 5.0625) = 0.990$ mL/s and $Q_2 = 5.01$ mL/s. Reducing both radii scales each conductance by 0.9^4, so the common pressure drop must increase by $1/0.9^4 = 1.52$. This assumes both flows remain developed and laminar and manifold losses remain negligible.

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2. Compare pressure and wall driving

A 2 mm film of viscosity 0.10 Pa·s is bounded by a fixed lower plate and an upper plate moving at 0.40 m/s. A downstream pressure fall produces $G = 3000$ Pa/m. (a) Find the two contributions to Q/b. (b) Find the midpoint speed. (c) What pressure gradient would produce zero net flux?

▼ **Superpose the profiles**

The moving-wall and pressure terms are $UH/2 = 4.0 \times 10^{-4}$ and $GH^3/(12\mu) = 2.0 \times 10^{-5}$ m²/s, totalling 0.00042 m²/s. At the middle, $u = U/2 + GH^2/(8\mu) = 0.215$ m/s. To cancel wall transport one needs $G = -6\mu U/H^2 = -60\,000$ Pa/m. The negative sign represents a pressure rise in the direction of plate motion.

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3. Design a reduced-scale tow test

A 2 m submerged object is towed at 1.5 m/s in water with $\nu = 10^{-6}$ m²/s. A geometrically similar 0.50 m model is available. (a) Find the required speed for equal Reynolds number in the same water. (b) For equal drag coefficient, compare model and prototype drag. (c) If surface waves dominate instead, find the speed for equal Froude number and the resulting Reynolds-number ratio.

▼ **Match a dimensionless balance**

Prototype Reynolds number is 3.0×10^6. At quarter length the required model speed is 6.0 m/s. Same density and drag coefficient imply $F_m/F_p = (6/1.5)^2(0.5/2)^2 = 1$. Froude matching would require

$1.5\sqrt{0.25} = 0.75$ m/s, giving $Re_m/Re_p = (0.75/1.5)(0.5/2) = 1/8$. The choice of matching rule changes the interpretation of the measured drag.

Which pressure difference establishes an accelerating uniform stream?

In the uniform core of a horizontal water test channel let $u = 0.60t$ in SI units for the first two seconds, with negligible transverse velocity and negligible viscous stresses in the core. Two pressure taps sampling the core at the same elevation are 1.50 m apart. Find the material acceleration and the pressure difference, then compare this result at $t = 2$ s with a steady core moving at the same instantaneous speed. Use $\rho = 998\,\mathrm{kg\,m^{-3}}$.

▼ **Show the worked solution**

Spatial gradients vanish, so the advective term is zero. During the ramp $Du/Dt = 0.60\,\mathrm{m\,s^{-2}}$, hence

$$\partial_x p = -998(0.60) = -598.8\,\mathrm{Pa\,m^{-1}}.$$

Integrating downstream gives $p_{\mathrm{up}} - p_{\mathrm{down}} = 898.2\,\mathrm{Pa}$. Both cases have $u = 1.20\,\mathrm{m\,s^{-1}}$ at the instant of comparison. In the steady uniform ideal core the local derivative also vanishes, so the tap difference is zero. A calculation using speed alone could not distinguish them. Actual wall friction, if significant, would demand an additional pressure gradient.

Revisit chapter 5 · How does a local pressure gradient accelerate a fluid particle?.

Explore the governing equations after solving the sheet: Viscous stresses, duct resistance and wall-driven flow; Dynamic similarity, boundary layers and model tests. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Chapter 7 · Vorticity, circulation and swirling flow

Distinguish rotation of a fluid element from an orbiting streamline and use radial momentum balance to predict a free-surface shape.

FROM FIRST PRINCIPLES

Derive vorticity from circulation around a small rectangle

In a smooth planar velocity field write velocity as (u, v) and consider a fixed small rectangle of sides dx, dy. Traverse it counterclockwise and add the component of velocity along each side times the side length. Uniform velocity cancels between opposite sides. A difference in vertical velocity across dx or horizontal velocity across dy can leave a net circulation.

$$d\Gamma = [v(x+dx, y) - v(x, y)]dy - [u(x, y+dy) - u(x, y)]dx.$$

Taylor expansion and division by the enclosed area give the local circulation density, or axial vorticity. Adding neighbouring rectangles cancels the internal boundaries and yields the finite-area formula.

$$\omega_z = \lim_{dA \to 0} \frac{d\Gamma}{dA} = \partial_x v - \partial_y u,$$

$$\Gamma = \oint_C \mathbf{u} \cdot d\boldsymbol{\ell} = \int_A \omega_z \, dA.$$

For solid-body rotation $u = -\Omega y$, $v = \Omega x$, so $\omega_z = 2\Omega$. A fluid element's local angular velocity is half its vorticity. This local rate can vanish even on a curved streamline because the motion of a parcel centre and the deformation around it are different observations.

Background · Vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

Its units are reciprocal seconds. Circulation around a closed loop integrates the corresponding vorticity flux when the interior is regular.

DERIVATION AND APPLICATIONS

Reconstruct the core of a vortex from circulation

A vortex in a stirred tank often rotates nearly as a solid body close to its axis and approximately as a potential vortex farther out. Model a steady planar axisymmetric flow with tangential speed v_θ, no radial velocity, and a circular core of radius a. Choose uniform axial vorticity $\omega_0 > 0$ in the core and zero

vorticity outside. For a circle of radius r, Stokes' theorem relates the tangential speed to the vorticity integrated over the enclosed disk.

$$2\pi r v_\theta(r) = \int_0^r \omega_z(s) 2\pi s \, ds = \pi \omega_0 \min(r^2, a^2).$$

Inside the core the circulation grows with area. Outside it, the entire core is already enclosed, so the same circulation is spread over a larger circumference. The speed is continuous and reaches its maximum at the core radius.

$$v_\theta(r) = \begin{cases} \omega_0 r/2, & r \le a, \\ \omega_0 a^2/(2r), & r \ge a. \end{cases}$$

For a horizontal flow of constant density ρ, radial momentum balance gives

$$dp/dr = \rho v_\theta^2 / r.$$

Integrate first from the centre to the core radius and then from the core to a distant radius where the swirl has decayed. The two regions make equal contributions to the centre-to-infinity pressure deficit.

$$p(\infty) - p(0) = \rho \left[\int_0^a \frac{\omega_0^2 r}{4} dr + \int_a^\infty \frac{\omega_0^2 a^4}{4 r^3} dr \right] = \frac{\rho \omega_0^2 a^2}{4}.$$

The idealised sharp change of vorticity at a replaces a smooth viscous transition. A finite vessel truncates the exterior integral, so the pressure reference radius must be specified when it lies close to the core.

WORKED EXAMPLE

Predict a core pressure deficit

A water vortex has $a = 30.0$ mm and $\omega_0 = 40.0$ s⁻¹. The peak tangential speed is 0.600 m/s and its exterior circulation is $\pi(40.0)(0.0300)^2 = 0.113$ m²/s. The predicted pressure difference from the centre to distant liquid at the same height is

$$1000(0.600)^2 = 360\,\text{Pa}.$$

Half of this rise occurs within the core.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Infer the core radius from two speed measurements

A second vortex has speed 0.200 m/s at $r = 10.0$ mm inside the core and 0.250 m/s at $r = 50.0$ mm outside. The inner reading gives $\omega_0 = 2(0.200)/0.0100 = 40.0$ s^{-1}. The outer reading gives $a = \sqrt{2rv_\theta/\omega_0} = 25.0$ mm, between the measuring radii as assumed. Its maximum speed should therefore be 0.500 m/s.

Interactive calculation · Change the inputs of this example

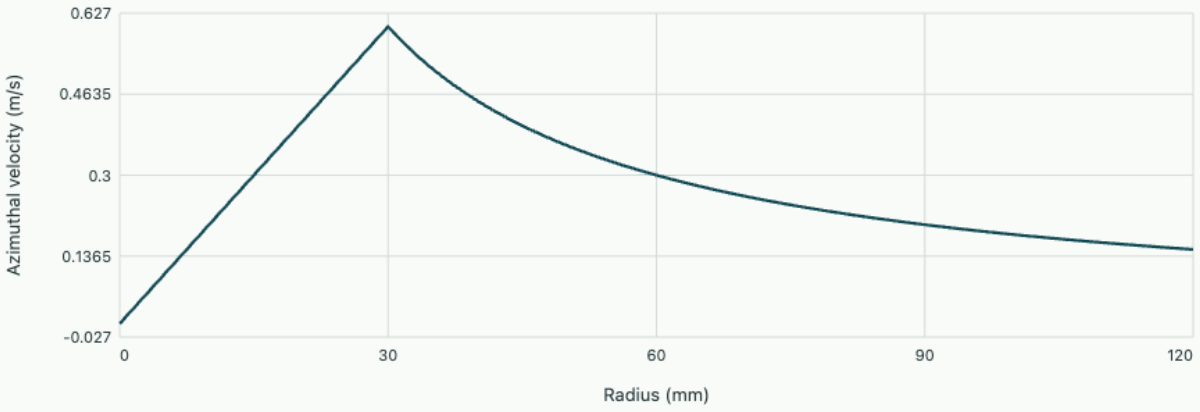
Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Core radius (mm) **30**

Core vorticity (1/s) **40**



Uniform-vorticity core and irrotational exterior

Peak tangential speed 0.600 m/s at the core edge. For water, far-field-to-axis pressure difference=360.000 Pa; no singular velocity is introduced at the axis.

Distinguish a particle orbit from local spin

Drop a small paddle wheel into a swirling tank. Its centre can travel around the tank axis whether or not the wheel itself spins relative to distant reference axes. To measure local rotation, consider two short perpendicular material lines through the same fluid element. In a planar velocity field (u, v), an initially

horizontal line rotates at $\partial v/\partial x$, and an initially vertical line rotates at $-\partial u/\partial y$, both with counterclockwise positive. Their mean is the local angular velocity, one half the out-of-plane vorticity.

$$\omega_z = \partial v/\partial x - \partial u/\partial y,$$

$$\Gamma = \oint_C \mathbf{v}\cdot d\mathbf{l} = \iint_A \omega_z \, dA.$$

The second relation is Stokes' theorem. The curve encloses A with consistent orientation and the velocity must be regular throughout that surface. Circulation is measured in square metres per second. In a region with zero vorticity a closed curve can still enclose a singular or otherwise excluded core; then the area needed for the usual zero-circulation argument does not lie entirely in that region.

Compare two angular velocity distributions

In axisymmetric swirl with only tangential velocity $v_\theta(r)$, circulation on a circular path is $2\pi r v_\theta$. Dividing its change between neighbouring rings by the annular area gives

$$\omega_z = (1/r)\, d(r v_\theta)/dr.$$

A forced vortex rotates like a rigid body, $v_\theta = \Omega r$, with vorticity 2Ω. At radius 0.10 m and angular speed 8 rad/s, its velocity is 0.80 m/s and the circular-path circulation is 0.503 m²/s.

In a free-vortex exterior $v_\theta = K/r$; the vorticity is zero at every regular point there although the streamlines are circular. If its velocity at the same radius is also 0.80 m/s, then $K = 0.080\,\mathrm{m^2/s}$. At radius 0.20 m its speed falls to 0.40 m/s, while in the forced vortex the speed rises to 1.60 m/s. Both flows have the same circulation at the original radius. A second measuring circle distinguishes them without an instrument that directly senses local shear.

Recover pressure and free-surface rise

For steady axisymmetric circular streamlines without radial or vertical motion the radial acceleration points inward with magnitude v_θ^2/r. Radial momentum balance requires

$$dp/dr = \rho v_\theta^2/r.$$

Vertical equilibrium gives

$$\partial p/\partial z = -\rho g.$$

Along a free surface at uniform ambient pressure their contributions cancel, so $dz/dr = v_\theta^2/(gr)$. For rigid rotation, integration gives

$$z(r) - z(0) = \Omega^2 r^2/(2g).$$

A cylindrical open tank of radius 0.20 m rotating at 8 rad/s has a predicted wall-to-centre rise of $64(0.20)^2/[2(9.81)] = 0.1305\,\mathrm{m}$. If no water spills, volume conservation places its former flat level halfway between the new centre and rim elevations for this paraboloid. Each shifts by about 65 mm in opposite directions. A nearly full tank will spill instead, and the average-level constraint then changes. A free vortex produces a different surface profile and its singular axis must be replaced by a physical core.

Construct irrotational flow around a cylinder

Outside a vortex core or boundary layer, a two-dimensional incompressible flow can often be approximated as irrotational. In a simply connected region irrotationality permits a velocity potential with $\mathbf{v} = \nabla\phi$, and continuity then requires

$$\nabla^2\phi = 0.$$

For a circular cylinder of radius a in a uniform stream U along the positive x direction, try $\phi = (Ar + B/r)\cos\theta$. The two radial factors solve Laplace's equation for the angular first harmonic. Matching the distant uniform potential $Ur\cos\theta$ gives $A = U$. No penetration requires radial velocity $\partial_r\phi = 0$ at $r = a$, giving $B = Ua^2$.

$$v_r = U(1 - a^2/r^2)\cos\theta,$$

$$v_\theta = -U(1 + a^2/r^2)\sin\theta.$$

The boundary has zero normal speed but a nonzero tangential speed. This is consistent with an inviscid slip condition. It does not satisfy the no-slip condition for an actual viscous wall, so predicted surface pressure is meaningful only insofar as outer flow remains attached and nearly irrotational. In a real cylinder wake, separation breaks the fore–aft symmetry of this ideal solution.

Applying steady Bernoulli relative to the uniform upstream stream gives the surface pressure coefficient $C_p = 1 - 4\sin^2\theta$. Its values change considerably around the cylinder even though integrating the pressure forces yields zero net drag and lift by symmetry. Zero potential-flow drag does not imply uniform pressure.

Add circulation and predict unequal side pressures

A mathematical cylinder excludes its interior from the fluid. Superpose a free-vortex velocity $\Gamma/(2\pi r)$ in the positive counterclockwise direction; this preserves the no-penetration condition while changing circulation around the body. Its sign adds to the negative surface velocity on one side and opposes it on the other. For a 50 mm radius cylinder in air at $U = 10\,\mathrm{m/s}$, choose $\Gamma = 0.50\,\mathrm{m^2/s}$ counterclockwise. The added surface speed is

$$0.50/[2\pi(0.050)] = 1.59\,\mathrm{m/s}.$$

At the top the signed tangential speed is –18.41 m/s, and at the bottom it is +21.59 m/s.

For air density 1.20 kg/m³ the dynamic pressure is 60 Pa. The top and bottom coefficients $1 - (v_\theta/U)^2$ are −2.39 and −3.66, respectively. The lower pressure is now at the bottom, producing downward lift. Integrate $-p\, n_y a\, d\theta$ around the circumference: the terms even between top and bottom cancel, leaving signed lift per span $L' = -\rho U \Gamma = -6.0\,\mathrm{N/m}$ for these axes. A rigid-body rotation of the entire fluid would give a different vorticity field; the superposed free vortex has zero local vorticity outside the excluded body. Physical circulation in an experiment depends on how the flow and surface boundary layers are established, and cannot be selected by the potential solution alone.

The stagnation points satisfy $\sin\theta = \Gamma/(4\pi a U)$, or 0.0796 here. They shift by about 4.6° from their zero-circulation positions without changing the fact that pressure is highest where speed vanishes. This offers a second observable for estimating circulation, independent of integrating the measured lift, provided the observed surface points reasonably follow the attached outer-flow model.

Chapter 8 · Boundary layers and momentum thickness

Connect a measured velocity deficit to wall drag and distinguish the physical edge of a boundary layer from its integral thicknesses.

FROM FIRST PRINCIPLES

Obtain wall drag from the streamwise momentum deficit

A thin boundary layer slows fluid next to a solid wall. Consider steady two-dimensional flow over a smooth flat plate with constant external speed U and pressure. Use unit span and constant density ρ. Choose a control volume ending at location x, with its upper surface a streamline outside the layer. Mass conservation relates the uniform inlet flow to the downstream velocity profile. Multiplying the discharged mass in a strip by the streamwise speed it has lost gives that strip's contribution to the momentum deficit.

$$dD = \rho\, u(y)[U - u(y)]\, dy,$$

$$D(x) = \rho U^2 \theta(x),$$

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy.$$

The integrated deficit equals the drag per unit span exerted by the wall on the flow in magnitude. Extending the plate by a small dx adds wall drag $\tau_w\, dx$, so differentiating yields

$$\tau_w = \rho U^2 \frac{d\theta}{dx},$$

$$C_f = \frac{\tau_w}{\rho U^2/2} = 2\frac{d\theta}{dx}.$$

These relations assume negligible pressure gradient. The thickness θ weights lost velocity by the mass flux actually present in that part of the profile.

DERIVATION AND APPLICATIONS

Relate displacement thickness to boundary-layer entrainment

Mass conservation requires a weak normal flow at the edge of a developing boundary layer. Consider steady, two-dimensional incompressible flow over an impermeable flat plate with constant external speed U. Let $u(x, y)$ be tangential velocity, $v(x, y)$ normal velocity and $y = 0$ the wall. Integrate continuity from the wall to a fixed measurement height H chosen well outside the layer throughout the region of interest. Since $v(x, 0) = 0$,

$$\partial_x u + \partial_y v = 0,$$

$$v(x, H) = -\frac{d}{dx}\int_0^H u(x, y)\, dy.$$

The integral would equal UH for a uniform velocity profile. The missing volume flux per unit span defines displacement thickness δ^* times U. For constant H, U, differentiating cancels the large uniform part and connects the missing flux directly to the weak external normal velocity.

$$U\delta^* = \int_0^H (U - u)\, dy,$$

$$v(x, H) = U\frac{d\delta^*}{dx}.$$

For an illustrative profile take $u/U = 2\eta - \eta^2$ up to $\eta = y/\delta = 1$ and $u = U$ above it. Its deficit is $(1 - \eta)^2$ and its integral is one third. When δ grows as the square root of distance from the leading edge, the normal velocity follows by differentiation.

$$\delta^* = \frac{\delta}{3},$$

$$\delta = C\sqrt{x} \implies v(x, H) = \frac{U\delta}{6x}.$$

The sign is positive because the low-speed layer occupies an increasing share of the fixed-height section. The value at fixed external height differs from the mass entering a control surface that itself follows the growing edge.

WORKED EXAMPLE

Estimate normal flow above a plate

At $x = 0.500$ m, a measured polynomial layer has $\delta = 6.00$ mm and $U = 4.00$ m/s. If it grows locally as $\sqrt{x}$, the predicted edge velocity is

$$4.00(0.00600)/(6 \times 0.500) = 0.00800\,\mathrm{m/s}.$$

Its ratio to the external tangential speed is only 0.00200. The displacement thickness is 2.00 mm, the height by which a uniform-speed stream would need to be displaced to produce the same missing flux.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Use two profiles to infer the average vertical flux

Independent surveys at $x = 0.200$ and 0.600 m find displacement thicknesses 1.20 and 2.00 mm. At $U = 3.00$ m/s the interval-average normal velocity is

$$3.00(0.000800)/0.400 = 0.00600\,\mathrm{m/s}.$$

Over a plate span of 0.500 m this corresponds to outward volume flow $0.00600(0.400)(0.500) = 1.20$ L/s through the fixed upper face, equal to the growth of the streamwise flux deficit.

Interactive calculation · Change the inputs of this example

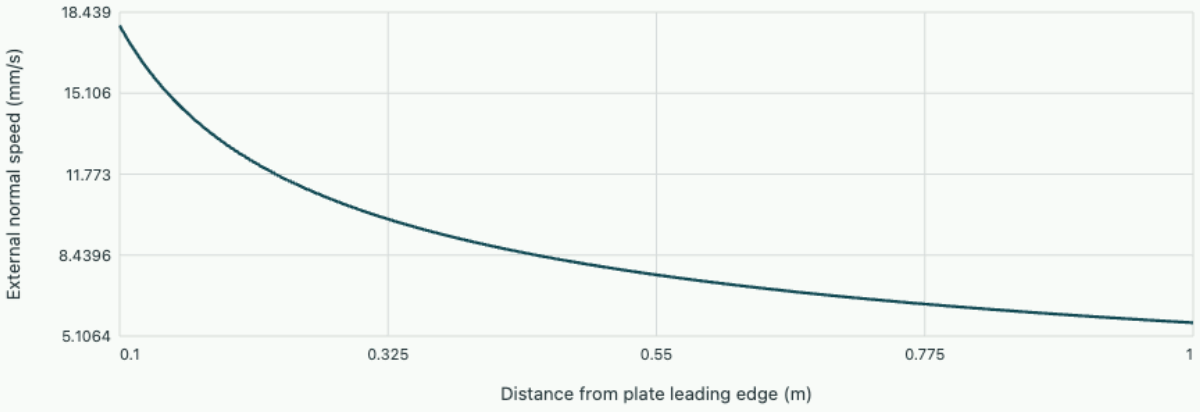
Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

External speed (m/s) 4

Layer thickness at0.5 m (mm) 6



Quadratic-profile displacement effect

At0.5 m, upward speed 8.000 mm/s. The displacement thickness is one third of the assumed boundary-layer thickness.

Start from this chapter. Balance streamwise advection against diffusion normal to the plate. Predict the thickness scaling, then measure both the 99% thickness and wall-friction coefficient at one specified station.

OneSimulation laboratory · Measure boundary-layer growth from a velocity profile

Viscosity matters where velocity changes rapidly

Flow past a smooth plate with a large length-based Reynolds number often has nearly uniform speed outside a thin layer next to the wall. No slip requires zero velocity at the stationary surface. Inside the thin region, transverse velocity gradients can be large enough for viscosity to balance streamwise inertia. If the free-stream speed is U, the streamwise distance is x and the layer thickness is δ, the leading inertial and viscous scales are U^2/x and $\nu U/\delta^2$. Their balance gives $\delta/x \sim 1/\sqrt{\mathrm{Re}_x}$, where $\mathrm{Re}_x = Ux/\nu$.

This scaling predicts how a laminar layer grows. Its numerical factor depends on how the thickness is defined. A common measured edge is the position reaching 99% of the external speed. Two integral measures instead quantify flow displaced by the layer and its lost momentum, and are less sensitive to picking a particular edge point.

$$\delta^* = \int_0^\infty (1 - u/U)\, dy,$$

$$\theta_m = \int_0^\infty (u/U)(1 - u/U)\, dy.$$

The thickness δ^* gives the equivalent missing volume flux per unit width divided by U. Momentum thickness θ_m weights that deficit by the actual local speed. Fluid right at the wall contributes strongly to displacement thickness but carries almost no streamwise momentum flux.

Relate the momentum deficit to wall shear

A steady control volume surrounding one face of a flat plate at zero incidence receives fluid from the uniform upstream stream. For constant external speed and pressure, the integrated wall force equals the deficit of streamwise momentum flux downstream after accounting for entrained fluid. Per unit plate width this gives

$$D'(x) = \rho U^2 \theta_m(x).$$

Differentiating gives local wall shear $\tau_w = \rho U^2\, d\theta_m/dx$. A growing deficit implies positive drag on the plate.

Suppose a velocity survey is represented by $u/U = 2\eta - \eta^2$ for $0 \le \eta = y/\delta \le 1$, and uniform flow outside. Integrating yields

$$\delta^*/\delta = \int_0^1 (1-\eta)^2 d\eta = 1/3.$$

Since $\int_0^1 (2\eta - \eta^2) d\eta = 2/3$ and $\int_0^1 (2\eta - \eta^2)^2 d\eta = 8/15$, the momentum thickness is $2\delta/15$.

Infer drag from a measured layer

At the end of a 0.60 m plate of width 0.40 m in air, let $U = 12\,\mathrm{m/s}$, $\rho = 1.20\,\mathrm{kg/m^3}$ and $\delta = 5.0\,\mathrm{mm}$ according to the fitted profile. Then $\theta_m = 0.667\,\mathrm{mm}$ and drag on the measured face is

$$1.2(12)^2(0.000667)(0.40) = 0.0461\,\mathrm{N}.$$

The mean shear over that face is

$$0.0461/(0.60 \times 0.40) = 0.192\,\mathrm{Pa}.$$

It need not equal the local shear at the measuring station. An adverse pressure gradient or separated flow changes the momentum balance; surface pressure forces and external acceleration must then be included before interpreting a wake or boundary-layer survey.

Predict a boundary-layer thickness before viewing a simulation

At the same measuring station the laminar scaling from the first section gives $\delta \propto \sqrt{\nu/U}$. Starting from a measured 35.84 mm layer at $x = 0.50$ m, predict its thickness after separately doubling free-stream speed and doubling kinematic viscosity. Which change should raise the dimensional wall stress if density and viscosity are held fixed while speed increases? Compare your thickness ratios with the two computed runs and distinguish the measured skin-friction coefficient from wall stress in pascals. Compare the flat-plate runs.

Reduce a partial differential equation to one similarity profile

For a laminar zero-pressure-gradient boundary layer along a flat plate, the leading equations are $u_x + v_y = 0$ and $uu_x + vu_y = \nu u_{yy}$. Assume external speed U constant, a stationary no-slip surface and a layer thin relative to streamwise distance. Introduce a streamfunction ψ through $u = \psi_y,\ v = -\psi_x$ to satisfy continuity identically. The scaling $\delta \sim \sqrt{\nu x/U}$ suggests $\eta = y\sqrt{U/(\nu x)}$ and $\psi = \sqrt{\nu U x}\, f(\eta)$. Differentiation gives $u = U f'$ and $v = \tfrac{1}{2}\sqrt{\nu U/x}(\eta f' - f)$.

When these forms enter the momentum equation, the terms containing $\eta f' f''$ from longitudinal and transverse advection cancel. Both the remaining inertial and viscous terms carry U^2/x. Dividing through gives the Blasius boundary-value problem

$$f''' + \tfrac{1}{2} f f'' = 0,$$
$$f(0) = f'(0) = 0,$$
$$f'(\infty) = 1.$$

The two wall conditions impose zero normal and tangential velocity. The outer condition matches the uniform stream. The unknown wall slope of the velocity profile, $f''(0)$, is determined by matching that distant condition; prescribing it independently would overdetermine the third-order equation. Numerical integration gives approximately 0.3321 for this definition of η. Conventions that include a factor of two in the similarity coordinate assign a different numerical constant but the same physical wall stress.

Predict a force measurement from the similarity solution

The wall shear is $\mu U f''(0) \sqrt{U/(\nu x)}$, giving $c_f(x) = \tau_w / (\rho U^2/2) = 0.664/\sqrt{\mathrm{Re}_x}$. Its integral over a plate of length L yields mean coefficient $\bar{c}_f = 1.328/\sqrt{\mathrm{Re}_L}$. The mean equals twice the local trailing-edge value because the integrable $x^{-1/2}$ shear is highest near the leading edge. Using a single trailing-edge wall gauge for the whole area would therefore underestimate the force.

Take a smooth plate 0.50 m long and 0.25 m wide in air at $U = 6.0\,\mathrm{m/s}$, $\nu = 1.5 \times 10^{-5}\,\mathrm{m^2/s}$, $\rho = 1.20\,\mathrm{kg/m^3}$. The trailing-edge Reynolds number is 200,000, and a nominally laminar calculation gives

$$c_f(L) = 0.00149, \; \bar{c}_f = 0.00297.$$

The one-face drag is

$$0.5(1.20)(6)^2(0.50)(0.25)(0.00297) = 0.00802\,\mathrm{N}.$$

The trailing-edge wall shear is just 0.0321 Pa, and the 99% thickness estimate $5L/\sqrt{\mathrm{Re}_L}$ is 5.59 mm. A support carrying both plate faces would register roughly twice the one-face drag if both flows are comparable, plus fixture and edge drag. A single net force measurement cannot by itself tell whether a discrepancy arose from transition, unequal face conditions or the support geometry.

The similarity solution also predicts how to collapse profiles measured at different stations. At $x = 0.125$ m the same value of η corresponds to half the wall-normal distance it does at 0.50 m, since the distance scales with $\sqrt{x}$. Meanwhile wall shear doubles and local skin-friction coefficient doubles. A traverse resolving only whole millimetres may describe the outer velocity defect at both positions but badly estimate the near-wall slope at the earlier one. Testing both the profile collapse and the independent integral drag provides a stronger check of the assumed laminar flow than comparing a single velocity at the outer edge.

Measure the momentum transported by turbulent fluctuations

In a smooth plane channel driven by a constant pressure drop, time averaging can hide the dominant transport mechanism. Consider steady statistics and a fully developed mean flow between parallel walls. Let $y = 0$ denote the lower wall and h the distance to the centreline. Write $u = U(y) + u'$, $v = v'$. Impermeability and incompressibility give zero mean transverse velocity. The average of the instantaneous flux of streamwise momentum across a horizontal surface is nevertheless $\rho\overline{uv} = \rho\overline{u'v'}$. Averaging the product retains the correlation between the two fluctuations.

For constant density and viscosity the mean streamwise balance becomes

$$0 = -d\bar{p}/dx + d(\mu\, dU/dy - \rho\overline{u'v'})/dy.$$

Set $G = -d\bar{p}/dx > 0$. Integrating from y to the centreline, where symmetry sets the total shear to zero, gives

$$\tau_{\mathrm{total}}(y) = \mu\,\frac{dU}{dy} - \rho\overline{u'v'} = G(h - y),$$

$$\tau_w = Gh.$$

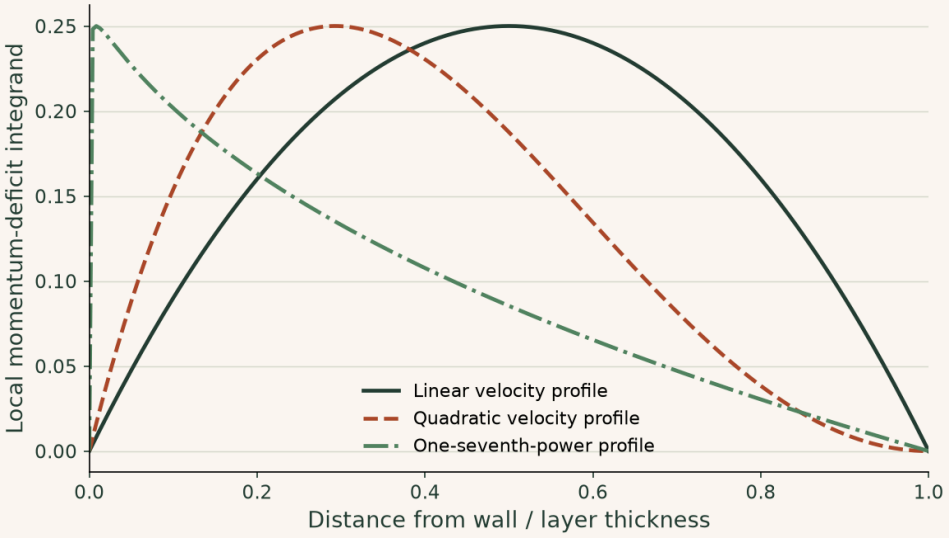
The right side is fixed by the driving pressure gradient. At a no-slip wall both fluctuating velocities vanish, so molecular viscosity carries the full shear there. Farther out a parcel moving rapidly downstream can move towards the lower wall ($u' > 0,\ v' < 0$). A slow parcel leaving that wall has the same negative product. Both transfer streamwise momentum towards the wall. With the signs above the turbulent contribution to the shear is positive.

For water with $\rho = 1000$ kg/m³ and $\mu = 0.0010$ Pa s, take $h = 0.020$ m and $G = 100$ Pa/m. The wall stress is 2.00 Pa. At $y = 0.0050$ m the total shear is 1.50 Pa. A measured slope of 150 s⁻¹ accounts for 0.150 Pa. The remaining 1.35 Pa implies $\overline{u'v'} = -1.35 \times 10^{-3}$ m²/s². The exact force balance gives a quantitative consistency check for velocity measurements or a simulation. Predicting the split from G alone would require a relation for the velocity correlations.

Practise on sheet 4 · Test a turbulent channel measurement. Further reading · University of Iowa · ME:7268 Turbulent Flow · Chapter 8, part 1 · Channel flow, Reynolds stress and force balance.

Estimate a boundary layer’s momentum deficit

Momentum thickness is $\theta = \int (u/U)(1 - u/U)\, dy$. Use three idealised trial profiles at the same outer thickness. Integrating the ordinate with respect to the plotted dimensionless height gives θ/δ.



The three areas are 1/6, 2/15 and 7/72. All profiles agree at both endpoints, yet they predict different momentum losses. The power-law profile is unsuitable for inferring wall shear from its limiting wall derivative.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

Change the exponent of the trial velocity profile. Equal endpoint velocities need not imply equal momentum deficit.

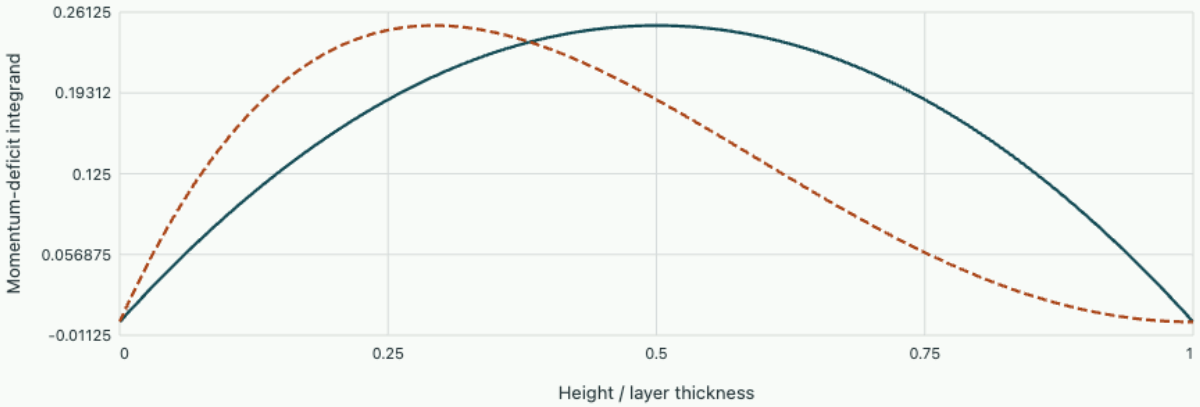
The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Power-law exponent **1**

Layer thickness (mm) **5**



Selected power profile Quadratic reference

Selected momentum thickness is 0.833 mm. The quadratic reference gives 0.667 mm. For exponent below one the wall derivative diverges.

Study the full derivation

Estimate wall drag by integrating a boundary-layer profile · Momentum, motion and transport in fluids

A velocity probe can resolve the momentum missing from a boundary layer more easily than the wall shear itself. Derive an integral momentum relation, estimate drag using a polynomial profile and determine when a pressure gradient or a measurement too far from the wall changes the answer.

Problem sheet 4 · Swirling flows and wall drag

Use a velocity distribution to infer local vorticity, pressure differences and the momentum lost to wall drag.

A rigidly spinning tank

Water in an open cylindrical tank of radius 0.15 m rotates as a solid body at 6 rad/s. Determine the circulation around the rim, local vorticity and wall-to-centre free-surface rise. Compare the vorticity with that of an ideal free-vortex exterior carrying the same circulation.

▼ **Reveal the worked solution**

The rim speed is $6(0.15) = 0.90\,\mathrm{m/s}$ and circulation is

$$2\pi(0.15)(0.90) = 0.848\,\mathrm{m^2/s}.$$

The vorticity is $2\Omega = 12\,\mathrm{s^{-1}}$. Integrating the radial and vertical pressure balance gives

$$\Delta z = 6^2(0.15)^2/(2g) = 0.0413\,\mathrm{m}.$$

A free vortex of the same circulation has zero local vorticity outside its core. The circulation measures what the loop encloses.

A linear layer profile

An idealised velocity survey gives $u/U = y/\delta$ to the layer edge and unity above. Derive displacement and momentum thicknesses. For $U = 8\,\mathrm{m/s}$, water density 1000 kg/m³, $\delta = 2\,\mathrm{mm}$ and plate width 0.15 m, estimate drag on the surveyed face to this station.

▼ **Reveal the worked solution**

Integrating $1 - \eta$ gives

$$\delta^* = \delta/2 = 1.0\,\mathrm{mm}.$$

Integrating $\eta(1-\eta)$ gives

$$\theta_m = \delta/6 = 0.333\,\mathrm{mm}.$$

Under constant external pressure and speed,

$$D = \rho U^2 \theta_m b = 3.20\,\mathrm{N}.$$

Only the one face to which the survey applies has been counted.

Infer a local shear stress

A measured zero-pressure-gradient momentum thickness follows $\theta_m = 0.0010\sqrt{x/(1\,\mathrm{m})}\,\mathrm{m}$. Air density is 1.2 kg/m³ and external speed 15 m/s. At $x = 0.25\,\mathrm{m}$, find local wall shear and integrated drag per unit width. Why do these two quantities have different units?

▼ **Reveal the worked solution**

At the station $\theta_m = 0.00050\,\mathrm{m}$ and $d\theta_m/dx = 0.0010$. Local shear is

$$\rho U^2 d\theta_m/dx = 0.270\,\mathrm{Pa}.$$

Drag per width is

$$\rho U^2 \theta_m = 0.135\,\mathrm{N/m}.$$

Local shear is force per area; integrating along plate length removes one inverse length.

Test a turbulent channel measurement

A water channel has half-height 15 mm and pressure gradient -80 Pa/m. At a probe 5 mm from the lower wall, measurements give $dU/dy = 100$ s⁻¹ and $\overline{u'v'} = -5.0 \times 10^{-4}$ m²/s². Use $\rho = 1000$ kg/m³ and $\mu = 0.0010$ Pa s. How well do the measurements close the momentum balance? What correlation would close it at the stated position?

▼ **Worked solution**

The expected total stress is

$$80(0.015 - 0.005) = 0.80\,\mathrm{Pa}.$$

The gradient gives 0.10 Pa and the measured covariance gives another 0.50 Pa. Together these account for 0.60 Pa. The shortfall is 0.20 Pa, or 25% of the pressure-based stress at the probe. The wall stress of 1.20 Pa is a different comparison value.

Holding the gradient fixed gives $\overline{u'v'} = -(0.80 - 0.10)/1000 = -7.0 \times 10^{-4}$ m²/s². Before changing a turbulence model, check the probe height, the averaging interval and whether the flow is fully developed. Changing the sign convention for just one term cannot resolve the shortfall.

Revisit chapter 8 · Measure the momentum transported by turbulent fluctuations.

Explore the governing equations after solving the sheet: Vorticity, circulation and swirling flow; Boundary layers and momentum thickness. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Infer viscosity from the edge of a boundary layer

Using the baseline 99% thickness 35.84 mm at $x = 0.50$ m and speed 1 m/s, calibrate the coefficient of $\delta = C\sqrt{\nu x/U}$ at $\nu = 10^{-4}$ m²/s. At the same station and speed a different run gives 51.03 mm. Infer its kinematic viscosity without measuring a wall gradient. Compare it with the input and identify one reason for a discrepancy.

▼ Work through the solution

The baseline gives

$$C = 0.03584/\sqrt{10^{-4}(0.50)} = 5.07.$$

At common x, U, the coefficient cancels in the ratio, so $\nu_2 = 10^{-4}(51.03/35.84)^2 = 2.03 \times 10^{-4}$ m²/s. The laboratory input was 2.00×10^{-4} m²/s. The similarity argument describes an ideal laminar layer. Estimating its edge from a discretised velocity profile also depends on interpolation at 99% of the free-stream velocity.

Use the OneSimulation laboratory to check your answer

Vary the corresponding laboratory reference interactively. Compare the recalculated observable with the recorded OneSimulation runs and the approximation used in this exercise.

Chapter 9 · Pressure integration, lift and drag

Resolve pressure and shear into integrated forces and infer drag independently from a far-wake velocity survey.

FROM FIRST PRINCIPLES

Resolve measured surface tractions into lift and drag

Choose a closed surface on a body and a unit normal $\mathbf{n}$ pointing from solid into fluid. Pressure acts inward on the body. Viscous stress acts along or normal to the surface depending on the flow. For fluid stress tensor $-pI + \boldsymbol{\tau}$, the total fluid force on the body is the sum of all local tractions times surface area.

$$\mathbf{F} = \int_A (-p\mathbf{n} + \boldsymbol{\tau}\mathbf{n})\, dA,$$
$$D = \mathbf{F} \cdot \mathbf{e}_D,$$
$$L = \mathbf{F} \cdot \mathbf{e}_L.$$

Here $\mathbf{e}_D$ points along the incident stream and $\mathbf{e}_L$ is the chosen transverse lift direction. A constant pressure reference can be subtracted because the normal integrates to zero over a closed surface. For a thin wing strip at small slope, upper and lower pressure contributions to lift reduce to their difference integrated over projected area.

$$C_L = \frac{L}{q_\infty S} \simeq \frac{1}{c}\int_0^c (C_{p,l} - C_{p,u})\, dx,$$
$$C_p = \frac{p - p_\infty}{q_\infty},$$
$$q_\infty = \tfrac{1}{2}\rho U_\infty^2.$$

The last expression assumes uniform spanwise loading and negligible viscous contribution to lift. Estimating drag can require the small axial components of pressure and the wall shear that this lift approximation omits.

DERIVATION AND APPLICATIONS

Determine the aerodynamic centre of pressure from surface data

A net lift measurement does not say where the load acts. Consider a thin two-dimensional panel of chord c and span b in steady flow of dynamic pressure q. Let $\xi = x/c$ increase from the leading edge and $\Delta C_p(\xi)$ be lower minus upper pressure coefficient. Assume small surface slopes and neglect shear and end effects so that each strip exerts a vertical force. Define moment as positive for an upward load acting aft of the leading edge. The integrated force and moment are

$$L = qbc \int_0^1 \Delta C_p \, d\xi,$$

$$M_{\mathrm{LE}} = qbc^2 \int_0^1 \xi \Delta C_p \, d\xi.$$

The centre of pressure x_{cp} replaces the distributed forces by one force with the same moment. When the total force is nonzero, dividing the two integrals eliminates dynamic pressure and span.

$$\frac{x_{\mathrm{cp}}}{c} = \frac{M_{\mathrm{LE}}}{cL} = \frac{\int_0^1 \xi \Delta C_p \, d\xi}{\int_0^1 \Delta C_p \, d\xi}.$$

For a linear distribution $\Delta C_p = A + B\xi$, perform both elementary integrations. A constant increment to the pressure distribution acts at mid-chord and can move the centre even as the original pressure gradient stays unchanged.

$$C_L = A + B/2,$$

$$\frac{M_{\mathrm{LE}}}{qbc^2} = \frac{A}{2} + \frac{B}{3},$$

$$\frac{x_{\mathrm{cp}}}{c} = \frac{A/2 + B/3}{A + B/2}.$$

Near zero net lift the position ratio can become very large. In that situation a force and a moment about a specified structural axis remain meaningful even when replacing them by one force is inconvenient.

WORKED EXAMPLE

Place the resultant from a triangular pressure trace

Let $A = 1.20$, $B = -1.20$, $c = 0.400$ m, $b = 1.00$ m and $q = 200$ Pa. Lift is $200(1)(0.400)(0.600) = 48.0$ N. Its moment is $200(1)(0.400)^2(0.200) = 6.40$ N m. Hence $x_{\mathrm{cp}} = 6.40/48.0 = 0.133$ m, exactly one third of the chord from the high-pressure-difference end.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Recognise a pure couple in a loading test

A second panel has $A = 1.00$, $B = -2.00$, $q = 300$ Pa, $b = 0.800$ m and $c = 0.500$ m. Positive and negative contributions cancel in the lift integral, giving zero total lift. The moment coefficient is $1/2 - 2/3 = -1/6$ and the moment is -10.0 N m. A force transducer alone would miss this load; two separated vertical reactions can balance the measured couple.

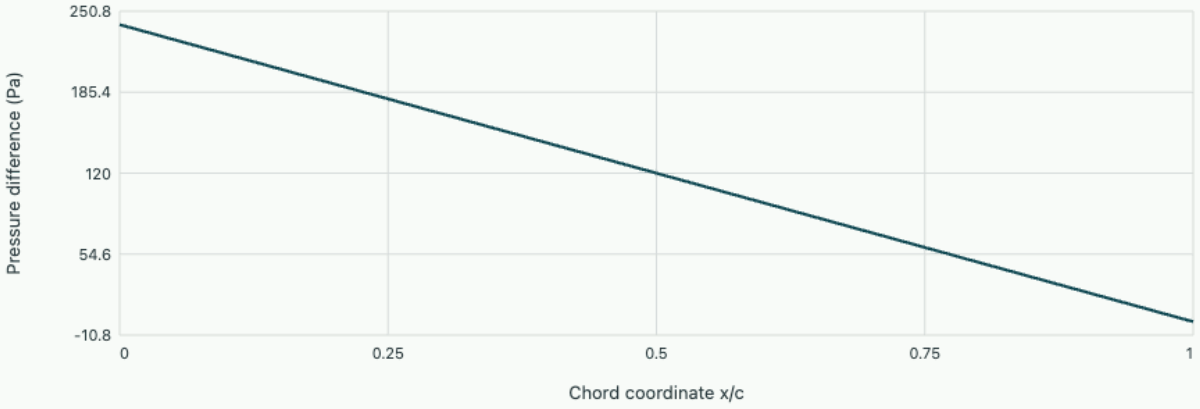
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Pressure-difference intercept A	1.2
Pressure-difference slope B	-1.2
Dynamic pressure (Pa)	200



Linear pressure trace

For1 m span and0.4 m chord: lift=48.000 N; leading-edge moment=6.400 N m. Centre of pressure=0.133 m from the leading edge.

Start from this chapter. The local wall shear is proportional to dynamic pressure times the local friction coefficient. Test how doubling speed can increase shear even while the dimensionless coefficient falls.

OneSimulation laboratory · Measure boundary-layer growth from a velocity profile

Build the resultant from surface measurements

A wind tunnel balance reports force components along and across the incoming flow. A pressure tap reports only local pressure. To connect the two, draw the surface outward normal from the solid into the fluid. A pressure load then acts on the solid in direction $-n$; fluid wall shear acts tangentially. Integrate

pressure traction and shear traction over the body and project the result onto drag and lift axes. A uniform ambient pressure integrates to zero on a closed surface, so gauge pressure is sufficient for this calculation.

For a thin wing at small incidence the upper and lower surfaces are almost horizontal. To leading order the upward pressure force is the planform integral of $p_{\mathrm{lower}} - p_{\mathrm{upper}}$. With dynamic pressure $q_\infty = \rho U^2/2$, define $C_p = (p - p_\infty)/q_\infty$. For rectangular planform and a pressure difference independent of span, the lift coefficient is approximately

$$C_L = \frac{L}{q_\infty S} \simeq \int_0^1 (C_{p,l} - C_{p,u})\, d(x/c).$$

The reference area S must be stated. A coefficient based on wing planform and a bluff-body coefficient based on frontal area cannot be compared as if they used the same force scale.

Integrate a pressure distribution

A wing panel has chord 0.30 m and span 0.80 m. Take the pressure-coefficient difference as a triangle, rising linearly from zero at the leading edge to 1.20 at $x/c = 0.25$ and decreasing linearly to zero at the trailing edge. The coefficient integral is the area under this triangle, $C_L = 0.60$. At $U = 20\,\mathrm{m/s}$ in air of density 1.20 kg/m³, $q_\infty = 240\,\mathrm{Pa}$. The estimated lift is $240(0.24)(0.60) = 34.56\,\mathrm{N}$.

For this spanwise-uniform distribution the centre of pressure follows from the first moment of the triangle. Its chord fraction is the mean of the three vertex abscissae, $(0 + 0.25 + 1)/3 = 0.4167$. The resultant acts about 0.125 m aft of the leading edge. The moment magnitude about that edge is $34.56(0.125) = 4.32\,\mathrm{N\,m}$. Two panels with the same total lift can therefore transmit different support moments if their pressure distributions differ.

Recover a drag force without surface taps

For a survey far enough downstream that static pressure has returned to the external value, integrate the streamwise momentum deficit across the wake. With essentially parallel flow, constant density and negligible outer boundary shear, drag per unit span is $D' = \rho \int u(U - u)\, dy$. This momentum survey includes the effect of both body pressure and surface shear.

For a simplified wake 40 mm thick with uniform $u = 16\,\mathrm{m/s}$ inside it and $U = 20\,\mathrm{m/s}$ outside, the inferred drag per span is $1.20(16)(4)(0.040) = 3.07\,\mathrm{N/m}$. Replacing the product by $\rho U(U - u)$ overestimates the deficit because it assigns the undisturbed mass flux to the wake. A survey close to the trailing edge often also needs a static pressure correction; insufficient lateral coverage can miss part of the deficit.

Relate finite-span lift to a second contribution to drag

A pressure integration on one two-dimensional section neglects a feature of every finite lifting surface. Bound circulation changes along the span and trailing vorticity produces downwash behind the wing. For attached lifting-line flow at small angles, downwash tilts the effective incoming stream downward by induced angle α_i. The section's aerodynamic force is approximately perpendicular to that local stream, so its projection along the original incoming direction creates induced drag $D_i \simeq L\alpha_i$. This contribution can remain even for an inviscid outer flow because momentum is carried away in a three-dimensional trailing wake.

For elliptical span loading the ideal lifting-line result is

$$\alpha_i = C_L/(\pi AR),$$

where aspect ratio $AR = b^2/S$, span b and planform area S. A span efficiency factor e accounts approximately for a non-elliptic distribution at a given lift, giving $C_{D,i} = C_L^2/(\pi eAR)$. With a locally constant parasite coefficient $C_{D,0}$, total drag at fixed required lift W becomes

$$D = qSC_{D,0} + \frac{W^2}{\pi eAR\, qS},$$

$$q = \rho U^2/2.$$

At low speed the necessary lift coefficient rises and induced drag dominates; at high speed parasite drag rises with dynamic pressure. Differentiating D with respect to qS gives minimum drag where the two terms are equal. Thus $C_{L,\mathrm{opt}} = \sqrt{\pi eARC_{D,0}}$. This optimum maximises lift-to-drag ratio for the adopted polar. Minimum power DU is a different operating point.

Select a glide speed and check the angle assumptions

A small unpowered survey glider has weight 120 N, area 1.20 m² and span 3.0 m. Use density 1.20 kg/m³, $e = 0.80$ and $C_{D,0} = 0.025$. Assume shallow steady glide so lift is approximately weight. Aspect ratio is

$$3^2/1.2 = 7.50,$$

$$\pi eAR = 18.85.$$

The optimum lift coefficient is

$$\sqrt{18.85(0.025)} = 0.686.$$

From $W = qSC_L$ the dynamic pressure is 145.8 Pa and the airspeed is

$$\sqrt{2(145.8)/1.20} = 15.6\,\mathrm{m/s}.$$

At this condition total drag coefficient is

$$0.025 + 0.686^2/18.85 = 0.0500.$$

Drag is 8.75 N, giving $L/D = 13.7$ and glide angle approximately $\arctan(1/13.7) = 4.17^\circ$. The assumption $L \simeq W$ is consistent to better than one percent. Doubling speed at unchanged weight reduces C_L by four, so induced drag falls by four while parasite drag rises by four. The original 4.38 N contributions become about 1.09 and 17.5 N. If an estimated optimum lies at or beyond the measured stall coefficient, the attached-flow polar and lifting-line argument no longer support that speed decision.

Banking the same glider changes the required lift. In a coordinated near-level turn with bank angle ϕ, the vertical lift component must support the weight, so $L = W/\cos\phi$. At 45° bank the required lift rises by $\sqrt{2}$. At unchanged speed the induced portion of drag then doubles, while the parasite portion is approximately unchanged under the same polar. This transfer calculation explains why a straight-flight drag estimate cannot be inserted unchanged into a turning-flight performance prediction.

The required lift coefficient at the higher bank angle must still remain inside the measured unstalled polar.

Settling in the creeping-flow limit

A small tracer bead in a viscous liquid can reach a regime in which inertial forces on the surrounding liquid are negligible. Estimate their ratio to viscous forces using $Re_d = \rho U (2a)/\mu$, with radius a. For $Re_d \ll 1$, the steady incompressible equations outside a sphere reduce to $\nabla p = \mu \nabla^2 \mathbf{u}$ and $\nabla \cdot \mathbf{u} = 0$. Apply no slip at the sphere and uniform speed U far away. Linearity then requires drag proportional to U.

The coefficient comes from surface traction. Fix the sphere in flow directed along positive z, and measure polar angle θ from that axis. The Stokes solution gives surface pressure relative to the far field $-3\mu U\cos\theta/(2a)$ and tangential shear $-3\mu U\sin\theta/(2a)$. The normal viscous stress vanishes there. Project pressure and shear onto the flow direction and use the area of a ring, $2\pi a^2 \sin\theta\, d\theta$. The pressure integral uses $\int_0^\pi \cos^2\theta \sin\theta\, d\theta = 2/3$, while the shear integral uses $\int_0^\pi \sin^3\theta\, d\theta = 4/3$. The two contributions are $2\pi\mu a U$ and $4\pi\mu a U$. Even in this highly viscous regime, pressure accounts for one third of the drag.

A spherical calibration bead settles far from the container walls. At terminal speed its apparent weight equals $6\pi\mu a U_t$, so

$$U_t = \frac{2(\rho_s - \rho) g a^2}{9\mu}.$$

For $a = 0.200\,\mathrm{mm}$, $\rho_s = 1250$, $\rho = 1000\,\mathrm{kg\,m^{-3}}$, and $\mu = 0.100\,\mathrm{Pa\,s}$,

$$U_t = 0.218\,\mathrm{mm\,s^{-1}}.$$

The resulting $Re_d = 0.000872$ justifies dropping convective inertia. After terminal speed is reached, 10 mm takes about 45.9 s. Doubling the radius at the same density difference quadruples speed and multiplies Re_d by eight. A size change therefore merits a fresh check on the approximation.

Practise on sheet 5 · Measure viscosity with a falling calibration bead. Further reading · Georgia Institute of Technology · PHYS 4421 course reading, B. Lautrup, Physics of Continuous Matter · Chapter 20, §20.2 Stokes' solution for a sphere.

Study the full derivation

Settling spheres, Stokes drag and a wall correction · Momentum, motion and transport in fluids

A falling-sphere viscometer infers viscosity from terminal speed. Derive the drag on a no-slip sphere, distinguish creeping flow from merely steady motion and use radius, vessel width and orientation checks to decide what the speed experiment actually measures.

Chapter 10 · Pump curves, pipe networks and operating points

Construct a hydraulic system curve and solve the operating point of a pump connected to serial or parallel flow paths.

FROM FIRST PRINCIPLES

Find pump discharge by matching added and required head

A pump supplies head that depends on its discharge and speed. The connected pipework requires head to overcome both terminal elevation or pressure differences and losses. Consider steady incompressible flow between large reservoirs with negligible endpoint velocities. For a fixed pump speed, conservation of mechanical energy at the operating point gives

$$H_p(Q) = H_{\text{static}} + h_L(Q),$$
$$H_{\text{static}} = z_2 - z_1 + (p_2 - p_1)/(\rho g).$$

For fittings and pipe sections with approximately constant loss coefficients, each loss varies as $V^2/(2g)$. In a single series path all sections carry the same Q, with $V_j = Q/A_j$. Adding the individual losses gives a system coefficient K.

$$h_L = \sum_j \frac{\zeta_j}{2gA_j^2} Q^2 = KQ^2.$$

If the measured pump characteristic is approximated by $H_p = H_0 - aQ^2$, solve the intersection explicitly on the positive-flow branch.

$$Q = \sqrt{\frac{H_0 - H_{\text{static}}}{a + K}}.$$

This solution needs $H_0 > H_{\text{static}}$. Branches in parallel share an endpoint head difference and their discharges add; those constraints must be imposed before using an equivalent system curve.

DERIVATION AND APPLICATIONS

Find the sensitivity of a pump operating point to a valve

A pump curve and a system curve determine their intersection. Their local slopes also determine how far that intersection moves when a valve setting changes. Model a fixed-speed pump by $H_p = H_0 - aQ^2$ and a network by $H_s = H_z + bQ^2$ in the intended flow direction. Here H_z is a static elevation or pressure head, $a, b > 0$ have units of head per squared discharge, and $H_0 > H_z$. The loss model assumes an approximately constant turbulent loss coefficient in the examined flow range.

$$H_0 - aQ^2 = H_z + bQ^2,$$

$$Q = \sqrt{\frac{H_0 - H_z}{a + b}}.$$

Differentiate implicitly with respect to the network coefficient at fixed speed and static head. A larger loss coefficient raises the system curve and reduces discharge. The fractional sensitivity is weaker when the pump itself has a steep characteristic.

$$\frac{dQ}{db} = -\frac{Q}{2(a + b)},$$

$$\frac{d\ln Q}{d\ln b} = -\frac{b}{2(a + b)}.$$

The actual head delivered by the pump changes as the flow moves along its curve. Differentiating the pump head with the result just obtained gives a positive slope with respect to b, an effect distinguishable by measuring both flow and pressure.

$$\frac{dH_p}{db} = -2aQ\frac{dQ}{db} = \frac{aQ^2}{a + b}.$$

These derivatives apply to small changes. Near zero available head a fractional uncertainty in the difference $H_0 - H_z$ can dominate, even if the separate head measurements are quite accurate.

WORKED EXAMPLE

Predict the effect of throttling a circulation line

Measure Q in L/s and head in metres. Take $H_0 = 20.0$, $H_z = 4.00$, $a = 0.250$ and $b = 0.750$ in consistent units. The flow is 4.00 L/s and pump head is 16.0 m. A ten-percent increase in b predicts -3.75 percent flow by the derivative, or 3.85 L/s. The exact new flow is $\sqrt{16/1.075} = 3.86$ L/s, close to the linear estimate.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Infer which curve limits adjustability

At a separate operating point the measured logarithmic flow sensitivity to the total network coefficient is -0.100. This implies $b/(a + b) = 0.200$ and therefore $a = 4b$. A reduction of b by 20.0 percent at fixed available head changes flow by the exact factor $\sqrt{5/4.8} = 1.0206$. The small 2.06 percent increase quantifies why a noticeable valve change can have little effect on discharge.

Interactive calculation · Change the inputs of this example

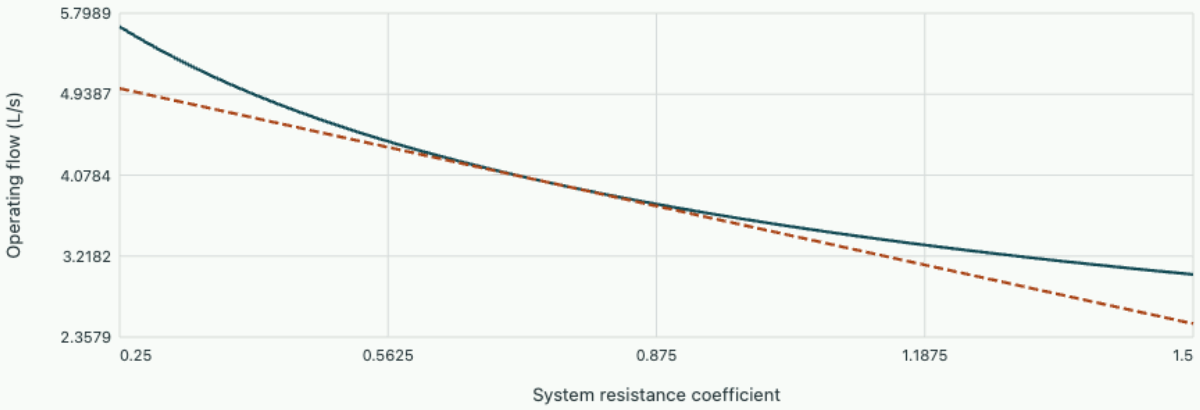
Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Pump quadratic coefficient **0.25**

Chosen system coefficient **0.75**



Exact intersection Local sensitivity about chosen coefficient

At selected coefficient, Q=4.000 L/s; logarithmic sensitivity=-0.375. Shutoff20 m, static lift4 m. The tangent is accurate only for small resistance changes.

Why a pump does not prescribe a unique flow

A centrifugal pump at fixed speed can deliver different combinations of flow and head. The connected network selects an operating point by requiring the head gained through the machine to match the static lift and losses elsewhere. For two large reservoirs, surface speeds are negligible and equal surface pressures cancel. A steady mechanical energy balance then gives

$$H_{\mathrm{pump}} = \Delta z + h_L.$$

With turbulent pipe resistance represented locally by constant loss coefficients, the losses can be written $h_L = KQ^2$ for positive flow.

This quadratic law is an approximation over a specified operating range. It follows because fittings and pipe friction have head losses proportional to velocity squared when the relevant coefficients vary little. In laminar pipe flow, the pressure loss at fixed viscosity is instead proportional to flow. A system curve extrapolated far outside its underlying Reynolds-number range can give a misleading operating point.

OneSimulation · When does reducing pump speed close the check valve?

An ideal check valve connects a pump to an open reservoir above its inlet. Let n denote the speed divided by its reference value and let Q be measured in litres per second. The pump supplies $20n^2 - 0.025Q^2$ metres of head. The pipe and reservoir require $H_s + 0.075Q^2$ metres. Forward flow requires a positive intersection.

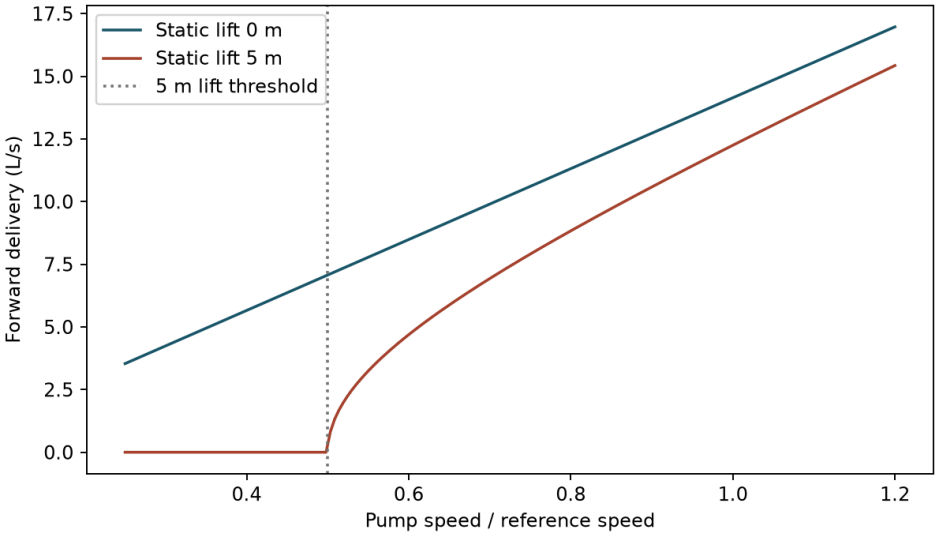
$$0.100Q^2 = 20n^2 - H_s.$$

OneSimulation inverts the total dynamic loss over the forward-flow interval. With a 5 m static lift, half speed just balances gravity at zero delivery. At reference speed the pump delivers 12.25 L/s. At three-quarter speed it delivers 7.91 L/s, considerably below three quarters of the reference delivery. Beneath the threshold the closed valve carries the pressure difference; the forward-flow equation no longer applies.

```
"""Intersect a pump curve with a pipe-system head curve at two lifts."""
import matplotlib.pyplot as plt
import numpy as np
from onesim.numerics.nonlinear.kernels.local_solvers.inversion.bracketed_inverse import
bisect_increasing

speed = np.linspace(0.25, 1.2, 181)  # fraction of reference speed
shutoff = 20. * speed**2             # pump shutoff head in m
pump_loss, pipe_loss = 0.025, 0.075   # m per (L/s)²
fig, ax = plt.subplots(figsize=(7, 4), constrained_layout=True)
for lift, colour in [(0., "#19596a"), (5., "#a5442e")]:
    available = shutoff - lift
    delivery = np.zeros_like(speed)
    active = available > 0
    def total_loss(flow):
        return (pump_loss + pipe_loss) * flow**2
    delivery[active] = bisect_increasing(total_loss, available[active], 0., 30.)
    assert np.allclose(shutoff[active] - pump_loss*delivery[active]**2,
                       lift + pipe_loss*delivery[active]**2)
    ax.plot(speed, delivery, color=colour, label=f"Static lift {lift:g} m")

ax.axvline(0.5, color="0.45", linestyle=":", label="5 m lift threshold")
ax.set(xlabel="Pump speed / reference speed", ylabel="Forward delivery (L/s)")
ax.legend()
fig.savefig("fluid-mechanics.png", dpi=170)
```



The zero-lift curve isolates quadratic friction. On the elevated line part of the available head must overcome gravity even at arbitrarily low delivery.

Download the Python example

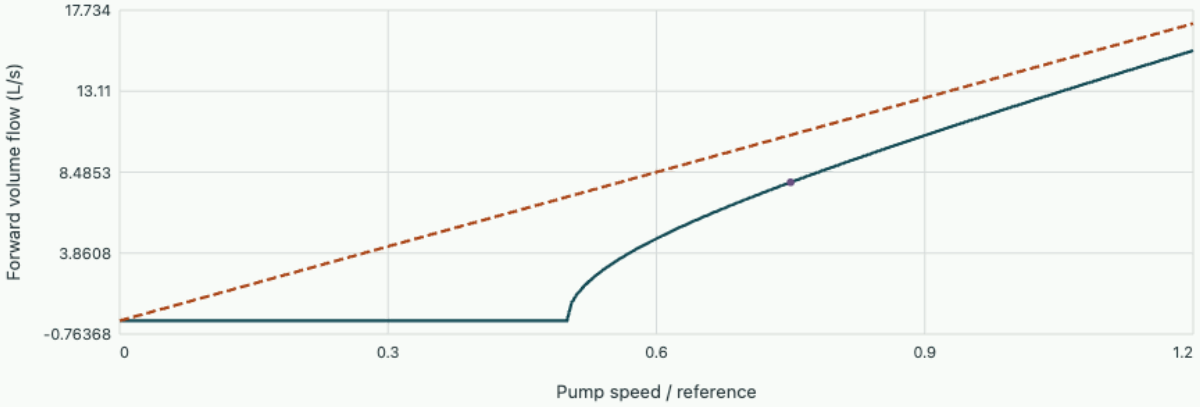
Change the inputs of this exact example and compare the recalculated curves with the recorded OneSimulation output above. The browser evaluates the stated analytical or modal reference.

Open this live calculation

Try the calculation

Static reservoir lift (m) **5**

Selected speed / reference **0.75**



Elevated reservoir, check valve | Zero static lift | Selected speed

Valve opens above speed ratio 0.500. Selected delivery 7.906 L/s. Below threshold the valve carries the remaining pressure difference and forward delivery is zero. Pump $20n^2 - 0.025Q^2$; system $Hs + 0.075Q^2$, heads in metres.

Combine resistances by their physical constraints

In series the same flow passes through each component and head losses add. Thus two quadratic loss coefficients combine as $K_s = K_1 + K_2$. Parallel branches between the same nodes have equal head loss while their flow rates add. For two branches with forward flow, $Q_i = \sqrt{h/K_i}$, giving

$$Q = \sqrt{h}\,(K_1^{-1/2} + K_2^{-1/2}),$$

$$K_{\parallel} = (K_1^{-1/2} + K_2^{-1/2})^{-2}.$$

A second identical branch divides the flow equally and reduces the quadratic coefficient of that section to one quarter of the single-branch value. Any upstream or downstream common pipe is still in series with the equivalent section and carries total flow.

Open a parallel return line

Use Q in litres per second and head in metres. A pump has $H_p = 30 - 0.20Q^2$; two open reservoirs differ in elevation by 6 m. Initially one pipe connects them with loss $0.40Q^2$. Equating the heads gives $30 - 0.20Q^2 = 6 + 0.40Q^2$, so $Q = \sqrt{40} = 6.32\,\mathrm{L/s}$ and $H_p = 22.0\,\mathrm{m}$. Hydraulic power delivered to water is

$$\rho g Q H_p = 1000(9.81)(0.00632)(22) = 1.36\,\mathrm{kW}.$$

Now add a second identical parallel pipe over the whole route. The system curve becomes $6 + 0.10Q^2$. The new flow is $\sqrt{80} = 8.94\,\mathrm{L/s}$ and pump head 14.0 m. Each pipe takes only 4.47 L/s, less than the original single-pipe flow. Total delivery increases by about 41% while hydraulic power falls to 1.23 kW. The input shaft power also depends on the efficiency at each operating point. If a flow controller holds the original delivery after the second branch opens, its valve or speed controller changes the operating curve and the free-running comparison no longer predicts the duty.

Determine which side of a branch controls the flow

A check valve on a return line permits only positive branch flow. If an added elevated reservoir imposes a static threshold of 5 m on that branch, the forward branch law becomes $Q_b = \sqrt{(H - 5)/K_b}$ only for driving head $H > 5$ m. Below that threshold the flow is zero. The resulting system curve changes form as the valve opens, so a single constant equivalent coefficient cannot describe both regimes.

Derive the first pressure change in a rapidly closed line

The operating point of a pump and pipe network is a steady state. A valve closure can generate a much higher transient pressure while the flow far upstream has barely changed. Consider a straight horizontal liquid-filled pipe of area A and initially uniform forward speed V. A rapid downstream closure sends a compression wave upstream at wave speed a. During a short interval dt, the wave reaches approximately

$a\,dt$ of pipe, bringing the fluid behind it to rest. Neglect friction over this short wavefront and take $a \gg V$.

The stopped mass is $\rho A a\,dt$, losing momentum $\rho A a\,dt\,V$. The pressure difference across the front supplies an impulse $A\Delta p\,dt$. Balancing magnitudes yields the Joukowsky rise $\Delta p = \rho a V$; for a partial rapid speed change use its magnitude $\rho a|\Delta V|$ with sign determined by whether the wave compresses or relieves the pipe. Small fluid compressibility and wall compliance both enter the actual wave speed. In a rigid tube $a \simeq \sqrt{K/\rho}$, but an expanding elastic pipe stores some of the volume change and lowers that speed.

Reflections matter after the wave has travelled to a boundary and back. For a reservoir a distance L away the first round-trip time is approximately $2L/a$. A valve that completes closure before the returning wave acts can produce the full first rise. A much slower closure allows incoming reflections to alter the pressure during the manoeuvre and requires a time-dependent network calculation.

Compare closure schedules against a pressure allowance

A water line is 300 m long with measured wave speed $a = 1000\,\mathrm{m/s}$, initial speed 1.5 m/s, density 1000 kg/m³ and initial valve gauge pressure 0.40 MPa. Rapid full closure predicts $\Delta p = 1000(1000)(1.5) = 1.50\,\mathrm{MPa}$, bringing the first local peak to about 1.90 MPa gauge. A valve closing in 0.15 s completes its action well inside the 0.60 s round-trip interval. A static energy estimate based only on $\rho V^2/2 = 1.13\,\mathrm{kPa}$ would miss the pipe-length momentum transient by orders of magnitude.

Suppose an operating specification allows a first-wave rise of 0.45 MPa. The maximum abrupt decrement in line speed consistent with it is

$$450000/[1000(1000)] = 0.45\,\mathrm{m/s}.$$

This provides a useful bound on a single rapid speed change. Dividing the full stop into several identical steps does not automatically satisfy the total pressure limit: reflected waves can add if the steps are poorly timed. A proposed 3 s controlled closure is slow compared with $2L/a$, so the full instantaneous formula no longer estimates its actual peak. Model the valve characteristic, boundary reflections, pump coast-down and possible sub-atmospheric pressure before choosing a schedule. Measuring pressure at the valve and a second station allows wave speed to be identified from the arrival-time difference.

Calculate a pipe resistance from roughness and flow rate

For fully developed turbulent flow in a round pipe the pressure gradient depends on both the roughness height and the viscous scale of the near-wall flow. Force balance on a length L gives

$$\Delta p\,\pi D^2/4 = \tau_w \pi D L.$$

Define the Darcy factor by $f_D = 8\tau_w/(\rho U^2)$, where U is the cross-sectional mean velocity. Combining the equations gives

$$h_f = f_D (L/D) U^2/(2g).$$

Use the same Darcy convention throughout. The Fanning factor is $f_D/4$.

For a straight, uniform, turbulent pipe the empirical Colebrook equation relates f_D to $Re = UD/\nu$ and equivalent sand roughness ϵ/D. Setting $x = 1/\sqrt{f_D}$ turns the implicit equation into a simple iteration:

$$x_{n+1} = -2 \log_{10} \left(\frac{\epsilon}{3.7 D} + \frac{2.51 x_n}{Re} \right).$$

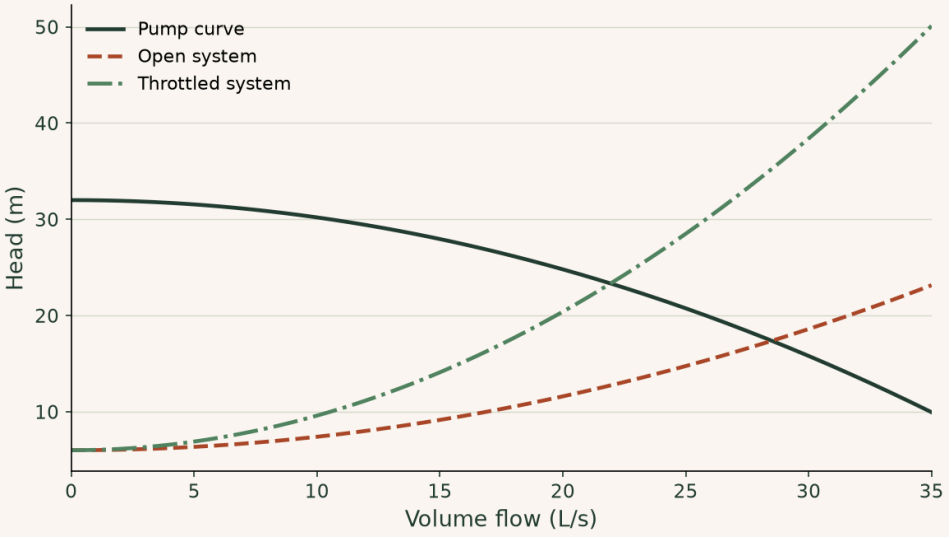
When roughness dominates the second term, the friction factor becomes nearly independent of Reynolds number. The smooth limit retains a Reynolds-number dependence. For fully developed laminar flow the separate law $f_D = 64/Re$ applies. Near transition a roughness estimate and a single turbulent curve alone cannot resolve the flow regime.

A building cooling loop carries $Q = 4.00\,\mathrm{L\,s^{-1}}$ of water through $L = 40.0\,\mathrm{m}$ of straight pipe with $D = 50.0\,\mathrm{mm}$, $\nu = 1.00 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}$ and $\epsilon = 0.050\,\mathrm{mm}$. Continuity gives $U = 4Q/(\pi D^2) = 2.037\,\mathrm{m\,s^{-1}}$ and $Re = 1.019 \times 10^5$. Starting at $x_0 = 7$, iteration converges to $x = 6.721$, hence $f_D = 0.02214$ and $h_f = 3.75\,\mathrm{m}$. If deposits raise equivalent roughness to $0.200\,\mathrm{mm}$ at unchanged bore and flow, the same calculation gives $f_D = 0.02948$ and $h_f = 4.99\,\mathrm{m}$. Thus the pump needs 1.24 m additional head for this pipe segment alone. At fixed pump speed the operating flow would generally change. That requires solving the pump and system curves together.

Practise on sheet 5 · Infer an equivalent roughness from two pressure taps. Further reading · Purdue University · ME 30900 Introduction to Fluid Mechanics · Pipe Flows – Pipe Systems, “Colebrook Formula” and the Moody plot.

Read the new operating point after throttling

Approximate a pump curve and the piping resistance by parabolas in flow measured in litres per second. Static lift is 6 m. Throttling raises the coefficient of the flow-squared loss, while speed and pump geometry stay fixed.



The original intersection is 28.5 L/s at 17.4 m. With the valve throttled the intersection is 21.9 L/s at 23.3 m. Computing the new loss at the old flow would miss the movement of the operating point.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

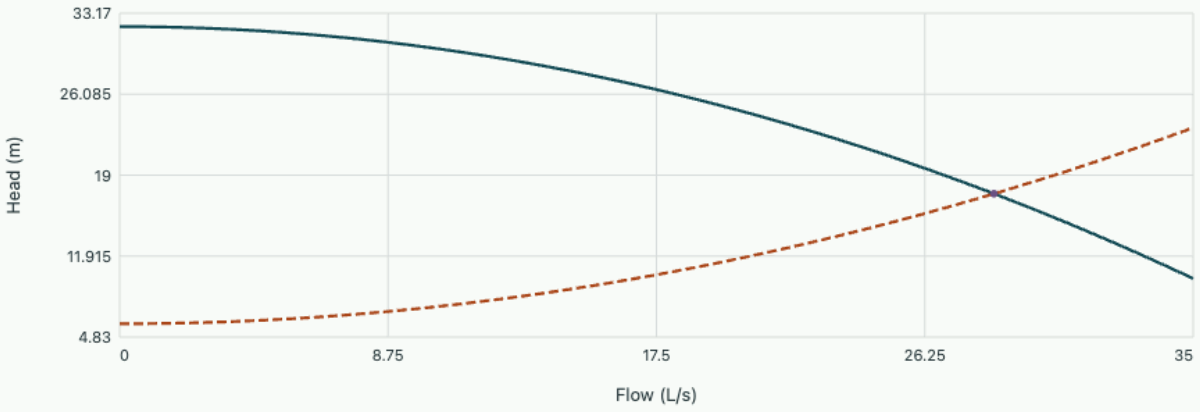
Throttle by increasing the system coefficient. Read the recomputed intersection instead of evaluating loss at the previous flow.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Shutoff head (m)	**32**
Static lift (m)	**6**
System coefficient (m per (L/s)²)	**0.014**



Operating flow 28.504 L/s and head 17.375 m. The intersection is recomputed after each change.

Problem sheet 5 · Surface forces and system curves

Determine loads from a pressure or wake survey and find the operating point after a network change.

Move the point of application

A rectangular 0.40 m chord, 1.0 m span panel has pressure-coefficient difference falling linearly from 1.4 at the leading edge to zero at the trailing edge. The dynamic pressure is 300 Pa. Estimate lift, centre of pressure and moment magnitude about the quarter-chord position.

▼ **Reveal the worked solution**

The average coefficient difference is 0.70, so

$$L = 300(0.4)(0.70) = 84\,\mathrm{N}.$$

The triangular load acts one third of the chord from the larger end, at 0.1333 m. The quarter-chord is 0.100 m from the leading edge, giving moment magnitude $84(0.0333) = 2.80\,\mathrm{N\,m}$. This is a leading-order thin-panel calculation.

Integrate a two-zone survey

Behind a long cylinder in air at density 1.25 kg/m³, a far-wake survey finds velocity 7 m/s in a central strip 20 mm wide and 9 m/s in two neighbouring strips each 15 mm wide. The undisturbed speed is 10 m/s. Estimate drag per span and the drag coefficient based on diameter 25 mm.

▼ **Reveal the worked solution**

The pressure-recovered momentum deficit is

$$D' = 1.25[7(10-7)0.020 + 9(10-9)0.030] = 0.8625\,\mathrm{N/m}.$$

The reference force per width is $0.5(1.25)(10)^2(0.025) = 1.5625\,\mathrm{N/m}$, giving $C_D = 0.552$. Counting only the central strip misses a substantial share of the deficit.

Open unequal branches

A pump delivers head $24 - 0.25Q^2$ metres with Q in litres per second. Static lift is 4 m. Parallel delivery branches have losses $1.0Q_1^2$ and $4.0Q_2^2$ metres. Find equivalent resistance, total delivery and flow split. Verify equal branch losses.

▼ **Reveal the worked solution**

The equivalent coefficient is

$$(1+1/2)^{-2} = 4/9.$$

The head balance gives $Q^2 = 20/(0.25 + 4/9) = 28.8$, hence $Q = 5.367\,\mathrm{L/s}$. Equal branch losses imply $Q_1 = 2Q_2$, giving 3.578 and 1.789 L/s. Either branch loses 12.8 m, so the pump supplies 16.8 m including static lift.

Infer an equivalent roughness from two pressure taps

A level, constant-bore pipe has $D = 60.0\,\mathrm{mm}$. At mean speed $2.00\,\mathrm{m\,s^{-1}}$, a 60.0 m fully developed length loses 5.00 m of water head. Take $\nu = 1.00 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}$. Determine the Darcy factor and infer the equivalent roughness. Explain how an unnoticed valve loss between the taps would affect the inferred number.

▼ **Show the worked solution**

$$f_D = \frac{2 g h_f D}{L U^2} = 0.024525,$$

$$Re = 1.20 \times 10^{5}.$$

Exponentiate the Colebrook relation to obtain

$$\epsilon = 3.7 D [10^{-1/(2\sqrt{f_D})} - 2.51/(Re\sqrt{f_D})].$$

With the measured factor this gives

$$1.128 \times 10^{-4}\,\mathrm{m} = 0.113\,\mathrm{mm}.$$

The hidden valve would raise the measured loss at this flow, raising both the assigned pipe factor and its inferred roughness. The result would then describe the installation at one operating point.

Revisit chapter 10 · Calculate a pipe resistance from roughness and flow rate.

Measure viscosity with a falling calibration bead

A spherical bead of radius 0.300 mm and density $1500\,\mathrm{kg\,m^{-3}}$ falls 12.0 mm in 20.0 s after its speed settles. The liquid density is $1000\,\mathrm{kg\,m^{-3}}$. Walls are distant. Infer dynamic viscosity, check the creeping-flow criterion, and estimate how a 2% overestimate in radius affects the reported viscosity if speed is measured independently.

▼ **Show the worked solution**

The measured terminal speed is

$$0.0120/20.0 = 6.00 \times 10^{-4}\,\mathrm{m\,s^{-1}}.$$

Solving the buoyancy–drag balance gives

$$\mu = 2(500)(9.81)(0.000300)^2/[9(0.000600)] = 0.1635\,\mathrm{Pa\,s}.$$

From the inferred viscosity $Re_d = 1000(0.000600)(0.000600)/0.1635 = 0.00220$, consistent with the assumed limit. Since $\mu \propto a^2$, the inferred viscosity is higher by $1.02^2 - 1 = 4.04\%$. Timing more precisely does not correct a radius calibration error.

Revisit chapter 9 · Settling in the creeping-flow limit.

Explore the governing equations after solving the sheet: Pressure integration, lift and drag; Pump curves, pipe networks and operating points. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Chapter 11 · Angular momentum and work in a rotor

Use absolute and blade-relative velocities to derive the ideal head developed by a rotating impeller.

FROM FIRST PRINCIPLES

Obtain rotor work from angular momentum and shaft power

A rotating impeller transmits work by changing the angular momentum carried by fluid. Choose a control volume around the rotor, with axisymmetric inlet and outlet sections. Let V_θ be absolute tangential velocity, r the representative radius, $\dot{m}$ steady mass flow and Ω rotor angular speed. With negligible torque on other boundaries, the torque exerted by the blades on the fluid is outgoing minus incoming angular momentum flux.

$$M = \dot{m}(r_2 V_{\theta 2} - r_1 V_{\theta 1}).$$

Mechanical power from a rotating shaft is torque times angular speed. Divide by mass flow and write $U_j = \Omega r_j$ for local blade speed. This gives Euler's turbomachinery work and its corresponding ideal head.

$$w_E = \frac{\Omega M}{\dot{m}} = U_2 V_{\theta 2} - U_1 V_{\theta 1},$$
$$H_E = w_E / g.$$

A velocity triangle relates the absolute velocity to the moving blade through $\mathbf{V} = \mathbf{U} + \mathbf{W}$, where $\mathbf{W}$ is relative velocity. A positive w_E adds energy to the fluid. Measured hydraulic head can be smaller owing to losses, even when the angular-momentum accounting itself is exact.

DERIVATION AND APPLICATIONS

Extract reaction torque from stationary guide vanes

A stator can carry torque while delivering no shaft power. Consider a steady incompressible annular flow passing through stationary guide vanes. Assume the inlet and outlet flow each have a representative radius, reasonably uniform tangential speed and negligible externally applied torques other than the vanes. Axisymmetric pressure on inlet and outlet planes contributes no torque about the axis. For mass flow $\dot{m}$ the instantaneous angular momentum per unit mass is rV_θ, so its outgoing minus incoming flux fixes the torque applied to the fluid.

$$M_{\text{vane}\to\text{fluid}} = \dot{m}(r_2 V_{\theta 2} - r_1 V_{\theta 1}),$$
$$M_{\text{fluid}\to\text{vane}} = -M_{\text{vane}\to\text{fluid}}.$$

The torque on the casing and the work on the stream are different measurements. A row rotating at angular velocity Ω performs shaft work per unit mass proportional to the change in angular momentum. The stationary row follows by setting $\Omega = 0$ even when the change in swirl is substantial.

$$w_{\text{shaft}} = \frac{\Omega M_{\text{vane}\to\text{fluid}}}{\dot{m}} = \Omega(r_2 V_{\theta 2} - r_1 V_{\theta 1}).$$

For a stator at constant height with equal axial velocities at inlet and outlet, reduction of swirl releases kinetic energy. In an ideal lossless row with zero shaft work, Bernoulli's balance transforms that kinetic energy into static pressure.

$$p_2 - p_1 = \frac{\rho}{2}(V_{\theta 1}^2 - V_{\theta 2}^2).$$

Viscous dissipation lowers the actual recovered pressure. It does not alter the angular-momentum balance when fluxes and all external torques are measured correctly. Residual outlet swirl therefore affects both casing torque and the pressure available downstream.

WORKED EXAMPLE

Predict the reaction of a water straightener

Water flows at 2.00 kg/s through vanes at representative radius 0.100 m. Its tangential speed decreases from 4.00 to 1.00 m/s in the same direction. The fluid pushes the casing with $2.00(0.100)(4.00 - 1.00) = 0.600$ N m. At $\rho = 1000$ kg/m³ the maximum ideal pressure gain is

$$1000(16 - 1)/2 = 7.50\,\text{kPa}.$$

The stationary casing absorbs the moment while transmitting zero mechanical power.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Recover an exit angle from casing torque

A second stator takes in air with $V_{\theta 1} = 12.0$ m/s at $r_1 = 0.150$ m. A 0.900 N m positive casing torque at $\dot{m} = 0.750$ kg/s leaves $r_2 V_{\theta 2} = 1.80 - 0.900/0.750 = 0.600$ m²/s. With $r_2 = 0.120$ m, outlet swirl is 5.00 m/s. At axial speed 20.0 m/s, the absolute flow direction is $\arctan(5/20) = 14.0^\circ$ from the axis.

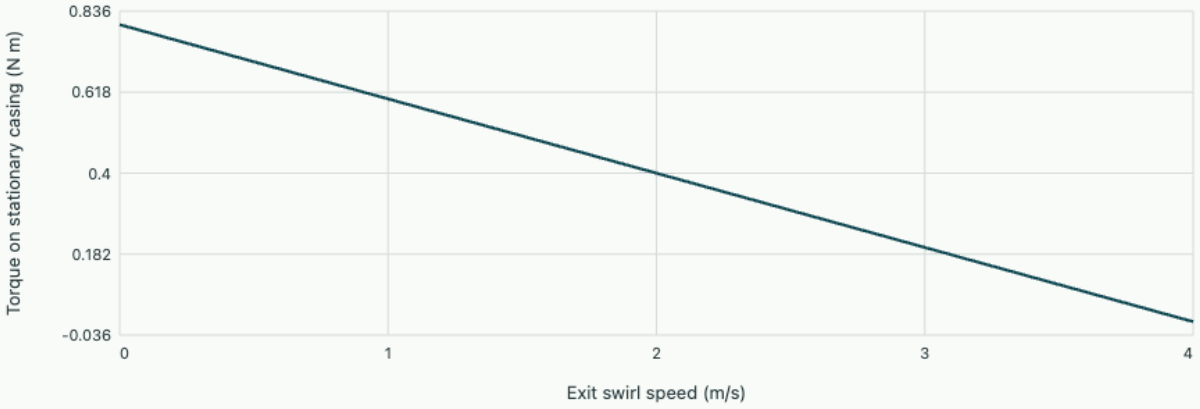
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Inlet swirl speed (m/s)	4
Mass flow (kg/s)	2
Representative radius (m)	0.1



Angular-momentum balance

At exit swirl1 m/s, casing torque=0.600 N m; ideal water pressure gain=7.500 kPa. Stationary vanes exchange no shaft work.

Convert a change of swirl into shaft torque

A fluid element can gain energy in a rotating impeller even if its inlet and outlet are at the same elevation. The rotor exerts a torque that changes the angular momentum carried by the flow. Draw a fixed control volume around the rotor. In a steady, single-inlet, single-outlet approximation, with representative mass-flow-weighted tangential velocities $V_{\theta 1}, V_{\theta 2}$, the axial torque on the fluid is $\dot{m}(r_2 V_{\theta 2} - r_1 V_{\theta 1})$. Positive tangential velocity points in the direction of shaft rotation.

Multiplying by shaft angular speed converts torque to power. Divide by the mass flow to obtain the Euler specific work. Writing blade speeds $U_i = \omega r_i$ avoids confusion between the angular velocity of the hardware and the velocity components of the fluid.

$$w_E = U_2 V_{\theta 2} - U_1 V_{\theta 1},$$
$$H_E = w_E / g.$$

A pump adding energy has positive w_E under this convention. A turbine removes energy from the fluid and develops shaft output corresponding to a decrease in the fluid’s angular momentum. A stationary guide vane may change swirl while doing no shaft work because its angular speed is zero.

Relate the blade angle to the exit flow

At each station absolute velocity is the vector sum of local blade velocity U and velocity W relative to the moving blade. The throughflow component of V equals that of W when the blade motion is purely tangential. For an ideal radial impeller with backward-curved blades, define outlet angle β_2 from the backward tangent towards the radial throughflow. Then $W_{\theta 2} = -V_{r2} \cot \beta_2$ and $V_{\theta 2} = U_2 - V_{r2} \cot \beta_2$. State this angle convention before reading a drawing; other conventions change the algebraic appearance.

Predict work and distinguish hydraulic delivery

Let the outer radius be 0.10 m, rotor speed 1800 rpm, outlet radial velocity 3.0 m/s, exit blade angle 30°, and inlet swirl zero. The tip speed is 18.85 m/s; the ideal absolute exit swirl is

$$18.85 - 3 \cot 30^\circ = 13.65 \,\mathrm{m/s}.$$

Hence specific work is 257.4 J/kg and head $257.4 / 9.81 = 26.24 \,\mathrm{m}$. For water at 8.0 L/s the mass flow is about 8.0 kg/s. Rotor torque on the fluid is $8 (0.10)(13.65) = 10.92 \,\mathrm{N\,m}$, and work rate is 2.06 kW.

If measurements show an actual delivered head of 21 m at this flow, hydraulic output is $8 (9.81)(21) = 1.65 \,\mathrm{kW}$. The ratio of actual head to the ideal blade-following estimate is 0.800. It combines departures from the ideal exit triangle with hydraulic losses in this simplified comparison. Mechanical bearing loss would additionally enter a comparison with measured shaft input power. If inlet guide vanes add positive inlet swirl, the subtracted $U_1 V_{\theta 1}$ term reduces specific work for the same exit conditions.

Connect the triangle to pump affinity

For geometrically similar operation at unchanged impeller diameter, a fixed ratio V_r / U holds the ideal outlet triangle shape constant. Flow then scales with shaft speed, head with speed squared and hydraulic power with speed cubed. A real installation with static lift need not move between similar operating points when speed changes.

Derive work from a jet in a moving bucket

A radial pump uses an impeller to add angular momentum to a confined flow. An impulse turbine reverses the sequence: a stationary nozzle first converts pressure head to a high-speed free jet, which transfers momentum to buckets at nearly ambient pressure. Let jet speed V_j be tangential at the runner and bucket speed U positive in that direction. Relative inlet speed is $V_j - U$. Suppose the bucket turns this relative stream through angle $\pi - \beta$ and its exit relative speed magnitude is $k(V_j - U)$, with $k \le 1$ accounting for relative-flow loss. The exit tangential absolute speed is

$$V_{\theta 2} = U - k(V_j - U)\cos\beta.$$

A succession of buckets intercepts the full continuous nozzle mass flow $\dot m = \rho Q$, unlike one isolated vane travelling away from a jet. Neglect leakage, windage and mechanical losses. Tangential runner force is $\dot m(V_j - V_{\theta 2})$, hence shaft work per kilogram and hydraulic efficiency relative to incoming jet kinetic energy are

$$w = U(V_j - U)(1 + k\cos\beta),$$

$$\eta_j = \frac{2U(V_j - U)(1 + k\cos\beta)}{V_j^2}.$$

Differentiating with respect to speed at fixed inlet jet and exit geometry gives $U = V_j/2$ for maximum power. At zero bucket speed force can be large but shaft power is zero. At bucket speed equal to jet speed there is no relative inlet stream to turn and therefore no useful momentum transfer. These two limits expose an incorrectly drawn exit velocity triangle.

Compare torque at the design and off-design speeds

A laboratory nozzle delivers 12 L/s of water at 30 m/s to a runner of pitch radius 0.20 m. Adopt $k = 0.85$ and $\beta = 15^\circ$, so

$$1 + k\cos\beta = 1.821.$$

At optimum speed $U = 15\,\mathrm{m/s}$ and $\omega = 75\,\mathrm{rad/s}$, about 716 rpm. The runner force is $12(30 - 15)(1.821) = 328\,\mathrm{N}$, torque 65.6 N m, and power $15(328) = 4.92\,\mathrm{kW}$. The jet supplied $12(30^2)/2 = 5.40\,\mathrm{kW}$, giving efficiency 91.0% under these idealised runner assumptions.

Load the shaft until bucket speed drops to 9 m/s. The force rises to $12(21)(1.821) = 459\,\mathrm{N}$ and torque rises to 91.8 N m, but power drops to $9(459) = 4.13\,\mathrm{kW}$. Increased measured torque therefore does not demonstrate improved conversion without the speed measurement. The exit tangential velocity becomes $9 - 0.85(21)\cos 15^\circ = -8.24\,\mathrm{m/s}$, indicating the bucket now reverses some absolute tangential momentum. Maintaining the assumed jet flow requires sufficient nozzle head independently of shaft loading; a restricted supply may change both jet velocity and flow rate during the test.

At ideal design speed the leaving water still has a radial or transverse velocity because β is nonzero. Its relative exit transverse component is $k(V_j - U)\sin\beta = 3.30\,\mathrm{m/s}$. The absolute tangential component

is $15 - 0.85(15)\cos 15^\circ = 2.69\,\mathrm{m/s}$, giving residual jet kinetic energy about 9.1 J/kg. The rest of the gap between inlet kinetic energy and shaft work includes dissipation associated with $k < 1$. It would be incorrect to attribute the entire efficiency shortfall to measured tangential exit velocity alone. Traversing velocity components or collecting the discharge separately provides a diagnostic of losses within the bucket.

Chapter 12 · Compressible nozzles and choking

Relate area change to Mach number and calculate the limiting mass flow through a converging nozzle.

FROM FIRST PRINCIPLES

Derive the area–velocity relation for an isentropic stream

Consider a steady, quasi-one-dimensional flow through a smooth duct of area A. Assume negligible viscosity, heat exchange and elevation change, and no shocks. Density ρ and speed V vary between stations. Mass conservation gives constant $\rho V A$, so taking logarithmic differentials relates three fractional changes.

$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0.$$

Streamwise momentum gives

$$dp + \rho V\, dV = 0.$$

For an isentropic change, sound speed satisfies $a^2 = (\partial p/\partial \rho)_s$ so the pressure differential can be eliminated in favour of density. With $M = V/a$ this yields

$$\frac{d\rho}{\rho} = -\frac{V\, dV}{a^2} = -M^2 \frac{dV}{V},$$

$$\frac{dA}{A} = (M^2 - 1)\frac{dV}{V}.$$

Subsonic flow accelerates in a contraction because its density variation is comparatively small. A supersonic stream accelerates in a diverging duct because density falls more rapidly. A smooth transition through $M = 1$ with nonzero velocity gradient requires a stationary area, usually the minimum area of a nozzle. Reservoir state and downstream pressure determine whether that sonic condition is actually reached.

DERIVATION AND APPLICATIONS

Recover subsonic Mach number from stagnation pressure

A forward-facing pressure probe brings a small sample of moving gas nearly to rest. Relating the resulting pressure to speed requires accounting for compressibility. Consider a calorically perfect ideal gas, constant heat-capacity ratio $\gamma > 1$, no heat input or shaft work, and subsonic flow brought reversibly to rest. Neglect the height change. Use $a^2 = \gamma R T$, Mach number $M = V/a$ and $c_p = \gamma R/(\gamma - 1)$. The steady energy balance gives

$$c_p T_0 = c_p T + \tfrac{1}{2} V^2,$$
$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2.$$

For an isentropic ideal gas, p/ρ^γ is constant. Using the gas law to eliminate density yields a pressure ratio as a power of the temperature ratio. This provides a relation between a stagnation tap and a separate static tap at the same upstream state.

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\gamma/(\gamma-1)} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)}.$$

Solve for the nonnegative speed magnitude. A static temperature measurement is needed to convert dimensionless Mach number into dimensional velocity. Expanding the pressure law for small Mach number recovers the familiar dynamic pressure as its leading term.

$$M = \sqrt{\frac{2}{\gamma - 1}\left[\left(\frac{p_0}{p}\right)^{(\gamma-1)/\gamma} - 1\right]},$$
$$p_0 - p = \tfrac{1}{2}\rho V^2 + O(p M^4).$$

Both pressures in the ratio must be absolute. A supersonic probe can form a shock and its recovered stagnation pressure then differs from the reversible upstream value.

WORKED EXAMPLE

Estimate air speed from two pressure readings

For air take $\gamma = 1.40$, $R = 287$ J/(kg K), $T = 290$ K, $p = 90.0$ kPa and $p_0 = 106$ kPa. Their ratio is 1.178, giving $M = \sqrt{5[(106/90)^{2/7} - 1]} = 0.489$. The sound speed is 341 m/s, so the flow speed is about 167 m/s. The incompressible pressure estimate using static density would be higher because the gas compresses while it decelerates.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Predict the pressure ratio of a calibration stream

A second air stream has measured speed 102 m/s and static temperature 288 K, yielding $a = 340$ m/s and $M = 0.300$. At static pressure 100 kPa, the stagnation rise is

$$100[(1 + 0.2(0.300)^2)^{3.5} - 1] = 6.44\,\mathrm{kPa}.$$

The leading incompressible expression predicts

$$0.5\gamma p M^2 = 6.30\,\mathrm{kPa}.$$

Their 0.14 kPa difference sets a scale for the pressure resolution needed to detect compressibility here.

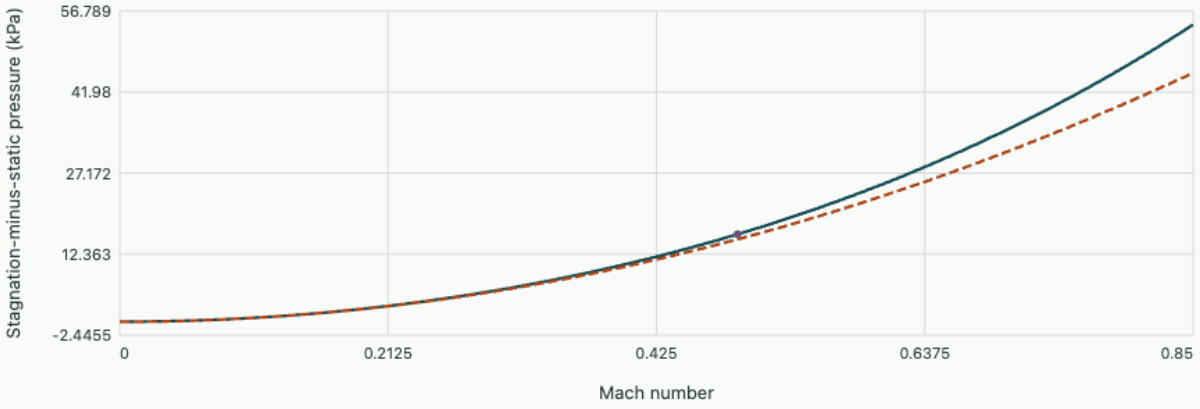
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Static pressure (kPa)	90
Pitot excess pressure (kPa)	16
Static temperature (K)	290



Inferred Mach=0.489; speed=167.0 m/s. The subsonic, shock-free pitot relation is used.

Incorporate expansion work and mass conservation

Gas accelerates through a nozzle by converting enthalpy into kinetic energy. At sufficiently high speeds, density changes appreciably and a constant-density volume balance fails. For steady one-dimensional adiabatic flow without shaft work or appreciable elevation change, $h + V^2/2 = h_0$. A calorically perfect

gas has $h = c_p T$; its local sound speed is $a = \sqrt{\gamma R T}$, and Mach number is $M = V/a$. Evaluate a at local static temperature, rather than at the upstream reservoir temperature.

Assume the acceleration is reversible, so $dp = a^2 d\rho$ along the isentrope. Differential momentum balance gives $dp = -\rho V dV$, hence $d\rho/\rho = -M^2 dV/V$. Inserting this in continuity $d\rho/\rho + dV/V + dA/A = 0$ gives the area–velocity relation

$$\frac{dA}{A} = (M^2 - 1)\frac{dV}{V}.$$

A subsonic stream accelerates through a converging passage. A supersonic stream accelerates through a diverging one. A smooth transition through sonic conditions requires a stationary area, conventionally a minimum-area throat for acceleration from subsonic to supersonic speed. A converging nozzle supplied from rest can first reach Mach one at its smallest area, the exit.

Obtain the throat state from the reservoir

In the upstream reservoir the flow is essentially at rest, so its static pressure and temperature are stagnation values p_0, T_0. Combining $c_p(T_0 - T) = V^2/2$ with $c_p = \gamma R/(\gamma - 1)$ gives

$$T/T_0 = [1 + (\gamma - 1)M^2/2]^{-1}.$$

Apply the ideal-gas isentropic pressure–temperature relation to find the critical pressure and temperature ratios at $M = 1$.

$$T^*/T_0 = 2/(\gamma + 1),$$

$$p^*/p_0 = [2/(\gamma + 1)]^{\gamma/(\gamma - 1)}.$$

With exit area fixed, lowering back pressure increases mass flow until the exit becomes sonic. Further reduction in back pressure does not change the ideal throat state for fixed reservoir conditions. Expansion downstream can then adjust the emerging jet to its surroundings.

Estimate a choked discharge

An insulated converging air nozzle has throat area $50\,\mathrm{mm}^2$ and reservoir conditions 300 kPa absolute and 300 K. Use $R = 287\,\mathrm{J/(kg\,K)}$ and $\gamma = 1.40$. The critical temperature is $300(2/2.4) = 250\,\mathrm{K}$, critical pressure $300(2/2.4)^{3.5} = 158.5\,\mathrm{kPa}$, density $158500/(287 \times 250) = 2.21\,\mathrm{kg/m^3}$ and critical speed $\sqrt{1.4(287)(250)} = 317\,\mathrm{m/s}$.

Atmospheric back pressure of 100 kPa lies below the critical pressure, so the ideal mass flow is

$$\rho^* a^* A^* = 2.21(317)(50 \times 10^{-6}) = 0.0350\,\mathrm{kg/s}.$$

A back pressure of 210 kPa would leave this converging nozzle unchoked, and the exit state must then be calculated from that back pressure. Real contraction and boundary-layer losses can reduce the flow through a calibrated orifice compared with the ideal nozzle prediction; a measured discharge coefficient accounts for those effects within its tested range.

Predict changes at a choked throat

At fixed reservoir pressure, the critical temperature is proportional to T_0, critical density to $1/T_0$ and critical speed to $\sqrt{T_0}$. Their product shows that the choked mass flow scales as $p_0 A^*/\sqrt{T_0}$ for the same gas. Warming the original reservoir from 300 to 360 K at unchanged absolute pressure therefore reduces the ideal mass flow by $\sqrt{300/360} = 0.913$, to about 0.0320 kg/s. Increasing throat area or reservoir pressure has a direct linear effect while the downstream pressure stays low enough to maintain choking.

Derive the admissible jump from conservation laws

Supersonic gas in a duct can decelerate abruptly through a normal shock. The shock is thin compared with the duct dimensions, so treat it as a stationary discontinuity of unchanged area. Take a calorically perfect gas, no heat or shaft work and negligible elevation change. Mass, streamwise momentum and stagnation enthalpy are conserved:

$$\rho_1 V_1 = \rho_2 V_2,$$

$$p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2,$$

$$c_p T_1 + V_1^2/2 = c_p T_2 + V_2^2/2.$$

Entropy increases through the shock, so the two end states cannot be joined with the isentropic pressure–temperature relation. Set $r = \rho_2/\rho_1 = V_1/V_2$. Momentum gives $p_2/p_1 = 1 + \gamma M_1^2 (1 - 1/r)$ and the gas law gives

$$T_2/T_1 = (p_2/p_1)/r.$$

Substitution into the energy equation yields two roots, one for unchanged flow and the compression solution

$$r = \frac{(\gamma+1)M_1^2}{2 + (\gamma-1)M_1^2},$$

$$M_2^2 = \frac{1 + (\gamma-1)M_1^2/2}{\gamma M_1^2 - (\gamma-1)/2}.$$

For $M_1 > 1$ this second state has higher density and lower Mach number. As $M_1 \to 1^+$, every state ratio tends to unity. For very large upstream Mach number the compression ratio approaches $(\gamma+1)/(\gamma-1)$ in the constant-specific-heat approximation, although real-gas changes eventually limit that approximation.

Calculate a diffuser’s stagnation-pressure loss

Air upstream of a normal shock has $M_1 = 2.0,\ p_1 = 80\,\mathrm{kPa}$ absolute and $T_1 = 250\,\mathrm{K}$. Use $\gamma = 1.4,\ R = 287\,\mathrm{J/(kg\,K)}$. The density ratio is

$$2.4(4)/(2 + 0.4(4)) = 2.667,$$

$$p_2/p_1 = 1 + 1.4(4)(1 - 1/2.667) = 4.50.$$

Hence $p_2 = 360\,\mathrm{kPa}$ and $T_2 = 250(4.50/2.667) = 421.9\,\mathrm{K}$. The downstream Mach number is

$$\sqrt{1.8/(5.6 - 0.2)} = 0.577.$$

Upstream speed is approximately 634 m/s and mass conservation gives about 238 m/s downstream.

The stagnation temperature upstream is $250[1 + 0.2(4)] = 450\,\mathrm{K}$; downstream it is $421.9[1 + 0.2(0.577)^2] = 450\,\mathrm{K}$ as required by energy conservation. Evaluate stagnation pressure separately from each local state by a hypothetical isentropic deceleration on its own side. Upstream $p_{01} = 80(1.8)^{3.5} = 626\,\mathrm{kPa}$, whereas downstream $p_{02} = 360(1.0667)^{3.5} = 451\,\mathrm{kPa}$. Recovery $p_{02}/p_{01} = 0.721$ shows that a large static pressure rise can accompany a 28% loss of stagnation pressure. For the same stagnation temperature the entropy rise is $-R\ln(0.721) = 94\,\mathrm{J/(kg\,K)}$, a positive value consistent with the irreversible deceleration.

A pitot tube facing the upstream supersonic flow forms its own shock ahead of the tip. The final stagnation pressure measured there corresponds ideally to isentropic deceleration after the probe’s shock, about 451 kPa here, rather than to the upstream isentropic stagnation pressure of 626 kPa. A conventional subsonic pitot reduction applied to the reading would therefore infer an incorrect upstream speed. By contrast, a static pressure tap placed sufficiently far upstream can still report the 80 kPa static state. The ratio $451/80 = 5.64$ illustrates why the probe configuration must be stated when comparing a pressure ratio with a compressible-flow prediction.

The downstream temperature calculation can also be checked using mass flux. The upstream density is $80000/[287(250)] = 1.12\,\mathrm{kg/m^3}$ and the measured downstream static values imply $360000/[287(421.9)] = 2.97\,\mathrm{kg/m^3}$. Multiplication by the corresponding velocities gives roughly $707\,\mathrm{kg/(m^2 s)}$ on both sides. A calculation conserving stagnation pressure instead would generally fail one of these basic balances even if its pressure trend looked plausible.

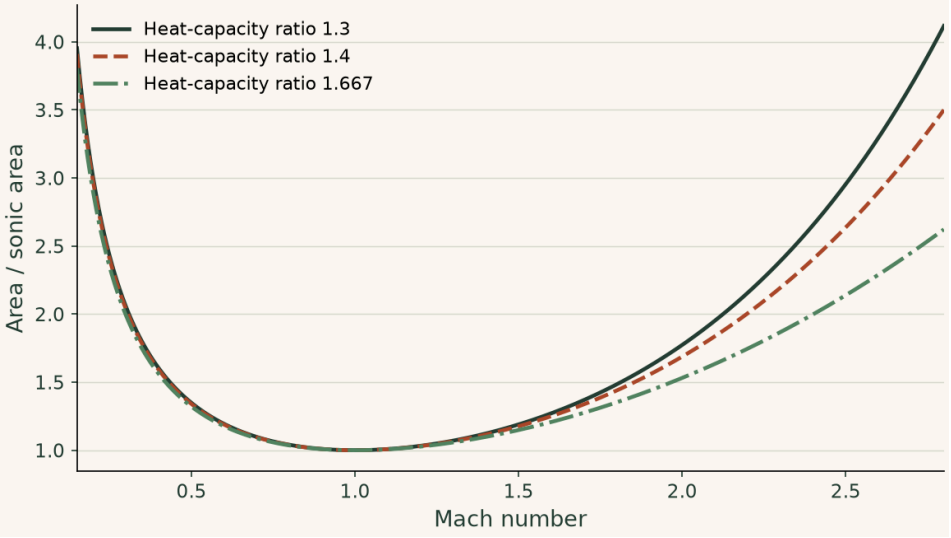
The shock also changes the area needed to choke a downstream passage. For a given gas the choked mass flow scales as $A^* p_0/\sqrt{T_0}$. Stagnation temperature is unchanged but stagnation pressure drops to 72.1% of its upstream value. Passing the same mass flow therefore requires a downstream critical area larger by $1/0.721 = 1.39$. This does not mean the physical duct jumps in area at the normal shock. The critical area is the sonic reference area for an isentropic stream tube drawn from each side, and the two hypothetical reference conditions differ because entropy has increased.

A converging test section downstream that was sized from the upstream stagnation pressure may consequently choke earlier than expected. The entropy and critical-area tests point to the same lost capacity from different directions: one uses the gas state, the other its maximum deliverable mass flux.

Together with conservation of actual mass flux at the shock, they make it possible to distinguish a real shock loss from a simple mismatch between static and total pressure instrumentation.

Find the two states that share a nozzle area ratio

For steady isentropic flow of a calorically perfect gas, combine continuity with stagnation enthalpy and the isentropic pressure law. The critical area A^* is the area giving Mach one for the same mass flow and stagnation state.



At $\gamma = 1.4$ and $A/A^* = 2$ the two solutions are approximately Mach 0.306 and 2.20. Geometry alone does not choose the branch. Reservoir and downstream conditions, and possible shock formation, complete the physical specification.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

Change heat-capacity ratio and target area. Track both Mach branches and their common sonic limit.

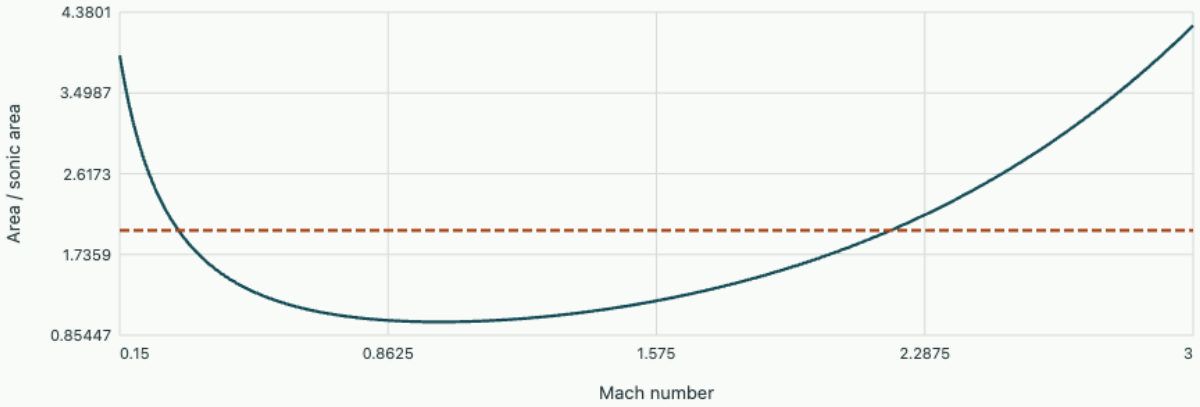
The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Heat-capacity ratio **1.4**

Required area ratio **2**



Isentropic area map Selected area

Subsonic Mach 0.306; supersonic Mach 2.197. Boundary conditions determine which branch can occur.

Problem sheet 6 · Rotor work and compressible discharge

Close angular momentum and nozzle balances while distinguishing measured performance from ideal reference states.

Add inlet pre-swirl

An impeller has $U_1 = 9,\ U_2 = 18$ m/s, exit swirl 12 m/s and positive inlet swirl 3 m/s. Fluid mass flow is 4 kg/s. Find Euler work, power transfer and head. What changes if inlet swirl is reversed at unchanged exit conditions?

▼ **Reveal the worked solution**

Specific work is $18(12) - 9(3) = 189\,\mathrm{J/kg}$; ideal power transfer is 756 W and head 19.27 m. With reversed inlet swirl the term becomes positive in the work difference, giving 243 J/kg, 972 W and 24.77 m. The change reflects a larger increase in the angular momentum of the fluid.

Reconstruct the velocity triangle

At an ideal radial impeller exit the blade speed is 24 m/s, radial fluid velocity 4 m/s and backward blade angle 45°. Calculate exit swirl, absolute speed magnitude and work at zero inlet swirl. If delivered head is 40 m, find the ratio of measured to ideal head.

▼ **Reveal the worked solution**

The blade-relative tangential component is $-4\cot 45^\circ = -4\,\mathrm{m/s}$, giving $V_{\theta 2} = 20\,\mathrm{m/s}$. The absolute speed is

$$\sqrt{20^2 + 4^2} = 20.40\,\mathrm{m/s}.$$

Work is $24(20) = 480\,\mathrm{J/kg}$, or 48.93 m ideal head. The measured head ratio is

$$40/48.93 = 0.818.$$

Compare two back pressures

Air at 400 kPa absolute and 360 K discharges through an ideal converging nozzle of exit area 30 mm². Use $\gamma = 1.4$, $R = 287\,\mathrm{J/(kg\,K)}$. Are back pressures 180 and 250 kPa choked? Find mass flow in the choked case and exit speed in the unchoked case.

▼ **Reveal the worked solution**

Critical pressure is $400(2/2.4)^{3.5} = 211.3\,\mathrm{kPa}$; 180 kPa chokes, 250 kPa does not. At choking $T^* = 300\,\mathrm{K}$, $\rho^* = 2.45\,\mathrm{kg/m^3}$, $a^* = 347.2\,\mathrm{m/s}$, hence $\dot{m} = 0.0256\,\mathrm{kg/s}$. At 250 kPa,

$$T = 360(250/400)^{0.4/1.4} = 314.8\,\mathrm{K}.$$

The energy balance with $c_p = 1004.5\,\mathrm{J/(kg\,K)}$ gives

$$V = \sqrt{2c_p(360 - 314.8)} = 301\,\mathrm{m/s}.$$

Find the operating point and account for the lost head

A pump between two open tanks supplies head $H_p = 24 - 0.020q^2$ metres, where q is measured in litres per second. The upper surface is 4.0 m higher, and pipe losses give $h_L = 0.030q^2$ metres. With water density 1000 kg/m³ and pump efficiency 0.75, find volume flow, electrical input assuming an ideal motor, and the share used to raise the water. Then throttle the discharge by adding $0.030q^2$ of valve loss.

▼ **Worked solution**

Equalising pump and system head gives $0.050q^2 = 20$, or q = 20.0 L/s. The head is 16.0 m, split into 4.0 m elevation and 12.0 m pipe loss. The fluid receives $\rho g Q H$ = 3.14 kW and the assumed motor supplies 4.19 kW. Only 0.785 kW raises gravitational potential. It is one quarter of the fluid power and about 18.8% of the input.

With the valve, $0.080q^2 = 20$ gives 15.81 L/s and a pump head of 19.0 m. Pipe and valve each dissipate 7.50 m. The new fluid power is 2.95 kW and useful lift power is 0.620 kW. Increasing the system resistance reduces flow even as the operating head rises.

An energy check sums elevation, pipe and valve contributions at the new flow. Extrapolating a fixed 75% efficiency beyond this example would need a measured efficiency curve. Also, tank surface velocities are negligible here; a discharge jet carrying appreciable kinetic energy would add an endpoint term.

Explore the governing equations after solving the sheet: Angular momentum and work in a rotor; Compressible nozzles and choking. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Chapter 13 · How a moving wall sends momentum into a liquid

A plate oscillating horizontally in a deep bath has no average velocity. It nevertheless dissipates energy continuously. A velocity measurement at two heights can reveal the fluid's viscosity.

Keep the unsteady term in a parallel flow

Let y increase upward from a very broad, horizontal plate. The liquid is Newtonian and incompressible with constant kinematic viscosity ν; the plate moves parallel to itself with speed $U\cos\omega t$. Ignore edges, any pressure gradient along the plate and transients left from starting the motion. By symmetry the velocity has a single component $u(y,t)$. A parcel has no vertical speed and meets no streamwise velocity variation, so convective acceleration vanishes even though the motion is time dependent. Horizontal momentum balance on a slice of thickness dy compares its acceleration with the shear-stress difference on its two faces.

$$\rho\,dy\,\partial_t u = \tau(y+dy) - \tau(y),$$

$$\tau = \mu\,\partial_y u,$$

$$\partial_t u = \nu\partial_y^2 u.$$

Velocity is prescribed at the plate by no slip and tends to zero far away. The same equation describes thermal diffusion with temperature in place of velocity. Here the quantity transferred is horizontal momentum, and a velocity gradient sets its transfer rate.

Read decay and phase from one complex root

Represent periodic velocity by the real part of $\widehat{u}(y)e^{i\omega t}$. Inserting it into the equation gives

$$\widehat{u}'' = (i\omega/\nu)\widehat{u}.$$

Since $(1+i)^2 = 2i$, the decaying branch of the solution has an exponent with equal real and imaginary parts.

$$\delta = \sqrt{2\nu/\omega},$$

$$u(y,t) = Ue^{-y/\delta}\cos(\omega t - y/\delta).$$

Every increase in height by δ divides the amplitude by e and delays the phase by one radian. This δ is a scale rather than a physical edge. The unbounded solution requires the top and sidewalls to be far enough away for the amplitude there to be negligible. At five penetration depths it is less than one percent of its value at the plate.

Worked example · interpret a submerged velocity probe

In an oil with $\nu = 5.0 \times 10^{-5}$ m²/s, set $f = 2.00$ Hz and $U = 0.080$ m/s. Using $\omega = 2\pi f$ gives $\delta = 2.82$ mm. At 3.00 mm the predicted amplitude is

$$0.080 e^{-3/2.82} = 0.0276\,\mathrm{m/s}.$$

The phase lag is 1.06 rad or 0.0846 s. A synchronized velocity snapshot can therefore show nearly stationary liquid while the wall speed is large. A single instantaneous ratio generally differs from the ratio of fitted amplitudes.

Worked example · power delivered to the wall layer

The wall exerts on the fluid a tangential traction $-\mu u_y(0,t)$. Its complex amplitude is $\mu U(1+i)/\delta$. Only the in-phase part contributes to power averaged over a cycle, since averaging a product of two cosines gives one half of the cosine of their phase difference. Thus

$$\overline{P}/A = \mu U^2/(2\delta).$$

For the same oil at density 900 kg/m³, $\mu = 0.045$ Pa s and the mean power is 0.0510 W/m². The positive value is consistent with viscous dissipation even though mean shear and mean speed separately vanish.

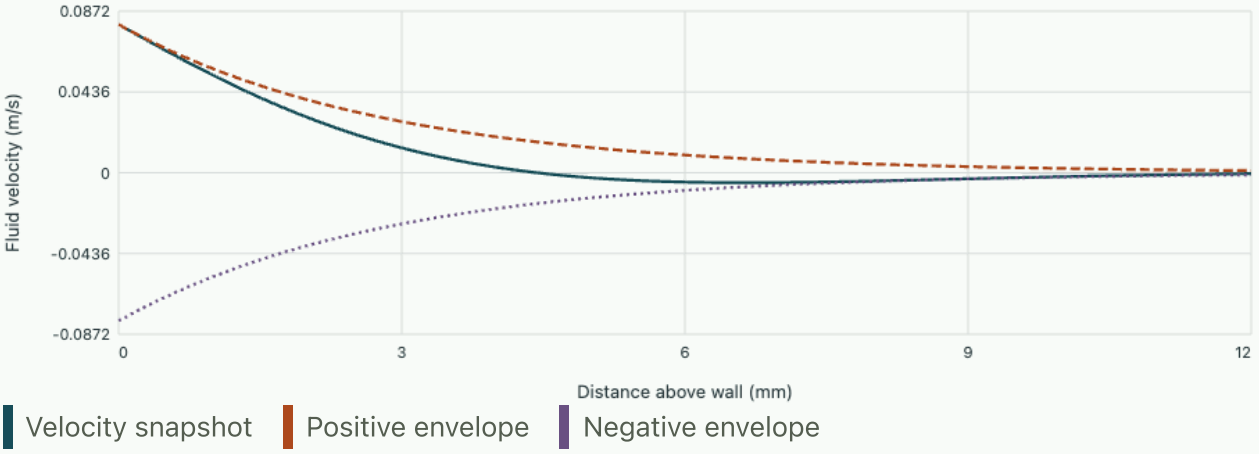
Freeze the phase at a quarter cycle and find the depth at which the velocity first changes sign away from the wall. Sweep the phase to distinguish that moving zero from the stationary amplitude envelope, then double the frequency and predict the new penetration depth.

Open this experiment online

Try the calculation

Wall frequency (Hz) 2

Cycle phase (°) 0



The penetration depth is 2.82 mm. The 3 mm probe has a speed amplitude of 0.0276 m/s and lags by 0.085 s.

A related OneSimulation calculation

The course flat-plate calculation provides a second momentum-diffusion example. Predict what doubling viscosity does to the measured layer thickness at a fixed downstream station, then compare the square-root scaling across the two geometries.

Measure boundary-layer thickness in OneSimulation

Tutorial · estimate a material property without a force sensor

At 1.00 Hz two probes spaced 2.00 mm vertically report fitted amplitudes in the ratio upper/lower = 0.500. Estimate δ, ν, and the phase difference. Identify a measurement that would falsify this model even if the amplitude ratio agreed.

▼ **Use both the magnitude and phase predictions**

The spacing divided by δ is $\ln 2$, giving $\delta = 2.89$ mm. Then $\nu = \omega\delta^2/2 = 2.62 \times 10^{-5}$ m²/s. The upper signal must lag by 0.693 rad, equivalent to 0.110 s. A substantial phase discrepancy suggests confinement, an unremoved transient or motion beyond a one-dimensional shear layer. Fitting both observables is a stronger test than assigning a viscosity to every amplitude pair.

Further reading · MIT 2.25, Stokes' second problem (PDF)

Chapter 14 · A cylinder in potential flow and the absent drag

A pressure field can vary by several dynamic-pressure heads around a body while producing exactly zero resultant drag in an ideal flow. Derive the field and identify what a real measurement must add.

Satisfy two boundary conditions with a potential

Consider steady two-dimensional flow of constant density far from a circular cylinder of radius a. Assume irrotational motion in the exterior, an inviscid fluid and zero circulation. Then $\mathbf{u} = \nabla\phi$ and incompressibility gives

$$\nabla^2\phi = 0.$$

The incoming uniform velocity $U\mathbf{e}_x$ has potential $Ur\cos\theta$. The cylinder cannot absorb fluid, so the radial velocity must vanish at $r = a$. Seek a correction with the same angular dependence and write $\phi = f(r)\cos\theta$. The polar Laplacian reduces to

$$r^2 f'' + r f' - f = 0,$$
$$f = Ar + B/r.$$

The coefficient of the growing term must be U for the far field. The derivative $f' = U - B/r^2$ vanishes at a if $B = Ua^2$. Taking radial and tangential derivatives then gives the complete external velocity field.

$$u_r = U(1 - a^2/r^2)\cos\theta,$$
$$u_\theta = -U(1 + a^2/r^2)\sin\theta.$$

The surface speed is zero at the upstream and downstream points. At either side it reaches $2U$. The tangency condition constrains the normal component alone. At a real viscous wall liquid immediately adjacent to the solid also satisfies zero tangential speed. The ideal surface speed should be interpreted as an outer-flow prediction.

From speed to pressure and from pressure to force

Along streamlines Bernoulli relates static pressure to speed at the same height. Since the incoming flow is uniform and irrotational the same Bernoulli constant applies throughout this ideal solution. Normalize the pressure change by $q = \rho U^2/2$. On the body,

$$C_p = (p - p_\infty)/q = 1 - 4\sin^2\theta.$$

Pressure force on an arc per unit span is $-p(\cos\theta, \sin\theta)a\,d\theta$. Points at θ and $\pi - \theta$ have identical pressures and opposite horizontal normals. The corresponding drag contributions cancel exactly. Top and bottom symmetry cancels lift. A real cylinder can have a separated wake with low rear pressure. Substituting the attached-flow pressure at the rear is the assumption that then fails.

Worked example · read three ports

Water of density 1000 kg/m³ flows at 0.80 m/s, so $q = 320$ Pa. At the upstream stagnation point the model predicts $p - p_\infty = +320$ Pa. At a point where the normal is 45° to the stream axis, $\sin^2\theta = 1/2$, so $C_p = -1$ and the reading is −320 Pa. On the side the predicted reading is −960 Pa. The front-to-side difference is 1280 Pa, four times the incident dynamic pressure. A pressure transducer connected to these two ports requires the correct geometric factor before being interpreted as a speed meter.

Worked example · convert a wake discrepancy into a drag

Approximate a measured departure from the ideal pressure by reducing pressure uniformly by 150 Pa on the downstream semicircle and leaving the upstream half unchanged. For a radius of 50 mm, the added downstream force per span is

$$150 a \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = 15.0 \,\mathrm{N/m}.$$

A span of 0.40 m gives 6.00 N before end effects. This integration illustrates why even a modest rear pressure error changes the net drag materially. Local low pressure on the sides contributes little horizontal force where the outward normals point almost vertically.

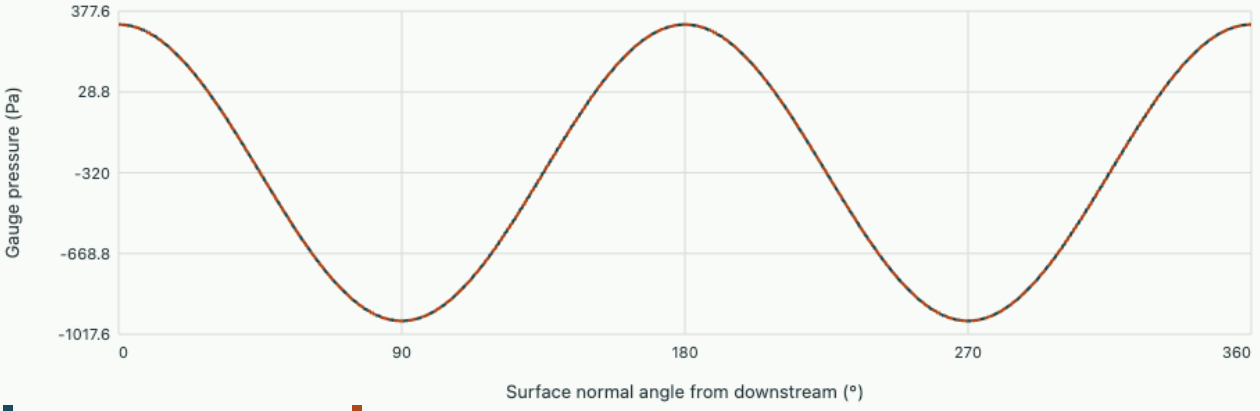
Begin with zero rear pressure deficit and check the pressure symmetry about the transverse axis. Introduce a 160 Pa downstream deficit at 0.80 m/s. Predict the pressure-drag coefficient from the projected area before reading the result.

Open this experiment online

Try the calculation

Incident speed (m/s) **0.8**

Rear pressure deficit (Pa) **0**



The ideal front-to-side difference is 1280 Pa. Lowering pressure on the downstream semicircle gives a pressure-drag coefficient 0.000 based on diameter. The added deficit is an imposed measurement model.

Tutorial · estimate speed and test the symmetry hypothesis

An ideal front-to-side differential is 720 Pa in water. Infer the incoming speed. Two further ports mounted at 60° and 120° report a difference of 90 Pa. Can a different incoming speed make that second difference consistent with the assumed model?

▼ **Separate calibration from model failure**

The front-to-side difference is $4q = 2\rho U^2$, giving $U = 0.600$ m/s. Both other ports have the same $\sin^2\theta = 3/4$. Their ideal pressure difference vanishes at every speed. A discrepancy at these ports therefore tests symmetry and the assumed attached field independently of the first calibration.

Further reading · MIT 2.25, Potential Flow Theory

Chapter 15 · Curvature pressure, wetting and capillary rise

The water height in a narrow glass tube is a measurement of an interface force. Starting from a surface energy explains both its direction and its dependence on tube radius.

Convert a surface-energy change into a pressure difference

A clean liquid–gas interface of fixed composition and temperature requires reversible work $\gamma\,dA$ to increase its area by dA. The surface tension γ therefore has equivalent units J/m² and N/m. Imagine a spherical drop changing radius by dR. Pressure inside does work while the outside pressure resists. Equating net pressure work to the increase in interface energy gives

$$(p_i - p_o)\,dV = \gamma\,dA,$$

$$p_i - p_o = \gamma \frac{8\pi R}{4\pi R^2} = \frac{2\gamma}{R}.$$

For a local patch with two principal radii the same argument gives $\Delta p = \gamma(1/R_1 + 1/R_2)$ with signed curvature and consistently chosen normal. A cylinder has only one nonzero curvature, so the spherical factor of two is inappropriate there. A thin soap bubble has two nearly equal interfaces and doubles the pressure jump for a given radius.

Background · Capillary pressure

$$\Delta p = \gamma(1/R_1 + 1/R_2)$$

Principal curvatures carry signs set by the chosen normal. A cylindrical interface contributes only one finite principal curvature.

Close the balance at the wall

Put a vertical tube of inner radius r into a bath and let θ be the contact angle measured through the liquid. Surface tension along the circular contact line has upward component $2\pi r\gamma\cos\theta$. The excess column over the external free surface has effective weight $(\rho_l - \rho_g)g\pi r^2 h$. Neglect meniscus volume relative to the column and assume equilibrium, a broad bath, a uniform tube and a well-defined contact angle.

$$h = \frac{2\gamma\cos\theta}{(\rho_l - \rho_g)gr}.$$

The sign of h changes when θ exceeds 90°. A smaller tube gives more contact-line length per unit cross-sectional area and therefore supports a taller column. The same equation follows from the pressure jump for a nearly spherical cap of radius $r/\cos\theta$ in magnitude and a hydrostatic pressure change. If gravity distorts the local cap considerably, the full interface shape must be solved. The capillary length $\sqrt{\gamma/(\Delta\rho g)}$ indicates when tube radius is small on the relevant scale.

Worked example · inspect a level reading

Take water with $\gamma = 0.072$ N/m, density difference 1000 kg/m³, radius 0.400 mm and contact angle 30°. The rise is

$$2(0.072)\cos 30^\circ / [1000(9.81)(0.0004)] = 31.8\,\mathrm{mm}.$$

The hydrostatic pressure difference is 312 Pa. The capillary length is about 2.71 mm, comfortably larger than the radius. A reader who assumes perfect wetting would infer 36.7 mm. The uncertainty in surface condition can therefore exceed a millimetre-scale ruler uncertainty.

Worked example · use a second liquid column as a test

In a tube of radius 0.500 mm a liquid of density 850 kg/m³ rises 10.0 mm. The product determined by this experiment is

$$\gamma\cos\theta = \Delta\rho g r h/2 = 0.02085\,\mathrm{N/m}.$$

If an independent measurement gives $\gamma = 0.0300$ N/m, then $\cos\theta = 0.695$, or $\theta = 46.0^\circ$. A second tube of half the radius should approximately double the height on identically prepared walls. It does not by itself separate contact angle from surface tension because the same product appears in both balances.

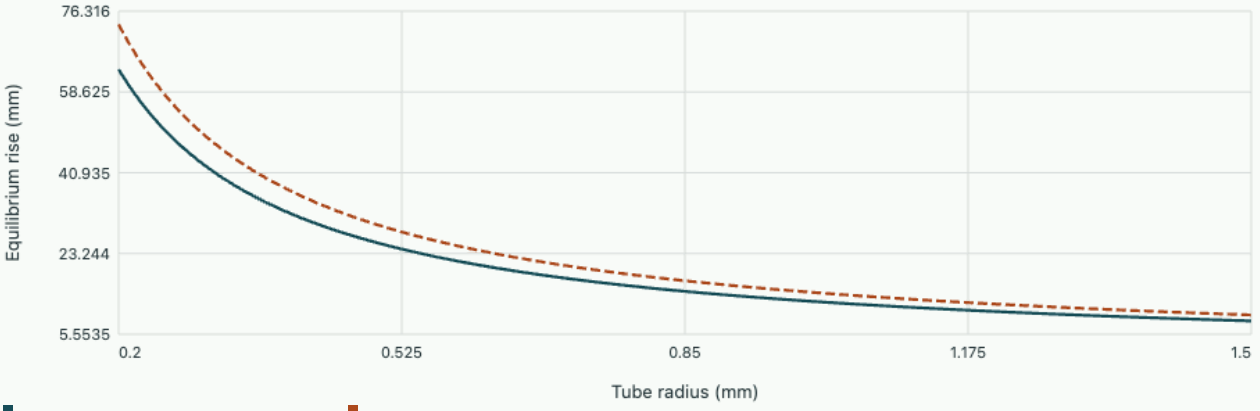
At a fixed radius rotate the contact angle through 90 degrees. Estimate the sign and magnitude just on either side of the crossing. Compare the effect of halving surface tension with doubling radius in the complete-wetting comparison.

Open this experiment online

Try the calculation

Contact angle (°) 30

Surface tension (mN/m) 72



At a radius of 0.400 mm the level change is 31.8 mm relative to the reservoir. Positive values indicate rise; negative values indicate depression.

Tutorial · compare a drop and a bubble

A clean water drop of radius 0.60 mm has $\gamma = 0.072$ N/m. Find its excess pressure. A spherical soap film of the same radius has tension 0.030 N/m at each of its two interfaces. Which object has the larger excess pressure? State a radius at which the soap bubble would match the drop.

▼ **Count each interface**

The drop has one interface and excess pressure $2(0.072)/0.00060 = 240$ Pa. The bubble has two interfaces and $4(0.030)/0.00060 = 200$ Pa. To match 240 Pa its radius must be 0.500 mm. The liquid volume between the bubble interfaces is treated as small enough for both interface radii to coincide.

Further reading · J. Bush, MIT 18.357, Lecture 5, stress boundary conditions and the static meniscus (PDF)

Study the full derivation

Gravity, capillarity and instability at a moving interface · Momentum, motion and transport in fluids

A flat interface can carry waves restored by gravity and surface tension. If its two liquids or gases move at different speeds, a band of those waves can grow. Derive the ideal threshold, identify the most vulnerable wavelength at onset and distinguish growth of an instability from advection of an otherwise stable disturbance.

Chapter 16 · Critical depth and the hydraulic jump

In a wide shallow channel the same discharge can pass as a shallow fast stream or as a deeper slow stream. A gradual hump and an abrupt jump connect depths through different conservation laws.

Minimise the energy needed to carry a discharge

Use a broad rectangular channel and let $q = Q/b$ be its volume flow per unit width. At a section with approximately uniform speed u and water depth h, mass conservation gives $u = q/h$. Take pressure to be hydrostatic over the depth and neglect losses between adjacent smooth sections. Measured relative to the local channel bottom, specific energy in metres of head is

$$E(h) = h + \frac{q^2}{2gh^2},$$

$$\frac{dE}{dh} = 1 - \frac{q^2}{gh^3} = 1 - \mathrm{Fr}^2.$$

The derivative vanishes at $h_c = (q^2/g)^{1/3}$, where $\mathrm{Fr} = u/\sqrt{gh} = 1$. The second derivative $3q^2/(gh^4)$ is positive. Critical depth is the unique minimum, with $E_c = 3h_c/2$. The speed $\sqrt{gh}$ is the speed of small long gravity waves relative to shallow water. When u exceeds it, even a disturbance travelling upstream relative to the liquid is carried downstream relative to the bed.

Background · Froude number

$$\mathrm{Fr} = U/\sqrt{gh}$$

For shallow rectangular flow with gravity waves this compares material speed to wave speed. Critical depth occurs when the two are equal.

Worked example · decide if a smooth rise can pass the flow

A channel carries $q = 0.40$ m²/s at depth 0.50 m. Speed is 0.80 m/s and specific energy is

$$0.50 + 0.80^2/(2g) = 0.5326\,\mathrm{m}.$$

Critical depth is 0.2536 m and the minimum specific energy is 0.3804 m. Since total head includes bed elevation, a local rise of up to about 0.152 m can be accommodated in the ideal fixed-upstream-state model. A higher rise requires the upstream water level or discharge to adjust. For a small rise on the subcritical branch the available specific energy decreases and depth decreases toward the critical value.

Keep momentum across a short rapid transition

A hydraulic jump is turbulent and strongly dissipative. Place control surfaces far enough upstream and downstream for pressure to be hydrostatic again. Across a short horizontal reach, neglect net bed shear compared with the momentum and end-pressure forces. The hydrostatic force on an end section is $\rho g h^2/2$ per unit width, obtained by integrating a triangular pressure profile. Momentum flux is $\rho q^2/h$. Requiring equal momentum functions gives

$$\frac{h_1^2}{2} + \frac{q^2}{g h_1} = \frac{h_2^2}{2} + \frac{q^2}{g h_2}.$$

Factor out $h_2 - h_1$ and take the distinct-depth root. With $r = h_2/h_1$, the remaining relation is

$$r(r+1) = 2\mathrm{Fr}_1^2.$$

The positive solution and associated energy drop are

$$r = \frac{\sqrt{1 + 8\mathrm{Fr}_1^2} - 1}{2},$$

$$E_1 - E_2 = \frac{(h_2 - h_1)^3}{4 h_1 h_2}.$$

The loss is positive for a jump from shallow to deep flow. Reversing the depths would require a gain in mechanical energy without a source, which excludes that continuation of the same algebraic equation.

Worked example · locate the conjugate depth

At a flume section let $h_1 = 0.100$ m and $u_1 = 2.00$ m/s. Then $q = 0.200$ m²/s and $\mathrm{Fr}_1 = 2.02$. The depth ratio is 2.40, giving $h_2 \simeq 0.240$ m and $u_2 \simeq 0.835$ m/s. Upstream energy is 0.304 m and downstream energy about 0.275 m. Momentum balance predicts a head drop near 0.0285 m. Its location along the channel also depends on the imposed tailwater level; the two-section balance itself contains no position variable.

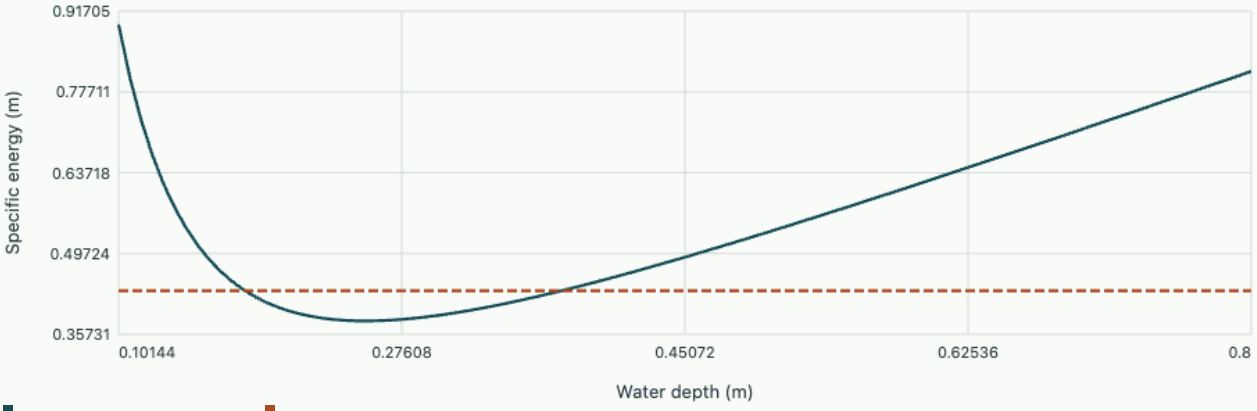
Follow the head over the bump until its horizontal line just touches the energy curve. Estimate the maximum admissible bed rise from the two vertical readings. Predict how changing discharge moves that tangency before changing the input.

Open this experiment online

Try the calculation

Discharge per width (m²/s) **0.4**

Bed rise (m) **0.1**



Critical depth is 0.254 m. The available head over the bump is 0.433 m versus a minimum of 0.380 m. The specified upstream state can pass this discharge.

Tutorial · infer discharge from two depth gauges

Two gauges straddle a stationary jump and report 0.12 and 0.30 m. Use the momentum relation to recover the discharge per unit width, determine which side is upstream and calculate the specific mechanical-energy loss.

▼ **Recover the flow and check its direction**

Factoring the momentum equality gives $q^2 = g h_1 h_2 (h_1 + h_2)/2$, hence $q = 0.2724$ m²/s. The shallow side has speed 2.27 m/s and Froude number 2.09. The deeper side has speed 0.908 m/s and Froude number 0.529. Taking the shallow side upstream yields energy loss $0.18^3/[4(0.12)(0.30)] = 0.0405$ m. Both discharge and the sign of dissipation support this ordering.

Further reading · W. Smyth, All Things Flow, §9.3, hydraulic jumps and bores

Chapter 17 · A viscous squeeze film as a rate-dependent support

Bring two smooth circular disks slowly together with a thin layer of liquid between them. The force needed depends strongly on gap size even when neither surface is elastic.

Decide which motion the gap permits

Take rigid parallel disks of radius R separated by $h(t) \ll R$, filled with an incompressible Newtonian liquid of viscosity μ. The lower disk is fixed and the upper disk approaches at speed $V = -\dot{h} > 0$. Assume an axisymmetric film discharging to ambient pressure at the edge, no slip on both disks and negligible inertia. The latter requires viscous stress to dominate transient and convective inertia, including the scale $\rho V h/\mu \ll 1$. Fluid displaced from the centre must travel sideways through the thin film. Across-gap derivatives dominate radial derivatives of velocity, and the leading pressure is uniform through the thickness.

$$\mu\, \partial_z^2 u_r = \frac{dp}{dr},$$

$$u_r(0) = u_r(h) = 0.$$

Two integrations give

$$u_r = (dp/dr) z(z-h)/(2\mu).$$

The pressure gradient must be negative for outward flow. Integrating velocity over the gap gives radial volume flow per circumferential length $q_r = -h^3 (dp/dr)/(12\mu)$. The cubic power comes from a velocity scale proportional to h^2 multiplied by the available gap h.

Balance the displaced volume inside every circle

The film inside radius r loses volume at rate $\pi r^2 V$. With no compressibility or leakage through the disks this must equal $2\pi r q_r$. Hence $q_r = rV/2$. Match to the viscous flow relation, integrate inward from $p(R) = p_a$, and sum the excess pressure over the wetted disk:

$$\frac{dp}{dr} = -\frac{6\mu V r}{h^3},$$

$$p(r) - p_a = \frac{3\mu V}{h^3}(R^2 - r^2),$$

$$F = \int_0^R [p(r) - p_a]\, 2\pi r\, dr = \frac{3\pi \mu V R^4}{2h^3}.$$

The pressure distribution is parabolic; its area average is half its centre value. A fixed-speed actuator sees a rapidly increasing load as the film narrows. Conversely a constant load makes the approach slower. An experiment must state which of speed or force is controlled before two records can be compared.

Worked example · halve a film thickness

Let $R = 20$ mm, $h = 0.20$ mm, $\mu = 0.10$ Pa s and $V = 0.10$ mm/s. The centre excess is

$$3(0.10)(10^{-4})(0.020)^2/(0.00020)^3 = 1500\,\mathrm{Pa}.$$

The net force is 0.942 N. Reducing the gap to 0.10 mm at unchanged speed raises pressure and force by eight, giving 12.0 kPa and 7.54 N. Reducing speed by eight at the thinner gap would recover the original force. The model consequently distinguishes a viscous support from a spring by repeating the same gap at different rates.

Worked example · integrate the approach at constant load

Rearrange the force law to $-dh/dt = 2Fh^3/(3\pi\mu R^4)$. Separation of variables gives

$$t = [3\pi\mu R^4/(4F)](h_f^{-2} - h_i^{-2}).$$

Under a constant 1.00 N load the example disk takes 2.83 s to go from 0.20 to 0.10 mm. The ideal time to reach zero mathematical thickness diverges. Real surfaces eventually develop solid contact at roughness peaks, and departures from parallel alignment can redirect much of the pressure distribution.

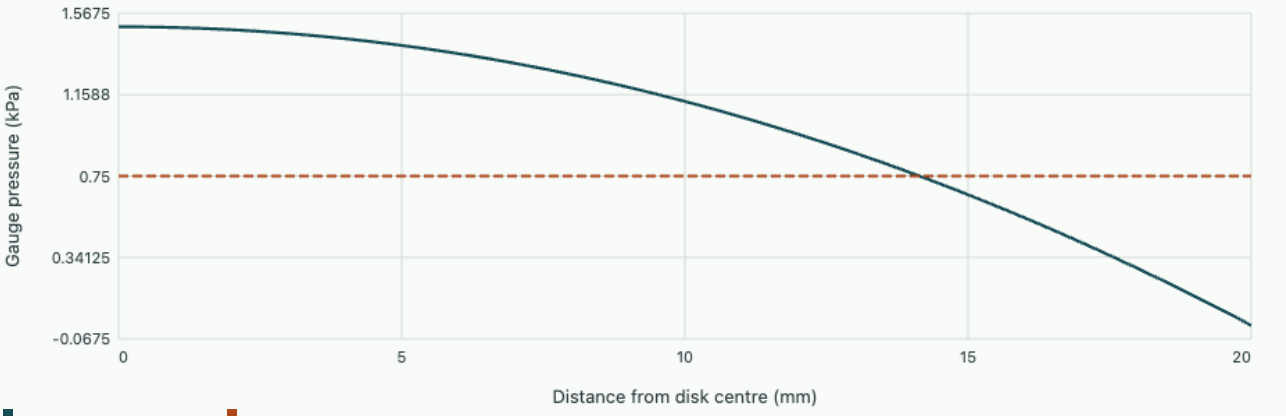
First halve the gap and then independently halve the approach speed. Predict how the centre pressure changes in each trial. Check the area-average line against the radial curve and explain why an unweighted average over radius gives a different value.

Open this experiment online

Try the calculation

Disk gap (mm) **0.2**

Approach speed (mm/s) **0.1**



The central excess pressure is 1.500 kPa and the integrated disk load is 0.942 N.

Tutorial · identify a viscosity and test the boundary assumption

At gap 0.15 mm, disks of radius 15 mm give 2.00 N at 0.25 mm/s. Infer the viscosity. Predict the load if a vented outer annulus reduces the effective wetted radius to 12 mm while the gap and velocity stay fixed.

▼ **Integrate over the wetted radius**

The inverse is $\mu = 2Fh^3/(3\pi V R^4) = 0.113$ Pa s. With the edge now open to ambient at 12 mm, force scales by $(12/15)^4 = 0.4096$, giving 0.819 N. Simply multiplying the original force by the area ratio would miss the change in peak pressure caused by the shorter flow path.

Further reading · MIT 2.25, §6.4, locally developed viscous flows and lubrication

Chapter 18 · How turbulent momentum transport produces a logarithmic velocity profile

The time-averaged velocity next to a wall looks smooth, even when individual velocity records fluctuate rapidly. Decompose the momentum flux, derive the wall scaling, and recover a shear stress from two measurements in the logarithmic region.

Where the turbulent shear stress comes from

For an incompressible constant-density fluid write $u = U + u'$ and $v = V + v'$, with uppercase velocities time averages and mean fluctuations zero. The flux of streamwise momentum carried in the wall-normal direction contains $\rho\,\overline{uv} = \rho(UV + \overline{u'v'})$. In a statistically steady parallel mean flow with $V = 0$, the apparent shear transporting streamwise momentum toward the wall becomes

$$\tau_{\text{total}} = \mu\frac{dU}{dy} - \rho\,\overline{u'v'}.$$

Upward moving near-wall fluid often carries lower-than-average streamwise momentum and descending fluid carries higher-than-average momentum. Both events have negative $u'v'$ in this sign convention, producing a positive turbulent shear. At a stationary impermeable no-slip wall fluctuations vanish and the entire wall stress is viscous, even when turbulence dominates a short distance away.

Match the inner scales and a constant-stress layer

Define $u_\tau = \sqrt{\tau_w/\rho}$, $y^+ = yu_\tau/\nu$ and $U^+ = U/u_\tau$. Close enough to the wall $dU/dy \simeq \tau_w/\mu$, so $U^+ \simeq y^+$. Farther out, provided y is still small compared with the outer flow thickness, take $\tau_{\text{total}} \simeq \tau_w$ and viscous stress small. Prandtl's mixing-length closure approximates the turbulent shear by $\rho\ell_m^2|U'|U'$. The near-wall dimensional assumption $\ell_m = \kappa y$ gives

$$\frac{dU}{dy} \simeq \frac{u_\tau}{\kappa y},$$

$$U^+ = \frac{1}{\kappa}\ln y^+ + B.$$

The constant B must come from matching or measurements. Use illustrative smooth-wall values $\kappa = 0.41,\ B = 5.0$. The line $U^+ = y^+$ applies in the viscous sublayer and the log line only in an overlap region (often beyond about $y^+ = 30$ and well inside the outer layer). In between lies a buffer region. Extending a logarithm to the wall would violate the no-slip condition. Roughness and adverse pressure gradients alter these comparisons.

Predict a difference, then use it as a stress measurement

Take water with $\rho = 1000$ kg/m³, $\nu = 10^{-6}$ m²/s and $u_\tau = 0.040$ m/s. Wall stress is 1.60 Pa. At $y = 0.05$ mm, $y^+ = 2$, so the viscous prediction is $U = 0.080$ m/s. At 1.0 mm and 4.0 mm, $y^+ = 40, 160$. If both points are in the overlap region, their velocity difference is $(0.040/0.41)\ln 4 = 0.1353$ m/s; B cancels.

The individual predicted speeds are about 0.560 and 0.695 m/s. A log-linear plot over many positions is a stronger test than fitting a line to just two values.

Suppose two probes 0.80 and 3.20 mm from a similar wall instead read a difference of 0.100 m/s. The inferred friction velocity is $0.41(0.100)/\ln 4 = 0.0296$ m/s and the inferred stress is 0.875 Pa. The lower probe then has $y^+ = 23.7$, which makes the logarithmic assumption questionable there. This check feeds the inferred parameter back into the condition for the inference. Repositioning both probes at larger wall distances, while staying away from the outer flow, supplies an independent test.

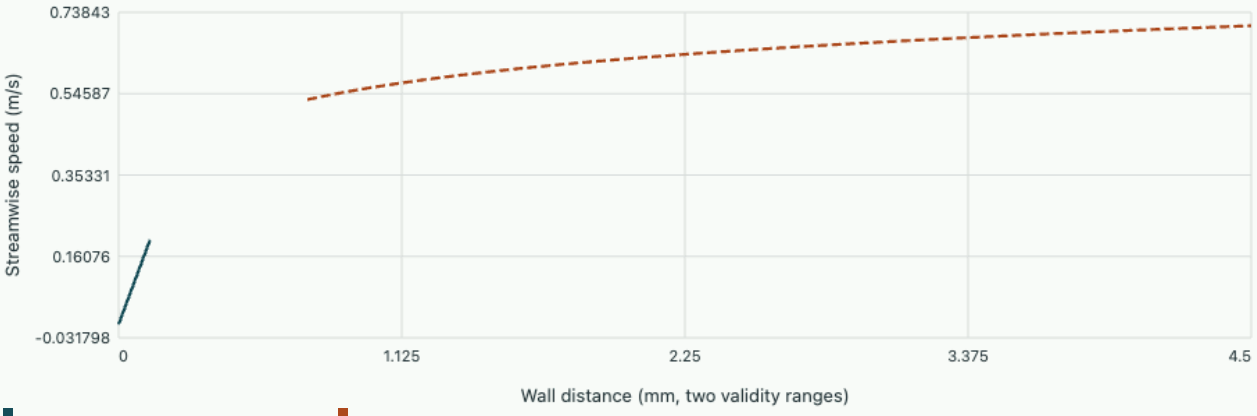
Increase viscosity while holding friction velocity fixed. Predict how the two displayed validity intervals move in millimetres. Use the reported wall distances in viscous units to decide whether a logarithmic fit through fixed probes is still plausible.

Open this experiment online

Try the calculation

Friction velocity (m/s) **0.04**

Kinematic viscosity (mm²/s) **1**



In water the wall stress is 1.600 Pa. At 1 and 4 mm, y⁺ is 40.0 and 160.0. Where both points lie in the overlap layer the predicted speed difference is 0.1352 m/s.

A related OneSimulation calculation

Use the laminar flat-plate calculation to compare two definitions of wall shear. Infer the wall gradient from viscosity and shear, then consider which additional velocity covariance enters an averaged turbulent momentum balance.

Read the OneSimulation flat-plate fields

Tutorial · compare pressure-drop and velocity-gradient estimates

A fully developed round pipe of radius 25 mm has $-dp/dx = 160$ Pa/m. Derive its wall shear by balancing axial pressure force and perimeter friction. Predict the increase between two overlap-layer

points whose distances have a ratio of three.

▼ **Derive the wall force and velocity increment**

For pipe length L the pressure force is $(-dp/dx)L\pi R^2$, balanced by $\tau_w 2\pi RL$, hence

$$\tau_w = (-dp/dx)R/2 = 2.0\,\mathrm{Pa}.$$

In water $u_\tau = 0.0447$ m/s. The expected speed increase is

$$u_\tau \ln 3/0.41 = 0.120\,\mathrm{m/s}.$$

The pressure-force result remains an exact mean balance under the fully developed assumptions. The profile comparison additionally depends on the closure and the placement of both probes.

Further reading · ETH Zürich · turbulence lecture, law of the wall and logarithmic layer

Study the full derivation

Estimate wall drag by integrating a boundary-layer profile · Momentum, motion and transport in fluids

A velocity probe can resolve the momentum missing from a boundary layer more easily than the wall shear itself. Derive an integral momentum relation, estimate drag using a polynomial profile and determine when a pressure gradient or a measurement too far from the wall changes the answer.

Chapter 19 · How gravity and surface tension select different wave speeds

A long crest and a millimetre-scale ripple on the same deep liquid can travel at different speeds. Derive their dispersion relation from two surface boundary conditions and distinguish the motion of a crest from the motion of a narrow packet.

Continue the surface motion down into the liquid

Use z positive upward, liquid occupying $z < 0$, and an undisturbed planar free surface at zero. Neglect viscosity and consider small-amplitude irrotational motion. The velocity is $\nabla\phi$; incompressibility requires

$$\nabla^2\phi = 0.$$

A horizontal Fourier component $e^{i(kx-\omega t)}$ with $k > 0$ therefore has vertical dependence e^{kz} in water deep compared with $1/k$. The other Laplace solution grows without bound as $z \to -\infty$ and is discarded. Let η denote surface elevation. Linearising the kinematic condition at z=0 gives $\eta_t = \phi_z$, since the surface moves with the normal fluid velocity.

Translate curvature and elevation into surface pressure

For small slope the liquid surface pressure above atmospheric is $-\gamma\eta_{xx}$, positive beneath a crest. Linear Bernoulli with the hydrostatic shift at the rising surface gives $\phi_t + g\eta - (\gamma/\rho)\eta_{xx} = 0$ at z=0. For amplitudes $\Phi, \hat{\eta}$, the kinematic condition gives

$$-i\omega\hat{\eta} = k\Phi.$$

Insert this in the dynamic condition to eliminate Φ:

$$\omega^2 = gk + \frac{\gamma}{\rho}k^3,$$

$$c_p = \frac{\omega}{k} = \sqrt{g/k + \gamma k/\rho},$$

$$c_g = \frac{d\omega}{dk} = \frac{g + 3\gamma k^2/\rho}{2\omega}.$$

For long deep-water gravity waves $c_g \simeq c_p/2$. In the opposite capillary limit $c_g \simeq 3c_p/2$. Thus tracking the envelope of a short group and timing an individual crest can give different speeds even with accurate instruments. Finite depth would multiply the right-hand side for ω^2 by $\tanh(kH)$; deep-water long-wave asymptotics still require $kH \gg 1$.

Find the slowest crests and time a wave packet

For water take $\rho = 1000$ kg/m³ and $\gamma = 0.072$ N/m. Differentiating c_p^2 with respect to k gives a stationary point at $k_* = \sqrt{\rho g/\gamma}$, so

$$\lambda_* = 2\pi\sqrt{\gamma/(\rho g)} = 17.0\,\mathrm{mm}.$$

Here gravity and capillarity contribute equally to ω^2; the minimum phase speed is 0.231 m/s. Also $d(kc_p)/dk = c_p + k\,dc_p/dk = c_p$ at this point. Thus group and phase velocities coincide locally.

At 100 mm wavelength the predicted phase and group speeds are 0.401 and 0.212 m/s. At 10 mm they are 0.247 and 0.307 m/s. Over a 0.50 m path a narrow 10 mm packet envelope takes about 1.63 s, whereas timing a crest across the same distance would give 2.03 s if it stayed observable. Surface contamination changes effective restoring and damping behaviour, and dispersion broadens a packet with appreciable bandwidth.

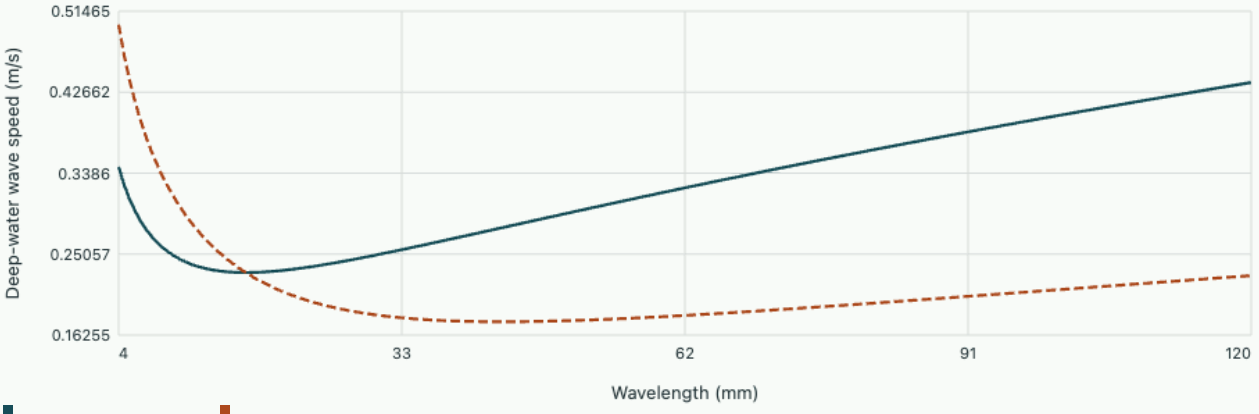
Find where phase and group speed meet, then halve the surface tension and predict the new wavelength of that crossing. Time an individual short crest and its packet over half a metre. Which arrives first on either side of the crossing?

Open this experiment online

Try the calculation

Surface tension (mN/m) **72**

Wavelength to time (mm) **10**



For a 10 mm wavelength a packet crosses 0.50 m in 1.63 s. Phase speed is 0.247 m/s. The minimum occurs at 17.02 mm.

Tutorial · recover surface tension from a measured frequency

A sinusoidal mode of wavelength 12.0 mm oscillates at 20.0 Hz in deep water of density 1000 kg/m³. Infer its surface tension using the full relation and estimate the error caused by discarding gravity. Which depth would make the deep-water approximation credible?

▼ **Solve for the constitutive parameter**

With $k = 523.6$ m⁻¹ and $\omega = 125.66$ rad/s, $\gamma = \rho(\omega^2 - gk)/k^3 \simeq 0.0742$ N/m. Omitting gravity gives about 0.110 N/m, roughly 48% high. At depth 10 mm, $kH = 5.24$ and $\tanh(kH)$ exceeds 0.9999, making the finite-depth correction very small in this model. Frequency, wavelength and depth should all be measured before attributing a discrepancy to surface chemistry.

Further reading · John Bush, MIT · lecture 8 on water waves, curvature and dispersion

Study the full derivation

Gravity, capillarity and instability at a moving interface · Momentum, motion and transport in fluids

A flat interface can carry waves restored by gravity and surface tension. If its two liquids or gases move at different speeds, a band of those waves can grow. Derive the ideal threshold, identify the most vulnerable wavelength at onset and distinguish growth of an instability from advection of an otherwise stable disturbance.

Chapter 20 · Watch a wetting front approach the capillary height

Dip two clean glass tubes of different inner radii into the same wetting liquid. A microscope following only the first few millimetres of rise can give a different impression of which tube fills more readily than a photograph taken at equilibrium. Start with the force on one meniscus, derive a transient relation including gravity, and learn which measurements can distinguish a change of viscosity from a change of surface condition.

Balance the surface force against a pressure difference

Let a be the tube radius and θ the advancing contact angle measured through the liquid. The vertical component of surface tension along a circular contact line is $2\pi a\sigma\cos\theta$. Dividing by the cross-sectional area equates this force to a capillary pressure magnitude $p_c = 2\sigma\cos\theta/a$. The liquid pressure immediately below a wetting meniscus is below the gas pressure. An equivalent derivation uses a spherical cap and the Young–Laplace pressure jump with radius of curvature $a/\cos\theta$. Both derivations require a meniscus whose transverse shape is set mainly by capillarity, so a should be small compared with the capillary length $\sqrt{\sigma/(\rho g)}$.

Take z to be the height of the meniscus above an effectively infinite open reservoir. The bottom and the gas above the meniscus share the same external atmospheric pressure. A static column can sustain the hydrostatic head $\rho g z$. Equating that head with p_c gives the equilibrium height $H = 2\sigma\cos\theta/(\rho g a)$. A wider tube has a smaller equilibrium rise. Viscosity does not enter this statement: after flow stops, there is no viscous pressure drop.

This equilibrium construction implicitly neglects the small volume in the curved meniscus, any entrance flow below the reservoir surface, and changes in the contact angle as the wall wets. These corrections matter when the rise is only a few tube radii. They can also matter in comparing two nominally identical capillaries cleaned in different ways. A practical measurement should define whether the microscope tracks the meniscus centre, a contact line or an optical threshold. An inconsistent definition looks like an offset in the inferred height.

Relate the instantaneous capillary force to wall shear

Farther inside a long circular tube the local pressure gradient sets a parabolic velocity profile for steady, fully developed laminar flow. Applying this relation quasi-steadily to a column of length z, the pressure loss is $8\mu z\,\bar{u}/a^2$. Conservation of liquid volume in a tube of uniform cross-section identifies $\bar{u} = \dot{z}$. The pressure remaining to drive viscous motion is the capillary jump less the hydrostatic head. Therefore

$$\frac{8\mu z}{a^2}\frac{dz}{dt} = p_c - \rho g z = \rho g(H - z).$$

The driving pressure falls as the column grows, and its hydraulic resistance grows at the same time. These two effects account for the long final approach to equilibrium. Introduce $B = \rho g a^2/(8\mu)$, which has dimensions of velocity. The resulting first-order equation is

$$\dot{z} = B(H - z)/z.$$

Close to the entrance it predicts an arbitrarily large speed when z tends to zero. The approximation has discarded fluid inertia, entrance losses and moving contact-line dynamics. Treat the early viscous law as an intermediate asymptotic once a wetted column exists. A good fit late in the record is no evidence for that divergent initial prediction.

To integrate the differential equation it is simplest to regard time as a function of height. Set $s = z/H$ and $\tau = H/B$. Then $dt/\tau = s\,ds/(1-s)$. Integrate from zero to s, choosing the nominal clock origin at vanishing length. The result is

$$\frac{t}{\tau} = -s - \ln(1-s), \qquad 0 \le s < 1.$$

The right side is increasing. The notebook uses bisection to invert it for height at a specified laboratory time. The two bracket endpoints have clear physical meanings, zero rise and the equilibrium rise. Near $s = 0$, direct subtraction of s from $-\ln(1-s)$ can discard significant digits. Evaluating the logarithm with `log1p` and using the power series for very small s preserve the small positive elapsed time.

Open editable notebook Download .ipynb

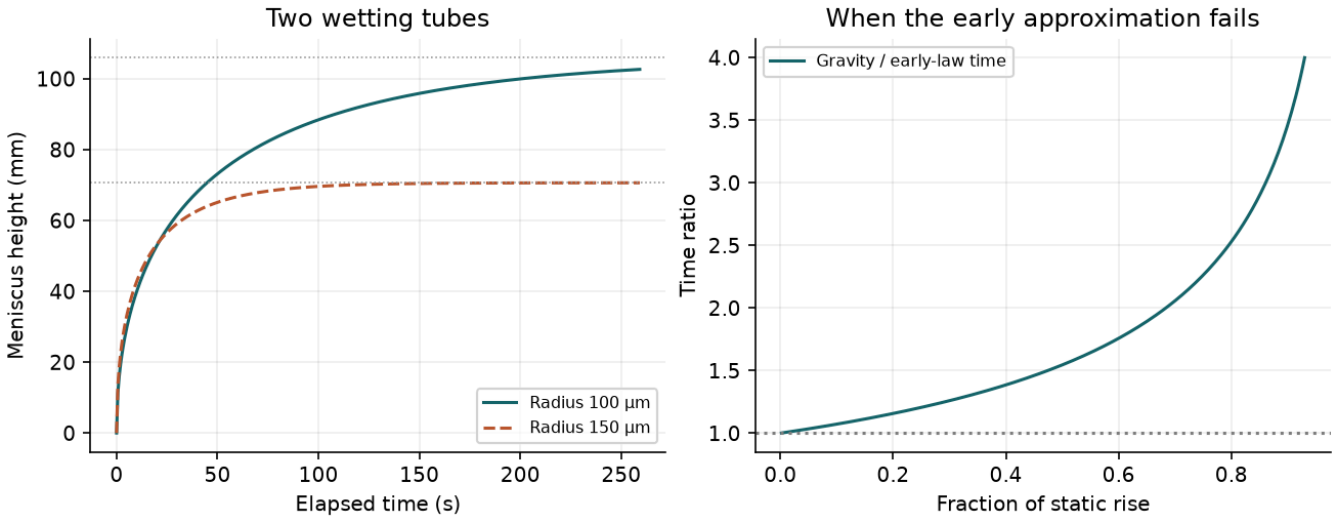
Read two distinct time laws from one expression

Expanding the logarithm gives

$$t/\tau = s^2/2 + s^3/3 + \cdots.$$

To leading order $z^2 = 2HBt = a\sigma\cos\theta\, t/(2\mu)$. At a common early time, a wider tube therefore has a larger height proportional to $\sqrt{a}$. It has less capillary pressure but its resistance per unit length falls rapidly with radius. An experiment that accelerates the initial uptake by increasing radius can simultaneously reduce the final available lifting height.

Near equilibrium write $s = 1 - \epsilon$. The implicit solution reads $t/\tau = -1 + \epsilon - \ln\epsilon$. To leading order $\epsilon \simeq \exp[-1 - t/\tau]$. The final height is approached exponentially and is never reached in finite time by this idealisation. Fitting a straight line in z^2 against t across this late part of the record will overestimate the viscosity if gravity is omitted, because it attributes the slower rise entirely to viscous resistance. A semilog plot of $H - z$ is a more appropriate late-time diagnostic, although uncertainty in H becomes more important as the difference shrinks.



A wider capillary can lead over a short interval yet finish lower. The time to a prescribed fraction of static rise departs markedly from a square-root estimate well before the column is near its final height.

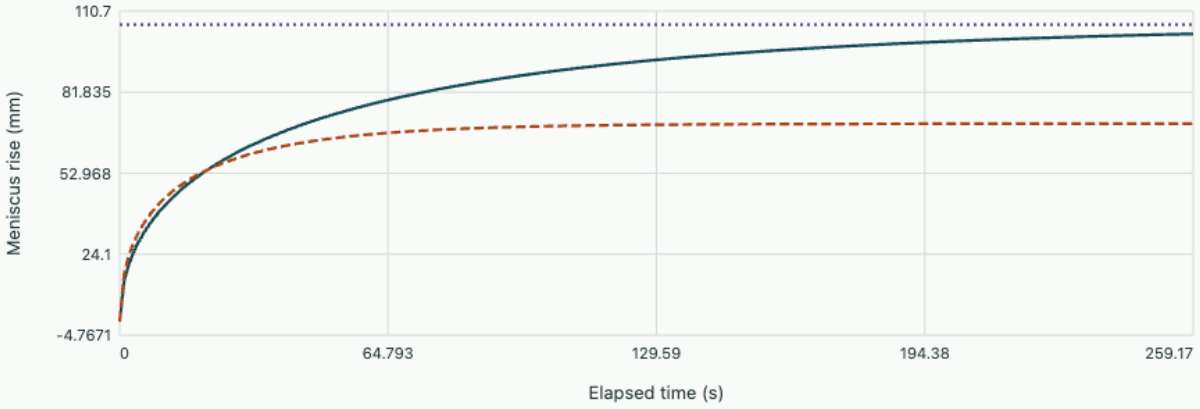
Interactive calculation · Capillary rise

Change the physical and numerical inputs of the notebook model. Compare the updated curves and balance checks with the recorded notebook figures. Each plot uses the same equations and assumptions documented in that notebook; its readout names any analytical or reduced-model reference.

Open this live calculation

Try the calculation

Tube radius (µm)	100
Viscosity (mPa s)	12
Contact angle (degrees)	30



Static rise 105.936 mm; time to half-height 20.023 s. Fixed surface tension0.060 N/m; density1000 kg/m³; inertia and dynamic contact angle are omitted.

Download this example's editable notebook

For $a = 100$ µm, $\mu = 0.012$ Pa s, $\rho = 1000$ kg/m³, $\sigma = 0.060$ N/m and $\theta = 30^\circ$, the predicted static height is 105.94 mm. The time scale is about 103.7 s. Reaching 25%, 50% and 90% of that height takes approximately 3.91, 20.02 and 145.40 s. The time to half height is not half the time to a specified near-equilibrium height. At half height the exact dimensionless time is

$$\ln 2 - 1/2 = 0.19315.$$

The leading square-root estimate gives only 0.125 and underestimates the time by about 35% relative to the exact result.

Use a static observation and a moving observation together

In a high-viscosity sample a single short-time image constrains only the combination $\sigma\cos\theta/\mu$ if all you have is the early growth rate. A change of surface chemistry and a change of viscosity can produce the same slope. A static-height measurement independently fixes $\sigma\cos\theta$ for a known radius, density and gravity. Combining it with the time to half height gives a viscosity estimate $\mu=\rho g a^2 t_{1/2}/[8H(\ln 2-1/2)]$. The two measurements have different parameter dependences, which is what makes the inverse calculation possible.

Consider the limitations of this estimate. A slowly falling reservoir shifts the elevation origin and changes the effective head. Evaporation from the front can drive a persistent flow even with an apparently stationary meniscus. A wall with contact-angle hysteresis can hold a range of static heights. Repeating the experiment after prewetting the tube changes which contact angle is relevant. Observing both the growing profile and the final plateau helps locate these effects. Reducing the residual of a single two-parameter fit can otherwise conceal them.

Estimate viscosity using only passage times at two marks

A camera need not capture the instant of immersion if the tube carries two calibrated height marks. Select marks at $z_1=0.25H$ and $z_2=0.50H$ using an independent long-time reading of H. If the camera clock has an unknown constant offset, subtract the recorded times. The offset cancels exactly. With $F(s)=-s-\ln(1-s)$ the model gives

$$t_2-t_1=\tau[F(0.50)-F(0.25)].$$

The bracket is about 0.15547. A measured interval of 16.11 s therefore gives $\tau=103.6$ s. Substituting this value into $\mu=\rho g a^2\tau/(8H)$ gives approximately 0.0120 Pa s for the stated dimensions. The two-mark construction does not cancel a time-varying camera rate or an error in the static height.

Where should the marks be placed? Two marks very close together make the absolute timing uncertainty large relative to the measured interval. Setting the second mark extremely near equilibrium can also be troublesome because a small height uncertainty then produces a large uncertainty in passage time. Differentiate the dimensionless time to obtain

$$F'(s)=s/(1-s).$$

At half height its slope is one, while at ninety percent it is nine. The same one-percent error in fractional height produces approximately nine times as much time uncertainty at the latter mark as at half height. Use the movie to estimate a passage-time uncertainty from the meniscus location itself, alongside the uncertainty of one video frame.

The notebook also feeds three specified camera times into OneSimulation’s vectorised bracketed inverse in a local installation. After identifying the corresponding heights, substitute them back into the time law. This checks the inversion independently of a visual match between two trajectories.

Exercises on scaling and interpretation

1. Enlarge the tube radius by 50% while holding all material parameters fixed. Predict the ratios of final height, initial square-root coefficient and time to half of the new height.

▼ **Work through the solution**

The static height is inversely proportional to radius, so the height ratio is $2/3$. The coefficient in $z = C\sqrt{t}$ scales as $\sqrt{a}$, giving $C_{\mathrm{new}}/C_{\mathrm{old}} = \sqrt{1.5} = 1.225$. The time scale behaves as $\tau \propto a^{-3}$. Reaching half of the smaller final height now takes about $20.02/(1.5)^3 = 5.93$ s. The two experiments reach different absolute heights at their respective half-height times.

---

2. In a tube of known radius the static rise is unchanged after a fluid is replaced, but the time to 50% height doubles. What quantity changes in this model?

▼ **Work through the solution**

At fixed density, gravity and radius the observed H fixes $\sigma\cos\theta$. The dimensionless time at 50% is fixed, so a doubling of $t_{1/2}$ doubles τ and hence doubles the viscosity. If density also changes, the measurements constrain the kinematic combination μ/ρ and the capillary combination $\sigma\cos\theta/\rho$, so an independent density measurement is needed to identify dynamic viscosity.

---

3. A plot of z^2 versus time is linear over the first recorded interval. Does that identify the contact angle uniquely?

▼ **Work through the solution**

Its slope is $a\sigma\cos\theta/(2\mu)$ in the gravity-free viscous regime. Knowing only the radius leaves several unknown factors. Independent measurements of viscosity and surface tension would give $\cos\theta$. Even then, a dynamic advancing angle could differ from the static angle, so compare with the plateau and inspect whether the inferred angle varies with the chosen fit interval.

Read the original capillary-flow argument

Washburn's paper The Dynamics of Capillary Flow develops the connection between viscous motion and capillary pressure. In comparing radii, first decide whether the experimental observable is a penetration length, a transported volume or a fraction of equilibrium height.

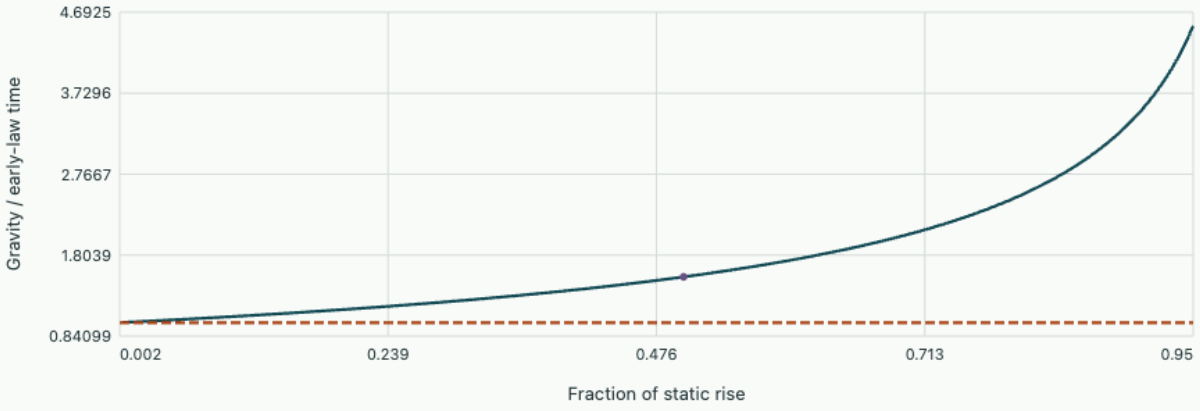
Interactive calculation · when gravity invalidates the early rise law

Select a target height and compare the exact implicit rise time with the early square-root approximation. The radius and viscosity affect dimensional times even though the dimensionless correction collapses onto one universal curve.

Open this live calculation

Try the calculation

Tube radius (µm)	100
Viscosity (mPa s)	12
Contact angle (degrees)	30
Selected fraction of static rise	0.5



At 50% of static rise (52.97 mm), gravity requires 20.02 s versus the early estimate 12.96 s. Radius, viscosity and wetting change these dimensional scales; the dimensionless correction is universal.

Download the editable notebook and compare both observables

Problem sheet 7 · A capillary that fills horizontally and climbs vertically

A transparent tube draws a wetting liquid from a large reservoir. Combine a static meniscus balance with a viscous pressure-drop law to predict a transient, recover a property from timing and determine where the simplest startup model fails.

1 · Predict the final height before timing the rise

The inner radius is 0.200 mm, surface tension 0.072 N/m, contact angle 30°, density 1000 kg/m³ and dynamic viscosity 0.010 Pa s. Neglect gas viscosity and assume the radius is small relative to the capillary length. Calculate the pressure head supplied by the meniscus and the final vertical rise.

▼ **Calculate pressure and static elevation**

A spherical-cap meniscus gives

$$\Delta p = 2\gamma \cos\theta / r = 624\,\mathrm{Pa}.$$

Hydrostatic pressure balances it at $H = \Delta p/(\rho g) = 63.6$ mm. Viscosity does not enter this equilibrium balance. A liquid with the same density and surface properties but twice the viscosity therefore arrives at the same final level under these assumptions.

Read the 0.20 mm capillary radius on the equilibrium curve

2 · Turn a pressure drop into a filling-time law

First lay the tube horizontally. Denote the filled length by x. In developed laminar flow the mean speed is $r^2\Delta p/(8\mu x)$. With constant tube area, the speed of the meniscus is that mean speed. Predict the time to fill 50 mm and the effect of doubling viscosity.

▼ **Integrate the resistance that grows with the filled length**

Equating $\dot{x}$ to the mean liquid speed gives

$$x\dot{x} = r^2\Delta p/(8\mu).$$

Integrating from negligible initial length gives

$$x^2 = \gamma r \cos\theta\, t/(2\mu).$$

The time for 50 mm is

$$2(0.010)(0.050)^2/[0.072(0.00020)\cos 30^\circ] = 4.01\,\mathrm{s}.$$

Doubling viscosity doubles the filling time. At x=25 mm the prediction gives about 0.0125 m/s, with Reynolds number based on diameter about 0.50. Laminar resistance is then plausible. Extrapolating its infinite startup speed to zero length would neglect entrance inertia and the formation of the velocity profile.

---

3 · Predict a measurable fraction of the vertical rise

Turn the tube upright. Only $\Delta p - \rho g x$ remains available for viscous losses. Derive a time formula for $0 < x < H$. Find the time to H/2 and compare with the time needed to reach that same length horizontally.

▼ **Solve the separable moving-front equation**

Write $K = \rho g r^2/(8\mu) = 0.004905$ m/s, which gives

$$\dot{x} = K(H/x - 1).$$

Integration of $dt = x\,dx/[K(H - x)]$ yields

$$t(x) = \frac{1}{K}\left[H \ln \frac{H}{H - x} - x\right].$$

At half height this becomes

$$(H/K)(\ln 2 - 1/2) = 2.50\,\mathrm{s}.$$

Horizontal filling over the same 31.8 mm takes 1.62 s. The logarithm diverges as $x \to H$, so a laboratory definition such as “within 1 mm of the final level” is necessary when reporting an arrival time.

---

4 · Recover viscosity independently of contact angle

Suppose a fresh tube of the same radius gives static rise 60.0 mm and fills 40.0 mm horizontally in 3.00 s. Infer the viscosity without fitting a contact angle. Which pair of measurements lets you test whether static and dynamic wetting agree?

▼ **Eliminate the unknown capillary pressure**

The vertical rise yields

$$\Delta p = \rho g H = 588.6\,\mathrm{Pa}.$$

Horizontal integration gives $\mu = r^2 \Delta p\, t/(4x^2) = 0.0110$ Pa s. Compare this inferred value with independent viscometry and record the meniscus angle during travel. A moving contact angle different

from the static value changes the pressure available during the horizontal run. A time trace at several lengths also tests the assumed $x^2 \propto t$ scaling.

Problem sheet 8 · Calibrate a rotational viscometer without a thin-gap assumption

An inner cylinder rotates inside a stationary cylindrical cup filled with a Newtonian liquid. Instruments often convert torque to viscosity with a gap approximation. Solve the actual cylindrical velocity field, use the measured torque to infer viscosity and determine how end drag or secondary flow would appear in a calibration series.

Find the velocity field from stress balance

Take inner radius $a = 15$ mm, outer radius $b = 20$ mm and active length $L = 80$ mm. The inner cylinder rotates at $\Omega = 5.0$ rad/s; the outer one is fixed. Assume steady axisymmetric flow with no radial or axial velocity, no slip and negligible end effects. Why is the shear rate not simply the derivative of tangential velocity? Derive the velocity between the two walls.

▼ **Compare neighbouring angular velocities**

A fluid rotating as a rigid body with $u_\theta = \Omega r$ has nonzero radial velocity gradient but zero shear deformation. The cylindrical shear rate that respects this fact is $r\,d(u_\theta/r)/dr$. The torque transmitted through a coaxial cylinder is $2\pi L r^2 \tau_{r\theta}$. In a steady purely azimuthal flow with no body torque this is independent of radius. For Newtonian viscosity μ, integrate $r^3\,d(u_\theta/r)/dr = \text{constant}$ to find

$$u_\theta = Ar + B/r.$$

Applying both wall speeds gives

$$u_\theta(r) = \frac{\Omega a^2}{b^2 - a^2}\left(\frac{b^2}{r} - r\right).$$

It returns $a\Omega = 0.075$ m/s at the inner wall and zero at the outer wall. Mid-gap speed is approximately 0.0344 m/s rather than the 0.0375 m/s of a straight line. A small-looking velocity discrepancy can have a larger effect on stress because stress contains a derivative and a curvature correction.

Infer viscosity from the measured torque

The measured sidewall torque after bearing drag subtraction is 0.646 mN m. Determine the viscosity, the magnitude of inner wall shear stress and the result a laboratory would get using a planar gap model. Identify which direction the resulting viscosity bias takes.

▼ **Integrate shear traction with the lever arm**

Differentiating the velocity solution yields

$$\tau_{r\theta} = -2\mu\Omega a^2 b^2/[(b^2 - a^2)r^2].$$

The liquid resists the inner cylinder, so the applied steady drive torque is positive and has magnitude

$$T = \frac{4\pi\mu L\Omega a^2 b^2}{b^2 - a^2}.$$

The measurement implies $\mu = 0.250$ Pa s. Inner stress is

$$T/(2\pi L a^2) = 5.71\,\mathrm{Pa}.$$

A planar gap estimate uses wall speed $a\Omega$ divided by $b - a$ as the shear rate. It predicts 0.424 mN m at the true viscosity and would infer 0.381 Pa s from the actual reading. The sign of bias follows from the cylindrical shear being larger than the assumed planar rate at the inner surface.

Reducing gap while keeping inner radius fixed makes both approximations approach one another. The trade-off is that fractional uncertainty in the actual clearance can become large relative to machining tolerances or eccentricity. Measuring the two radii independently to good absolute precision need not yield an accurately known very small difference.

Check the power balance and use an active-length sweep

Predict the mechanical power entering the liquid. Show that it equals the viscous dissipation integrated over the annulus. How could torque at two lengths separate the intended sidewall response from an approximately length-independent end contribution?

▼ **Integrate power and isolate the intercept**

Driving power is $T\Omega = 3.23$ mW. Local dissipation density is $\mu[r\,d(u_\theta/r)/dr]^2$. Multiplying by $2\pi r L\,dr$ leaves an integral proportional to $\int_a^b dr/r^3$, giving exactly the drive power after substitution of both radii. This check tests both the factor of two and the sign convention in the torque derivation. It also predicts a small but continuous heat load whose importance depends on the sample's thermal control.

If lengths 40 and 80 mm yield measured torques 0.373 and 0.696 mN m, the slope is 8.08 mN m per metre and extrapolated end intercept 0.050 mN m. The inferred 80 mm sidewall contribution is 0.646 mN m as assumed. This correction relies on similar end geometry and sidewall flow at both lengths. A varying free surface, slip, secondary recirculation or contact-line torque can change that intercept with immersion.

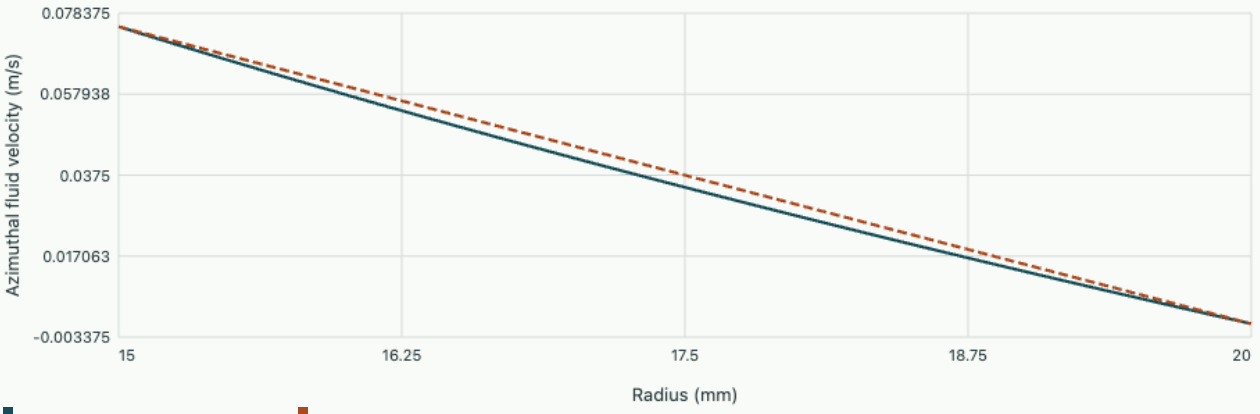
Open this experiment online

Try the calculation

Outer radius (mm) **20**

Inner rotation (rad/s) **5**

Viscosity (Pa s) **0.25**



The lateral torque is 0.646 mN m and inner wall shear magnitude is 5.71 Pa. End effects are excluded.

Distinguish non-Newtonian behaviour from a failing flow geometry

The graph lets you change viscosity while monitoring the profile shape and computed torque. Explain what an observed nonlinear relationship between torque and angular speed would mean and what it would fail to identify by itself.

▼ **Test independent model assumptions**

In the assumed Newtonian steady regime, wall speeds set the entire velocity shape independently of viscosity, while torque scales linearly with viscosity and rotation rate. At fixed geometry, a nonlinear torque curve can suggest a shear-dependent viscosity. It can also result from heating, a moving contact line, wall slip, inertia-driven secondary circulation or a changing parasitic bearing contribution. Repeat at controlled temperature and multiple active lengths, acquire upward and downward speed sweeps and measure reversibility after full equilibration.

The inner rotating configuration can develop Taylor vortices above an onset that depends on geometry and fluid properties. A torque departure accompanied by axial cellular motion violates the assumed single azimuthal component. A velocity profile measured near the cylinder centre and away from its ends supplies information unavailable in one total torque. In the benign low-speed calibration one can also use a reference liquid of known viscosity and repeat for different gaps; collapse onto the finite-gap prediction is a stronger test than fitting a single instrument factor.

MIT Continuum Electromechanics · shear flow and a Couette viscometer

Notebook practice · Watch a wetting front approach the capillary height

Momentum, motion and transport in fluids

Derive a flow from its conservation laws and boundary conditions. Quantify how length scales and material properties decide which approximation can predict a measurement.

1. ## Gravity, capillarity and instability at a moving interface

   A flat interface can carry waves restored by gravity and surface tension. If its two liquids or gases move at different speeds, a band of those waves can grow. Derive the ideal threshold, identify the most vulnerable wavelength at onset and distinguish growth of an instability from advection of an otherwise stable disturbance.

2. ## Settling spheres, Stokes drag and a wall correction

   A falling-sphere viscometer infers viscosity from terminal speed. Derive the drag on a no-slip sphere, distinguish creeping flow from merely steady motion and use radius, vessel width and orientation checks to decide what the speed experiment actually measures.

3. ## Why shear disperses a tracer faster than molecular diffusion

   A short dye pulse in a straight tube first stretches into a filament and much later resembles a broadened one-dimensional cloud. Derive the long-time Taylor–Aris coefficient, predict the width of a measured pulse and identify the mixing time needed before fitting the familiar one-dimensional approximation.

4. ## Estimate wall drag by integrating a boundary-layer profile

   A velocity probe can resolve the momentum missing from a boundary layer more easily than the wall shear itself. Derive an integral momentum relation, estimate drag using a polynomial profile and determine when a pressure gradient or a measurement too far from the wall changes the answer.

5. ## How long does a moving wall take to set a fluid layer in motion?

   Start the top of a narrow gap suddenly and follow the velocity, volume flow and wall stresses as viscous diffusion builds a Couette profile.

6. ## How far upstream does a downstream water level matter?

   A depth gauge downstream of a change in bed level measures a local state. Predict the depth one kilometre upstream by combining energy, cross-section geometry and a resistance law, then fit the roughness against independent uniform-flow readings.

7. ## How can a constant-area duct choke?

   A nozzle reaches a sonic state by changing area. A straight pipe can approach the same Mach number through friction or through heating. The two cases conserve different combinations of stagnation properties.

Momentum, motion and transport in fluids · 1. Gravity, capillarity and instability at a moving interface

A flat interface can carry waves restored by gravity and surface tension. If its two liquids or gases move at different speeds, a band of those waves can grow. Derive the ideal threshold, identify the most vulnerable wavelength at onset and distinguish growth of an instability from advection of an otherwise stable disturbance.

Impose kinematic conditions on both fluids

Consider two immiscible inviscid incompressible fluids of effectively infinite depth. The lower fluid has density $\rho_1 > \rho_2$; base velocities U_1, U_2 are uniform and parallel to a horizontal interface at $z = 0$. Let its small displacement be $\eta e^{i(kx-\omega t)}$ with $k > 0$. Assume irrotational disturbances within each layer, so their potentials satisfy Laplace's equation. Decay away from the interface gives forms $A_1 e^{kz}$ below and $A_2 e^{-kz}$ above. An interfacial vortex sheet is still allowed because tangential base velocities differ.

A material point on either side stays on the perturbed interface in the normal direction. Applying $(\partial_t + U_j \partial_x)\eta = \partial_z \phi_j$ gives

$$A_1 = -i(\omega - kU_1)\eta/k,$$
$$A_2 = +i(\omega - kU_2)\eta/k.$$

Pressure perturbations from linear Bernoulli include both the unsteady potential and hydrostatic change at the displaced surface. Equating pressure jump to the surface-tension curvature force yields

$$\rho_1(\omega - kU_1)^2 + \rho_2(\omega - kU_2)^2 = (\rho_1 - \rho_2)gk + \sigma k^3.$$

Gravity restores long waves for the stated stable density ordering, while surface tension restores short waves. Check the equal-velocity limit: adding the same velocity to both layers merely Doppler-shifts the wave frequency and cannot make a stable interface unstable.

Complete the square and locate the first unstable wavelength

Define density-weighted velocity $\bar{U} = (\rho_1 U_1 + \rho_2 U_2)/(\rho_1 + \rho_2)$. The squared frequency relative to $k\bar{U}$ becomes the restoring right-hand side divided by total density minus $\rho_1\rho_2 k^2 (U_1 - U_2)^2/(\rho_1 + \rho_2)^2$. A negative value gives one exponentially growing and one decaying branch. In that regime the real frequency still advects with $k\bar{U}$, while growth rate is the square root of the negative discriminant's magnitude. A moving pattern alone is insufficient evidence of instability.

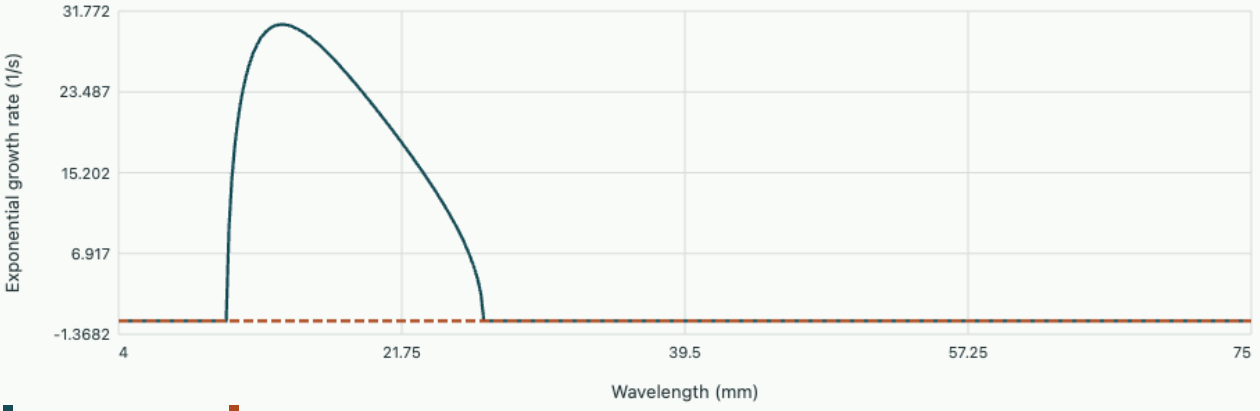
For each k, the squared critical relative speed is proportional to $(\rho_1 - \rho_2)g/k + \sigma k$. Differentiating gives the minimum at $k_* = \sqrt{(\rho_1 - \rho_2)g/\sigma}$. For densities 1000 and 1.2 kg/m³ and tension 0.072 N/m the onset wavelength is 17.0 mm, with ideal threshold 6.66 m/s. At 7.0 m/s the growth rate at this same wavelength is about 27.6 per second, an e-fold interval of 36 ms. The wavelength giving maximum growth above threshold can differ from the onset wavelength.

Open the live experiment

Try the calculation

Relative flow speed (m/s) 7

Surface tension (mN/m) 72



Selected flow At 90% of selected speed

The ideal minimum threshold is 6.66 m/s at wavelength 17.03 mm. At that wavelength selected growth is 27.6 per second.

Make a quantitative observation under controlled assumptions

A suitable numerical experiment starts from the same small disturbance amplitude for a set of Fourier modes and compares early exponential growth with the dispersion relation. If an initial ripple amplitude doubles from 0.10 to 0.20 mm in 25 ms, its estimated growth is $\log 2/0.025 = 27.7$ per second. Measure several early time windows and ensure amplitude remains small compared with wavelength. Nonlinear saturation, finite channel depth, viscosity, actual shear-layer thickness and wave generation at an inlet all change the late or measured growth. The ideal air–water threshold is not a prediction for wind-wave onset with an arbitrary atmospheric velocity profile.

Distinguish a convected disturbance from growth at one place

An unstable wave in the temporal calculation has a real wavelength and a complex frequency. Its visible crests are carried along by the density-weighted mean flow while its envelope grows. A fixed camera viewing a short section may see a larger wave on one side than on the other without observing a temporal increase at a fixed material phase. An inlet wavemaker, reflection, a varying channel depth and convective transport can each change the spatial pattern. Follow the same crest group with a time-stamped image sequence and inspect its early amplitude before the interface becomes strongly distorted.

As a simple measurement example, suppose a tracked narrow band grows from 0.20 mm to 0.40 mm in 0.10 seconds. The inferred temporal exponent is the natural logarithm of two divided by the elapsed time, or 6.93 per second. If noise adds a constant apparent amplitude floor, fitting logarithms of the

earliest images can suppress the inferred slope. Measure a quiet reference sequence and repeat the estimate over several windows. Once the slope falls substantially as amplitude increases, nonlinear saturation may have begun. Extrapolating the early exponential through the late stage then overestimates growth.

At the theoretical onset the predicted growth rate vanishes. This is the least convenient location for a finite observation interval because a weakly unstable wave may scarcely grow before it leaves the field of view. Compare a sequence of speeds on both sides of threshold and plot the squared measured growth rate against squared relative velocity at one chosen wavelength. The ideal dispersion relation gives a locally linear relation and suggests a more stable intercept estimate than judging the first visibly wavy frame.

Tutorial · Recover surface tension from the onset wavelength

At the same densities, an ideal calculation with a different liquid predicts the minimum threshold at wavelength 12.0 mm. Estimate surface tension. What independent measurement checks the model without reusing the wavelength?

▼ **Invert the onset condition**

With $k_* = 2\pi/0.012 = 524$ per metre, $\sigma = (998.8)(9.81)/k_*^2 = 0.0357$ N/m. Insert that value into the threshold-speed equation and compare with a measured critical relative velocity. Wavelength alone fixes the ratio of buoyancy and tension terms in this model; matching a second independent onset observable is a stronger test. Verify that the observed wavelength corresponds to first onset rather than the fastest mode at a substantially larger speed.

Tutorial · Distinguish finite growth from a short observation

A mode has predicted growth 5.0 per second but is visible in a camera field for only 30 ms as it advects. Estimate its amplitude increase. Could an apparently constant amplitude establish stability?

▼ **Compare residence and growth times**

The predicted factor is $e^{5(0.03)} = 1.162$, just sixteen percent. If amplitude noise, illumination variation or uncertainty in initial conditions is comparable, such a record cannot distinguish weak growth from neutral propagation. Measure along the convecting packet or extend the observable path, then fit log amplitude with calibrated spatial gain. A lack of measurable growth over one short camera region places only an upper bound on its rate.

Tutorial · Suppress tension in the mathematical model

Set surface tension to zero while retaining a sharp velocity discontinuity and stable gravity. What happens to the high-wavenumber threshold and why would finite-mesh simulations disagree as they are refined?

▼ **Examine the highest resolved modes**

The required squared speed then falls as $1/k$, so any nonzero velocity jump can destabilise sufficiently short waves. In the vortex-sheet idealisation their high-k growth rate scales with k. Refining a numerical mesh allows faster modes previously excluded by the grid. A physically smooth shear layer, surface tension, viscosity or other physical regularisation must be specified if the small-scale evolution is important. Numerical smoothing can control a calculation but its length scale is then part of the assumed model, not a measured material parameter.

MIT Interfacial Phenomena · surface waves and Kelvin–Helmholtz instability

Momentum, motion and transport in fluids · 2. Settling spheres, Stokes drag and a wall correction

A falling-sphere viscometer infers viscosity from terminal speed. Derive the drag on a no-slip sphere, distinguish creeping flow from merely steady motion and use radius, vessel width and orientation checks to decide what the speed experiment actually measures.

Balance pressure and viscosity around a sphere

Let a rigid sphere of radius a move with small relative speed U in a Newtonian fluid. In the sphere frame impose a uniform far flow $U\mathbf{e}_z$, no slip at its surface and incompressibility. If particle Reynolds number $\rho U(2a)/\mu$ is small, drop convective inertia, obtaining $-\nabla p+\mu\nabla^2\mathbf{u}=0$. Dimensional reasoning predicts a drag proportional to $\mu a U$ but cannot determine its prefactor. That prefactor depends on the no-slip surface and three-dimensional geometry.

The axisymmetric Stokes solution has $u_r=U\cos\theta[1-3a/(2r)+a^3/(2r^3)]$,

$$u_\theta=-U\sin\theta[1-3a/(4r)-a^3/(4r^3)],$$

$$p-p_\infty=-3\mu U a\cos\theta/(2r^2).$$

Substitution verifies zero surface velocity and the uniform far limit. The pressure contribution in the streamwise direction integrates as the surface integral of $-(p-p_\infty)\cos\theta$, giving $2\pi\mu a U$. Tangential viscous traction supplies $4\pi\mu a U$. Their sum gives

$$F_D=6\pi\mu a U,$$

$$C_D=\frac{24}{Re_D},$$

$$Re_D=\frac{2\rho a U}{\mu}.$$

A seemingly simpler potential-flow solution satisfying only impermeability misses the viscous traction and cannot reproduce the drag. Conversely a calculation that integrates just wall shear misses one third of this drag even at negligible inertia. The signs depend on whether one computes force of fluid on sphere or sphere on fluid; the magnitude above resists relative motion.

Balance buoyancy and find the terminal motion scale

A sphere of density $\rho_s>\rho$ has buoyancy-corrected weight $(4\pi a^3/3)(\rho_s-\rho)g$. Equating drag and this weight yields

$$U_t=2a^2(\rho_s-\rho)g/(9\mu).$$

Take $a=0.50$ mm, density difference 1200 kg/m³ and viscosity 0.50 Pa s. The unbounded terminal speed is 1.31 mm/s. For fluid density 1000 kg/m³, $Re_D=0.00262$, supporting a steady creeping-flow approximation. A 30 mm timed distance then takes about 22.9 s.

A coarse transient model uses sphere inertia alone, $m_s\dot{U} + 6\pi\mu a U = W_{\mathrm{eff}}$, predicting relaxation time $m_s/(6\pi\mu a)$. Actual transient hydrodynamics also contains added mass and a viscous-history contribution. Measuring a steady late portion is much safer than fitting the initial release with the simple exponential when extracting viscosity. Measure speed over successive equal intervals and check that it has become constant, away from both the release and vessel bottom.

Compare the viscous penetration time with the oscillating-wall experiment

A slow particle can still strongly interact with the container

A stationary no-slip cylindrical vessel increases drag on a sphere translating along its centreline. For small size ratio $\lambda = a/R_{\mathrm{tube}}$, the standard leading mobility correction is $U_{\mathrm{tube}}/U_{\infty} \simeq 1 - 2.1044\lambda + \cdots$. At $\lambda = 0.05$ the measured speed is already roughly ten percent smaller. Inserting that diminished speed into an unbounded drag law biases inferred viscosity upward. A small Reynolds number does not make walls irrelevant; it only addresses inertia relative to viscosity.

Measure terminal speed from a recorded trajectory

Choose a transparent column wide enough that the sphere remains many radii from its wall in the analysed window. Convert pixel distances to physical length using a scale in the same plane as the particle centre. Refraction through a curved vessel can otherwise give different apparent distances at the wall and centre. Fit position against time over several windows beginning after release. A stable slope with changing window length supports the terminal-speed interpretation; a trend in that slope can indicate acceleration, a fluid current or proximity to the bottom.

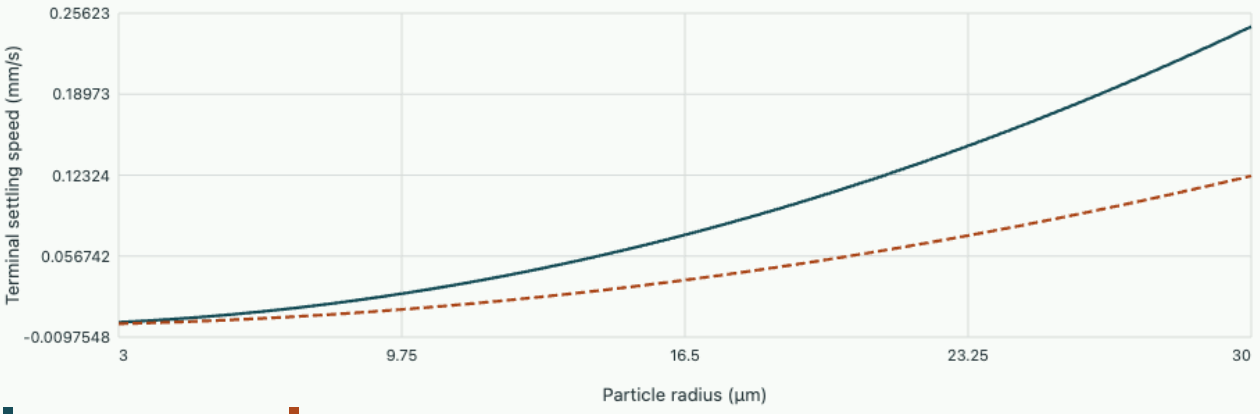
Open the live experiment

Try the calculation

Particle radius (µm) **10**

Liquid viscosity (mPa s) **4**

Density excess (kg/m³) **500**



For the selected radius the Stokes prediction is 0.0272 mm/s, giving diameter Reynolds number 0.000136. Compare this number with unity before trusting the approximation.

The plot overlays the selected viscosity with a run at twice that viscosity. At fixed particle size their ideal terminal speeds differ by a factor of two. The horizontal axis invites a different comparison: the same settling speed in the more viscous fluid requires a radius larger by a factor of the square root of two. Use the numerical Reynolds number in the readout as a validity test. Increasing particle size tenfold raises terminal speed a hundredfold but diameter Reynolds number a thousandfold, so the apparent ease of tracking a larger particle has a fluid-mechanical cost.

A density inference can be especially fragile near neutral buoyancy. For example, if particle and liquid densities are 1020 and 1000 kilograms per cubic metre, an absolute uncertainty of two kilograms per cubic metre in each input is small relative to either density yet large relative to their difference. If independent it gives a standard uncertainty about 2.8 kilograms per cubic metre in the excess, or fourteen percent. Compare with the microscope radius calibration before attributing observed variation to the viscosity. Vertical temperature gradients can drive background circulation whose speed is comparable with a very slowly settling particle.

Tutorial · Estimate viscosity and its radius uncertainty

A sphere of radius 0.40 mm and density contrast 1000 kg/m³ falls at 0.80 mm/s far from walls. Infer viscosity and propagate an independent two-percent radius uncertainty, neglecting other errors.

▼ **Invert the terminal balance**

The inferred viscosity is $2(0.00040)^2(1000)(9.81)/[9(0.00080)] = 0.436$ Pa s. Its radius contribution to relative uncertainty is approximately four percent because radius is squared. If density contrast is estimated by subtracting two similar measured densities, its relative uncertainty can be much larger than either density's relative error. Repeat with a sufficiently well-characterised reference sphere and measure liquid temperature since viscosity can vary markedly with it.

Tutorial · Could a speed mismatch be wall drag?

For a 0.50 mm radius sphere and a 10 mm radius tube, the measured terminal speed is 1.17 mm/s rather than the 1.31 mm/s unbounded prediction. Check the leading wall correction and propose a test that does not assume the listed viscosity is exact.

▼ **Compare a vessel-size sequence**

The radius ratio is 0.050, so the leading correction predicts a factor 0.895 and speed 1.17 mm/s. Repeat in two or more wider tubes at unchanged temperature and fit speed against inverse tube radius. Extrapolation toward zero inverse radius estimates unbounded mobility without requiring the listed viscosity for the extrapolation. Off-axis migration, wall roughness, bottom confinement and finite-size higher terms must still be checked. One match to a leading coefficient does not prove perfect centring.

Tutorial · Brownian jitter or sustained background flow?

A much smaller tracer shows fluctuations in falling displacement and an unexpected common horizontal drift. How would you estimate its settling speed without attributing every displacement increment to gravity?

▼ **Separate first and second displacement moments**

Brownian motion adds variance increasing with lag in a diffusive interval but no mean drift in a homogeneous quiescent bath. Background flow adds a mean vector common to passive tracers. Measure many trajectories, estimate the fluid's mean velocity with a nearly neutrally buoyant marker and subtract it before fitting mean vertical drift. Fit the residual displacement variance separately to estimate diffusion and compare with the Einstein–Stokes prediction. A static temperature gradient can generate convection or thermophoresis, so repeat at several positions or orientations before assigning a gravity-only terminal balance.

MIT Advanced Environmental Fluid Mechanics · Stokes flow past a sphere

Momentum, motion and transport in fluids · 3. Why shear disperses a tracer faster than molecular diffusion

A short dye pulse in a straight tube first stretches into a filament and much later resembles a broadened one-dimensional cloud. Derive the long-time Taylor–Aris coefficient, predict the width of a measured pulse and identify the mixing time needed before fitting the familiar one-dimensional approximation.

Cross-sectional averaging leaves a correlated flux

Consider steady fully developed laminar Poiseuille flow of mean speed U in a circular tube of radius a. The axial profile is $u(r) = 2U(1 - r^2/a^2)$. A dilute passive solute of molecular diffusivity D is carried by this velocity and diffuses in all directions, with no solute flux at the wall. Section-averaging its advection–diffusion equation leaves a flux $\overline{uc}$. Splitting into mean and deviation gives $U\bar{c} + \overline{u'c'}$. It is the latter covariance that disperses the pulse beyond its molecular diffusion.

At times long compared with a transverse mixing scale of order a^2/D, and for a slowly varying axial average, take $c' \simeq f(r)\partial_z \bar{c}$. Substituting into the leading radial balance gives

$$D r^{-1}(r f')' = u - U = U(1 - 2r^2/a^2).$$

Regularity at the centre and zero wall derivative give

$$f' = (U/D)(r/2 - r^3/(2a^2)).$$

Integrate again and choose a constant so $\overline{f} = 0$. The section-integrated correlation is most easily evaluated by multiplying the radial equation by f and integrating by parts:

$$\overline{u'f} = -D\overline{(f')^2} = -\frac{U^2 a^2}{48 D},$$

$$D_{\mathrm{eff}} = D + \frac{U^2 a^2}{48 D}.$$

The negative sign means an average concentration gradient drives a net down-gradient dispersive flux. The molecular diffusivity in the denominator is not a paradox. Faster radial diffusion lets a tracer sample slow and fast streamlines more frequently and reduces long-time axial velocity memory. Taking D arbitrarily close to zero at a fixed observation time violates the long transverse-mixing requirement before the formula could diverge.

Predict a pulse width and its applicability window

For $a = 100$ micrometres, $U = 0.50$ mm/s and $D = 1.0 \times 10^{-9}$ square metres per second, $Ua/D = 50$ and $D_{\mathrm{eff}} = 5.31 \times 10^{-8}$ square metres per second. Radial mixing time a^2/D is about 10 s, corresponding to an advected length 5 mm. After 100 s, under long-tube and narrow-initial-pulse assumptions, the axial rms width is $\sqrt{2 D_{\mathrm{eff}} t} = 3.26$ mm while the mean pulse has travelled 50 mm. A velocity measured from width alone is ambiguous without an independent molecular diffusivity and a check of the time regime.

A detector at a fixed axial station records concentration versus time. Converting a small spatial width to a temporal width by division by U is a local approximation when the pulse is narrow compared with distance travelled and continues broadening slowly during its passage. An absorbing detector or outlet collecting solute flux can weight the distribution differently from a nonintrusive optical section-average. State whether the instrument measures local intensity, section-average concentration or total mass flux.

Recover the parabolic base profile in the viscous-duct chapter

Background · How does a finite injection pulse change the inferred dispersion?

A tracer valve introduces a plug of finite axial width. In the long-time regime with linear transport, the measured concentration is the convolution of the released profile with the spreading kernel. After normalising both to unit mass their variances add, just as variances add for independent displacements.

$$\sigma_x^2(t) = \sigma_{x,0}^2 + 2D_{\mathrm{eff}}t,$$

$$D_{\mathrm{eff}} = \frac{\sigma_x^2(t_2) - \sigma_x^2(t_1)}{2(t_2 - t_1)}.$$

A uniform 12 mm plug has initial variance $12^2/12 = 12$ mm². A one-time estimate forced through zero at $t = 10$ s therefore overestimates D_{eff} by 0.60 mm²/s. Two complete spatial images can eliminate the intercept if both are taken in the Taylor-dispersion regime. A camera that clips one tail does not record the full variance.

Read dispersion from the second moment of an image

After a short pulse has entered the straight observation tube, a camera integrates fluorescence across its diameter. Subtract a background frame and flat-field the illumination, integrate across the complete width, then compute the axial centroid and variance with concentration as weights. A constant gain cancels in both normalised moments. Saturating the brightest central pixels does not cancel: it reduces the measured peak more than the tails and biases the width. A window too narrow to capture the tails produces the opposite error. Shift and widen the analysis window as the pulse travels.

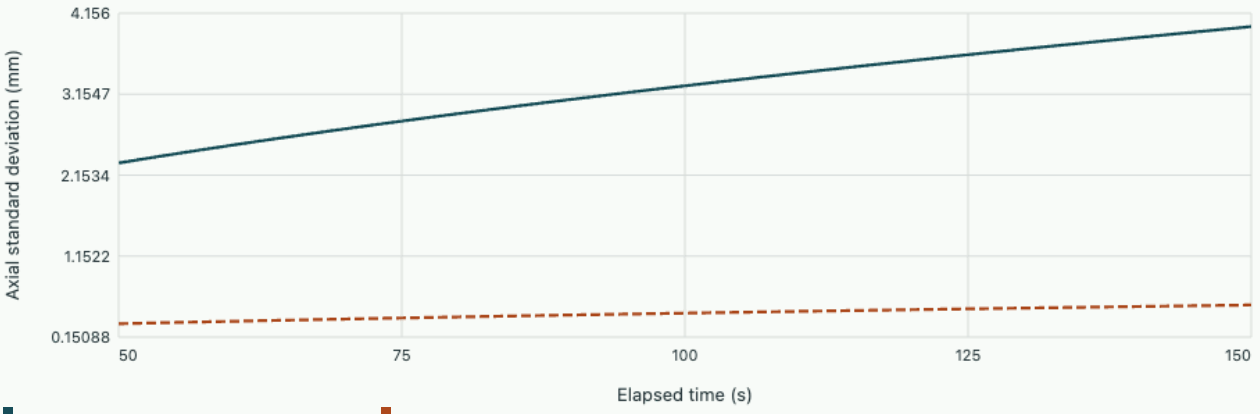
Open the live experiment

Try the calculation

Tube radius (µm) **100**

Mean velocity (mm/s) **0.5**

Molecular diffusivity (nm²/ns) **1**



Radial mixing scale is 10.00 s and effective diffusivity is 5.308e-8 m²/s. Ten mixing times after injection the effective axial rms width is 3.26 mm.

The figure starts several radial mixing times after injection. Its left endpoint represents an already broadened pulse; the model was derived in this late regime. For an injection with appreciable initial variance, add that variance to the predicted diffusive growth instead of forcing the variance line through zero. With a constant axial imaging blur, variances likewise add under convolution with a normalised response. In that case two late measurements can identify a slope even if the intercept includes both injection width and camera blur. Recovering the initial width from the intercept would require an independent calibration.

The radius and velocity sliders are independent. To model a pump holding flow rate fixed when the radius is doubled, divide velocity by four as well. The effective enhancement then decreases by four and the radial mixing time increases by four. Comparing observations at the same absolute time could inadvertently move one case outside the long-time regime. Compare at a stated multiple of each radial mixing time and also record the corresponding distances needed along the tube.

Tutorial · Predict what a second pulse station should see

Two nonintrusive imaging stations observe the same well-mixed pulse after mean transit times 60 and 120 seconds under the example conditions. Estimate the increase in spatial variance. Why does differencing help with uncertainty in initial injection width?

▼ **Use the long-time variance slope**

The difference is $2D_{\mathrm{eff}}(120-60)=6.37\times10^{-6}$ square metres, or 6.37 square millimetres. An initial injection variance and any common Gaussian imaging blur add approximately constant offsets to the spatial variance and cancel in this difference. If the image point spread differs between stations or the first station is still within the shear-dominated entrance transient, cancellation fails. Check more than two stations to test linear variance growth before inferring a dispersion coefficient.

Tutorial · Compare a thinner tube at fixed volumetric rate

Halve the radius while keeping volumetric flow rate and molecular diffusivity fixed, assuming fully developed laminar flow remains valid. What happens to mean velocity, transverse mixing time, dispersive enhancement and required pressure gradient?

▼ **Keep the imposed quantity fixed**

Mean speed increases by four because area is quartered. Transverse mixing time falls by four. The enhancement $U^2a^2/(48D)$ increases by four. For Poiseuille flow at fixed volume rate the required axial pressure gradient scales as a^{-4}, hence increases by sixteen. At fixed mean velocity, in contrast, the enhancement would decrease by four. A claim that narrower tubes always improve or always suppress dispersion is incomplete without naming the operating constraint.

Tutorial · Interpret a non-Gaussian short-time profile

At a distance reached in only one second the optical image shows a long leading filament and a concentration tail near the wall. Does this invalidate the measured molecular diffusivity? Sketch the limiting motion if radial diffusion were absent during that interval.

▼ **Compare the observation time with the radial mixing time**

One second is only a tenth of the example transverse diffusion scale. Material near the centre moves near $2U$, while material close to a no-slip wall is almost stationary. An initially uniform cross-section pulse therefore separates by streamline velocity, covering a broad range of axial positions. A single effective Gaussian diffusion law has not yet emerged. Compare direct advection–diffusion with an independently measured initial profile, then check the long-time variance at larger distances. Wall adsorption would create additional lag even after radial mixing and can be tested by changing wall chemistry or solute type.

MIT Advanced Fluid Dynamics · section 4.7, dispersion and the Aris solution for a circular pipe

Momentum, motion and transport in fluids · 4. Estimate wall drag by integrating a boundary-layer profile

A velocity probe can resolve the momentum missing from a boundary layer more easily than the wall shear itself. Derive an integral momentum relation, estimate drag using a polynomial profile and determine when a pressure gradient or a measurement too far from the wall changes the answer.

Define displacement and momentum thickness separately

For steady incompressible two-dimensional flow over a smooth wall, let $U_e(x)$ denote the local speed just outside a thin attached boundary layer and $u(x,y)$ the tangential speed inside it. Displacement thickness $\delta^* = \int_0^\infty (1 - u/U_e)dy$ measures missing volume flow per unit outer speed. Momentum thickness $\theta = \int_0^\infty (u/U_e)(1 - u/U_e)dy$ measures a different deficit. Both have dimensions of length, while their ratio $H = \delta^*/\theta$ characterises profile shape. Neither thickness is defined by a particular percentage edge speed.

Integrating the streamwise boundary-layer momentum equation and using continuity to eliminate the edge-normal flux yields

$$\frac{\tau_w}{\rho U_e^2} = \frac{d\theta}{dx} + (2 + H)\frac{\theta}{U_e}\frac{dU_e}{dx}.$$

The edge pressure gradient follows outer inviscid flow through $dp_e/dx = -\rho U_e\, dU_e/dx$. Under uniform outer speed the equation reduces to

$$d\theta/dx = C_f/2,$$

$$C_f = \tau_w/(\rho U_e^2/2).$$

With an adverse pressure gradient the second term is negative, and an increasingly thickening layer can still have very small or negative wall shear. Applying the uniform-speed relation outside its conditions would overestimate shear near separation.

Derive an approximate flat-plate law

Assume a laminar zero-pressure-gradient layer with $u/U = 2\eta - \eta^2$ for $0 < \eta = y/\delta < 1$ and outer speed beyond. The profile satisfies no slip at the wall, reaches the edge speed and has zero first derivative at its nominal outer edge. Integrating gives $\delta^* = \delta/3$, $\theta = 2\delta/15$, while the wall gradient gives

$$\tau_w = 2\mu U/\delta.$$

Substitute into the momentum integral and integrate from the leading edge:

$$\delta = \sqrt{\frac{30\nu x}{U}},$$

$$C_{f,x} = \frac{4}{\sqrt{30 Re_x}},$$

$$Re_x = \frac{U x}{\nu}.$$

The coefficient 0.730 from this approximate profile differs from the exact Blasius laminar result 0.664 while reproducing the square-root scaling. Profile choice closes an otherwise exact integrated balance and creates a measurable modelling error. It also treats the local leading-edge region outside its proper thin-layer asymptotics. Refining numerical integration of the same polynomial cannot eliminate the profile approximation.

For air with $U = 5.0$ m/s, $\nu = 1.5 \times 10^{-5}$ square metres per second, $x = 0.20$ m and $\rho = 1.2$ kg/m³, $Re_x = 6.67 \times 10^{4}$ and predicted $\delta = 4.24$ mm. The approximate local $C_f = 0.00283$ gives $\tau_w = 0.0424$ Pa. If a pressure-gradient or transition changes the profile appreciably, measure the actual shape and its x variation rather than applying this laminar estimate.

Compare the local velocity profile and wall stress with the boundary-layer chapter

Use a probe traverse to measure a momentum deficit

A small velocity probe is moved away from a smooth plate in increments of known height. Calibrate zero height and measure outer velocity independently. Convert the measured values to ratios with the outer velocity, integrate the displacement and momentum kernels using the same station coordinates, and repeat downstream. Even if the probe misses the point where the speed first reaches 99 percent of the outer value, the two integrals can remain informative. A contaminated outer-velocity estimate, however, leaves a persistent offset in the integrands across the entire outer part of the traverse.

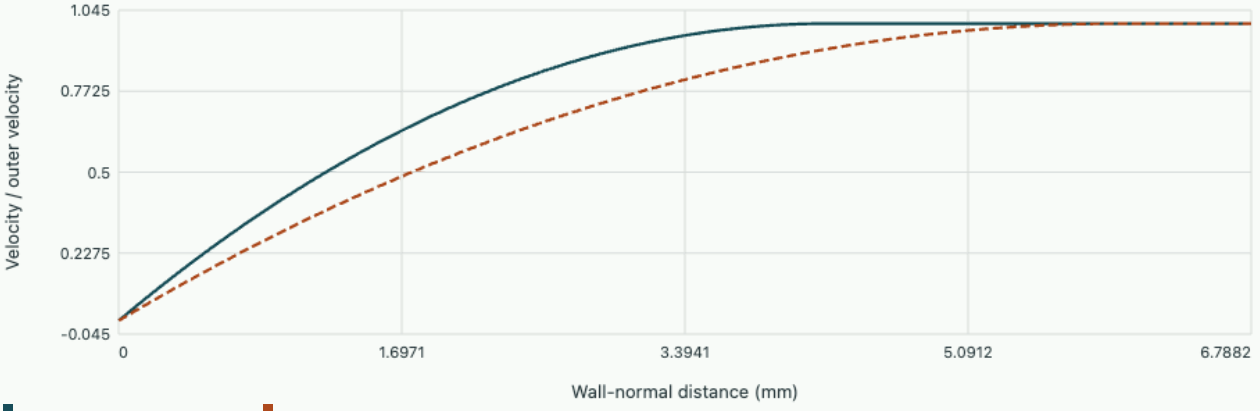
Open the live experiment

Try the calculation

Outer speed (m/s) **5**

Kinematic viscosity (mm²/s) **15**

Plate station (m) **0.2**



Selected station Twice the distance downstream

The quadratic trial layer has thickness 4.24 mm, displacement thickness 1.41 mm and local friction coefficient 0.00283. Smooth, laminar, zero-pressure-gradient conditions are assumed.

The model overlays a profile at a selected station and at twice that station distance. In a zero-pressure-gradient laminar layer both curves collapse if wall-normal position is divided by its own thickness. The outer velocity is the same in both plots and the wall slopes differ by a factor of the square root of two. This shows why a constant free-stream speed does not imply constant wall shear. Wall shear depends on the gradient normal to the wall, and diffusion spreads the velocity change over a progressively thicker layer.

For an independent check take a measured momentum thickness of 0.55 mm at a station and 0.66 mm at a station 60 mm downstream. Their finite difference gives a slope of 0.00183 and hence a mean skin-friction coefficient of 0.00367 under uniform outer speed. A measurement error of 0.02 mm at each station, treated as independent, corresponds to roughly one quarter of the measured difference. Improving one individual profile is not enough if the station separation is too small to resolve a change. Increasing separation improves signal but averages over more streamwise development. Repeating over several separation distances exposes that tradeoff and can reveal transition within the interval.

If the same probe is used at both stations, a common calibration gain can cancel in the normalised velocity ratios. An error in the zero wall distance affects the inner region differently. Repeat the integration after deliberately shifting the nominal wall location by its measured positioning uncertainty.

Tutorial · Infer drag from two traverse stations

A constant-speed incompressible layer has momentum thickness 0.30 mm at one station and 0.50 mm 0.15 m farther downstream. With $U = 5.0$ m/s and $\rho = 1.2$, estimate mean wall shear over the interval

and total drag on a strip of width 0.20 m.

▼ **Integrate the momentum relation**

Mean shear is

$$\rho U^2 \Delta\theta/\Delta x = 1.2(25)(0.00020)/0.15 = 0.040\,\mathrm{Pa}.$$

The area is 0.030 square metres, so drag is 1.20 mN. This estimate uses the complete momentum deficit and does not require differentiating a noisy near-wall velocity. It does require both profiles to cover the entire defect region and share a calibrated outer speed.

Tutorial · Check an apparent separation claim

At one location $H = 2.5, \theta = 1.0$ mm, $U_e = 4.0$ m/s, $dU_e/dx = -8.0$ per second and $d\theta/dx = 0.010$. Infer wall shear sign. Would the same momentum-thickness growth guarantee that sign in zero pressure gradient?

▼ **Retain the pressure-gradient term**

The dimensionless right side is $0.010 + 4.5(0.001/4)(-8) = 0.0010$, still positive but only one tenth of the zero-gradient estimate. In this example both cases have positive shear, yet a slightly stronger adverse gradient would reverse the indicated sign. Near zero the differences among several measured terms magnify relative uncertainty. A near-wall direction measurement or oil-film shear indicator can help distinguish a true sign reversal from fitted profile noise.

Tutorial · Detect a probe that misses the outer deficit

A traverse stops at the first point where measured speed reaches 97 percent of the apparent outer value. How might this bias displacement and momentum thickness, and how would you design a convergence check?

▼ **Integrate the remaining positive tail**

For an attached monotone profile both integrands are positive below the true outer speed, so prematurely truncating usually underestimates both thicknesses. Near the edge their integrands become similar because $u/U_e \simeq 1$; even a small speed defect over a broad tail can contribute appreciable area. Extend the traverse and compare integrals as the cutoff rises. Independently estimate true undisturbed U_e: a slight bias in that denominator can create a false nonzero tail over the entire outer integration region. Repeat at finer near-wall spacing too, separately testing quadrature and physical domain coverage.

MIT Aerodynamics of Viscous Fluids · boundary layers and momentum-integral methods

Momentum, motion and transport in fluids · 5. How long does a moving wall take to set a fluid layer in motion?

Start the top of a narrow gap suddenly and follow the velocity, volume flow and wall stresses as viscous diffusion builds a Couette profile.

Remove the final state before separating variables

Take two large, parallel plates separated by H, with $y = 0$ at the fixed plate. The liquid starts from rest. At $t = 0$ the upper plate begins to move at constant speed U. For an incompressible Newtonian liquid with no imposed pressure gradient, the unidirectional velocity $u(y, t)$ has zero convective acceleration. The streamwise momentum balance on a thin slice then reduces to

$$\rho\, \partial_t u = \partial_y(\mu\, \partial_y u).$$

With constant viscosity, kinematic viscosity $\nu = \mu/\rho$ is the momentum diffusivity.

Use $\eta = y/H$ and $F = \nu t/H^2$. The linear profile $u/U = \eta$ solves the eventual steady equation and both wall conditions. Subtract it by setting $w = u/U - \eta$. Both boundaries now require $w = 0$, which selects eigenfunctions $\sin(n\pi\eta)$ for $w_F = w_{\eta\eta}$. Each amplitude decays as $\exp(-n^2\pi^2 F)$. The interior initially has $w = -\eta$. Multiplying by a sine and integrating from zero to one gives

$$b_n = -2\int_0^1 \eta \sin(n\pi\eta)\, d\eta = 2(-1)^n/(n\pi).$$

Combining the parts gives

$$\frac{u}{U} = \eta + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi\eta) e^{-n^2\pi^2 F}.$$

For any positive time, higher modes decay rapidly. At the initial top corner the sudden change in surface velocity conflicts with the initially resting liquid. The mathematical wall gradient there diverges as $t \downarrow 0$. A finite starting acceleration would regularize that prediction. Fluid well away from that wall sees little change until the diffusion distance, of order $\sqrt{\nu t}$, becomes comparable with its distance to the moving surface.

Distinguish stored momentum from transmitted shear

Integrating the profile over the gap eliminates the even sine modes. The dimensionless volume flow per unit span becomes

$$q/(UH) = 1/2 - (4/\pi^2) \sum_{n\ \text{odd}} e^{-n^2\pi^2 F}/n^2.$$

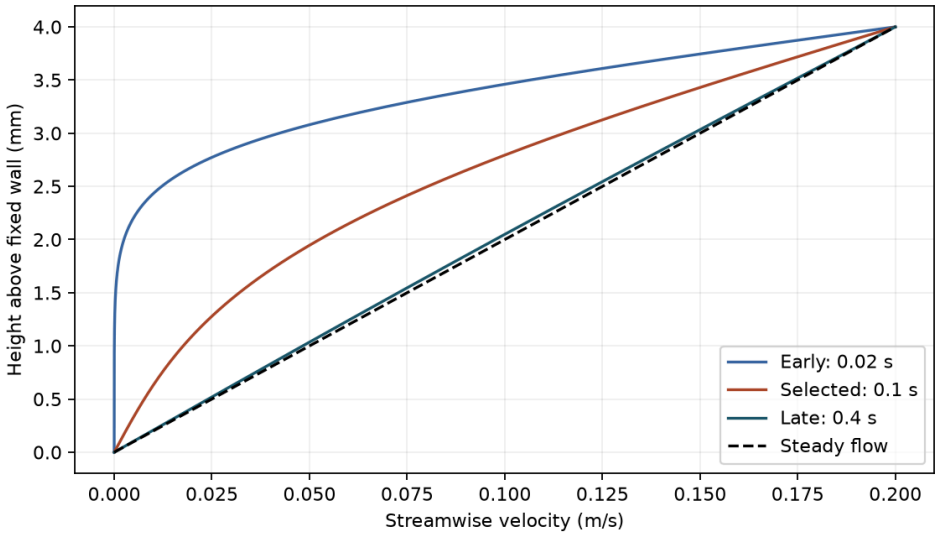
Differentiating at the walls gives

$$\tau_{\mathrm{top}}/(\mu U/H) = 1 + 2\sum_{n} e^{-n^2\pi^2 F},$$

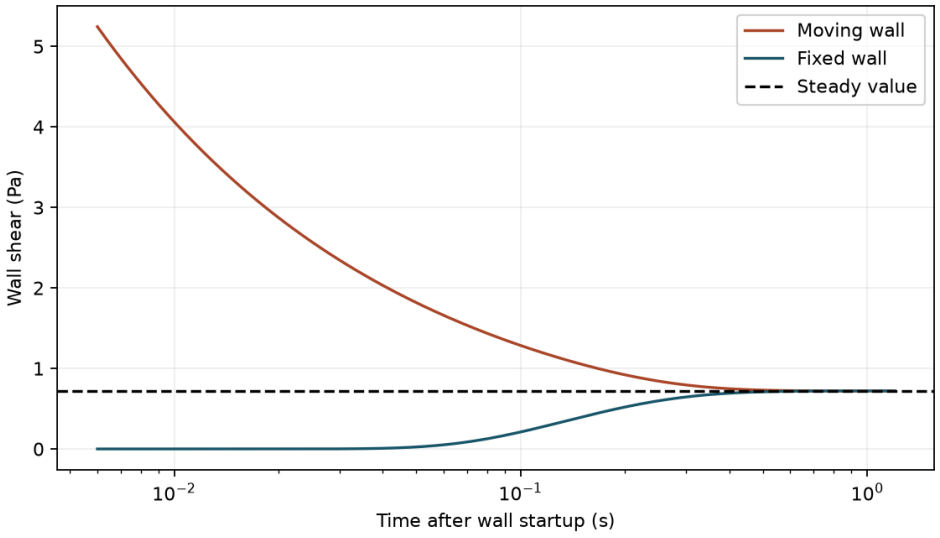
$$\tau_{\mathrm{bottom}}/(\mu U/H) = 1 + 2\sum_{n} (-1)^n e^{-n^2\pi^2 F}.$$

Here both symbols report μu_y evaluated in the positive y direction. The forces on the fluid act in opposite directions, so the net streamwise force per unit plate area is $\tau_{\mathrm{top}} - \tau_{\mathrm{bottom}}$.

Set $H = 4$ mm, $\nu = 1.6 \times 10^{-5}$ m^2/s, $\rho = 900$ kg/m^3 and $U = 0.20$ m/s. The gap diffusion time is 1 s. At 0.1 s the flow is $0.34894UH$, already 69.8% of its final value. The wall gradients still differ sharply, measuring 1.7843 and 0.2929 times the eventual shear of 0.72 Pa. Integrating the original momentum equation over the layer verifies that their difference accelerates the stored streamwise momentum, $\rho\, dq/dt$.



The top wall fixes the velocity at one end immediately. The curvature in the interior tracks how momentum is still entering the liquid.



Initially the driving wall accelerates the upper layer. The fixed wall feels an appreciable force later, even as the force required at the driving wall declines.

Problem 1 · Predict the flow-settling time

Estimate when the flow reaches 99% of its final value. Why is one mode enough for this estimate?

▼ **Work through the solution**

Divide the missing flow by its final value to obtain $(8/\pi^2)\sum_{n\ \mathrm{odd}} e^{-n^2\pi^2 F}/n^2$. Its leading term reaches 0.01 at $F = \ln[8/(0.01\pi^2)]/\pi^2 = 0.44532$, or 0.44532 s with the given gap. At this time the next contributing mode contains $e^{-9\pi^2 F}/9$, below 10^{-17}. At much earlier times several modes reconstruct the thin moving layer.

Problem 2 · Double the gap

The plate speed and liquid stay fixed while H doubles. Compare the 99% settling time, steady shear, and steady flow per unit width in the two gaps.

▼ **Work through the solution**

The dimensionless solution reaches the same settling criterion at the same F. Since $t = FH^2/\nu$, the wider gap takes four times as long, about 1.78 s. The steady gradient is U/H, so its shear halves to 0.36 Pa. The steady flow is the area under a triangle of height U and width H, namely $UH/2$. It doubles to 8.0×10^{-4} m^2/s.

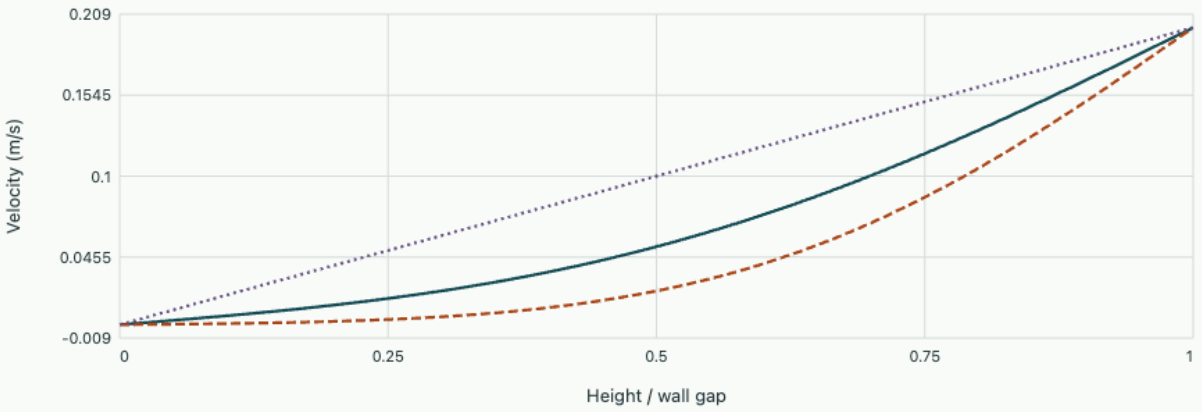
Interactive calculation · Startup couette

Change the physical and numerical inputs of the notebook model. Compare the updated curves and balance checks with the recorded notebook figures. Each plot uses the same equations and assumptions documented in that notebook; its readout names any analytical or reduced-model reference.

Open this live calculation

Try the calculation

Wall gap (mm)	**4**
Wall speed (m/s)	**0.2**
Fourier time νt/h²	**0.1**



Selected time | Earlier, half Fourier time | Steady Couette profile

For ν=1.6×10⁻⁵ m²/s, selected time 0.100 s. The100-mode series satisfies no-slip at both walls; startup is instantaneous.

Download this example's editable notebook

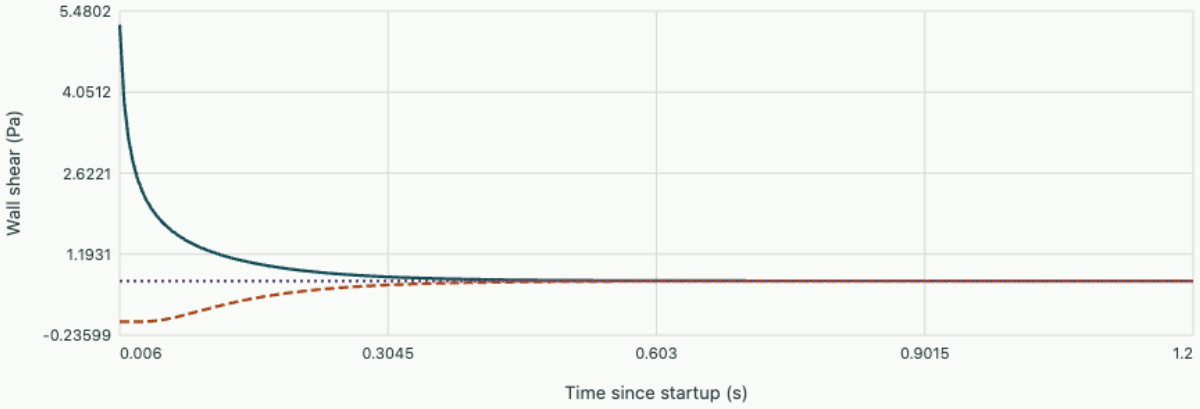
Interactive calculation · Startup shear

Change the physical and numerical inputs of the notebook model. Compare the updated curves and balance checks with the recorded notebook figures. Each plot uses the same equations and assumptions documented in that notebook; its readout names any analytical or reduced-model reference.

Open this live calculation

Try the calculation

Wall gap (mm)	4
Wall speed (m/s)	0.2
Kinematic viscosity (mm²/s)	16



Diffusion time 1.000 s; steady shear 0.720 Pa at density900 kg/m³. The initial moving-wall stress is singular, so the plotted range starts at finite time.

Download this example's editable notebook

Open editable notebook Download .ipynb

Momentum, motion and transport in fluids · 6. How far upstream does a downstream water level matter?

A depth gauge downstream of a change in bed level measures a local state. Predict the depth one kilometre upstream by combining energy, cross-section geometry and a resistance law, then fit the roughness against independent uniform-flow readings.

Separate normal depth and critical depth

Consider steady flow at discharge Q in a long prismatic rectangular channel of width b. Depth y sets area $A = by$, wetted perimeter $P = b + 2y$ and hydraulic radius $R = A/P$. For a long reach of uniform depth, weight down the slope balances wall shear over the wetted perimeter. If $S_0 = -dz_b/dx$ is the bed slope, the force balance gives $\tau_0 = \rho g R S_0$. A resistance relation from measurement closes this balance. In SI units the Manning convention is

$$Q = \frac{1}{n} A R^{2/3} \sqrt{S_f},$$

$$S_f = \left(\frac{nQ}{AR^{2/3}} \right)^2.$$

Normal depth y_n solves $S_f = S_0$. Roughness n has units s m$^{-1/3}$. The equivalent Chézy form $U = C\sqrt{RS_f}$ has $C = R^{1/6}/n$ for the same prediction. Critical depth follows from specific energy $E = y + Q^2/(2gA^2)$ at fixed discharge. Differentiating gives $1 - Q^2 b/(gA^3) = 0$ at y_c. The critical condition contains no roughness. For $b = 4$ m, $Q = 6$ m³/s, $S_0 = 0.001$ and $n = 0.030$, $y_n = 1.556$ m while $y_c = 0.612$ m. This bed is mild for the specified discharge.

Derive an evolution equation in space

Assume hydrostatic pressure, negligible local losses, unit velocity-energy coefficient and a sufficiently gentle surface slope. Along increasing downstream distance the energy head drops by S_f per metre. Differentiating $H = z_b + y + Q^2/(2gA^2)$ and using $dA/dy = b$ gives

$$\frac{dy}{dx} = \frac{S_0 - S_f(y)}{1 - Fr^2(y)},$$

$$Fr^2 = \frac{Q^2 b}{gA^3}.$$

A downstream weir may impose a subcritical stage. For $y > y_n > y_c$ we have $S_f < S_0$ and $Fr < 1$, so $dy/dx > 0$. The water is deeper toward the control and approaches normal depth upstream. Integrate toward decreasing x starting at the measured stage. Replacing R by y assumes a very wide channel, which the 4 m section does not justify at these depths. Close to $Fr = 1$ the denominator signals the limit of a smooth slowly varying profile. A jump requires its own momentum treatment.

Separate resistance calibration from the boundary condition

The notebook first fits one roughness to four synthetic uniform discharge measurements and then holds it fixed for two downstream controls. The fitted $n = 0.03001$ gives depths of 1.605 m and 1.745 m one kilometre upstream of outlet depths of 1.8 and 2.2 m. The downstream 0.4 m difference has decayed substantially and still leaves an observable upstream effect.

Interactive experiment · Change a downstream control and recompute its upstream reach

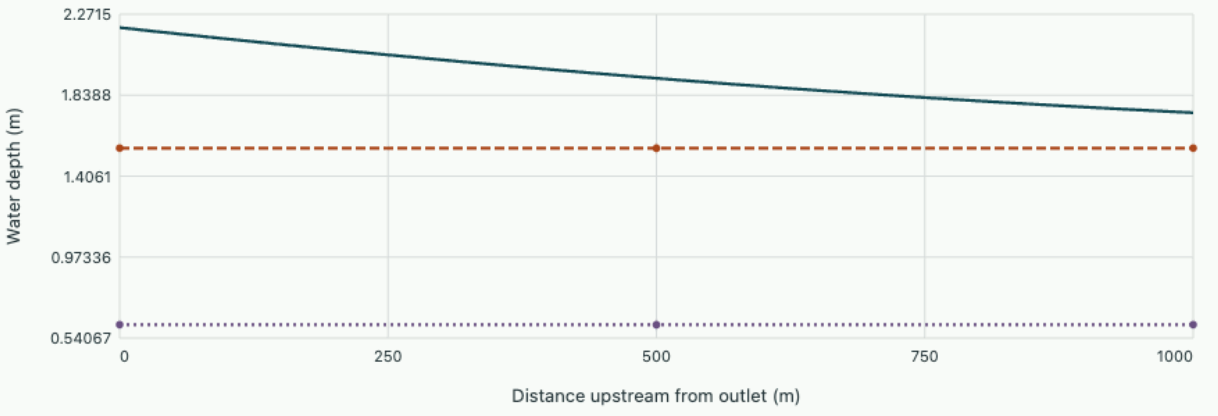
The live calculation integrates the depth equation upstream with fourth-order Runge–Kutta steps of 2 m. Keep the default discharge and roughness, move the outlet stage from 1.8 to 2.2 m and compare depths 1 km upstream. Next change roughness: normal depth moves, while critical depth stays fixed. Changing discharge moves both reference depths.

The channel remains 4 m wide with bed slope 0.001. All available input combinations stay subcritical, where a downstream stage can influence the upstream reach. The plotted coordinate increases upstream, so its derivative has the opposite sign to the downstream equation. The notebook below fits roughness with OneSimulation before integrating two profiles; the browser lets you explore the consequences of prescribing that coefficient directly.

Open this experiment online

Try the calculation

Discharge (m³/s)	6
Manning roughness (s/m⅓)	0.03
Outlet depth (m)	2.2



At 1 km upstream: 1.745 m. Normal depth 1.556 m; critical depth 0.612 m. Outlet Froude number 0.147. Fixed 4 m width and 0.001 bed slope; upstream RK4 steps are 2 m. The selected range stays subcritical.

Continue in the OneSimulation notebook · Download its Python model

OneSimulation experiment · Two downstream stages produce different upstream profiles at the same calibrated roughness

Run the notebook with OneSimulation 0.2.0, NumPy, SciPy and Matplotlib. Change the input data and compare your run with the recorded result.

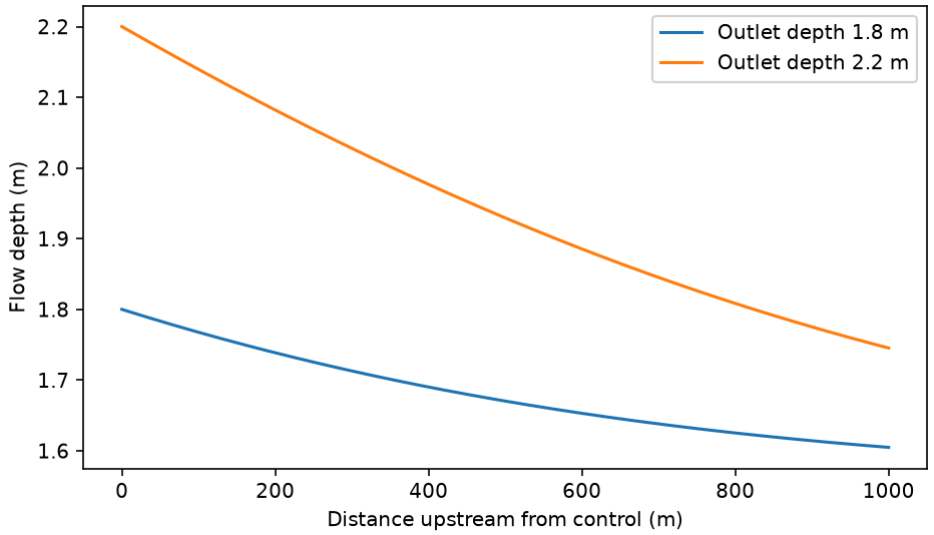
```
"""Infer Manning roughness from uniform flow, then predict a subcritical backwater."""
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp
from onesim.numerics.nonlinear import LeastSquares

width, slope, discharge, gravity = 4.0, 0.001, 6.0, 9.81
measured_depth = np.array([0.7,0.85,1.0,1.15])  # m
def section(depth):
    area = width*depth
    return area, area/(width+2*depth)
def uniform(depth, roughness):
    area, radius = section(depth)
    return area*radius**(2/3)*np.sqrt(slope)/roughness
measured_flow = uniform(measured_depth,0.030) + np.array([0.02,-0.03,0.02,-0.01])
fit = LeastSquares().solve(lambda n: uniform(measured_depth,n[0])-measured_flow,
                           [0.025], bounds=([0.010],[0.08]))
roughness = fit.x[0]
def depth_gradient(x, y):
    area,radius = section(y)
    friction = (roughness*discharge/(area*radius**(2/3)))**2
    froude_squared = discharge**2*width/(gravity*area**3)
    return (slope-friction)/(1-froude_squared)

fig, ax = plt.subplots(figsize=(6.4,3.7), layout="constrained")
print(f"fitted Manning n = {roughness:.5f} s/m^(1/3)")
for boundary in [1.8, 2.2]:
    x = np.linspace(0,-1000,250)
    sol = solve_ivp(depth_gradient,[0,-1000],[boundary],t_eval=x,rtol=1e-9,atol=1e-10)
    assert sol.success
    print(f"outlet {boundary:.1f} m: depth 1 km upstream = {sol.y[0,-1]:.3f} m")
    ax.plot(-sol.t,sol.y[0],label=f"Outlet depth {boundary:.1f} m")
ax.set(xlabel="Distance upstream from control (m)",ylabel="Flow depth (m)")
ax.legend()
plt.show()
```

Reference run

```
fitted Manning n = 0.03001 s/m^(1/3)
outlet 1.8 m: depth 1 km upstream = 1.605 m
outlet 2.2 m: depth 1 km upstream = 1.745 m
```



Two downstream stages produce different upstream profiles at the same calibrated roughness

Download editable notebook · Download Python script

Why two depth measurements do not always identify roughness

Near normal depth, the water surface changes slowly. Bed-slope bias, discharge error and roughness can produce similar residuals over a short reach. Calibrating against uniform measurements at more than one discharge gives an independent constraint before fitting the backwater.

Exercises on hydraulic controls

Two channels have the same discharge and width. Only the Manning coefficient is changed. Which characteristic depth changes?

▼ **Worked solution**

Critical depth is fixed by discharge and geometry. Increasing roughness lowers discharge capacity at fixed depth and slope, so normal depth must increase to carry the same discharge. An altered resistance can move the flow into a different profile class.

A subcritical section has $y = 2.0$ m in the example. Estimate the slope before integrating.

▼ **Worked solution**

Here $A = 8$ m², $R = 1$ m, $S_f = (0.03 \times 6/8)^2 = 0.0005063$ and $Fr^2 = 36(4)/(9.81 \times 512) = 0.0287$. Thus $dy/dx = 0.000508$ m/m. Extrapolating this local slope for a kilometre ignores its variation with depth.

Further reading · Gerhart, Hochstein and Gerhart, Munson, Young and Okiishi's Fundamentals of Fluid Mechanics, 9th ed., Sections 10.3–10.6.

Momentum, motion and transport in fluids · 7. How can a constant-area duct choke?

A nozzle reaches a sonic state by changing area. A straight pipe can approach the same Mach number through friction or through heating. The two cases conserve different combinations of stagnation properties.

Choose the wall process before eliminating variables

Take a steady one-dimensional calorically perfect gas with $G = \rho u$ constant, no shaft work and constant pipe area. Write $M = u/\sqrt{\gamma RT}$, $h_0 = c_pT + u^2/2 = c_pT_0$. An adiabatic wall fixes T_0 even when wall friction creates entropy. A frictionless wall which supplies heat changes T_0 while preserving the axial momentum flux $p + \rho u^2$. Measure static and stagnation variables independently to distinguish these processes.

Integrate the frictional momentum loss

For hydraulic diameter D define the Darcy factor by $\tau_w = f_D\rho u^2/8$. A slice has perimeter-to-area ratio $4/D$, so momentum balance is $dp + \rho u\,du + f_D\rho u^2 dx/(2D) = 0$. Differentiate constant G and T_0 to obtain $d\rho/\rho = -du/u$ and $dT/T = -(\gamma-1)M^2 du/u$. Substitute in $dp/p = d\rho/\rho + dT/T$ and $dM/M = du/u - dT/(2T)$:

$$\frac{f_D dx}{D} = \frac{4(1-M^2)}{\gamma M^3[2+(\gamma-1)M^2]}\,dM.$$

Integrate from inlet M_1 to outlet M_2. Defining the remaining distance to sonic conditions by $F(1) = 0$ gives

$$\frac{f_D L}{D} = F(M_1) - F(M_2),$$

$$F(M) = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma}\ln\frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}.$$

For air with $\gamma = 1.4$, $M_1 = 0.4$, $M_2 = 0.7$, the two function values are 2.30849 and 0.20814. With $D = 0.050$ m and constant $f_D = 0.020$, the required length is 5.25 m. The maximum length at the stated inlet condition is 5.77 m. A longer duct requires different inlet conditions or mass flow. A quoted Fanning coefficient is one quarter of this Darcy coefficient.

Remove friction and allow heat transfer

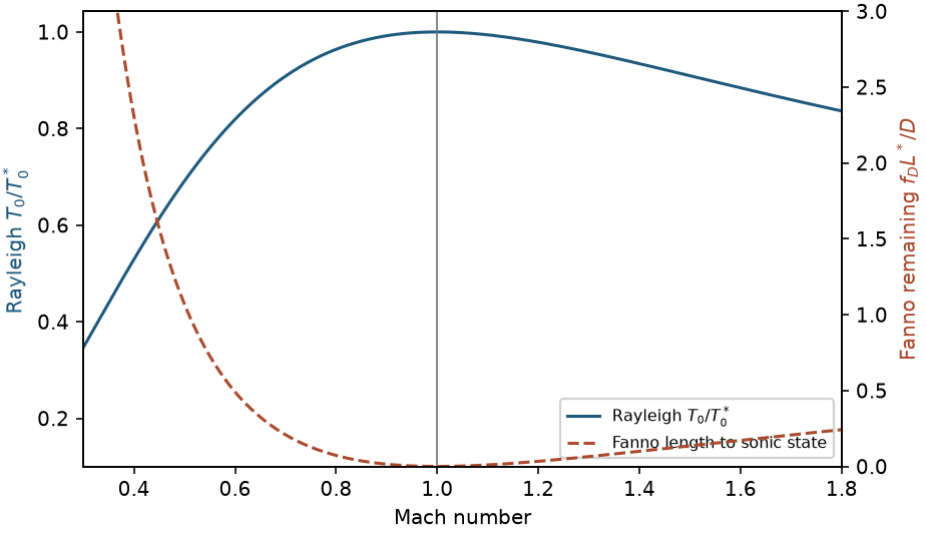
The constant momentum flux becomes $p(1+\gamma M^2) = p^*(1+\gamma)$, where the star denotes the sonic state on this same momentum line. Use mass conservation and the ideal-gas law to get $T/T^* = [(1+\gamma)M/(1+\gamma M^2)]^2$. Multiply by the ratio between stagnation and static temperatures at each state to obtain

$$\frac{T_0}{T_0^*} = \frac{2(1+\gamma)M^2[1+(\gamma-1)M^2/2]}{(1+\gamma M^2)^2}.$$

For the same inlet and outlet Mach numbers this ratio rises from 0.52903 to 0.90850. If $T_{01} = 350$ K, the outlet stagnation temperature is 601.1 K. The heat added per unit mass for $c_p = 1005$ J/(kg K) is approximately 252 kJ/kg. The sonic-line maximum $T_0^* = 661.6$ K places an upper bound on additional heat at the fixed inlet state. In the friction case the stagnation temperature remained at 350 K throughout.

Why both subsonic cases accelerate

Mass flux per area is fixed. The change of pressure and temperature required by momentum and wall conditions reduces density on these subsonic paths. Velocity rises. Supersonic and subsonic branches require explicit tracking when inverting a sonic-reference relation.



Separate wall-process models plotted at the same Mach number. The blue Rayleigh curve uses the left axis, the dashed Fanno curve the right axis. Both have a sonic reference at Mach one.

Interactive calculation · Approach a sonic state by two different wall processes

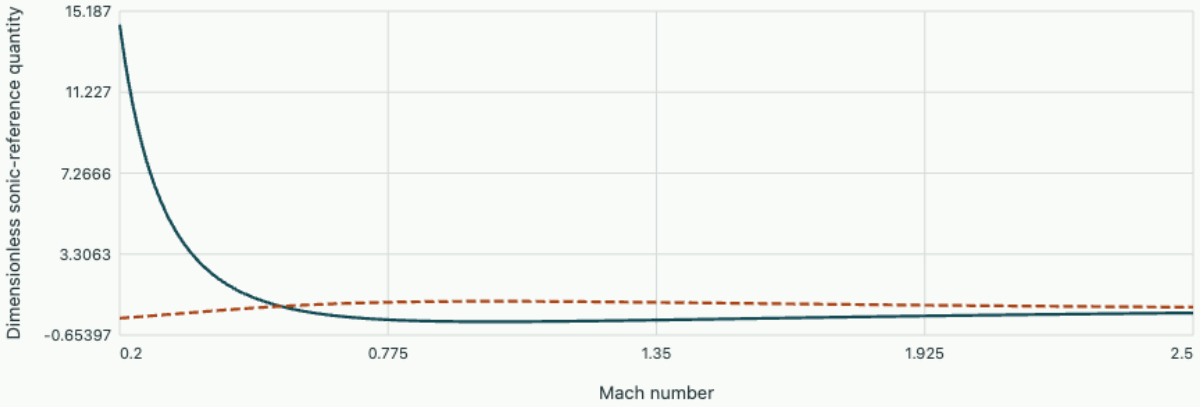
Vary the heat-capacity ratio and inlet Mach number. Compare the remaining friction-length budget with the heat addition required to reach the Rayleigh sonic reference. The two dimensionless ordinates have different definitions.

Open this live calculation

Try the calculation

Heat-capacity ratio **1.4**

Inlet Mach number **0.6**



From inlet Mach 0.600, Fanno remaining fD L/D=0.491; Rayleigh heat addition to choke is 0.221 times inlet cp T0. The curves describe two different wall processes and use independent dimensionless quantities.

Exercises on distinguishing mechanisms

A frictional pipe with the worked inlet already carries gas at Mach 0.7 at a pressure tap. How much more straight pipe reaches the mathematical choking limit?

▼ **Worked solution**

The remaining friction length is $DF(0.7)/f_D = 0.520$ m. This calculation assumes the same Darcy factor up to the sonic state.

A well-insulated pipe test gives a large stagnation-pressure loss but nearly identical measured stagnation temperatures. Could a frictionless Rayleigh model with steady net heating explain both

quantities?

▼ **Worked solution**

Net heating necessarily raises h_0. The nearly unchanged stagnation temperature supports adiabatic frictional loss or another adiabatic irreversible process. Static temperatures alone do not distinguish acceleration from wall heat.

Further reading · Gerhart, Hochstein and Gerhart, Munson, Young and Okiishi's Fundamentals of Fluid Mechanics, 9th ed., Sections 11.8–11.9.

Energy, entropy and mechanical response

Close a steady-flow energy balance, then determine which pressure or temperature change reveals an irreversible process. Connect a film force to a moving bearing.

1. Open systems, enthalpy and steady-flow devices
2. Entropy, irreversibility and useful-work loss
3. Useful work and exergy destruction
4. Diffusion with a stationary carrier gas
5. Hydrodynamic lubrication and sliding bearings
6. From cut forces to local equilibrium
7. Action density, stress waves and the boundary that reflects them
8. From binary collisions to the Boltzmann equation

Open systems, enthalpy and steady-flow devices

Engineering thermodynamics and heat transfer

Use an inlet-outlet balance to distinguish shaft work, kinetic-energy conversion and pressure reduction.

FROM FIRST PRINCIPLES

Derive the steady-flow energy balance including flow work

Surround a turbine, nozzle or mixing chamber with a stationary control surface. In addition to transporting its internal, kinetic and gravitational energies, an entering fluid is pushed across the inlet by upstream pressure. A mass increment dm occupies volume $v\,dm$ and requires pressure work $pv\,dm$ to enter. The outlet carries the corresponding work away. Combining internal energy u and flow work defines enthalpy:

$$h = u + pv,$$

$$e_{\text{stream}} = h + V^2/2 + gz.$$

Here V denotes speed, v specific volume and z elevation. In steady operation, sum these stream contributions and include heat into and shaft work out of the control volume. For one inlet and one outlet with equal mass rate, rearrangement gives

$$\dot{Q} - \dot{W}_s = \dot{m}\,[h_2 - h_1 + (V_2^2 - V_1^2)/2 + g(z_2 - z_1)].$$

All terms have units of power. A nozzle can accelerate flow without shaft work by lowering static enthalpy. A throttle with no heat, no shaft work and negligible speed and elevation changes is instead approximately isenthalpic. Omitting the pressure work while also using internal energy would miss both conclusions.

Background · Specific enthalpy

$$h = u + pv$$

Specific quantities are per unit mass. Enthalpy accounts for internal energy and flow work when mass passes through a control surface.

DERIVATION AND APPLICATIONS

Convert a nozzle energy balance into a mass-flow limit

A steady nozzle converts static enthalpy into bulk kinetic energy. A converging passage has a maximum possible mass flow at a specified inlet stagnation state. Derive this for a steady, one-dimensional,

adiabatic ideal gas with constant c_p and γ, reversible acceleration and negligible inlet-reservoir velocity. Write reservoir conditions as T_0, p_0 and the local static conditions as T, p. With no shaft work the energy balance gives

$$c_p T_0 = c_p T + u^2/2,$$
$$u = \sqrt{2 c_p T_0 (1 - T/T_0)}.$$

Isentropic acceleration links temperature and pressure. Let $r = T/T_0$, so $p/p_0 = r^{\gamma/(\gamma-1)}$. The mass flux is ρu and ideal-gas density is $p/(RT)$. Substitution gives a function of just one local variable:

$$G(r) = \frac{p_0}{R T_0} \sqrt{2 c_p T_0}\, r^{1/(\gamma-1)} \sqrt{1 - r}.$$

To find its maximum differentiate $\ln G = (\ln r)/(\gamma - 1) + \ln(1 - r)/2 + \text{constant}$. Equating the two derivative terms gives the sonic temperature and pressure ratios:

$$r_* = \frac{2}{\gamma + 1},$$
$$p_*/p_0 = \left(\frac{2}{\gamma + 1}\right)^{\gamma/(\gamma-1)},$$
$$u_*^2 = \gamma R T_*.$$

The flow through a converging exit is choked when downstream pressure is low enough for this state to be reached. Further lowering downstream pressure changes the flow after the exit; the ideal mass flux at that exit remains fixed. The model omits wall friction, heat transfer and discharge-coefficient corrections.

WORKED EXAMPLE

Size a sonic air metering aperture

Change the inputs and test this calculation.

For $p_0 = 200$ kPa absolute, $T_0 = 300$ K, $\gamma = 1.40$ and $R = 287$ J/(kg K), the critical state has $T_* = 250$ K and $p_* = 105.7$ kPa. Then $\rho_* = 1.47$ kg/m³ and $u_* = 317$ m/s, giving $G_* \simeq 467$ kg/(m² s). A 20.0 mm² effective aperture meters about 9.33 g/s if the downstream pressure allows choking.

SECOND APPLICATION

Decide whether a higher receiver pressure matters

Change the inputs and test this calculation.

Raise the receiver pressure to 160 kPa. Its ratio to reservoir pressure is now 0.800, above the critical 0.528. An ideal converging nozzle discharges at this pressure with $r = 0.800^{(0.4/1.4)} = 0.938$. The

energy equation gives $u \approx 193$ m/s, static density about 1.98 kg/m³, and mass flux about 382 kg/(m² s). Through the same aperture the predicted flow falls to 7.64 g/s. A model assuming choked flow at both settings would miss this change.

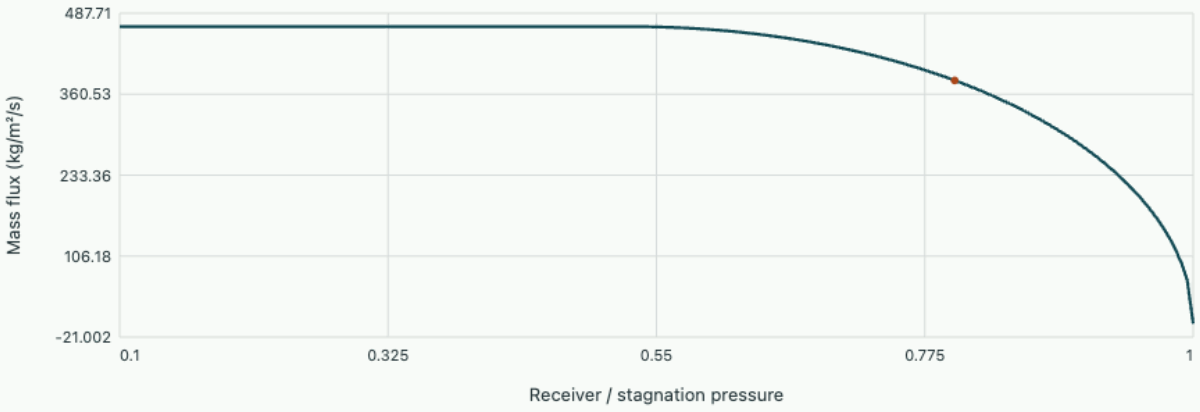
Try it · Nozzle

Set the inputs to the values in the worked example, then vary one quantity while holding the others fixed. Compare the curve and numerical balance with the second application. The calculation uses the assumptions derived above; its readout identifies bounds such as choking, phase-region changes and inconsistent measurements.

Open this live calculation

Try the calculation

Stagnation pressure (kPa)	200
Stagnation temperature (K)	300
Receiver / stagnation pressure	0.8



Converging ideal nozzle | Selected receiver

Critical pressure ratio 0.528282; choked mass flux 466.712 kg/(m² s). A 20 mm² aperture gives 7.64291 g/s at the selected receiver pressure. The converging exit is unchoked.

Account for fluid entering and leaving

A closed-system piston can be followed with its gas. A continuously operating turbine has fresh fluid crossing its boundary every second. Enclose the hardware and its internal flow in a stationary control volume. Flowing mass carries internal energy u, kinetic energy $v^2/2$ and gravitational energy gz per kilogram. Upstream pressure also does work pushing fluid in, while downstream pressure opposes its exit. Per unit mass this flow work is p/ρ, motivating the specific enthalpy $h = u + p/\rho$.

Adding all incoming and outgoing energy rates gives, for one inlet, one outlet and constant stored energy,

$$\dot{Q} - \dot{W}_s = \dot{m}\left[h_2 - h_1 + (v_2^2 - v_1^2)/2 + g(z_2 - z_1)\right].$$

Here $\dot{W}_s$ is shaft power out of the control volume. Pressure work at the flowing boundaries is already contained in enthalpy; adding it once more would count it twice. One inlet and outlet have the same steady mass rate even when their volumetric flow rates differ dramatically because gas expands.

Read the dominant energy conversion

In a short, insulated nozzle there is no shaft, the elevation difference is small and falling enthalpy raises velocity. The property $h + v^2/2$ is then constant, often called stagnation enthalpy. In an insulated turbine with small inlet and outlet speeds, the enthalpy drop instead appears mainly as positive shaft work. An insulated compressor takes work in, raising enthalpy. An insulated throttling valve has neither shaft power nor a large net change of bulk kinetic energy between distant ports, so its inlet and outlet enthalpies are approximately equal. These components can all drop pressure during expansion, yet their energy effects differ.

For ideal-gas air with nearly constant c_p, an enthalpy difference may be approximated by $c_p \Delta T$. Keeping h in kJ/kg while using $v^2/2$ in J/kg is a common factor-of-one-thousand error. At 100 m/s the kinetic energy is 5 kJ/kg. That estimate often decides whether an inlet velocity may be omitted from a turbine work calculation.

Evaluate heat loss and exhaust speed

A turbine passes 2.0 kg/s of gas from $h_1 = 820$ to $h_2 = 600$ kJ/kg. Inlet speed is 40 m/s, exhaust speed 160 m/s and the casing loses 10 kW of heat. Elevations are equal. The gas loses $2(820 - 600) = 440$ kW of enthalpy flow. Its kinetic-energy rate grows by $2(160^2 - 40^2)/2000 = 24.0$ kW. The unit delivers $440 - 24 - 10 = 406$ kW shaft power. Counting all 440 kW as shaft output would overstate the result by both the residual exhaust energy and the heat leak.

A separate nozzle using gas at $h_1 = 820$ kJ/kg and 40 m/s could reach $h_2 = 800$ kJ/kg without a shaft if its speed rose to $\sqrt{40^2 + 2(20\,000)} = 204$ m/s. This estimate is an energy condition. The required area ratio, pressure drop and possibility of choking also depend on compressible-flow relations.

Extend the same balance to a mixer

Two insulated water lines mix at negligible speed change. One supplies 0.20 kg/s at 360 K and the other 0.30 kg/s at 300 K. A mass balance gives 0.50 kg/s at exit. With equal, constant c_p and no shaft, summing the two inlet enthalpy rates yields an outlet temperature $[0.20(360) + 0.30(300)]/0.50 = 324$ K. The temperature lies between the inlet values, closer to the higher mass rate. The mixer conserves

energy while creating entropy. The next lecture explains how both statements can hold simultaneously and why reversing the mixer without another effect is impossible.

Interpret a measured stagnation temperature

For an insulated diffuser with negligible shaft work, reducing speed from 200 to 40 m/s raises static enthalpy by $(200^2 - 40^2)/2 = 19.2\,\mathrm{kJ/kg}$. For air with $c_p = 1.005\,\mathrm{kJ/(kg\,K)}$ that corresponds to a 19.1 K static temperature rise. Stagnation enthalpy $h + V^2/2$ stays constant even if internal friction produces entropy. A stationary temperature probe that slows the arriving air may approach a stagnation reading, so comparison with a predicted static temperature requires knowing where and how the measurement was taken.

Minimise compression work by selecting an intermediate pressure

A continuous air supply can be compressed in two stages and cooled at intermediate pressure. For a steady adiabatic compressor with negligible kinetic and potential changes, shaft work input is the stream enthalpy rise. With an ideal gas, constant c_p, γ, and a reversible stage,

$$w = c_p T_{\mathrm{in}} [(p_{\mathrm{out}}/p_{\mathrm{in}})^{a} - 1],$$

where $a = (\gamma - 1)/\gamma$. This follows by integrating $ds = c_p\,dT/T - R\,dp/p = 0$, then applying $dh = c_p\,dT$. If both stages use the same isentropic efficiency, divide each ideal rise by that efficiency.

Assume complete intercooling to the original inlet temperature T_1, no pressure losses and prescribed first and last pressures. Put $r_1 = p_i/p_1,\ r_2 = p_3/p_i$, so $r_1 r_2 = p_3/p_1 = r$. Total ideal work is $c_p T_1 [r_1^a + (r/r_1)^a - 2]$. Differentiate with respect to $\ln r_1$; the derivative is $c_p T_1 a (r_1^a - r_2^a)$. The sum has a minimum at equal ratios, giving intermediate pressure $p_i = \sqrt{p_1 p_3}$. This is a geometric mean of absolute pressures. A gauge-pressure ratio would give a completely different design.

Compare two cooled stages with a single compression

Supply air at 100 kPa and 300 K and deliver it at 900 kPa absolute. For $\gamma = 1.4,\ c_p = 1.005\,\mathrm{kJ/(kg\,K)}$, an ideal single stage gives $T_3 = 300(9^{0.2857}) = 562.0\,\mathrm{K}$ and work about 263 kJ/kg. Two equal-ratio stages operate through 300 kPa. Each reaches $300(3^{0.2857}) = 410.6\,\mathrm{K}$ before its outlet or intercooler, requiring $1.005(110.6) = 111.2\,\mathrm{kJ/kg}$. Total work is about 222.4 kJ/kg, roughly 15.5% lower. The intercooler must remove 111.2 kJ/kg between the stages under the ideal complete-cooling assumption.

With equal stage efficiencies of 0.80, each real enthalpy rise becomes 139.0 kJ/kg and the intermediate discharge temperature is about 438 K. Cooling back to 300 K then removes 139.0 kJ per kilogram, with a corresponding load of 27.8 kW at 0.20 kg/s. The equal-ratio choice survives because both stage work expressions acquired the same multiplier. An intercooler that can cool only to 330 K increases the weight of second-stage compression work; the new optimum shifts towards a larger first-stage pressure ratio. A

design comparing only compressor shaft power would also need to account for coolant pumping, fan power and pressure losses before choosing the number of stages.

For an incomplete intercooler the optimal shift can be calculated. Take stage two inlet temperature $T_m = 330$ K independently maintained and identical efficiencies. The variable portion of total work is proportional to $T_1 r_1^a + T_m (r/r_1)^a$. Differentiating with respect to log first-stage ratio gives $T_1 r_1^a = T_m r_2^a$, or $r_1/r_2 = (330/300)^{1/a} = 1.40$. At overall ratio nine,

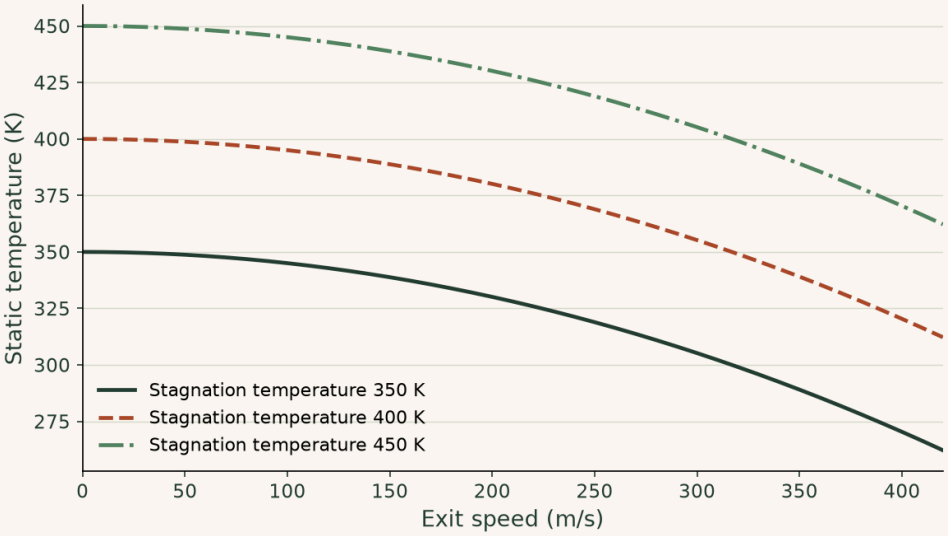
$$r_1 = \sqrt{9(1.40)} = 3.54,\ r_2 = 2.54.$$

Compressing farther while the air is colder is energetically preferable within this simplified comparison. These ratios presume the cooler can maintain 330 K over its revised duty and that both discharge temperatures stay within their allowable ranges. A compressor map may impose further flow and speed limits beyond the thermodynamic minimum.

At fixed pressure endpoints, the measured intermediate pressure alone would thus be insufficient for diagnosing whether the cooler or a compressor caused an observed power increase.

How much static temperature does a nozzle convert to speed?

Air flows steadily through an insulated nozzle with negligible shaft work and elevation change. Take $c_p = 1005$ J/(kg K). Divide the conserved stagnation enthalpy $c_p T + u^2/2$ by c_p to find the exit temperature.



At 300 m/s the static temperature is about 44.8 K below stagnation temperature, regardless of pressure in this enthalpy calculation. Predicting exit pressure as well would require an entropy or loss model.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

Change stagnation temperature and heat capacity independently. The speed-related temperature drop follows the energy balance.

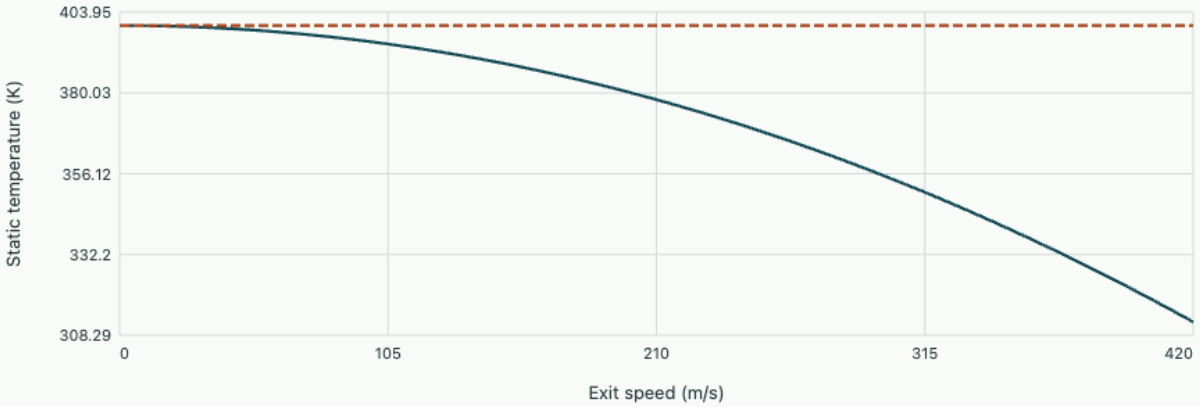
The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Stagnation temperature (K) **400**

Specific heat (J/kg K) **1005**



Selected stagnation state Inlet stagnation temperature

At 300 m/s the static temperature drop is 44.776 K. Pressure needs a separate entropy/loss relation.

Entropy, irreversibility and useful-work loss

Engineering thermodynamics and heat transfer

Check a proposed process with the second law and quantify the effect of a finite temperature difference.

FROM FIRST PRINCIPLES

Separate entropy transfer from entropy generated inside a boundary

For a reversible differential process the thermodynamic definition of entropy is

$$dS = \delta Q_{\mathrm{rev}}/T.$$

Irreversible changes reach the same state-function difference but can generate entropy internally. For a closed system exchanging heats Q_j across boundary portions at known local temperatures T_j, the second law gives an experimentally testable inequality:

$$S_2 - S_1 = \sum_j \int \frac{\delta Q_j}{T_j} + S_{\mathrm{gen}}, \qquad S_{\mathrm{gen}} \geq 0.$$

To evaluate the left side independently of the actual path, use a reversible path between the same equilibrium states. For an ideal gas the fundamental relation $Tds = du + p\,dv$, together with $du = c_v dT$ and $pv = RT$, yields

$$s_2 - s_1 = c_v \ln(T_2/T_1) + R \ln(v_2/v_1)$$

for constant specific heat. Inserting this property change into the balance isolates entropy generation of the actual process. A gas might cool and lose entropy while the combined gas plus surroundings generates entropy. For heat exchanged across a wall separating two reservoirs, the two reservoirs' boundary transfer terms have opposite signs and different temperature denominators.

Background · Entropy generation

$$\dot{S}_{\mathrm{gen}} = \dot{S}_{\mathrm{system}} - \sum \dot{Q}_j/T_j - \dot{S}_{\mathrm{mass,in}} + \dot{S}_{\mathrm{mass,out}}$$

Use boundary temperatures at which heat actually crosses. All flows, storage and units must refer to the same control volume.

DERIVATION AND APPLICATIONS

Count the entropy generated when two ideal gases mix

Two separated ideal gases can have the same temperature and pressure, yet their mixing increases entropy. Consider distinct nonreacting species in insulated rigid compartments. Remove the separator and let them equilibrate in the common volume. Initially both sides have temperature T and pressure p, and no shaft work is performed. The total internal energy of ideal gases depends only on temperature, so the final equilibrium temperature stays T. Denote mole numbers by n_i and total volume by V.

For each species, the entropy change can be calculated along a reversible isothermal expansion into its accessible final volume, even though the actual mixing is irreversible. Integrating $\delta Q_{\mathrm{rev}}/T = p_i\,dV/T = n_i R\,dV/V$ gives

$$\Delta S_i = n_i R \ln(V/V_i).$$

The initial volumes were proportional to mole numbers because p and T were common. Thus $V_i/V = n_i/\sum_j n_j = y_i$, the final mole fraction. Add the species changes and use zero boundary entropy transfer:

$$\Delta S_{\mathrm{mix}} = S_{\mathrm{gen}} = -R\sum_i n_i \ln y_i.$$

For two gases with y and $1-y$, the generation per mole of mixture is maximal when both are equally abundant. The derivative and its zero are

$$\frac{d}{dy}\frac{\Delta S_{\mathrm{mix}}}{n} = R\ln[(1-y)/y] = 0 \quad\Longrightarrow\quad y = \tfrac{1}{2}.$$

The entropy change tends to zero when either fraction tends to one. If the compartments contain the same gas and the initial temperatures and pressures already agree, removing the partition does not create distinguishable species; assigning artificial labels would predict a spurious mixing term.

WORKED EXAMPLE

Estimate the lost work of mixing two samples

Change the inputs and test this calculation.

Mix 1.00 mol of one ideal gas and 3.00 mol of another at 300 K. The mole fractions are 0.250 and 0.750. The generated entropy is

$$-8.314[\ln 0.25 + 3\ln 0.75] = 18.7\ \mathrm{J/K}.$$

Relative to an environment at 300 K, the destroyed work potential is $300(18.7) = 5.61$ kJ. No energy has disappeared from the insulated vessel.

SECOND APPLICATION

Compare equal and strongly unequal amounts

Change the inputs and test this calculation.

For a fixed total of 4.00 mol, an equal mixture generates $4R\ln 2 = 23.1$ J/K. With only 0.0400 mol of one species the fractions become 0.0100 and 0.990. The generation falls to

$$4R[-0.01\ln 0.01 - 0.99\ln 0.99] = 1.86\,\mathrm{J/K}.$$

Each rare molecule gains access to much more volume, but there are few such molecules; both contributions must be counted before comparing samples.

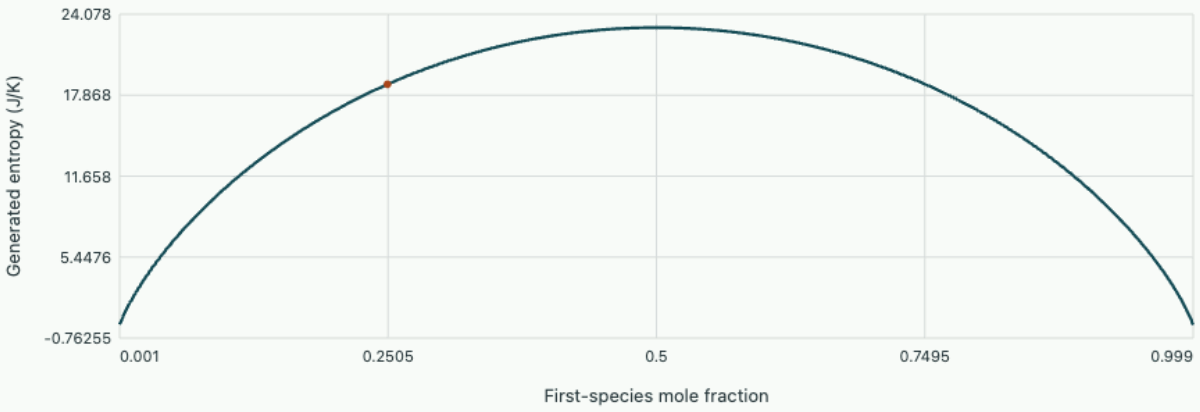
Try it · Mixing

Set the inputs to the values in the worked example, then vary one quantity while holding the others fixed. Compare the curve and numerical balance with the second application. The calculation uses the assumptions derived above; its readout identifies bounds such as choking, phase-region changes and inconsistent measurements.

Open this live calculation

Try the calculation

Total amount (mol)	4
First-species mole fraction	0.25
Environment temperature (K)	300



Two distinct ideal species | Selected mixture

Entropy generation 18.7021 J/K; destroyed work potential T0 Sgen = 5.61062 kJ. Equal initial temperature and pressure; the same-species case has no mixing term.

Energy bookkeeping is not sufficient

In an insulated two-body experiment, heat lost by one body is gained by the other. That energy balance alone would allow heat to move spontaneously from a colder body to a hotter one. Entropy supplies the missing direction constraint. It is a property of a state. For a simple equilibrium system a reversible differential heat transfer defines $dS = \delta Q_{\mathrm{rev}}/T$. An irreversible process between the same endpoints has the same entropy difference; the actual heat transfer divided by the body temperature need not equal that difference.

For a closed system, the integrated entropy balance reads $\Delta S = \int \delta Q/T_b + S_{\mathrm{gen}}$, where T_b is the absolute temperature at the boundary where the heat crosses. Entropy generation satisfies $S_{\mathrm{gen}} \geq 0$. A

system can lose entropy by exporting it in heat or in a mass stream. A negative change in the entropy of a single hot body therefore need not violate the second law.

Background · Why does a small temperature difference produce a second-order loss at fixed conductance?

Let a link of constant conductance G join reservoirs held at $T + \Delta T/2$ and $T - \Delta T/2$. In steady operation the heat current is $G\Delta T$. Subtract the entropy leaving the hot reservoir from that entering the cold reservoir.

$$\dot{S}_{\mathrm{gen}} = G\Delta T\left(\frac{1}{T - \Delta T/2} - \frac{1}{T + \Delta T/2}\right) = \frac{G(\Delta T)^2}{T^2 - (\Delta T/2)^2}.$$

At fixed conductance, halving a small difference approximately quarters the entropy production per second, while it also halves the heat current. If a design must keep the heat rate fixed, halving the difference requires nearly twice the conductance and only halves the production. This distinction matters when comparing a bigger exchanger with the original one.

Calculate a property change with a convenient path

For a calorically perfect ideal gas, combine $T\,ds = du + p\,dv$ with $du = c_v\,dT$ and $p = RT/v$. This gives

$$ds = c_v\,dT/T + R\,dv/v.$$

Replacing v using the ideal-gas law yields the alternative $ds = c_p\,dT/T - R\,dp/p$. Integrate between the initial and final equilibrium states. These are property relationships, even when the actual path between states includes friction, throttling or free expansion.

An ideal gas passing through an insulated valve with equal inlet and outlet speeds has $h_2 = h_1$, so its temperatures are equal in the constant-c_p approximation. Dropping air from 400 to 100 kPa gives $s_2 - s_1 = -0.287\ln(100/400) = 0.398$ kJ/(kg K). With no heat transfer or other streams, the positive entropy change per kilogram is entirely generated. A reversible insulated expansion turbine would generally produce a lower outlet temperature and shaft work.

Trace entropy across two reservoirs

An insulated conductor transfers 600 W from a reservoir at 400 K to another at 300 K. Its stored energy and entropy are steady. The hot reservoir loses entropy at $600/400 = 1.50$ W/K, while the cold one gains $600/300 = 2.00$ W/K. The total generation rate is 0.50 W/K. Heat rate is the same at both ends; entropy transfer in that heat differs because the temperatures differ.

$$\dot{S}_{\mathrm{gen}} = \dot{Q}\left(\frac{1}{T_c} - \frac{1}{T_h}\right).$$

If the surroundings defining the available-work reference are at $T_0 = 300$ K, the corresponding destruction of available work is $T_0 \dot{S}_{\mathrm{gen}} = 150$ W. For these particular reservoirs a reversible engine supplied with 600 W at 400 K could have produced $600(1 - 300/400) = 150$ W before rejecting the remainder. The conductor instead rejects all 600 W to the cold sink. Energy did not disappear; an opportunity for work was lost.

Locate sources and choose a boundary

A heat exchanger can have little external heat leak yet generate appreciable entropy internally through finite temperature differences and pressure losses. Write $\dot{S}_{\mathrm{gen}} = \sum_{\mathrm{out}} \dot{m} s - \sum_{\mathrm{in}} \dot{m} s - \sum_j \dot{Q}_j / T_{b,j}$ for steady operation. Both streams belong inside the same control volume when testing internal heat exchange. If the boundary encloses only one stream, heat crossing the wall carries entropy with it and must appear explicitly. The largest reduction in entropy generation per extra unit of area or pumping effort is a design question. The thermodynamic balance identifies losses without alone determining the economically preferred geometry.

Compare the conditions on a valve and a turbine

Air passes through a steady insulated throttling valve from 400 to 100 kPa with negligible changes of speed. With no shaft work the first law gives equal inlet and outlet enthalpy. For ideal air this implies equal temperature. Its entropy change follows from $\Delta s = c_p \ln(T_2/T_1) - R \ln(p_2/p_1) = 0.287 \ln 4 = 0.398\,\mathrm{kJ/(kg\,K)}$. At 0.10 kg/s the entropy generation is 0.0398 kW/K. An ideal adiabatic turbine undergoing the same pressure drop has constant entropy and delivers work as its temperature falls. Pressure ratio alone therefore cannot determine the temperature change across an unfamiliar device.

Obtain a work bound for two finite bodies

The reversible efficiency $1 - T_c/T_h$ describes operation between reservoirs held at fixed temperatures. If the two bodies are finite, each increment of transferred heat changes the temperatures and the instantaneous efficiency. Consider two incompressible blocks in a common insulated enclosure. Their heat capacities C_h, C_c are taken as constant. A reversible engine operates between them and exports mechanical work until no temperature difference remains. The working fluid returns to its starting state after each incremental cycle, so conservation of entropy for this reversible composite system determines the final temperature.

$$C_h \ln(T_f/T_h) + C_c \ln(T_f/T_c) = 0,$$
$$T_f = T_h^{C_h/(C_h+C_c)} T_c^{C_c/(C_h+C_c)}.$$

Energy conservation gives maximum work $W_{\max} = C_h T_h + C_c T_c - (C_h + C_c) T_f$ for consistent internal-energy references. The reversible final temperature is a weighted geometric mean. If the two blocks are simply put in thermal contact without work output, first-law conservation instead gives the weighted arithmetic mean. The arithmetic mean is larger for unequal initial temperatures; additional energy remains as sensible heat after irreversible contact. These different final temperatures matter when comparing two ways of using the same finite resources.

In a real engine running at finite heat-transfer rates there are temperature differences between each block and the working fluid. Total entropy then increases. For a specified common final temperature the entropy change of the blocks is fixed by state properties; measuring it indicates the departure from the ideal reversible construction without needing the working-fluid cycle.

Compare direct contact and a reversible engine quantitatively

Take a hot assembly of heat capacity 2.0 kJ/K at 450 K and a cooler assembly of 1.0 kJ/K at 300 K. Solving the log balance gives

$$T_f = (450^2\, 300)^{1/3} = 393.1\,\mathrm{K}.$$

The hot assembly releases about $2(450 - 393.1) = 113.8\,\mathrm{kJ}$ and the cold one receives 93.1 kJ. Their difference gives approximately 20.7 kJ of maximum work. Acting as though the initial temperatures stayed at 450 and 300 K throughout would assign an incorrectly high efficiency to later parts of the transfer.

By direct contact the common temperature would be $(2 \times 450 + 300)/3 = 400\,\mathrm{K}$. The entropy generated is

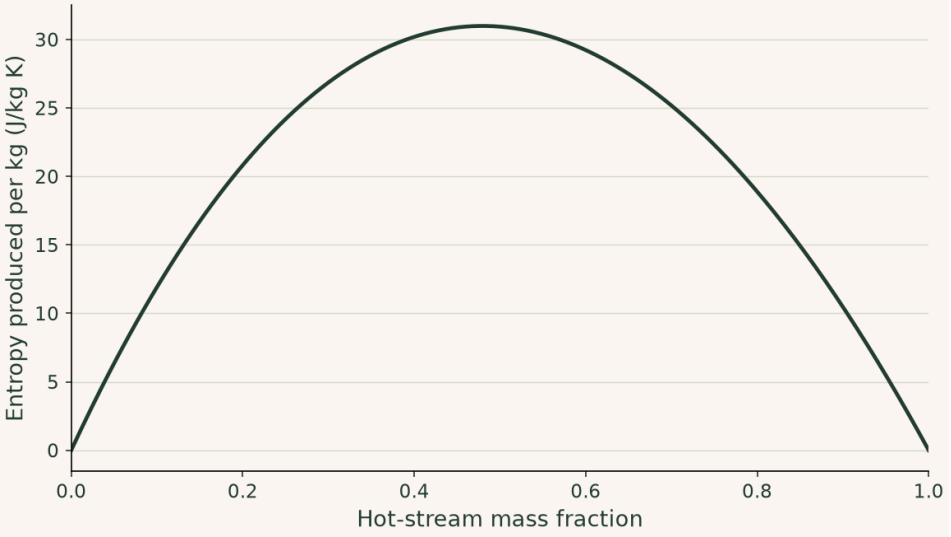
$$2 \ln(400/450) + \ln(400/300) = 0.0521\,\mathrm{kJ/K}.$$

If instead a device delivers a measured 12 kJ of work before the blocks reach a common temperature, energy fixes that temperature as $(1200 - 12)/3 = 396\,\mathrm{K}$. The measured process then generates $2 \ln(396/450) + \ln(396/300) = 0.0220\,\mathrm{kJ/K}$, a positive value between the two limiting cases. Entropy generation multiplied by an environmental temperature gives destroyed exergy relative to that environment; comparing the finite-body work outputs also requires acknowledging their different final stored energies.

The geometric-mean result has an informative limiting case. If the cold assembly is so large that C_c/C_h tends to infinity, its temperature remains effectively fixed, and the engine work reduces to $C_h[(T_h - T_c) - T_c \ln(T_h/T_c)]$. This is the integral of the reversible work recovered as the finite hot block approaches a constant sink. It is smaller than applying the initial Carnot efficiency to all of the hot block’s released heat.

Which mixture generates the most entropy?

Mix streams of the same incompressible liquid at 370 K and 290 K, at equal pressure, with constant $c_p = 4180$ J/(kg K). The energy balance fixes the outlet temperature. Evaluate the outlet entropy minus the mass-weighted inlet entropy using $c_p \ln T$.



Either pure stream gives zero entropy production. An equal-mass mixture settles at 330 K and produces about 30.9 J/(kg K). The maximum is near, but not exactly at, equal fractions because temperature appears inside a logarithm.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

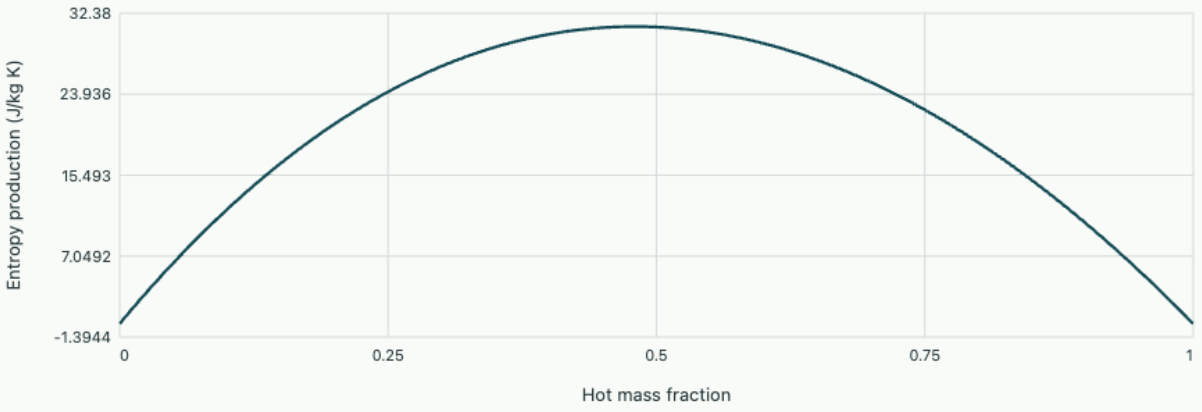
Move the two inlet temperatures toward each other and watch entropy production vanish throughout the mixture range.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Hot stream (K)	**370**
Cold stream (K)	**290**
Liquid heat capacity (J/kg K)	**4180**



Adiabatic mixture

Equal mass fractions reach 330.000 K and generate 30.935 J/(kg K). Endpoints are unmixed and generate zero.

Useful work and exergy destruction

Engineering thermodynamics and heat transfer

Evaluate the useful-work potential of a steady stream and distinguish energy conservation from the destruction of useful-work potential.

FROM FIRST PRINCIPLES

Derive maximum useful work relative to an environment

For a closed body changing toward equilibrium with surroundings at T_0, p_0, consider the reversible path that gives the largest possible useful work. Neglect bulk kinetic and gravitational energy. With heat Q_0 positive into the body and total boundary plus shaft work W positive out, the first law gives

$$W = U_i - U_f + Q_0.$$

Reversible heat exchange with this environment requires

$$Q_0 = T_0(S_f - S_i).$$

Part of total work simply displaces ambient pressure, so

$$W_{\text{useful,rev}} = U_i - U_f + T_0(S_f - S_i) - p_0(V_f - V_i).$$

Choose the final state as the dead state, denoted by subscript zero. Grouping the available initial differences defines nonflow physical exergy:

$$B = (U - U_0) + p_0(V - V_0) - T_0(S - S_0).$$

If the process generates entropy, the same first and second law combination lowers the useful work by $T_0 S_{\text{gen}}$. The expression depends on the specified surrounding state and assumes no other heat reservoirs or chemical changes. A stream crossing a control volume carries its pressure flow work as well, giving a distinct enthalpy-based expression for flow exergy.

Background · Exergy destruction

$$\dot{X}_{\text{dest}} = T_0 \dot{S}_{\text{gen}}$$

The environmental reference temperature enters in kelvin. Irreversibility destroys useful-work potential even when the total energy balance closes.

DERIVATION AND APPLICATIONS

Derive the work potential of a compressed flowing gas

A stream can deliver useful work by cooling or by expanding, even when no fuel or chemical reaction is involved. Choose a steady reference environment at T_0, p_0 with the same gas composition. Consider one kilogram taken reversibly from its inlet state to the environmental state. Neglect kinetic and potential energy. Let q_0 be heat supplied from the environment, positive into the device, and w shaft work delivered. The steady energy balance includes flow work in enthalpy:

$$w = h - h_0 + q_0.$$

The only external heat exchange occurs with the environment. Reversibility then gives $s_0 - s = q_0/T_0$, eliminating the unmeasured heat transfer. The maximum work is the specific physical flow exergy

$$b = (h - h_0) - T_0(s - s_0).$$

For an ideal gas with constant c_p and R, insert $dh = c_p\,dT$ and $ds = c_p\,dT/T - R\,dp/p$. Integrate relative to T_0, p_0:

$$b = c_p[(T - T_0) - T_0 \ln(T/T_0)] + RT_0 \ln(p/p_0).$$

The pressure term survives at ambient temperature. Near $T = T_0$ the thermal term is quadratic in the temperature difference because energy and the ambient heat contribution cancel to first order. The formula describes work relative to a specified reference and assumes the inlet and outlet composition is the same. An actual dissipative path destroys part of the reversible work potential.

WORKED EXAMPLE

Estimate the work available from warm compressed air

Change the inputs and test this calculation.

Take air at $T = 360$ K, $p = 300$ kPa entering an environment at $T_0 = 300$ K and $p_0 = 100$ kPa. With $c_p = 1.005$ and $R = 0.287$ kJ/(kg K), the thermal contribution is $1.005[60 - 300 \ln 1.2] = 5.33$ kJ/kg. The pressure contribution is $0.287(300) \ln 3 = 94.6$ kJ/kg. Total available work is approximately 99.9 kJ/kg, mostly associated with pressure.

SECOND APPLICATION

Account for loss in an ambient-temperature throttle

Change the inputs and test this calculation.

Let 0.100 kg/s of ideal air throttle adiabatically from 300 to 100 kPa, both at 300 K. Enthalpy stays constant, no shaft work is extracted, and the exergy destruction is $0.100(94.6) = 9.46$ kW. Independent

entropy accounting gives $\dot{S}_{\mathrm{gen}} = \dot{m} R \ln 3 = 0.0315$ kW/K. Its product with the ambient 300 K recovers the same 9.46 kW. A device balancing energy perfectly can therefore expend substantial work potential.

Try it · Flow exergy

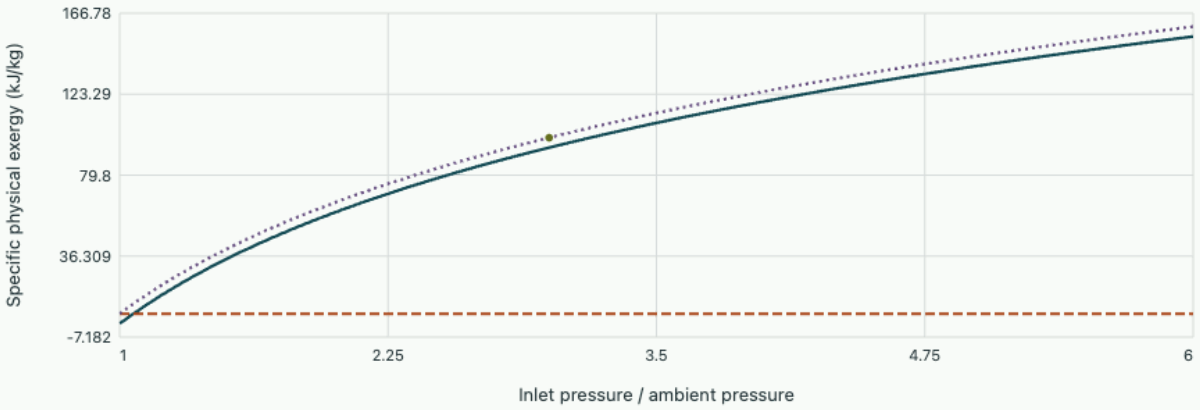
Set the inputs to the values in the worked example, then vary one quantity while holding the others fixed. Compare the curve and numerical balance with the second application. The calculation uses the assumptions derived above; its readout identifies bounds such as choking, phase-region changes and inconsistent measurements.

Open this live calculation

Try the calculation

Inlet temperature (K) **360**

Selected pressure / ambient pressure **3**



Pressure contribution · Thermal contribution · Total physical flow exergy · Selected inlet

Environment 300 K, 100 kPa. Thermal 5.33005, pressure 94.5905, total 99.9206 kJ/kg. At ambient temperature an ideal-gas throttle from this pressure to ambient destroys 9.45905 kW at 0.1 kg/s without changing enthalpy.

How much work is lost when two finite bodies touch?

Two insulated metal blocks start at different temperatures. A small cyclic heat engine could transfer heat from one to the other and deliver shaft work until the temperatures equalise. Simply bringing the blocks into thermal contact also gives a common final temperature, but it conserves a different quantity. Calculate the two end states before estimating how much work the initial temperature difference made available.

Assume constant positive heat capacities C_h, C_c, no phase change, no external heat exchange and no mechanical work by either block. In direct passive contact no outside work is produced. The total energy change is therefore zero, giving $T_{\mathrm{mix}} = (C_h T_h + C_c T_c)/(C_h + C_c)$. If the intervening engine instead operates reversibly, each small entropy transfer leaving the hot block equals the entropy entering the cold one. At the final common temperature T_f,

$$C_h \ln(T_f/T_h) + C_c \ln(T_f/T_c) = 0,$$

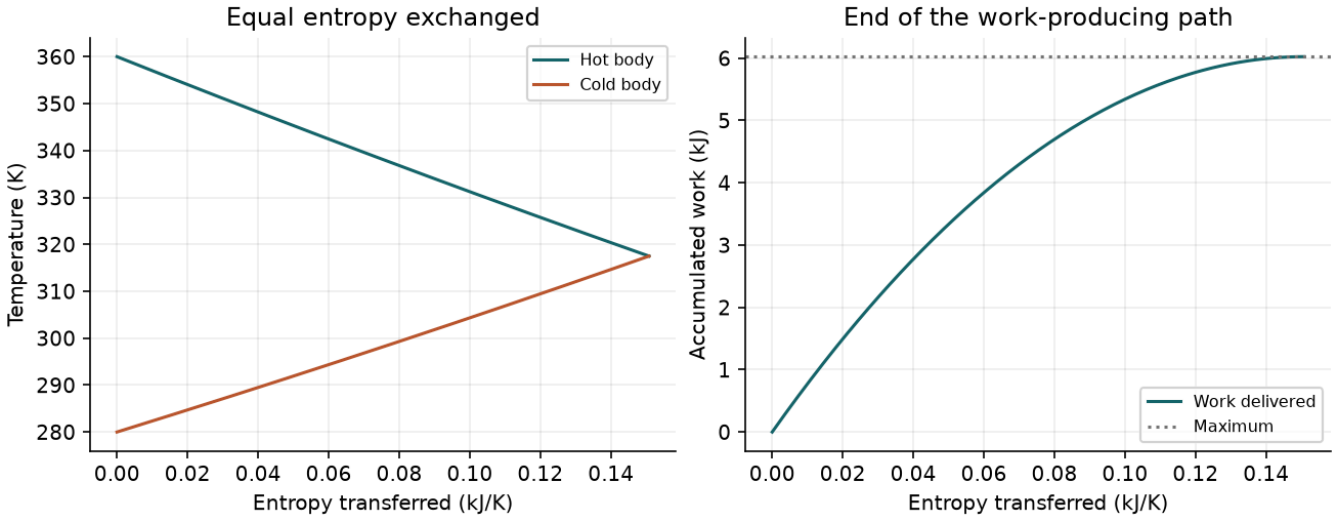
$$T_f = \exp\left[\frac{C_h \ln T_h + C_c \ln T_c}{C_h + C_c}\right].$$

Here all temperatures are absolute. More formally the logarithms can be taken of ratios to the same reference temperature; that reference cancels. The weighted geometric mean lies below the weighted arithmetic mean. The reversible final state thus contains less internal energy than the passive final state. The difference has emerged as work,

$$W_{\max} = (C_h + C_c)(T_{\mathrm{mix}} - T_f).$$

An irreversible engine generates entropy and ends at a higher common temperature with smaller available work.

Take two blocks with capacities of 1.2 kJ/K, initially at 360 K and 280 K. Passive contact ends at 320.0 K; the reversible engine ends at $\sqrt{360 \times 280} = 317.49$ K and delivers 6.02 kJ. Passive contact generates 0.01890 kJ/K of entropy. To visualise the reversible path, let s be the entropy transferred from one block into the other. The instantaneous temperatures are $T_h(s) = T_h(0)e^{-s/C_h}$ and $T_c(s) = T_c(0)e^{s/C_c}$. The next increment of work is $(T_h - T_c)ds$, so the signed area between the two temperature curves reproduces the total work.



The slope of the work curve is the temperature gap. It vanishes at the reversible end state even though the engine delivered finite work earlier.

The scaling becomes particularly clear for equal capacities C when the starting temperatures are $T_m + \Delta$ and $T_m - \Delta$. The mixed temperature is T_m, while the reversible final temperature is $T_m\sqrt{1 - (\Delta/T_m)^2}$. Expanding the square root for small temperature gaps gives

$$W_{\max} \simeq C\Delta^2/T_m.$$

For $T_m = 320$ K and $\Delta = 40$ K this is 6.0 kJ, close to the exact 6.02 kJ. Halving the temperature gap at the same mean temperature gives roughly a quarter of the work. The energy initially contained in the warmer block above the mean temperature scales only linearly with the gap, so that energy alone is a poor measure of usable work.

If the surroundings never exchange heat with either block, should their temperature appear in this maximum-work calculation?

▼ **Work through the solution**

The reversible engine can use only the two specified bodies. Their initial states and conservation of their total entropy determine how far the work extraction goes. A different problem allows ultimate equilibration with an external reservoir at T_0. In that enlarged problem the standard exergy destruction associated with the irreversible contact is $T_0 S_{\mathrm{gen}}$. These work comparisons have different permitted end states. The notebook keeps the enclosure insulated and checks the work using the energy balance on both bodies throughout the exchange.

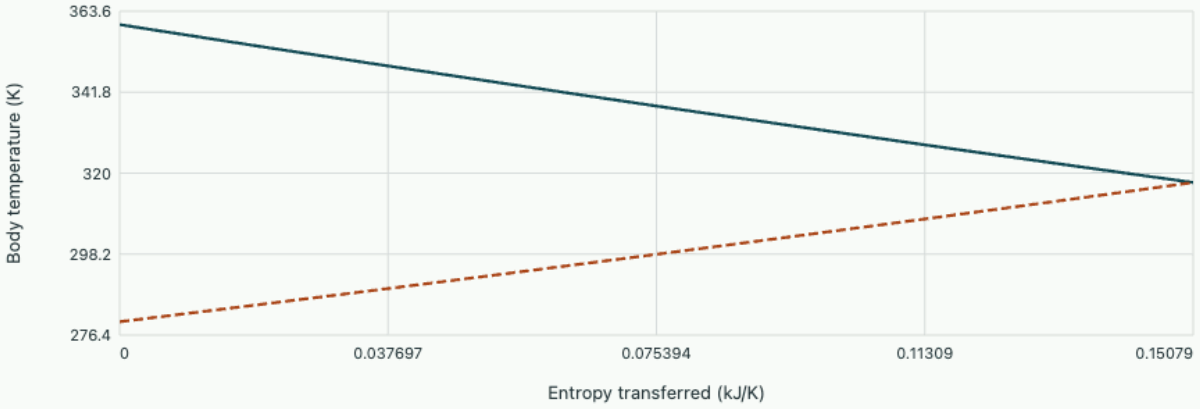
Interactive calculation · Finite source temperatures

Vary the notebook inputs and compare both plotted observables. The same equations generate the browser curves and the notebook reference; fixed material, geometry and model assumptions are stated in the readout.

Open this live calculation

Try the calculation

Hot initial temperature (K)	360
Cold initial temperature (K)	280
Hot / cold heat capacity	1



Common reversible temperature 317.490 K; maximum work 6.024 kJ. Cold heat capacity1.2 kJ/K. Direct contact reaches a warmer arithmetic-weighted mean and generates entropy.

Download the editable notebook for this example

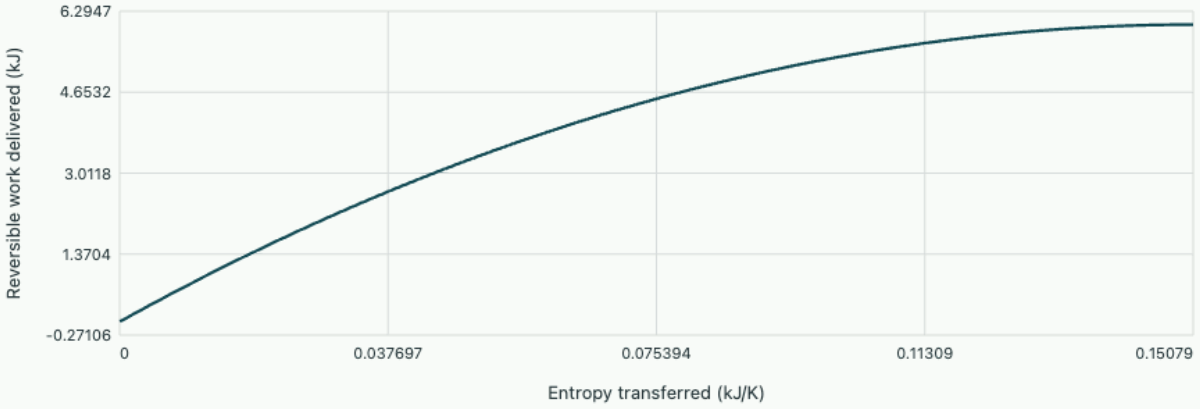
Interactive calculation · Finite source work

Vary the notebook inputs and compare both plotted observables. The same equations generate the browser curves and the notebook reference; fixed material, geometry and model assumptions are stated in the readout.

Open this live calculation

Try the calculation

Hot initial temperature (K)	360
Cold initial temperature (K)	280
Hot / cold heat capacity	1



Accumulated reversible work

Common reversible temperature 317.490 K; maximum work 6.024 kJ. Cold heat capacity1.2 kJ/K. Direct contact reaches a warmer arithmetic-weighted mean and generates entropy.

Download the editable notebook for this example

Open editable notebook Download .ipynb

Specify the surroundings before rating a resource

A warm stream contains energy relative to a chosen reference. The amount that could become useful work depends on the surroundings available to receive heat. If stream and environment reach mutual thermal and mechanical equilibrium, there is no further physical work potential associated with their difference. This equilibrium is the physical dead state at T_0, p_0. Exclude chemical reactions here and consider a steady flow with constant composition.

Imagine a reversible device that takes a stream from its current state to the dead state while interacting thermally only with the environment. The steady first law relates delivered work to changes in stream

enthalpy, kinetic and potential energy, and heat. The reversible entropy balance gives heat received from the environment $q_0 = T_0(s_0 - s)$. Eliminating that heat gives the specific flow exergy

$$\psi = (h - h_0) - T_0(s - s_0) + V^2/2 + g(z - z_0).$$

Use consistent units for the kinetic and potential terms. The entropy contribution reduces the fraction of enthalpy that a reversible device can turn into useful work. At the dead state, with the same velocity and elevation as the reference, that work potential vanishes.

Cool a liquid to the environment

Take a stream of liquid water at approximately ambient pressure, 360 K, with negligible motion and elevation change. It flows at 0.20 kg/s towards a large environment at 300 K. Assume $c_p = 4.18\,\mathrm{kJ/(kg\,K)}$ constant. Integration gives

$$h - h_0 = c_p(T - T_0),$$
$$s - s_0 = c_p \ln(T/T_0).$$

The stream's specific physical exergy is therefore

$$4.18[60 - 300\ln(360/300)] = 22.2\,\mathrm{kJ/kg}.$$

Its enthalpy difference is much larger, 250.8 kJ/kg.

Cooling all the way to 300 K makes at most $0.20(22.2) = 4.44\,\mathrm{kW}$ of useful work available, even though the stream releases 50.16 kW of heat. Direct heat exchange with the environment delivers no shaft work. The environment entropy gain is $50.16/300 = 0.1672\,\mathrm{kW/K}$, while the stream loses $0.20(4.18)\ln(360/300) = 0.1524\,\mathrm{kW/K}$. The difference, about 0.0148 kW/K, is the entropy production rate. Multiplication by T_0 gives the lost 4.44 kW of work potential.

Include heat crossing another temperature boundary

Heat Q supplied reversibly by a reservoir at a uniform boundary temperature T_b carries exergy $Q(1 - T_0/T_b)$. This follows by comparing its energy with the environmental heat required to discharge the associated entropy Q/T_b. A heat input of 10 kW from a 600 K reservoir in a 300 K environment carries 5 kW of exergy. A transfer from a finite cooling stream has changing source temperature and requires integration, as in the water example. Irreversibility in the device destroys exergy at rate $T_0\dot{S}_{\mathrm{gen}}$; directing heat to a useful secondary process can retain part of the exergy that direct cooling would destroy.

Include the work potential of pressure

For an ideal gas already at T_0, an elevated pressure still gives physical flow exergy $RT_0 \ln(p/p_0)$. Air at 300 K and twice environmental pressure has about $0.287(300)\ln 2 = 59.7\,\mathrm{kJ/kg}$ of pressure-related work potential although its ideal-gas enthalpy equals that at the dead state.

Derive the least work needed to heat a finite stream

Heat pumps are commonly compared by heat delivered per unit of electrical work. A second-law comparison additionally accounts for the temperature at which heat is delivered. Consider a steady stream heated from T_i to T_o at nearly environmental pressure, with constant heat capacity rate $C = \dot{m}c_p$. A reversible heat pump extracts heat from surroundings at T_0 and delivers it through a temperature-matched sequence of infinitesimal condensers. For one increment $dQ = C\,dT$ at stream temperature T, the reversed Carnot relation gives

$$dW_{\min} = dQ(1 - T_0/T).$$

Integrating over the stream yields

$$W_{\min} = C[(T_o - T_i) - T_0 \ln(T_o/T_i)].$$

This is the increase of its physical flow exergy, and is smaller than evaluating the Carnot work factor at the final high temperature for the whole heat duty. A device that condenses at a single temperature above the warmest outlet generates entropy during delivery to the colder portions. The result assumes that the source of heat is the environment at T_0; a colder source would introduce an additional resource requirement.

For steady equipment receiving only electricity and environmental heat, with negligible parasitic flows, a meaningful second-law efficiency is the stream’s exergy increase divided by actual electrical input. Electric power is entirely work potential, whereas a kilowatt of heat near room temperature carries only a small fraction of a kilowatt of exergy.

Compare a heat pump with a resistance heater

Water flows at 0.10 kg/s from 293 K to 333 K in a 293 K environment. With $c_p = 4.18\,\mathrm{kJ/(kg\,K)}$, heat duty is $0.418(40) = 16.72\,\mathrm{kW}$. The exergy increase is $0.418[40 - 293\ln(333/293)] = 1.05\,\mathrm{kW}$ approximately. This relatively small figure reflects the modest average temperature lift. A well-insulated heat pump drawing 4.20 kW from the electrical supply has heating COP $16.72/4.20 = 3.98$ and exergy efficiency about 25%. Its surroundings provide 12.52 kW of heat under the stated energy balance.

A resistance heater supplying the identical water service draws 16.72 kW when its losses to the room are negligible. It is nearly 100% efficient on the simple electrical-energy-to-heat basis, yet its exergy efficiency is only about 6.3%. For the heat pump the implied destruction of work potential is $4.20 - 1.05 = 3.15\,\mathrm{kW}$; for resistance heating it is about 15.67 kW. Electricity used by a circulating pump, external heat leaks or a source below environmental temperature belongs in the appropriate

device boundary before drawing a practical comparison. The heat-duty equality ensures that the machines are judged against the same delivered temperature service.

For the resistance case the entropy balance can be checked without using the exergy formula. Electrical work carries no entropy across the control surface, and assume no heat leakage. The stream increases its entropy at $0.418 \ln(333/293) = 0.0535\,\mathrm{kW/K}$. Multiplying by the 293 K environment gives 15.7 kW destroyed, consistent with the difference between input work and stream exergy gain. This agreement depends on keeping the heater and heated stream in the same chosen boundary.

Diffusion with a stationary carrier gas

Engineering thermodynamics and heat transfer

A solvent evaporates into a gas layer. Its molecules diffuse relative to the mixture, while evaporation also induces a bulk molar flow. Work through the species balances to see why the flux ceases to scale linearly with surface concentration.

Choose the reference frame for a flux

Let A be the evaporating species and B an insoluble carrier. A planar film of thickness L lies between the evaporating surface, at mole fraction y_s, and a well-mixed reservoir at $y_\infty < y_s$. Keep total pressure, temperature, total molar concentration $c = p/(RT)$ and binary diffusivity D uniform. In a stationary laboratory frame the total species flux equals diffusion relative to the molar-average velocity plus transport by that velocity. If positive z points towards the reservoir, write

$$N_A = -cD\frac{dy_A}{dz} + y_A(N_A + N_B).$$

Steady conditions in one dimension make N_A, N_B independent of z. Carrier B cannot enter the liquid and has zero net flux there. This fixes $N_B = 0$ throughout the film. B still diffuses towards the interface relative to the moving mixture. Its diffusive flux balances its outward convective flux. The constraint $N_A + N_B = 0$ would instead describe equimolar counterdiffusion between two exchanging species.

Separate variables and enforce both boundary concentrations

Collect the terms in N_A, divide by $1 - y_A$ and integrate from the surface to the reservoir. The remaining integral is a logarithm of the carrier mole fraction.

$$N_A(1 - y_A) = -cD\frac{dy_A}{dz},$$

$$N_A = \frac{cD}{L}\ln\frac{1 - y_\infty}{1 - y_s}.$$

$$1 - y_A(z) = (1 - y_s)\left(\frac{1 - y_\infty}{1 - y_s}\right)^{z/L}.$$

The concentration profile curves because a given net flux requires a different concentration gradient at different carrier fractions. A dilute expansion of the logarithm gives

$$N_A \simeq (cD/L)(y_s - y_\infty).$$

The exact-to-dilute flux ratio exceeds unity when the surface emits A. The discrepancy becomes measurable long before the carrier disappears. At fractions very close to unity, the assumptions of a stagnant, isothermal film also deserve new scrutiny.

Small and large amounts of vapour at the surface

At 300 K and 1 bar, $c = 40.09$ mol/m³. Take $D = 2.0 \times 10^{-5}$ m²/s and $L = 5$ mm. The coefficient $cD/L = 0.1604$ mol/(m² s). With a vapour-free reservoir and $y_s = 0.05$, the logarithm is 0.05129 and the predicted flux is 0.00823 mol/(m² s). A linear gradient would give 0.00802, about 2.5% lower.

For $y_s = 0.50$, the flux rises to 0.111 mol/(m² s), versus 0.0802 for the dilute estimate. At mid-depth the exact fraction is $1 - \sqrt{0.5} = 0.293$, while linear interpolation gives 0.25. An experimental check can therefore use either a mass-loss measurement or a local concentration measurement. Evaporative cooling changes the saturation mole fraction at the surface, so a measured surface temperature matters even if room temperature seems stable.

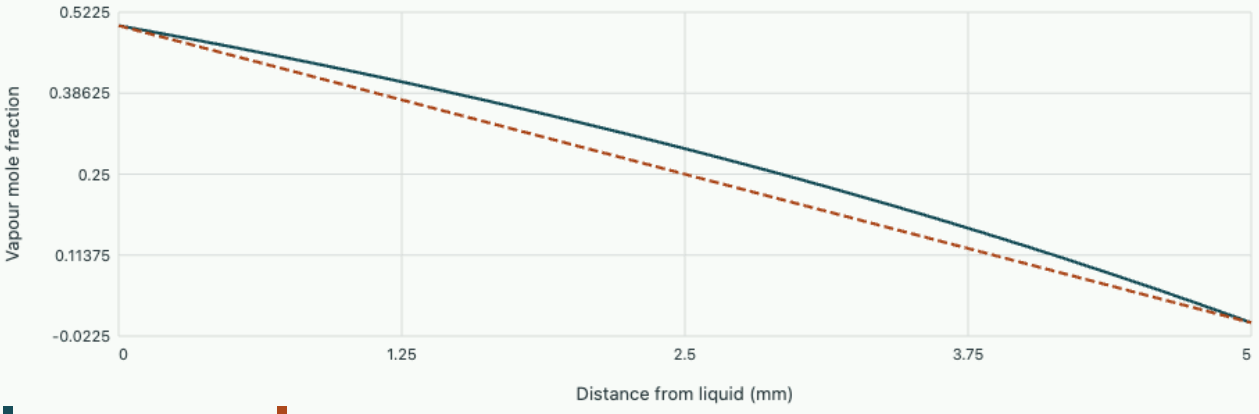
Use a vapour-free reservoir and increase the surface fraction from 0.10 to 0.80. Estimate which midpoint concentration lies farther from linear interpolation. Compare the resulting flux errors and identify the physical transport term responsible.

Open this experiment online

Try the calculation

Surface vapour mole fraction **0.5**

Reservoir vapour mole fraction **0**



The evaporation flux is 0.1112 mol/(m² s). The dilute estimate is 0.0802 mol/(m² s).

Tutorial · recover the effective film thickness

At the same temperature and diffusivity the surface mole fraction is 0.30, the reservoir fraction is 0.05 and a balance reports $N_A = 0.050$ mol/(m² s). Estimate L. What film thickness would be inferred if the outward molar flow were ignored?

▼ **Show the two fits**

Solving the exact expression gives

$$L = cD\ln(0.95/0.70)/N_A = 4.90\,\mathrm{mm}.$$

The linear fit yields

$$L = cD(0.25)/N_A = 4.01\,\mathrm{mm}.$$

The latter fit makes the film thinner to compensate for omitted advection. In an experiment that effective thickness would predict the wrong flux when the surface composition changes appreciably.

Further reading · Lienhard and Lienhard · A Heat Transfer Textbook, chapter 11 on mass transfer by diffusion

Hydrodynamic lubrication and sliding bearings

Machine elements and contact mechanics

Derive the flow balance in a converging lubricant film and determine why the sliding surface can support a load without solid contact.

FROM FIRST PRINCIPLES

Derive the flux through a moving lubricant film

Consider two broad bearing surfaces separated by a thin gap h. Let the lower surface move at speed U and the upper one remain still. Assume incompressible Newtonian lubricant, no slip, negligible inertia and a gap changing only slowly in the streamwise direction. Pressure is nearly constant through the thickness. Locally, the streamwise momentum balance reduces to a pressure gradient balanced by viscous shear curvature.

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{dp}{dx},$$
$$u(0) = U,$$
$$u(h) = 0.$$

Integrating twice gives a linear velocity contribution from sliding and a parabolic contribution from the pressure gradient. Integrating that velocity across the gap gives the discharge per unit width.

$$u(y) = U(1 - y/h) - \frac{1}{2\mu}\frac{dp}{dx} y(h - y),$$
$$q = \frac{Uh}{2} - \frac{h^3}{12\mu}\frac{dp}{dx}.$$

In a steady film without leakage or normal surface motion the same q crosses every section. A varying gap changes the sliding contribution, forcing a pressure gradient to preserve that flow. The integral of gauge pressure over the bearing pad gives its normal load capacity.

DERIVATION AND APPLICATIONS

Relate recess pressure to restrictor resistance in a hydrostatic pad

A hydrostatic bearing can sustain load at zero sliding speed because an external supply drives lubricant through a restriction. Consider a flat circular thrust pad with a central recess of radius a and outer land radius b. Let the parallel land clearance h be uniform and much smaller than its radial extent. Use incompressible Newtonian oil of viscosity μ, steady laminar radial outflow, no squeeze motion, ambient gauge pressure at b, and negligible pressure drop inside the recess. The pressure-driven flow per circumferential width follows the parallel-plate law.

$$Q = -2\pi r \frac{h^3}{12\mu}\frac{dp}{dr},$$

$$p(r) = \frac{6\mu Q}{\pi h^3}\ln\frac{b}{r}.$$

At the recess edge $p(a) = p_r$. The land therefore behaves as a flow resistance R_L proportional to the inverse cube of clearance. A linear feed restrictor R_f drops supply pressure p_s to p_r. Equal flow passes through the feed and land, giving a hydraulic pressure divider.

$$R_L = \frac{6\mu}{\pi h^3}\ln(b/a),$$

$$p_r = QR_L = p_s\frac{R_L}{R_f + R_L}.$$

Integrate pressure over the uniform recess and logarithmic land. The logarithm integral contains an inner-boundary term that cancels the associated part of the recess load, leaving a simple effective area.

$$W = \pi a^2 p_r + 2\pi\int_a^b r p(r)\,dr = \frac{\pi(b^2 - a^2)}{2\ln(b/a)}p_r.$$

Closing the pad increases land resistance, raises recess pressure and increases force opposing closure. When the land resistance is already very large, recess pressure approaches supply pressure and the available load saturates.

WORKED EXAMPLE

Predict load at a balanced restriction

For $a = 10.0$ mm and $b = 20.0$ mm, effective area is

$$\pi(0.0200^2 - 0.0100^2)/(2\ln 2) = 6.80\times10^{-4}\,\mathrm{m}^2.$$

If $R_L = R_f$ at the operating clearance and $p_s = 0.800$ MPa, recess pressure is 0.400 MPa and supported load is 272 N. This load is present without any relative sliding of the two faces.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Infer a clearance change from pressure

A different bearing has $p_r/p_s = 0.600$ initially and 0.750 after a load change. Resistance ratios $R_L/R_f = p_r/(p_s - p_r)$ therefore increase from 1.50 to 3.00. Since land resistance varies as h^{-3}, the final gap is $2^{-1/3} = 0.794$ of its initial value. An initial 25.0 µm film becomes 19.8 µm, assuming unchanged oil viscosity and feed resistance.

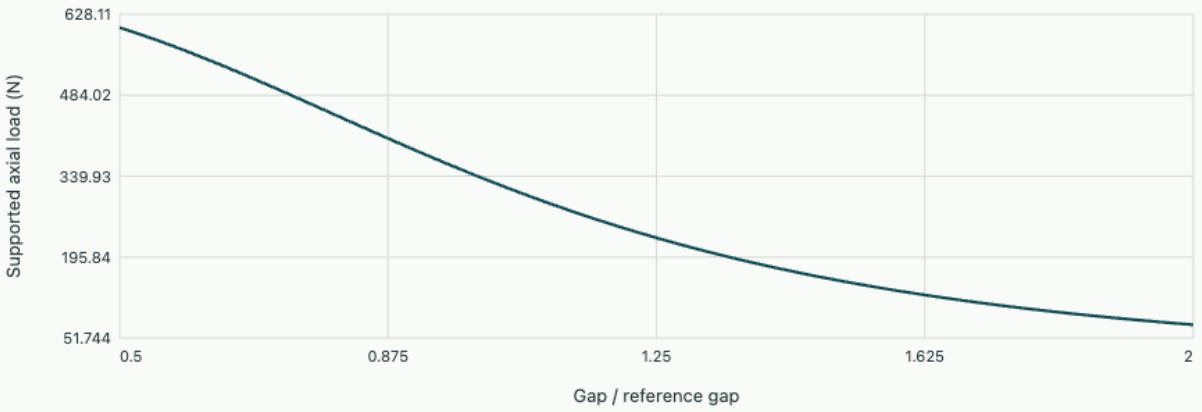
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Supply pressure (MPa)	1
Feed/land resistance at reference gap	1
Reference gap (µm)	20



Fed circular bearing

At reference gap 20 µm, supported force 339.9 N and recess pressure 0.500 MPa. The land hydraulic resistance varies as gap^{-3}.

Where lubricant pressure comes from

A shaft turning inside a plain bearing drags lubricant around with its surface. A load displaces the journal from the bore centre, creating a film whose gap narrows in one direction. Wall motion carries oil into this wedge, and a pressure gradient redistributes the flow so the same fluid rate passes successive sections. Pressure integrated over the bearing can then support a load. Surface motion alone in a uniform gap between surfaces with equal end pressures cannot produce this load in the ideal steady model.

Consider a broad slider bearing rather than the full journal geometry. Let a lower wall move at speed U; a fixed upper wall forms a slowly varying small gap $h(x)$. Assume incompressible Newtonian lubricant, constant viscosity, negligible inertia and negligible leakage through the sides. Pressure is nearly uniform across the thickness. The axial momentum equation reduces to $\mu d^2u/dy^2 = dp/dx$, and no slip gives

$$u(0) = U,\ u(h) = 0.$$

$$u = U(1 - y/h) - \frac{1}{2\mu}\frac{dp}{dx}\, y(h - y),$$

$$q = \frac{Uh}{2} - \frac{h^3}{12\mu}\frac{dp}{dx}.$$

Mass conservation makes q per unit width constant along the bearing. Where the local dragged flow $Uh/2$ exceeds q, pressure rises downstream. Where it falls short, pressure falls and supplies a forward pressure-driven contribution. The transition locates a pressure maximum.

Solve a two-depth ideal bearing

Consider a stepped pad with two equal 10 mm long regions, gaps $h_1 = 0.20$ and $h_2 = 0.10\,\mathrm{mm}$ in the sliding direction. Both ends have ambient pressure, $U = 1.0\,\mathrm{m/s}$, and $\mu = 0.05\,\mathrm{Pa\,s}$. Ignore the local loss at the abrupt step and treat each region as fully developed. Integration of $dp/dx = 12\mu(Uh/2 - q)/h^3$ across both regions gives zero net pressure change. With equal lengths this fixes

$$q = \frac{U}{2}\,\frac{h_1^{-2} + h_2^{-2}}{h_1^{-3} + h_2^{-3}} = 5.56 \times 10^{-5}\,\mathrm{m^2/s}.$$

In the first region $Uh_1/2 = 1.00 \times 10^{-4}\,\mathrm{m^2/s}$. The positive pressure gradient is $3.33 \times 10^{6}\,\mathrm{Pa/m}$, giving an interface gauge pressure of 33.3 kPa. The second region has an equal opposite gradient. The resulting triangular pressure distribution integrates to a force per unit width of $0.5(0.020)(33333) = 333\,\mathrm{N/m}$. A nominal 50 mm width would thus support approximately 16.7 N under the negligible-side-leakage assumption.

Understand failure at low speed

With zero sliding speed, equal end pressures imply zero steady gauge pressure everywhere in this model, so the load capacity vanishes. Start-up and shutdown therefore differ from established hydrodynamic running. A separately pressurised hydrostatic bearing can carry load at zero shaft speed. If a solution predicts large negative absolute pressures in a diverging region, actual oil may cavitate and the full-film boundary conditions must change. Viscous heating can also lower the local viscosity. Film thickness, thermal balance and supply conditions all matter when transferring the simple slider result to a journal in service.

Predict a viscosity change

For the same slider geometry and speed with equal prescribed end pressures, the calculated flux is independent of viscosity but all gauge pressure gradients are proportional to viscosity. Halving viscosity in the ideal stepped example therefore halves its load capacity at unchanged film thickness.

Derive load carried by a closing lubricant film

Sliding motion is not the only way to develop pressure in a thin film. Two parallel circular plates moving towards each other expel viscous oil radially, producing a squeeze-film pressure. Let plate radius be R, uniform gap $h(t)$, and approach speed $v = -dh/dt > 0$. Assume incompressible Newtonian oil, no slip, negligible inertia, rigid parallel plates, negligible edge resistance and ambient pressure at the rim. The gap is much smaller than the radius. Neither plate moves radially, so depth-integrated radial flux per unit circumference is $q_r = -h^3(dp/dr)/(12\mu)$, the stationary-wall pressure-driven slit relation.

Inside radius r, incompressibility requires outward flow through its circumference to equal the volume displaced by plate motion: $2\pi r q_r = \pi r^2 v$. Thus $q_r = rv/2$ and $dp/dr = -6\mu r v/h^3$. Integrating inward from ambient at the edge gives

$$p(r) - p_a = \frac{3\mu v}{h^3}(R^2 - r^2),$$

$$W = \int_0^R (p - p_a) 2\pi r \, dr = \frac{3\pi\mu R^4 v}{2h^3}.$$

The maximum pressure is at the centre, twice the area average because the pressure shape is parabolic. The load grows very steeply as the gap narrows at imposed speed. At imposed load, however, the approach slows. This distinction mirrors the need to separate displacement-controlled and force-controlled contact tests. A stationary gap with no relative normal motion produces no squeeze-generated pressure under the stated ambient boundary condition.

Estimate the time for a thrust pad to approach its stop

A circular pad of radius 25 mm settles under a constant additional normal force of 100 N on oil of dynamic viscosity $0.15\,\mathrm{Pa\,s}$. Ignore inertia after the short initial transient. Rearranging the load law gives

$$-dh/dt = 2Wh^3/(3\pi\mu R^4).$$

Between initial and final gaps, integration gives

$$t = [3\pi\mu R^4/(4W)](h_f^{-2} - h_i^{-2}).$$

For $h_i = 150\,\mu\mathrm{m}$ and $h_f = 50\,\mu\mathrm{m}$ this yields 0.491 s.

At an intermediate 100 micrometre gap the instantaneous speed is

$$2(100)(10^{-4})^3/[3\pi(0.15)(0.025)^4] = 0.000362\,\mathrm{m/s}.$$

The centre gauge pressure is $2W/(\pi R^2) = 102\,\mathrm{kPa}$, independent of gap here because the load is prescribed and the approach speed adapts. Halving the gap at unchanged load reduces approach speed eightfold. The ideal solution takes infinite time to reach zero thickness, which should not be interpreted as assurance against surface contact: roughness, oil leakage paths, tilt and elasticity enter before an arbitrarily small gap is reached. On separation the sign of the calculated pressure reverses and cavitation or air ingress may prevent the predicted tensile liquid pressure from developing.

A load-cell test can independently discriminate squeeze action from simple dry sliding resistance. At fixed small gap and sufficiently slow normal motion the predicted force is proportional to approach speed, with a slope scaling as h^{-3}. Closing the gap from 100 to 80 micrometres should multiply the force slope by $(100/80)^3 = 1.95$. The transient test must either measure instantaneous gap or be run with an imposed motion, since holding load constant causes speed and gap to evolve together. Comparing forces at the same clock time on two different tests would not isolate the gap dependence.

From cut forces to local equilibrium

Engineering mechanics

A support reaction tells you the total force entering a part. To predict what an inclined internal surface carries, keep track of both the orientation of the cut and the direction of the force across it.

A force that depends on the surface you choose

Imagine a small triangular wedge cut out of a thin, slowly loaded plate. Its sloping face has unit outward normal $\mathbf{n} = (n_x, n_y)$ and area A. The two faces normal to the coordinate axes have areas An_x and An_y when both components are positive. Name the internal force per unit area on a face its traction. Opposite faces of the same cut carry equal and opposite tractions. Gravity and acceleration act on volume, which shrinks faster than area as the wedge shrinks. Dividing its force balance by A therefore gives a relation that no longer contains its size.

$$\mathbf{t}(\mathbf{n}) = n_x \mathbf{t}(\mathbf{e}_x) + n_y \mathbf{t}(\mathbf{e}_y) = \boldsymbol{\sigma}\mathbf{n}.$$

The columns of the stress matrix are the two coordinate-face tractions. Their linear combination supplies the force on any inclined face. A small square additionally needs zero net moment. To leading order, the shear forces on opposite faces form two couples of opposite sign and equal lever arm. Their balance requires $\sigma_{xy} = \sigma_{yx}$ for an ordinary material without internal body couples.

> **Background · Traction**
>
> $$\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma}\mathbf{n}$$
>
> Specify the plane normal before projecting the force per area into normal and shear parts. A stress component is not a net force until integrated over an area.

Why a gradient of stress carries a body force

Now use a tiny rectangular element of side lengths dx, dy and unit thickness. The right-face contribution to horizontal force is $\sigma_{xx}(x + dx, y)dy$; the left face contributes its negative evaluated at x. Their difference is $\partial_x \sigma_{xx}\, dx\, dy$ to leading order. Resolve the upper and lower shear tractions in the same way. Add a prescribed horizontal force per volume b_x and divide by volume. Repeating in the vertical direction produces two independent equations.

$$\partial_x \sigma_{xx} + \partial_y \sigma_{xy} + b_x = 0,$$

$$\partial_x \sigma_{xy} + \partial_y \sigma_{yy} + b_y = 0.$$

These equations say nothing yet about how much the material stretches. They can test an assumed stress field before the elastic modulus has even been measured. A candidate must also reproduce the applied boundary tractions. Satisfying the interior equations alone leaves many possible stresses.

Worked example · the grip on a diagonal bonding plane

At a measurement point let $\sigma_{xx} = 60, \sigma_{yy} = 20, \sigma_{xy} = 10$ MPa. The outward normal is $(0.6, 0.8)$. Matrix multiplication gives $\mathbf{t} = (44, 22)$ MPa. Projection along the normal gives $0.6(44) + 0.8(22) = 44.0$ MPa tension. Projection on the unit tangent $(-0.8, 0.6)$ gives -22.0 MPa. Reversing the tangent changes the sign of reported shear. Reversing the surface normal reverses the traction vector itself. For a uniform 100 mm² patch the force has components 4.40 and 2.20 kN. Using only normal stress would miss the tangential load that the adhesive must transmit.

Worked example · a specimen with a known stress gradient

Stand a bar upright, support it at its bottom and leave the top free. Use upward y, length L, constant density ρ, and tension positive. The vertical equation reads $d\sigma_{yy}/dy - \rho g = 0$. Integrating from the free top gives

$$\sigma_{yy} = \rho g (y - L).$$

For a 3.0 m aluminium bar of density 2700 kg/m³ the base stress is −79.5 kPa. Its upward support traction is positive because the bottom face has downward outward normal. A uniform cross-sectional area of 200 mm² gives a base reaction of 15.9 N, equal to the weight. The local and global checks involve different quantities but must agree.

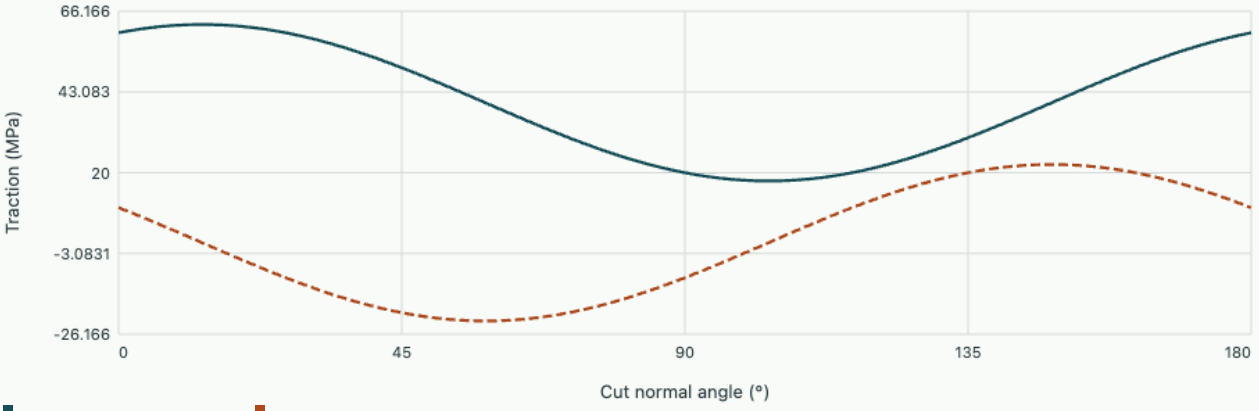
Find the angles where the shear traction passes through zero. Compare each with an extremum of the normal traction. Change the sign of the imposed xy shear, sketch the new angles before moving the slider, and check whether the two extreme normal stresses change.

Open this experiment online

Try the calculation

Normal stress xx (MPa) 60

Shear stress xy (MPa) 10



On the cut with normal (0.6, 0.8), the normal traction is 44.00 MPa and the signed shear is -22.00 MPa.

Tutorial · test a candidate before using a material law

In a region with x, y in metres, propose $\sigma_{xx} = ax,\ \sigma_{xy} = -ay,\ \sigma_{yy} = 0$, where $a = 2.0$ MPa/m. Does the field need a body force? Find the traction at $x = 0.30,\ y = 0.20$ m on a face normal to positive x.

▼ **Check the balance and the boundary reading**

The horizontal divergence is $a - a = 0$. The vertical divergence is zero because the shear has no x variation. No body force is needed. The face sees the first column of the matrix, $(0.60, -0.40)$ MPa. A traction-free exterior at that location would reject this field even though it satisfies both interior equations.

Further reading · A. F. Bower, Applied Mechanics of Solids, §2.3, internal forces and stress

Action density, stress waves and the boundary that reflects them

Analytical mechanics from constraints to invariants

A chain of coupled masses becomes an elastic field when the wavelength is long compared with the spacing. Derive the field equation and its boundary traction from the same variation, follow energy and stress across a junction, and decide whether a measured pulse is limited by material waves or by numerical dispersion.

Pass from a discrete rod to an action density

Let equal masses of spacing a move longitudinally by $u_j(t)$, with nearest-neighbour spring constant k_s and mass m_s per site. Their Lagrangian is $\sum_j [m_s \dot{u}_j^2/2 - k_s(u_{j+1}-u_j)^2/2]$. For variations smooth on the spacing, $(u_{j+1}-u_j)/a \to u_x$ and $\sum_j a\, f_j \to \int f(x)\,dx$. Hold the line density $\rho A = m_s/a$ and axial rigidity $EA = k_s a$ fixed. The continuum Lagrangian density per unit reference length is

$$\mathcal{L} = \rho A\, u_t^2/2 - EA\, u_x^2/2,$$

with A area, ρ volume density and E Young modulus. Both terms have units of joules per metre. This is a slender-rod model with small strain, primarily uniaxial stress and wavelengths long compared with the cross-sectional dimensions.

Vary the field in the action over $[0, L] \times [t_1, t_2]$. Integration of $\rho A\, u_t \delta u_t$ by parts in time, with fixed endpoint fields, produces $-\rho A\, u_{tt}\delta u$. Integration of $-EAu_x\delta u_x$ by parts in space produces $+\partial_x(EAu_x)\delta u$ and a remaining term $-[EAu_x\delta u]_0^L$. Thus the interior equation is $\rho A u_{tt} = \partial_x(EAu_x)$, and the end force conjugate to displacement is axial force $N = EAu_x$. Adding external work $F_L(t)\delta u(L)$ at the right face yields

$$N(L) = F_L.$$

A prescribed displacement instead imposes $\delta u(L) = 0$, leaving its reaction as an output. Keeping this boundary term prevents prescribing displacement and independent traction simultaneously at the same end of a second-order rod.

Determine the energy flux and characteristic impedance

Assume time-independent coefficients. Multiply the field equation by u_t and rearrange to obtain

$$\partial_t[\rho A u_t^2/2 + EAu_x^2/2] + \partial_x[-EAu_x u_t] = 0.$$

The energy flux is $-Nu_t$, with positive sign denoting energy moving towards increasing x. The minus sign follows the convention of positive tensile N: positive mechanical power entering through a right face is Nu_t, which is inward with respect to the rod on its left. This identity controls energy transport across both smooth material changes and sharp joints.

For a homogeneous right-moving wave $u = F(x - ct)$, $c = \sqrt{E/\rho}$, particle velocity $v = u_t = -cF'$ and axial force $N = EAF' = -Zv$, where $Z = \rho Ac = EA/c$. A left-moving wave has $N = +Zv$. The impedance is force divided by particle velocity and has units $\mathrm{kg/s}$. If one instead uses stress, impedance is ρc per unit area. Doubling the area of the same material keeps wave speed unchanged but doubles the force–velocity impedance. A transit-time measurement and a reflection measurement therefore provide independent material or geometry information.

For a rod with $E = 70\,\mathrm{GPa}$, $\rho = 2700\,\mathrm{kg/m^3}$ and $A = 1.0 \times 10^{-4}\,\mathrm{m^2}$, $c \approx 5092\,\mathrm{m/s}$ and $Z \approx 1375\,\mathrm{kg/s}$. A right-going compressive pulse with particle speed $0.020\,\mathrm{m/s}$ carries axial force $-27.5\,\mathrm{N}$, compressive stress $0.275\,\mathrm{MPa}$, and instantaneous power $Zv^2 = 0.550\,\mathrm{W}$ across a section. A rectangular approximation lasting $100\,\mu\mathrm{s}$ carries about $55\,\mu\mathrm{J}$ and occupies $0.509\,\mathrm{m}$. The pulse speed is kilometres per second; the material particles move only millimetres per second. Treating these as the same velocity would spoil force, impulse and energy estimates.

Match a bonded interface and distinguish amplitude from power

At a perfect massless bond displacement and axial force are continuous. For an incident right-going velocity amplitude v_i in rod 1, let v_r be reflected left-going velocity and v_t transmitted right-going velocity. Velocity continuity gives

$$v_i + v_r = v_t.$$

Force continuity gives

$$-Z_1 v_i + Z_1 v_r = -Z_2 v_t.$$

Thus

$$r_v = \frac{Z_1 - Z_2}{Z_1 + Z_2},$$

$$t_v = \frac{2Z_1}{Z_1 + Z_2},$$

$$\mathcal{R} = r_v^2,$$

$$\mathcal{T} = \frac{Z_2}{Z_1} t_v^2 = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2}.$$

The power fractions add to one for real positive impedances. The reflected force coefficient has opposite sign to the reflected velocity coefficient because the wave direction reverses. At a free end $Z_2 \to 0$, reflected velocity equals incident velocity, so end velocity doubles while total force vanishes. At a rigid end $Z_2 \to \infty$, velocity reverses and force doubles. The sign a strain gauge reports can therefore differ from that of a velocity-sensitive optical measurement at the same reflection.

A step from the worked rod into one of the same material with twice the cross-sectional area gives $Z_2 = 2Z_1$, $r_v = -1/3$, $t_v = 2/3$, and power fractions $1/9$ and $8/9$. A reduced transmitted velocity amplitude does not imply that most energy is reflected. If the interface includes a concentrated mass

M_b, force jumps instead obey $N_2 - N_1 = M_b \dot{v}_b$ with signs determined by forces exerted on that mass from either side. At high frequency the reaction of that mass can mimic a much stiffer termination. Frequency-dependent reflection is then an expected boundary effect, rather than evidence that the bulk modulus varies with frequency.

Connect continuum modes to their lattice predecessor

For a periodic chain the exact discrete equation is

$$m_s \ddot{u}_j = k_s(u_{j+1} - 2u_j + u_{j-1}).$$

Substitute $u_j = U e^{i(kja - \omega t)}$ to obtain

$$\omega = 2\sqrt{k_s/m_s}\,|\sin(ka/2)|.$$

In the first positive Brillouin zone, $\omega/(ck) = \sin(ka/2)/(ka/2)$ and group velocity $d\omega/dk = c\cos(ka/2)$, where $c = a\sqrt{k_s/m_s}$. The continuum rod retains only the leading low-ka term. A wave with eight sites per wavelength has phase speed about $0.9745c$ and group speed $0.9239c$. A narrowband packet's peak can arrive later than the continuum prediction even when a single crest is timed fairly accurately.

A finite-difference code approximating the continuum rod can produce the same spatial dispersion with its mesh spacing in place of a physical lattice spacing. Distinguish them experimentally or numerically by changing the computational mesh while holding the physical structure fixed. Lattice effects persist as the numerical grid is refined; artificial spatial dispersion diminishes. A separate time-step study is needed if a time integrator also disperses the wave. Comparing both peak arrival and phase at two frequencies tests more of the dynamics than fitting a single transit time.

Derive transverse beam waves from curvature energy

A slender beam bending laterally is governed by a different elastic energy from the axial rod. Let $w(x,t)$ denote small transverse deflection of its neutral axis. Under the Euler–Bernoulli assumption that cross sections remain plane and normal to that axis, the axial strain a distance z from the neutral axis is approximately $-z w_{xx}$. Integrating the local elastic energy $E\epsilon_{xx}^2/2$ across the section gives $EI_A w_{xx}^2/2$ per unit length, where $I_A = \int z^2\,dA$ is a geometrical second moment of area, with units m^4. It is distinct from a mass moment of inertia. The kinetic contribution uses line density ρA as before.

Varying $\mathcal{L} = \rho A w_t^2/2 - EI_A w_{xx}^2/2$ requires integrating twice by parts in space. With constant coefficients the interior variation gives

$$\rho A w_{tt} + EI_A w_{xxxx} = 0.$$

The boundary terms are $-[EI_A w_{xx}\delta w_x] + [(EI_A w_{xx})_x \delta w]$. Thus slope variations pair with bending moments and displacement variations with shear forces, with signs set by the chosen face orientation. A

clamped end prescribes both deflection and slope; a free end sets moment and shear to zero. A simply supported end prescribes deflection and zero bending moment. Using only one boundary condition per end, as for the axial rod, leaves the fourth-order beam problem underdetermined.

A sinusoid $e^{i(kx-\omega t)}$ gives

$$\omega = \sqrt{EI_A/(\rho A)}\, k^2.$$

Phase speed grows proportionally to wavenumber and group speed is twice phase speed in this model. A narrow transverse pulse in a long beam consequently spreads even in a perfectly homogeneous non-dissipative continuum. By contrast the axial rod was nondispersive in its long-wavelength continuum approximation. At high enough frequency the simple bending model omits rotary inertia, shear deformation and eventually three-dimensional elastic behaviour. Extrapolating its unbounded speed to arbitrary k would use the slender, low-frequency approximation far beyond its derivation.

For a simply supported beam with length L, the sine modes satisfy both end conditions and give

$$\omega_n = (n\pi/L)^2 \sqrt{EI_A/(\rho A)}.$$

For a rectangular section of width w_b and thickness d in the bending direction, $I_A = w_b d^3/12$ and $A = w_b d$. Doubling thickness doubles all the natural angular frequencies at fixed L, E, ρ, whereas doubling width alone leaves the ideal frequencies unchanged. The stiffness increases and so does the inertia. A beam with $L = 0.50\,\mathrm{m}$, $d = 5.0\,\mathrm{mm}$ and the earlier aluminium properties has $\sqrt{EI_A/(\rho A)} \approx 7.35\,\mathrm{m^2/s}$, giving fundamental $46.2\,\mathrm{Hz}$. The second bending mode is near four times that frequency, a clear difference from the approximately harmonic axial mode sequence.

Match a force pulse to a support spring and dashpot

A right termination exerts force on the rod through a dashpot with mechanical impedance Z_L. For a real positive dashpot impedance the reflected velocity fraction is $r = (Z - Z_L)/(Z + Z_L)$. Taking $Z_L = Z$ gives zero reflection at the termination even though energy leaves a finite rod: the dashpot dissipates the arriving power. With $Z_L = 3Z$ one obtains a reflected velocity of $-1/2$ of the incident value, a reflected energy fraction one quarter and absorbed power fraction three quarters. A free or rigid end also reflects energy, but it absorbs no net power in the ideal limit. A measured weak reflection alone is therefore not proof of weak damping inside the rod.

For a pure end spring of stiffness k_L attached to a rigid support, the velocity impedance is $Z_L = k_L/(i\omega)$ under $e^{i\omega t}$. It is imaginary, so the reflection coefficient has unit magnitude at every frequency and a frequency-dependent phase. At low frequency the spring impedance is large and the end looks nearly fixed. At high frequency its impedance is small and the end looks nearly free within the lumped approximation. An impulsive signal contains many frequencies and can return with a changed waveform even without any transmitted or dissipated energy. A support combining a resistor-like dashpot and a spring gives both amplitude loss and phase variation.

A measurement with strain gauges at two positions can separate forward and backward waves if their path delays are resolved. In a narrow frequency band the two records are linear combinations of advancing and returning phase factors. If the gauge spacing corresponds to an integer number of half wavelengths, these combinations become linearly dependent and the separation is ill-conditioned. Change the spacing, use a broadband pulse, or combine a velocity measurement with a strain measurement at one station so that the two characteristic directions enter with opposite signs. The relation $v = (v_+ + v_-)$, $N = Z(-v_+ + v_-)$ then gives a direct local decomposition.

Derive sound propagation from mass and momentum conservation

The displacement-action argument has a fluid analogue. Take an initially uniform fluid of density ρ_0 and pressure p_0, moving little from equilibrium in one spatial dimension. Write small perturbations ρ', p', v. Linearised mass conservation is

$$\partial_t \rho' + \rho_0 \partial_x v = 0.$$

Momentum balance for a short fluid parcel is $\rho_0 \partial_t v = -\partial_x p'$. If compression over an acoustic cycle is approximately isentropic, $p' = c_s^2 \rho'$ with $c_s^2 = (\partial p/\partial \rho)_s$. Differentiate the first equation in time, eliminate the velocity derivative with momentum balance, and obtain $\partial_t^2 p' = c_s^2 \partial_x^2 p'$.

The thermodynamic derivative matters because pressure can respond differently when compression is slow enough for heat exchange. For a perfect gas with ratio of specific heats γ, isentropic perturbations give

$$c_s^2 = \gamma p_0 / \rho_0.$$

Taking an isothermal derivative would give p_0/ρ_0 instead. For $p_0 = 101\,\mathrm{kPa}$, $\rho_0 = 1.20\,\mathrm{kg/m^3}$, and $\gamma = 1.40$, the isentropic speed is about $343\,\mathrm{m/s}$, while the isothermal calculation gives $290\,\mathrm{m/s}$. A measured transit time can discriminate between these constitutive conditions. Wall heat exchange in a very narrow tube can alter which approximation is appropriate over a chosen frequency band.

For a right-moving sound wave $p' = \rho_0 c_s v$, with the pressure positive in compression. This differs in sign from the tensile-positive axial force used for the rod, so match physical force conventions before borrowing formulas across the two examples. The acoustic power per area is $p' v$. For sinusoidal pressure with rms amplitude $1.0\,\mathrm{Pa}$ in the stated gas, the mean intensity is

$$p_{\mathrm{rms}}^2 / (\rho_0 c_s) = 2.43 \times 10^{-3}\,\mathrm{W/m^2}.$$

A peak amplitude of $1.0\,\mathrm{Pa}$ would yield half this number. Writing rms or peak explicitly prevents a routine factor-of-two error when comparing a microphone measurement with an energy flux.

Determine whether a measured transit reveals a modulus or a geometry

Suppose two axially loaded slender rods have measured wave speeds agreeing within their uncertainties. That alone determines only the ratio E/ρ under the slender-rod assumptions. Measuring force–velocity impedance $Z = \rho A c$ and cross section as well gives $\rho = Z/(Ac)$ and $E = Zc/A$. With $Z = 1375\,\mathrm{kg/s}$, $c = 5092\,\mathrm{m/s}$ and $A = 1.0 \times 10^{-4}\,\mathrm{m}^2$, the inversion gives

$$\rho = 2700\,\mathrm{kg/m^3},$$

$$E = 70.0\,\mathrm{GPa}.$$

If area were overestimated by five percent, both inferred quantities would be underestimated by almost five percent while their ratio remained correct.

Now examine a thick elastic solid under a narrow compressional disturbance with wavelengths short compared with its transverse dimensions. Lateral deformation can be constrained by surrounding material. In an isotropic three-dimensional continuum the longitudinal plane-wave modulus is $\lambda + 2\mu_s$, where λ, μ_s are Lamé coefficients, rather than the uniaxial-stress Young modulus of a slender rod. The shear-wave speed satisfies $c_T^2 = \mu_s/\rho$ and longitudinal speed $c_L^2 = (\lambda + 2\mu_s)/\rho$. Measuring both, together with density, can recover the two independent elastic moduli. Using only one longitudinal speed to infer Young modulus without specifying the deformation geometry can give a physically incorrect value.

For $\rho = 2500\,\mathrm{kg/m^3}$, $c_L = 5000\,\mathrm{m/s}$ and $c_T = 3000\,\mathrm{m/s}$, these relations give

$$\mu_s = 22.5\,\mathrm{GPa},$$

$$\lambda = 17.5\,\mathrm{GPa}.$$

The implied Young modulus $E = \mu_s(3\lambda + 2\mu_s)/(\lambda + \mu_s) = 54.8\,\mathrm{GPa}$, while $\rho c_L^2 = 62.5\,\mathrm{GPa}$. Both numbers represent material stiffness, but they enter different mechanical boundary conditions. A long thin specimen of the same material should have an axial speed closer to $\sqrt{E/\rho} = 4680\,\mathrm{m/s}$. Comparing the bulk and rod experiments tests the constitutive and geometrical assumptions behind the measured speed.

Derive orthogonality when a rod changes material along its length

For a rod with positive varying density $\rho(x)$, area $A(x)$ and modulus $E(x)$, a separated free oscillation $u(x,t) = \phi(x)\cos\omega t$ satisfies

$$-[EA\phi']' = \omega^2 \rho A \phi.$$

Suppose two modes ϕ_n, ϕ_m satisfy the same homogeneous fixed or free boundary conditions and the correct continuity of displacement and axial force across every perfectly bonded interface. Multiply the first equation by ϕ_m, the second by ϕ_n, subtract and integrate. Interior terms produced by integration by parts cancel pairwise, including the interface flux terms, and the endpoint terms vanish. Thus

$$(\omega_n^2 - \omega_m^2) \int \rho A \phi_n \phi_m \, dx = 0.$$

Mode shapes with distinct frequencies are orthogonal under the mass-weighted inner product, not generally under a simple average of displacements measured at uniformly spaced stations. An initial velocity distribution $v_0(x)$ projects onto mode n with coefficient $\int \rho A v_0 \phi_n \, dx / \int \rho A \phi_n^2 \, dx$. In a rod with a heavy section, the same imposed velocity there stores more energy and carries more modal weight. An experiment interpreting raw camera amplitudes without line-density weighting can misassign energy among modes. Normalising each measured mode shape to unit maximum does not fix the mass weighting needed for energies.

Multiplying one eigenvalue equation by its own shape gives

$$\omega_n^2 = \int EA(\phi_n')^2 dx / \int \rho A \phi_n^2 dx.$$

For an admissible trial function satisfying prescribed displacement conditions, the same quotient bounds the fundamental squared frequency from above when rigid translations are excluded. Consider a uniform fixed–free rod and trial $\phi = x/L$. The numerator is EA/L and denominator $\rho A L/3$, giving $\omega_{\mathrm{trial}} = \sqrt{3}\, c/L$. The exact fundamental is $\pi c/(2L)$, so this one-parameter shape overestimates frequency by about 10.3%. It need not satisfy the free-end traction condition itself because that is a natural condition of the variational minimiser; it does need to satisfy the essential fixed-end displacement.

For a free–free rod the constant displacement mode has zero frequency. Using an arbitrary positive trial function without removing its mass-weighted mean can drive a quotient towards zero by adding a large constant. To estimate the first deforming mode impose orthogonality to the rigid translation, $\int \rho A \phi \, dx = 0$. This has an immediate mechanical interpretation: the deformation mode should not translate the centre of mass of an isolated rod. The null mode and its mass-weighted constraint reappear in numerical eigenproblems built from discretised stiffness and mass matrices.

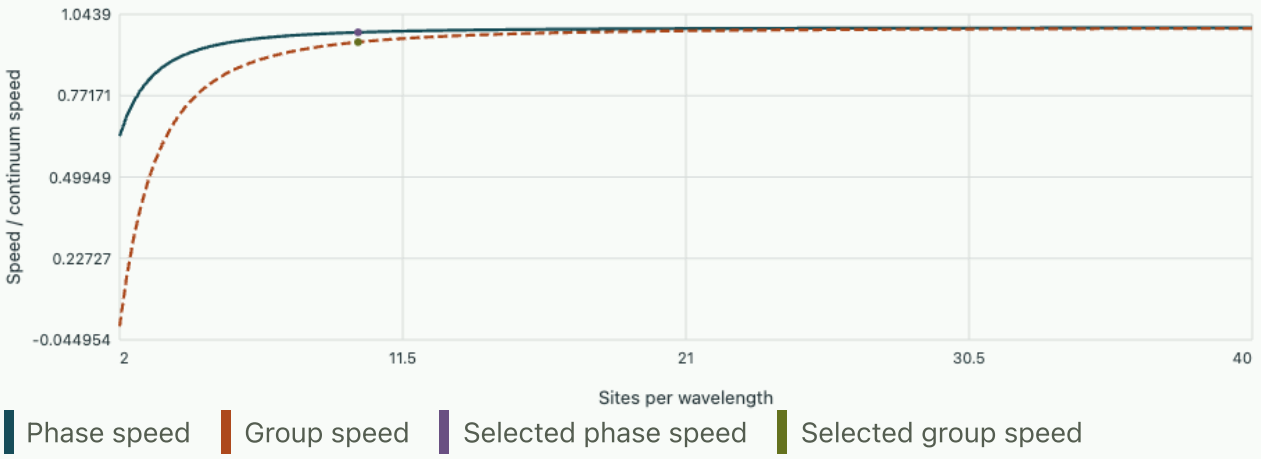
Find where a discrete elastic chain ceases to look continuous

Shorten the wavelength relative to the lattice spacing. Compare the phase and group velocities with those inferred from a continuum action.

Open this experiment online

Try the calculation

Sites per wavelength 10



Frequency and phase speed are 98.36% of the continuum prediction. Group speed is 95.11%.

Track work and impulse in a one-dimensional pulse experiment

A finite displacement pulse provides a useful warning about integrating strain and velocity records. A right-going strain pulse of one sign can carry net momentum $\int \rho A v \, dx$. Because $v = -c u_x$, its integrated material momentum is $-\rho A c[u(+\infty) - u(-\infty)]$ at a fixed time. A localised displacement bump that returns to the same displacement on both sides has zero net ordinary material momentum, even though its energy is positive and travels to the right. Its strain and particle-velocity records contain positive and negative parts. A unipolar velocity pulse leaves behind a displacement step, produced by a source that has supplied net impulse.

For a right-moving rectangular velocity pulse of amplitude $0.020\,\mathrm{m/s}$, duration $100\,\mu\mathrm{s}$ in the worked rod, the local displacement jump measured in time is $2.0\,\mu\mathrm{m}$. Its impulse magnitude across a section is $Zv\Delta t = 2.75 \times 10^{-3}\,\mathrm{N\,s}$, while its transmitted energy is

$$Zv^2\Delta t = 55.0\,\mu\mathrm{J}.$$

A second equal pulse of opposite velocity sign launched later cancels the net displacement and material momentum but doubles the transmitted energy. Thus integrating a signed velocity signal and integrating its square distinguish different properties of the same experimental record.

A displacement sensor has a small linear drift during a burst. Which inferred observables are particularly sensitive?

▼ **Work through the solution**

Differentiating adds a constant offset to particle velocity. Integrating that apparent velocity over a long window produces a growing spurious displacement jump and momentum estimate. Squaring it adds both

a cross term with the physical velocity and a positive baseline energy. Measure a pre-pulse baseline, choose an integration interval covering the physical pulse, and test that the result is stable when the window is moderately extended. An independent strain gauge gives another characteristic combination; comparison with $N = -Zv$ can expose drift or an overlapping reflection.

Can a finite-element rod model with a plausible first eigenfrequency still fail an impulse test?

▼ Work through the solution

The first frequency mostly probes a long-wavelength weighted stiffness-to-mass ratio. A brief pulse contains higher modes and is sensitive to temporal resolution, dispersion, boundary tractions and the way discrete source work enters. Verify modal frequencies at several wavelengths, check the pulse's positive energy and signed impulse separately, and compare the measured incoming and reflected characteristic fluxes. A single standing-wave eigenvalue cannot validate all of those physically independent balances.

Recover a jump condition at an interface with a point mass

For a concentrated mass M_b joining two rods include $M_b u_t(x_b)^2/2$ as a separate contribution to the Lagrangian. Its time variation contributes $-M_b u_{tt}\delta u$ at the joint. The left and right rod boundary terms contribute respectively $-N_1\delta u$ and $+N_2\delta u$. Requiring stationarity for an independently varying shared displacement gives exactly $N_2 - N_1 = M_b u_{tt}$. This derivation identifies both the sign and the constitutive assumption of a perfectly bonded, nonslipping joint.

For identical rods and a sinusoidal drive at angular frequency ω, the reflected velocity coefficient becomes $r = -i\omega M_b/(2Z + i\omega M_b)$ under the stated phasor convention. At $\omega M_b = 2Z$ it reflects half the power and transmits half. The mass itself absorbs no cycle-averaged energy, though it stores kinetic energy during a cycle. For $Z = 100\,\mathrm{kg/s}$ and $M_b = 0.10\,\mathrm{kg}$, the equal-power frequency is about $318\,\mathrm{Hz}$. A real joint that loses appreciable power at that frequency needs a damping or radiation term in addition to its measured mass.

Tutorial on stress waves

A gauge $0.80\,\mathrm{m}$ from a free end records the echo of a sharp disturbance in the worked rod. What echo delay and strain sign should it expect?

▼ Work through the solution

The extra path from gauge to end and back is $1.60\,\mathrm{m}$, giving $1.60/5092 = 0.314\,\mathrm{ms}$. A free end has zero net axial force, so the reflected force and strain have opposite signs to the incident ones. The velocity reflection keeps its sign. Finite pulse width and any change of material between gauge and end can obscure that simple result.

A bonded step has measured reflected velocity coefficient -0.20 and no detectable dissipation. Infer its impedance ratio and transmitted energy fraction.

▼ **Work through the solution**

Rearranging $r_v = (1 - r)/(1 + r)$ gives

$$r = Z_2/Z_1 = (1 - r_v)/(1 + r_v) = 1.50.$$

The energy reflection is 0.04 and transmission 0.96. The velocity transmission is 0.80, whose square alone is only 0.64. Multiplying by the transmitted-to-incident impedance ratio recovers the correct energy fraction.

A rod model with a prescribed displacement at its right end reports nonzero work there, even though the variational boundary term vanished in deriving its equation. Is there a contradiction?

▼ **Work through the solution**

Vanishing of $N\delta u$ follows because the allowed virtual displacement at a prescribed boundary is zero. Actual boundary velocity can be nonzero when the prescribed displacement varies in time. The actuator supplies power Nu_t at the right boundary. Compare the time-integrated input with the change of stored energy plus any losses or outgoing energy rather than identifying an admissible variation with actual displacement.

Compare a finite chain with its long-wavelength continuum.

Fließbach includes string vibrations, beam bending and sound in his continuum mechanics treatment.

From binary collisions to the Boltzmann equation

Statistical mechanics from microscopic counting to material response

An initially anisotropic velocity distribution can relax even though every individual collision conserves energy and momentum. Describe a dilute gas with a phase-space density, derive the structure of its collision term and prove which distribution can stop producing entropy. Then examine the assumptions behind that proof and the measurements most likely to reveal their failure.

What the one-particle function counts

Let $f(\mathbf{r}, \mathbf{v}, t) d^3r\, d^3v$ be the expected number of particles in a small phase-space cell. Integrating over velocities gives number density n, and the mean velocity is $\mathbf{u} = n^{-1} \int \mathbf{v} f \, d^3v$. Define the peculiar velocity $\mathbf{c} = \mathbf{v} - \mathbf{u}$; its second moment sets local kinetic energy after removing the organised flow. A gas can have large bulk speed but low temperature.

Follow a particle ballistically over a short interval under acceleration $\mathbf{a} = \mathbf{F}/m$. With no collisions, phase-space probability is advected along its trajectories. For a standard position-dependent force, phase-space flow has zero divergence, giving the streaming operator $\partial_t f + \mathbf{v} \cdot \nabla_r f + \mathbf{a} \cdot \nabla_v f$. Magnetic Lorentz acceleration also has zero velocity-space divergence. A dissipative velocity-dependent drag generally requires the divergence form $\nabla_v \cdot (\mathbf{a} f)$ instead. It should not silently be inserted into the Hamiltonian streaming expression.

We now assume a dilute classical gas of identical particles with short-ranged binary elastic collisions, approximated as local on the spatial scale resolved by f. Different particles approaching collision are assumed uncorrelated apart from their individual one-particle distributions at that position. This molecular-chaos approximation is an assumption about incoming pairs, not a claim that the outgoing particles remain uncorrelated. It is the step that allows the evolution to close on a one-particle function.

Count pairs crossing a scattering area

Consider a tagged incident velocity $\mathbf{v}$ and a possible partner $\mathbf{v}_1$. Their relative speed $g = |\mathbf{v} - \mathbf{v}_1|$ converts a differential scattering cross section $d\sigma/d\Omega$ into the swept interaction volume per unit time. A binary collision rotates the relative velocity in the centre-of-mass frame while preserving its magnitude. With equal masses, set $\mathbf{V} = (\mathbf{v} + \mathbf{v}_1)/2$ and $\mathbf{g} = \mathbf{v} - \mathbf{v}_1$. The partner and tagged postcollision velocities are $\mathbf{V} \pm \mathbf{g}'/2$, where $|\mathbf{g}'| = |\mathbf{g}|$. This parametrisation enforces momentum and kinetic energy conservation before choosing a scattering direction.

The loss term counts incoming pairs with velocities $\mathbf{v}, \mathbf{v}_1$. The gain counts inverse collisions that produce the specified velocity. Microreversibility of elastic scattering and invariance of the integration measure allow the compact expression

$$C[f](\mathbf{v}) = \int d^3v_1 \, d\Omega \, g \frac{d\sigma}{d\Omega} \left[f(\mathbf{v}') f(\mathbf{v}_1') - f(\mathbf{v}) f(\mathbf{v}_1) \right].$$

Both products refer to the same position and time. A change in which pair is called primed merely swaps gain and loss if one reverses the parametrisation consistently. This formula uses a consistent distinct-

partner convention for the full scattering measure. When counting the total number of collision events per volume rather than the loss of tagged particles, divide the sum over all ordered pairs by two. Confusing those two observables gives an erroneous collision frequency.

For hard spheres of molecular diameter d, the total geometric cross section is πd^2. In an equilibrium Maxwellian the mean relative speed is $\sqrt{2}$ times the mean single speed for identical masses. The mean free path under the usual hard-sphere convention is $\ell = 1/(\sqrt{2} n \pi d^2)$. For $d = 0.36$ nm, $n = 2.4 \times 10^{25}$ per cubic metre and molecular mass 28 unified atomic units at 300 K, $\ell = 72.4$ nm and $\bar{v} = 476$ m/s. Their ratio gives a characteristic flight time 0.152 ns.

Take a weak moment of the collision operator

For a test function $\psi(\mathbf{v})$, multiply the collision term by ψ and integrate over tagged velocity. Exchange the identities of both particles and then exchange incoming and outgoing velocities in the gain integral. The result expresses the change of the pair's total test-function value:

$$\int \psi C[f] \, d^3v = \frac{1}{2} \int d^3v \, d^3v_1 \, d\Omega \, g \frac{d\sigma}{d\Omega} f f_1 [\psi' + \psi'_1 - \psi - \psi_1].$$

For $\psi = 1$, $m\mathbf{v}$ and $mv^2/2$ the square bracket vanishes collision by collision. Integrating the kinetic equation gives exact local conservation of mass, momentum and energy with appropriate streaming fluxes, regardless of how close the distribution is to Maxwellian. Conservation alone leaves a hierarchy: the momentum flux depends on the pressure tensor and the energy flux on a heat flux, both higher velocity moments.

A numerical collision solver should check all these invariants separately. Equal masses allow a direct test with velocities $(300, 100, 0)$ and $(-100, 100, 0)$ m/s. Their centre velocity is $(100, 100, 0)$ and relative velocity has magnitude 400 m/s. Rotating it from x to y gives outgoing velocities $(100, 300, 0)$ and $(100, -100, 0)$. Total velocity and total squared speed agree before and after, but the directional kinetic energy has changed. A field of such collisions can isotropise the pressure tensor while leaving total energy fixed.

Derive a sign-definite entropy production

Define the local kinetic H density as $\int f \log(f/f_\star) d^3v$ with a constant reference of matching units. The reference only changes a conserved number contribution. For a homogeneous isolated gas the collision contribution to its time derivative follows by applying the weak form with $\psi = \log f$. Average the result with its incoming–outgoing interchange. Setting $A = f f_1$, $B = f' f'_1$, one obtains

$$\frac{dH}{dt} = -\frac{1}{4} \int d^3v \, d^3v_1 \, d\Omega \, g \frac{d\sigma}{d\Omega} (A - B) \log(A/B) \le 0.$$

For positive A, B, logarithm and difference have the same sign; appropriate limits handle zero occupation. Thus the Boltzmann entropy $-k_B H$ does not decrease from these local binary collisions. In

an inhomogeneous region, advection additionally transports entropy through the boundaries. A local cell's kinetic entropy can drop because entropy leaves the cell, while collision production remains nonnegative.

Equality requires $ff_1 = f'f_1'$ for collisions connected by the scattering kernel. Equivalently $\log f + \log f_1$ is a collision invariant. For sufficiently rich elastic scattering in three dimensions the smooth additive collision invariants are linear combinations of particle number, momentum and energy. Therefore $\log f = a + \mathbf{b}\cdot\mathbf{v} + cv^2$, with $c < 0$ for normalisation. Completing the square and matching moments yields the local Maxwellian:

$$f_M = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp[-m|\mathbf{v}-\mathbf{u}|^2/(2k_B T)].$$

A Maxwellian at each location makes the collision term zero locally, but spatial gradients usually leave a nonzero streaming term. Such a distribution can evolve because neighbouring cells exchange particles. Conversely, a truly global equilibrium in an external conservative potential includes the appropriate spatial Boltzmann factor. In one-dimensional equal-mass elastic collisions particles merely exchange velocities, so the assumed richness of three-dimensional scattering fails; an arbitrary velocity histogram need not thermalise through those collisions.

Assess the kinetic and hydrodynamic descriptions separately

Define Knudsen number $Kn = \ell/L$ using the length over which the distribution or boundary condition changes. In the earlier gas $Kn = 7.2 \times 10^{-5}$ for a 1 mm chamber feature but 0.072 for a 1 micrometre channel gap. The same binary collision model can remain dilute in both, yet a no-slip bulk hydrodynamic closure may be much less accurate in the microchannel. A mean free path small relative to the overall device length is insufficient if a sharp boundary layer develops on a much shorter scale.

A useful relaxation-time collision approximation is $C_{\mathrm{BGK}} = -(f - f_M)/\tau$, with f_M computed from the local conserved moments of f. This enforces their conservation and models return to local Maxwellian without evaluating an angular integral. If an externally fixed Maxwellian is used irrespective of actual local density, momentum and energy, the approximation can remove these quantities spuriously. A single fitted relaxation time also need not reproduce both stress and heat transport for a real gas.

A particle simulation resolving binary encounters must choose spatial cells small enough to represent local variation, and event or collision timesteps short against relevant flight or collision times. Statistical noise decreases with independent sampled pairs; refining cells while keeping total simulated particle count fixed can make local collision estimates noisier. Compare conservation, relaxation of anisotropic moments and a transport observable under separate refinements of geometry and sample count. A correct mean pressure alone is a weak test of a collision operator.

Compare the flux-weighted velocities of escaping particles with the bulk velocity distribution

Take the next moment and exhibit the source of viscosity

Write momentum density as $\rho u_i = m \int v_i f \, d^3v$. The flux of i-momentum through a surface with normal in direction j is $m \int v_i v_j f \, d^3v = \rho u_i u_j + P_{ij}$, where $P_{ij} = m \int c_i c_j f \, d^3v$. The scalar pressure is one third of its trace in three dimensions. Multiplying the kinetic equation by $m v_i$ and using collision invariance yields the momentum balance with $\partial_j P_{ij}$ explicitly. No constitutive assumption has yet set the off-diagonal components to zero or made them proportional to strain rate.

For a local Maxwellian, the odd moments of peculiar velocity vanish and $P_{ij} = n k_B T \delta_{ij}$. Now consider slow shear with mean velocity $u_x(y)$, and approximate collisions by a relaxation time τ. In a local coordinate frame moving with the fluid write $f \simeq f_M - \tau v_y \partial_y f_M$ to first order in gradient. At constant density and temperature,

$$\partial_y f_M = (m c_x / k_B T) f_M \partial_y u_x .$$

The off-diagonal momentum flux correction is therefore a fourth Gaussian velocity moment:

$$P_{xy} = -\tau \frac{m^2}{k_B T} \partial_y u_x \int c_x^2 c_y^2 f_M d^3v = -n k_B T \tau \, \partial_y u_x .$$

The integral factorises into variances $k_B T / m$ in the two directions. Under the kinetic pressure-tensor convention the result has a minus sign, because random motion transports high x-momentum toward lower mean velocity. The viscous part of the Cauchy stress is commonly written with the opposite sign convention. State which tensor enters the balance before declaring a predicted shear stress negative. The dynamic viscosity of this simple model is $\eta = p \tau$.

If $p = 1.0 \times 10^5$ Pa and the fitted stress relaxation time is 1.8×10^{-10} s, this gives 1.8×10^{-5} Pa s. At a shear rate of 10^3 per second the dissipative stress magnitude is 0.018 Pa, tiny compared with equilibrium pressure and with dimensionless shear rate $\tau |\partial_y u_x| = 1.8 \times 10^{-7}$. A stress measurement must resolve a small anisotropic contribution on top of a large isotropic pressure. A pressure gauge alone cannot test this kinetic viscosity relation.

In a dilute hard-sphere gas the mean collision time scales inversely with density. The simultaneous pressure factor then makes dynamic viscosity approximately independent of density at fixed temperature in the dilute continuum regime. The kinematic viscosity, η / ρ, still changes strongly. A gas made sufficiently rarefied no longer supports ordinary no-slip continuum flow through a fixed channel, so an apparent pressure-dependent flow resistance can coexist with an essentially density-independent bulk dynamic viscosity.

Infer accommodation from a wall momentum measurement

The interior collision operator does not prescribe the boundary condition at a solid wall. Specular reflection reverses normal velocity while preserving tangential velocity in the wall frame. Diffuse thermal re-emission from a fully accommodating stationary wall instead draws outgoing velocities from a wall-temperature distribution appropriate to the outward flux and erases incident tangential drift. A real clean

or rough surface may lie between these limits. The boundary scattering model matters particularly when particles encounter the wall as often as other particles.

For a simple moving plate in a dilute gas at the same temperature, suppose a fraction α of incident particles accommodates tangentially to plate speed U, and the rest reflects specularly. To leading order in small tangential drift, accommodated particles carry away an average tangential momentum increment mU if incident gas has zero mean tangential speed. The incident number flux on one plate face is $n\bar{v}/4$. Thus the leading tangential drag magnitude on the plate is $\alpha m U n \bar{v}/4$ per area in this ideal free-molecular geometry.

Take $n = 10^{21}$ per cubic metre, molecular mass 4.65×10^{-26} kg, $\bar{v} = 480$ m/s, $U = 2.0$ m/s and $\alpha = 0.80$. The predicted shear is about 8.93×10^{-3} Pa per exposed face. The effect scales linearly with density here, unlike the dilute continuum viscosity at a fixed macroscopic velocity gradient. Both results derive from momentum transport, but the first describes particles crossing a wall-bounded collisionless region and the second local shear sustained by many intermolecular collisions. Compare them only after specifying the Knudsen number and wall geometry.

To test the accommodation model, reverse the plate motion to remove static force offsets, vary density across a regime that remains free molecular and change the exposed surface while keeping area fixed. A single measurement gives the combination of accommodation fraction, incident number flux and velocity difference. It cannot separate a changing incident gas drift from a changing material accommodation coefficient. Measuring the upstream velocity distribution or using opposed plates with known relative motion supplies additional constraints.

A discrete collision quartet demonstrates the H theorem numerically

A single allowed reversible collision connects two incoming velocity cells to two outgoing cells. Suppose cell-volume and scattering factors have been normalised so that its net flux is $K(A - B)$, with $A = f_1 f_2$, $B = f_3 f_4$. It removes particles from cells one and two and adds them to three and four when $A > B$. Conservation requires the representative cell velocities to satisfy the same pair momentum and energy constraints used for a physical collision. Assigning arbitrary four velocities may conserve the number of particles but will generally create energy.

Choose the velocities from the earlier rotated-pair example and occupancies $f_1 = 4$, $f_2 = 1$, $f_3 = 2$, $f_4 = 1$ in a common dimensionless reference. Then $A = 4$, $B = 2$ and the forward flux is $2K$. Its contribution to dH/dt is $-2K \log 2$, negative. The exact continuous rate tends to zero as the products equalise. A forward Euler step larger than $f_2/(2K)$ would make the second occupancy negative and destroy the logarithm's meaning. Nonnegative entropy production for the continuum collision model does not guarantee it for an arbitrary numerical time step.

An implicit or positivity-preserving update can avoid this failure, but still requires independent tests of conserved moments and of the correct equilibrium. For this isolated quartet the condition $f_1 f_2 = f_3 f_4$ alone leaves many possible stationary cell distributions. A network of collision quartets sufficiently connecting the velocity space is needed to enforce a Maxwellian-like equilibrium. On a coarse discrete velocity lattice too few permitted momentum- and energy-conserving quartets may leave artificial

invariants. A code can display exactly conserved energy and monotonically decreasing H while relaxing to the wrong distribution because its collision graph is incomplete.

A velocity-selective aperture tests collision equilibration

A thin aperture feeding a vacuum produces a flux-weighted distribution from a local Maxwellian reservoir. A detector placed downstream generally selects a finite angular cone and may respond differently to fast and slow particles. Consider two narrow speed bins with the same width in speed and the same angular acceptance. For an effusive Maxwellian their flux ratio contains a cube of speed multiplied by a Boltzmann exponential. A bulk sampler instead has a square of speed. Confusing these prefactors shifts a temperature inferred from ratios even if the gas truly is locally equilibrated.

For bins at speeds v and $2v$, and choosing $mv^2/(2k_BT) = 1$, the effusive high-to-low ratio is

$$8e^{-3} = 0.398.$$

The bulk number-density ratio at those speeds would be $4e^{-3} = 0.199$. A downstream detector measuring particles present in a fixed optical observation volume may weight by travel time, introducing an additional approximate inverse speed that moves the response closer to density weighting. Distinguish arrivals counted per time from particles counted during an exposure.

Changing aperture diameter is an independent control. A broad opening or high upstream density can produce collisional expansion and a directed jet, altering both angular and speed distributions. Measure pressure and beam shape across aperture sizes and compare the small-aperture limit before attributing a non-Maxwellian downstream spectrum to a failure of local thermal equilibrium throughout the reservoir.

A flight-time measurement tests free paths and speed selection separately

Imagine a dilute molecular tracer introduced into a uniform bath. For a deliberately simplified constant collision-rate model with rate r, probability of no collision by time t is e^{-rt}. For tracers conditioned to have one speed v, the distribution of collision-free distance is then exponential with mean v/r. If the collision rate itself depends on speed through the average relative velocity against the bath, the conditional mean free path is $v/r(v)$. Mixing all initial speeds need not produce a single exponential in distance, even if every conditional waiting time is exponential.

Place a detector a distance L downstream that accepts only trajectories maintaining their initial direction. For a narrow injected speed class, the unscattered fraction is approximately $e^{-r(v)L/v}$ if any collision removes it from the acceptance. For a class at 500 m/s and mean free path 0.20 m, a detector at 0.10 m receives an unscattered fraction 0.607; at 0.30 m it receives 0.223. A ratio of the two readings removes the unknown injection strength and recovers the attenuation coefficient if detector acceptances and geometric factors match.

If forward-scattered trajectories remain inside the acceptance, the apparent attenuation is smaller than the total collision rate. Angularly narrowing the receiver should approach the total-removal model for

suitable scattering. A finite source pulse and finite time-of-flight bin also select velocities even if the source initially emitted a Maxwellian. Fit survival separately in narrow speed bins first. A nonexponential whole-beam attenuation then need not imply that the microscopic collision process has memory; it can arise purely from an unresolved distribution of rates.

A simulation can distinguish these alternatives by logging every collision event and comparing the full collision count with the detector-defined removed count. First test conditional survival for a prepared monochromatic beam, then add a known distribution of incident speeds and calculate the mixture prediction explicitly. Finally include a finite angular acceptance and integrate the differential cross section over the appropriate retained directions. These steps separate the kinetic collision operator, the initial distribution and the observation rule.

The same logic applies to a relaxation experiment reporting a single characteristic mean free time. The decay time of a directional flux depends on how much momentum is deflected per collision, not simply on how often particles meet. Strongly forward-peaked scattering can have a large total cross section and a much smaller momentum-transfer cross section weighted by $1-\cos\theta$. A measured attenuation and a measured viscosity may consequently constrain different moments of precisely the same differential scattering process.

For example, if every collision deflects a test particle through a small fixed angle of ten degrees in the relevant frame, the momentum-transfer weighting is

$$1-\cos 10^{\circ}=0.0152.$$

The corresponding transport cross section can then be far smaller than the total cross section even when collisions are frequent. A velocity-randomising approximation that uses the total collision time as its stress-relaxation time would predict much faster momentum relaxation than this angular picture supports. Actual stress and energy transport involve additional angular weights and velocity averages, so this numerical comparison is a scale diagnostic rather than an exact viscosity formula. Measure several angular moments of the differential scattering kernel or compare at least two independently sensitive transport observables before assigning a universal relaxation time to a real collisional gas.

Follow an H functional during a reversible elementary collision

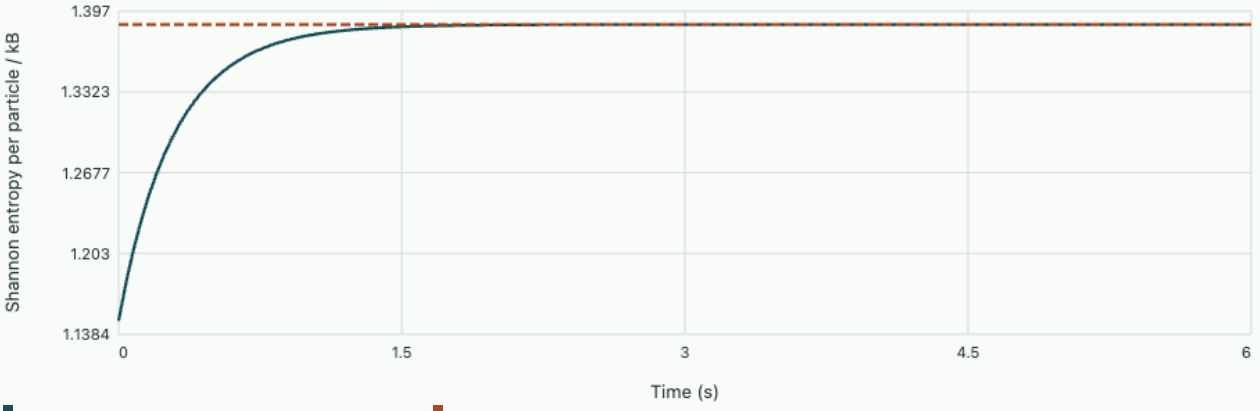
Prepare unequal populations in two velocity pairs. Count forward and reverse collision rates and compare the sign of entropy change with the population imbalance.

Open this experiment online

Try the calculation

Initial fraction in velocity pair A **0.83**

Collision rate coefficient (s⁻¹) **0.8**



Four-state collision ensemble Equal populations

Initial clockwise-minus-reverse pair imbalance x²−(1−x)² = 0.66. Initial d(S/kB)/dt = 0.83721 s⁻¹.

Tutorial · Find the equilibrium temperature of an anisotropic preparation

A homogeneous gas has zero mean flow and component kinetic temperatures 600, 300 and 150 K, defined by $m\langle v_i^2\rangle = k_B T_i$. In an isolated elastic gas, predict its equilibrium temperature and pressure tensor change. Could a thermostat holding the velocity variance in one direction at its original value permit the same inference?

▼ **Retain all three conserved energy contributions**

Conservation of total kinetic energy implies $T_f = (600 + 300 + 150)/3 = 350$ K. Initially the diagonal kinetic pressure components are nk_BT_i; ultimately each becomes $nk_B(350\,\mathrm{K})$. Their sum is unchanged. A component singled out by a thermostat can exchange work or heat with its environment. Its final common temperature, if one exists, must be obtained from the full energy balance including that exchange. Isotropisation cannot simply be assumed to retain the initial isolated mean.

Tutorial · Estimate the error in a collision-frequency inference

A detector reports a mean speed of 480 m/s and a geometric collision cross section 4.0×10^{-19} square metres at $n = 2.0 \times 10^{25}$ per cubic metre. Estimate the collision rate per tagged particle and total events per unit volume per second for an equilibrium identical-particle gas. What changes in a molecular beam with almost parallel velocities?

▼ **Use the relative rather than the single-particle speed**

The equilibrium mean relative speed is $\sqrt{2}(480) = 679$ m/s. The rate per tagged particle is $ng\sigma \simeq 5.43 \times 10^{9}$ per second. Multiplying by $n/2$ gives event rate 5.43×10^{34} per cubic metre per second. In a nearly monoenergetic collimated beam the relative speed can be much less than the mean laboratory speed. Evaluate its pair-distribution average explicitly; translating every velocity by a common vector leaves all collision rates unchanged at fixed distribution of relative velocities.

Tutorial · Does reversing all velocities contradict the H theorem?

In a dilute-gas simulation, reverse all particle velocities at a chosen instant with exact precision. An ideal reversible microscopic integrator would retrace the prior trajectory. Explain how this outcome can coexist with a kinetic equation predicting nondecreasing entropy, and propose a comparison that isolates the missing information.

▼ **State which distributions are controlled by the reversal**

The reversed incoming pairs carry precise correlations inherited from earlier outgoing collisions. Their two-particle distribution generally does not factor into the product of their one-particle distributions just before future collisions. The molecular-chaos step used to derive the collision operator therefore fails for this deliberately prepared history. Microscopic Hamiltonian evolution and the coarse one-particle closure make different state specifications.

Prepare a second gas with the same reversed one-particle velocity histogram but with spatial and partner correlations randomised while conserving total momentum and energy. Compare the two subsequent anisotropy histories. The kinetic equation has the same initial one-particle input for both, while a microscopic simulation can distinguish them through higher-order correlations. Imperfect numerical reversal introduces a separate error; test exactness and time-step dependence before interpreting a failure to retrace as a fundamental thermodynamic observation.

Cambridge Kinetic Theory · the Boltzmann equation, collision invariants and hydrodynamics

Solving transport while conserving mass

Follow control-volume reasoning onto a grid. Compare diffusion, interface fluxes and advective remapping. Check how a travelling perturbation moves at different resolutions.

1. Flux differences for an elliptic balance
2. Finite volumes and moving fronts
3. Time-dependent diffusion
4. Interfaces, contact resistance and Robin data
5. Cell Péclet number and monotonicity
6. What a periodic grid does to a decaying Fourier mode
7. Moving a cell average without inventing or losing material
8. Splitting a stiff relaxation from a slowly varying drive
9. Transport coefficients from equilibrium fluctuations

Flux differences for an elliptic balance

Numerical partial differential equations

Assemble a conservative boundary-value problem from face fluxes, trace the effect of boundary conditions and establish a discrete energy identity.

FROM FIRST PRINCIPLES

Derive a cell equation by integrating the physical balance

Consider steady one-dimensional conduction in a rod with area $A(x)$, conductivity $k(x) > 0$ and heating per unit length $s(x)$. Define heat current F positive to the right. Conservation in any finite interval requires that the current leaving equals the current arriving plus generation. Integrating over cell i fixes a discrete equation without yet approximating a derivative:

$$F_{i+1/2} - F_{i-1/2} = \int_{x_{i-1/2}}^{x_{i+1/2}} s(x)\,dx.$$

Each shared face must use the same current in its two adjacent cells. To close a face between constant-material regions with no intervening source, integrate Fourier's law from one cell centre to the other. The heat current is constant on that interval, so it can be taken outside the integral. Defining the face resistance $R_{i+1/2}$ makes the sign of the temperature drop explicit:

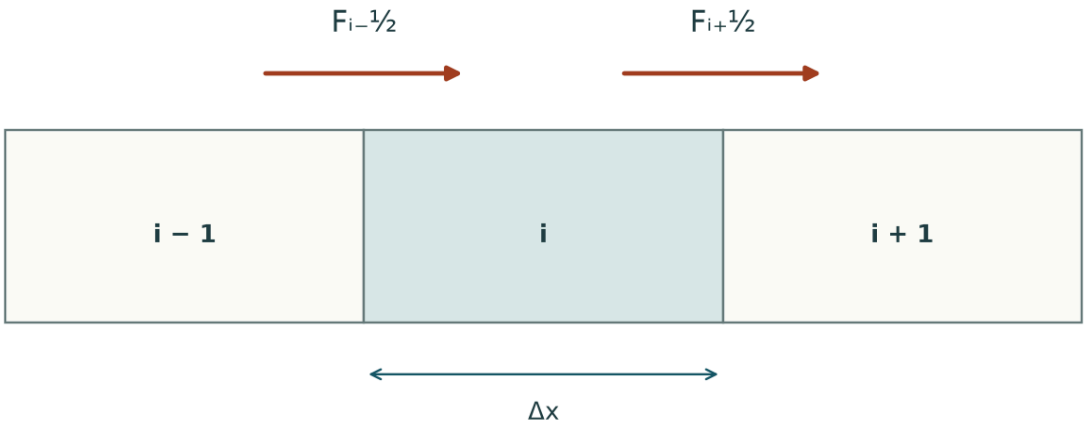
$$\begin{aligned} R_{i+1/2} &= \int_{x_i}^{x_{i+1}} \frac{dx}{k(x)A(x)}, \\ T_{i+1} - T_i &= -F_{i+1/2}\,R_{i+1/2}, \\ F_{i+1/2} &= \frac{T_i - T_{i+1}}{R_{i+1/2}}. \end{aligned}$$

On a uniform homogeneous mesh this reduces to $kA(T_i - T_{i+1})/h$. Across a material change the half-cell resistances must instead be added. Substituting face currents into the cell balances produces the matrix system. Boundary temperatures, prescribed currents or film laws determine which additional boundary equations close it.

Background · Conservative face flux

$$|C_i|\dot{u}_i + \sum_{f \subset \partial C_i} F_{if} = |C_i| s_i$$

Use the same face flux with opposite orientations in neighbouring cells. Interior contributions then cancel in the global balance.



Why an internal face must enter its neighbours with opposite signs
A positive rightward flux brings material into cell i through its left face and removes material through its right. Dividing the difference by Δx gives the rate of change of the cell average. Summing over cells cancels each shared flux once; only the two outside faces survive.

DERIVATION AND APPLICATIONS

Why a flux-only boundary requires a compatibility condition

An insulated or flux-controlled steady diffusion calculation can fail even when each individual face law is correct. Consider a one-dimensional bar of unit area, positive conductivity k, and cell sources s_i per unit length. Define $F_{i+1/2}$ as heat current positive to the right everywhere, including both boundaries. At steady state the current leaving the right face minus the current entering the left face equals the total source in that cell:

$$F_{i+1/2} - F_{i-1/2} = s_i \Delta x_i.$$

Summing over cells cancels every shared interior current. No detail of the face-gradient approximation can change the remaining global condition:

$$F_R - F_L = \sum_i s_i \Delta x_i.$$

With only prescribed fluxes, the discrete gradient also cannot determine a temperature offset. For a connected linear system with positive face transmissibilities, its assembled quadratic form is a sum of squared temperature differences:

$$\mathbf{T}^{\mathsf{T}} K \mathbf{T} = \sum_{\text{interior faces}} g_{ij} (T_i - T_j)^2,$$

$$K\mathbf{1} = 0.$$

The nullspace consists precisely of constant temperatures. A compatible right-hand side gives a family of solutions; prescribe one reference temperature or a weighted mean to select a member. An incompatible right-hand side gives no steady solution. Physically, the imbalance would accumulate energy. Removing one algebraic equation or pinning a temperature without checking global balance can inadvertently create an unreported heat sink.

WORKED EXAMPLE

Reconstruct temperatures from compatible flux data

Take three equal cells with source totals $1, 2, 1$ W, left incoming current $F_L = 2$ W and right outgoing current $F_R = 6$ W. Internal face currents are successively 3 and 5 W. With both conductances equal to 2 W/K, the temperature drops are 1.5 and 2.5 K from left to right. For a prescribed arithmetic mean of 300 K the temperatures are 301.833, 300.333 and 297.833 K. Adding a common constant would leave every face measurement unchanged.

SECOND APPLICATION

Diagnose an unbalanced insulated test

Seal both ends so $F_L = F_R = 0$ but retain the same sources. Their total is 4 W. If the combined heat capacity is 200 J/K, the heat-capacity-weighted mean temperature must rise at $4/200 = 0.0200$ K/s. A steady solver reporting fixed temperatures under those conditions should be checked for a hidden imposed temperature or a dropped residual. The missing four watts have a direct transient interpretation.

Place fluxes where conservation uses them

Consider steady conduction on $0 < x < L$ with heat flux $q = -k(x)T'$ and volumetric generation f. Taking unit cross-sectional area, steady energy balance gives $q' = f$. On uniform nodes separated by h, integrate the balance over a control segment centred on x_i:

$$q_{i+1/2} - q_{i-1/2} \simeq h f_i.$$

A shared face must have the same numerical flux in both adjacent equations, otherwise summing local equations can create an artificial internal source or sink.

Use $q_{i+1/2} = -k_{i+1/2}(T_{i+1} - T_i)/h$. Then row i of the discrete operator has diagonal $(k_{i-1/2} + k_{i+1/2})/h^2$ and off-diagonals $-k_{i\pm1/2}/h^2$. For positive k the off-diagonals are negative and the diagonal balances them. With zero source, an unconstrained interior value is a weighted average of its neighbours. This relation explains why a source-free discrete conductor with fixed end temperatures cannot develop an interior temperature larger than both ends.

Compute the face law across an interface

For smooth k an arithmetic average of neighbouring nodal coefficients provides a second-order face approximation on a uniform grid. A sharp material interface calls for a different physical calculation.

Consider two half-segments in series with constants k_L, k_R and no interface storage or contact resistance. The same flux crosses both halves. Adding their temperature drops gives $T_L - T_R = q[h/(2k_L) + h/(2k_R)]$, hence $k_{\text{eff}} = 2/(1/k_L + 1/k_R)$. This is a harmonic average for equal half-widths.

With $k_L = 1,\ k_R = 9$ W/(m·K) and a 10 K drop over a 0.1 m pair, the effective conductivity is 1.8 W/(m·K) and the flux is 180 W/m². Using the arithmetic mean would give 500 W/m², severely underestimating the resistance of the poorer conductor. In the limit $k_L \to 0^+$, the physical flux tends to zero while that arithmetic estimate remains finite. For unequal layer thicknesses use the actual layer resistances rather than either unweighted average.

Examine uniqueness and the role of the boundary

For homogeneous fixed temperatures and an interior vector u extended by zero at both ends, grouping adjacent row contributions gives

$$h \sum_i u_i (A_h u)_i = \sum_{\text{faces}} \frac{k_{i+1/2}}{h} (u_{i+1} - u_i)^2.$$

Each term is nonnegative. Vanishing energy requires a constant vector, and the fixed ends force that constant to be zero. Thus the interior matrix is symmetric positive definite. This reasoning justifies CG and gives a uniqueness argument that does not depend on inspection of a particular numerical matrix. With only prescribed heat fluxes a spatial constant instead remains undetermined. A solution then requires overall heat compatibility and one temperature reference or a mean constraint.

As a check, take constant $k = 1$, $L = 1$, zero end temperatures and generation 2. The exact profile $T = x(1 - x)$ has central value 0.25. Central differences recover its second derivative exactly, so all uniform grid nodes satisfy the balance without truncation error. This successful check verifies signs, boundary handling and the quadratic patch. To measure a convergence order use a smooth field with a nonzero leading truncation term, such as a sine profile, and compare at common physical positions.

Background · Can fixing one node hide an incompatible heat balance?

With specified flux at both ends, a steady source must exactly equal the net heat leaving the domain. After moving boundary terms to the right, the discrete equations have the form $KT = b$, where K is symmetric and its rows sum to zero. Consequently

$$\mathbf{1}^T b = \mathbf{1}^T K T = 0.$$

Pinning a temperature removes the undetermined additive constant. If the vector b fails this sum test, however, the pinned solve creates a reaction at that node. Physically that reaction is an unmodelled heat supply or heat sink. After solving, restore the original equation at the pinned node and inspect its residual. It exposes exactly the load that would have to be supplied to maintain the supposed steady state.

Why fixing one boundary value controls the solution

Suppose the left end of a conducting rod is held at zero temperature and the right end is insulated. A finite system of flux balances may be invertible on one mesh. We would also like a bound which survives refinement. In the continuous problem, use the space $V = \{v \in H^1(0,L) : v(0) = 0\}$. Its members have square-integrable first derivatives and a well-defined endpoint value. Every admissible difference between two temperature fields belongs to this same space. The fundamental theorem of calculus and Cauchy–Schwarz give

$$|v(x)|^2 = \left| \int_0^x v'(s)\,ds \right|^2 \le x \int_0^x |v'(s)|^2\,ds, \qquad \|v\|_2 \le \frac{L}{\sqrt{2}} \|v'\|_2.$$

The second inequality follows by integrating the first and bounding its inner integral by the full derivative norm squared. Thus a small temperature gradient forces the temperature itself to be small in this particular space. Take a possibly discontinuous conductivity with $0 < k_0 \le k(x) \le k_1$ almost everywhere and a square-integrable heat source f. Testing the balance and integrating once gives $a(u,v) = \ell(v)$ for every $v \in V$, with

$$a(u,v) = \int_0^L k u' v'\,dx,$$

$$\ell(v) = \int_0^L f v\,dx.$$

The upper bound on conductivity gives

$$|a(u,v)| \le k_1 \|u'\|_2 \|v'\|_2.$$

The lower bound gives $a(v,v) \ge k_0 \|v'\|_2^2$, called coercivity. The endpoint inequality bounds $|\ell(v)|$ by $L\|f\|_2\|v'\|_2/\sqrt{2}$. These are the hypotheses of the Lax–Milgram theorem on the Hilbert space with norm $\|v'\|_2$. Consequently the weak solution exists and is unique. For two loads, subtract the weak equations and test with their solution difference. Cancelling one derivative norm gives

$$\|(u-\tilde{u})'\|_2 \le L\|f-\tilde{f}\|_2/(\sqrt{2}\,k_0).$$

Every conforming discrete subspace inherits the same estimate.

How large a change can a load error produce?

In scaled units let $L = 1,\ k = 2,\ f = 3$. Integration of $-2u'' = 3$, first from the insulated end and then from the fixed end, gives

$$u' = 1.5(1-x),$$

$$u = 1.5x - 0.75x^2.$$

Its derivative norm is

$$1.5/\sqrt{3} = 0.8660.$$

The theorem guarantees at most $3/(2\sqrt{2}) = 1.0607$. Raising the source everywhere by 0.12 changes that norm of the difference by $0.06/\sqrt{3} = 0.03464$, within the guaranteed 0.04243. The estimate also applies to nonuniform perturbations of the same load norm. If both ends are insulated, constants have zero derivative and can be added freely. A net heat input then rules out steady balance altogether.

Eliminate boundary values with the correct sign

With four intervals on a unit rod, fixed left temperature 10, fixed right temperature 20 and zero source, multiply each row by h^2/k. The three unknowns then satisfy a tridiagonal system with diagonal two, off-diagonal minus one and load $(10, 0, 20)^T$. Moving a fixed value to the right changes its sign, producing positive boundary contributions here. Solving gives the linearly spaced interior temperatures 12.5, 15 and 17.5. Every discrete face reports the same flux for constant k.

This deliberately simple case isolates boundary elimination and sign conventions. Reversing only one load sign can generate a plausible-looking smooth curve with entirely wrong end-to-end transport. After checking the values, also sum all interior balances and verify cancellation of the internal face contributions. The resulting global statement includes only boundary fluxes and any source integrated over the interior control volumes.

Prove a discrete maximum principle from the face balances

Let positive conductances $G_{i-1/2}$ and $G_{i+1/2}$ connect each internal node to its neighbours, with both end temperatures specified. In the absence of internal sources its balance reads $G_{i-1/2}(u_i - u_{i-1}) + G_{i+1/2}(u_i - u_{i+1}) = 0$. Dividing by the sum of the conductances makes u_i a convex combination of its two neighbours. If a strict global maximum occurred inside the domain above both prescribed boundary values, its neighbours would each have to equal this maximum. Repeating along the connected grid would carry the same value to a boundary, contradicting the assumption.

The same reasoning applied to the negative field bounds the minimum. With a nonnegative internal heat source s_i on the right, an interior minimum below both boundaries would have both outgoing temperature differences nonpositive. Their positive weighted sum cannot equal a positive source, and if the sum is zero, propagation again carries the minimum to the boundary. A positive source may raise an interior maximum while preserving the lower comparison bound. Negative face conductances or a disconnected component invalidate the argument; symmetry and a small algebraic residual do not by themselves restore it.

Compare two thermal loads without inverting their matrices

Connect internal nodes 1 and 2 to end temperatures zero and ten, using consecutive conductances 1, 2 and 4 W/K. With no source the equations are

$$3u_1 - 2u_2 = 0,$$
$$-2u_1 + 6u_2 = 40.$$

The first gives $u_1 = 2u_2/3$, and the second gives

$$u_2 = 60/7,\ u_1 = 40/7.$$

Both values lie between the boundary temperatures. The common leftward heat flow has magnitude $40/7$ W, checked independently from each of the three temperature drops times its conductance.

Add a heat source of 7 W at node one. The difference v between new and old solutions has zero boundary values and satisfies

$$3v_1 - 2v_2 = 7,\ \ -2v_1 + 6v_2 = 0.$$

The second relation gives $v_1 = 3v_2$, so

$$v_2 = 1,\ v_1 = 3.$$

Both internal states warmed, consistent with the comparison principle applied to the difference. The additional outflow at the left boundary is $1 \cdot 3 = 3$ W and at the right is $4 \cdot 1 = 4$ W, accounting for the full applied source.

This provides two independent implementation tests. Conservation checks that the boundary power change equals the injected power. Monotonicity checks the signs of all changes in the field. A sign error in a face contribution can violate one of these checks even while the computed algebraic residual for the incorrectly assembled matrix is tiny. Both tests use physically accessible quantities and remain meaningful when no exact closed-form solution of a complex geometry is available.

The argument extends to a connected graph, since a weighted average at a maximising node requires equality along every positive-conductance edge. This observation is useful for testing an unstructured implementation. Deliberately disconnecting an entirely insulated component produces a free constant there, so boundary comparisons cannot control that component until a temperature reference or independent constraint is supplied.

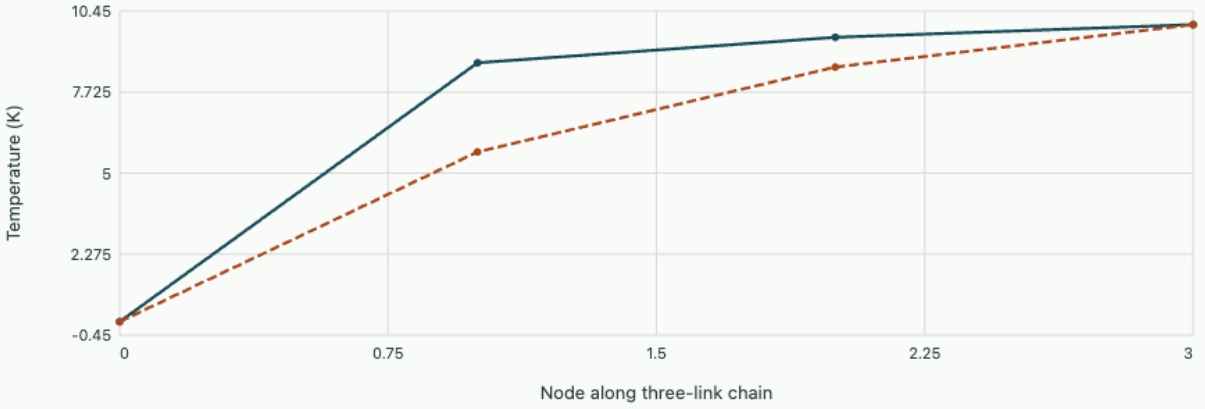
Change parameters · Flux differences for an elliptic balance

Change one input at a time and compare the recalculated curves with the argument above. The browser evaluates this small model locally; the accompanying numerical values state its assumptions and reference checks. Reset restores the reference inputs, and the data can be downloaded as CSV.

Open this calculation with adjustable inputs

Try the calculation

Middle conductance (W/K)	2
Source at first interior node (W)	7
Right boundary temperature (K)	10



Additional boundary outflow 7 W equals the inserted 7 W. Both interior temperature increments are nonnegative.

Finite volumes and moving fronts

Numerical partial differential equations

Derive an upwind transport update by conservation, interpret the Courant number as a fraction of cell contents and determine what conservation says about front speed.

FROM FIRST PRINCIPLES

Derive a conservative update from storage in a cell

Consider the one-dimensional conservation law $u_t + \partial_x f(u) = 0$ with no distributed source. Let $\bar{u}_i$ denote the average over cell i of width Δx. The integral of u over the cell is what the discrete variable stores. Integrate first over the cell and then over one time step. Only fluxes crossing the two faces can change its inventory:

$$\Delta x(\bar{u}_i^{n+1} - \bar{u}_i^n) = -\int_{t_n}^{t_{n+1}} [f_{i+1/2}(t) - f_{i-1/2}(t)]\, dt.$$

Replace each time-averaged face flux by one numerical function $\hat{f}$ of neighbouring states, using the same value from both sides. For constant positive advection speed a, the state reaching a face comes from upstream. With piecewise-constant cell data and $C = a\Delta t/\Delta x \le 1$ this gives

$$\hat{f}_{i+1/2} = a\bar{u}_i^n,$$
$$\bar{u}_i^{n+1} = (1 - C)\bar{u}_i^n + C\bar{u}_{i-1}^n.$$

The update conserves total mass on a periodic grid because every internal face cancels. Nonlinear fluxes also need a rule selecting the physically admissible interface state, especially when a jump separates two different initial cell values.

Background · Courant number

$$C = |a|\Delta t/\Delta x$$

For scalar advection this compares a transported distance with a cell width. An admissible range depends on the spatial method and time integrator jointly.

DERIVATION AND APPLICATIONS

Quantify the quadratic quantity lost by an upwind update

A conservative finite-volume method for advection preserves the sum of cell contents under periodic boundary conditions. It need not preserve their contrast. Consider the constant-speed transport equation $u_t + a u_x = 0$ with $a > 0$, uniform cells of width Δx and periodic indexing. Exact translation preserves $\int u^2\,dx$. The first-order upwind scheme transports a fraction $C = a\Delta t/\Delta x$ of a cell's content across its right boundary per step:

$$u_i^{n+1} = (1 - C)u_i^n + C u_{i-1}^n.$$

For $0 \le C \le 1$ this is a convex combination and cannot create a value outside its two inputs. To see precisely how much variation is removed, square one update and complete a difference square:

$$[(1 - C)x + Cy]^2 = (1 - C)x^2 + Cy^2 - C(1 - C)(x - y)^2.$$

Sum over all cells. Periodicity allows the shifted sum of old squared values to be relabelled. The exact discrete identity, with $E^n = (\Delta x/2)\sum_i (u_i^n)^2$, becomes

$$E^{n+1} - E^n = -\frac{\Delta x}{2} C(1 - C)\sum_i (u_i^n - u_{i-1}^n)^2.$$

This loss vanishes for spatially constant data and for $C = 1$, when the update shifts by one complete cell. Strong cell-to-cell jumps dissipate much more of the quadratic quantity than smooth profiles sampled with the same grid. Total mass still telescopes exactly; the two balance laws diagnose different aspects of accuracy.

WORKED EXAMPLE

Follow one step on four periodic cells

Take $\Delta x = 1$, $C = 1/2$ and initial averages $(1, 1, 0, 0)$. The updated values are $(0.5, 1, 0.5, 0)$. Both arrays sum to two. The quadratic energy falls from 1.00 to 0.750. There are two unit jumps at the start, and the loss formula gives $(1/2)(1/2)(1/2)(2) = 0.250$, accounting for the full change.

SECOND APPLICATION

Use an alternating pattern as an instability detector

For $(1, -1, 1, -1)$ each neighbour has opposite sign, so one step multiplies the amplitude by $1 - 2C$. At $C = 1/2$ all variation disappears in one step although the exact wave would keep its amplitude. At $C = 1.10$ the factor is -1.20 and energy grows by 1.44. The sign reversal in $C(1 - C)$ therefore predicts growth at this resolved wavelength. Conservation of the zero mean still holds in both experiments and alone cannot detect the instability.

Store quantities over cells

Consider a passive concentration satisfying $u_t + (au)_x = 0$ with positive constant speed a on a periodic line. Let $\bar{u}_i = h^{-1} \int_{x_{i-1/2}}^{x_{i+1/2}} u \, dx$ be its cell mean. Integrating the conservation law in space and time shows that every change in cell inventory comes from fluxes through its two faces. If neighbouring cells use the same face flux, summing all updates cancels every internal transfer exactly. On a periodic line the total stays fixed, even for a discontinuous concentration.

For positive speed, material reaching the right face during a short step comes from the cell to its left. Approximate the outgoing state by that cell's mean, giving $F_{i+1/2} = a\bar{u}_i$. Forward Euler in time yields

$$\bar{u}_i^{n+1} = (1 - C)\bar{u}_i^n + C\bar{u}_{i-1}^n,$$

where $C = a\Delta t/h$. For $0 \le C \le 1$ this is a convex average, preserving the range of neighbouring values. At $C = 1$ the entire array shifts exactly by one cell per step. If $a < 0$, information comes from the other side of the face and the upwind direction must reverse.

Identify the source of artificial spreading

Start with one cell mean equal to one and zeros everywhere else. With $C = 1/2$, one step gives adjacent means $1/2, 1/2$. A second step gives $1/4, 1/2, 1/4$, while the total remains one. Exact advection over two such steps shifts the original pulse by one cell with no change in shape. The scheme has conserved mass yet broadened the pulse. Peak height or variance therefore tests information not supplied by a global balance.

For smooth u, Taylor-expand the upwind update in space and time. Replace the leading u_{tt} using the exact transport equation, $u_{tt} = a^2 u_{xx}$. The leading modified equation is

$$u_t + a u_x = [ah(1 - C)/2]u_{xx} + \cdots.$$

The bracket acts as numerical diffusivity. It is positive for the monotone range and vanishes at $C = 1$. This local smooth expansion helps interpret observed spreading, but the Taylor expansion itself does not hold across a jump.

Choose a flux when speed depends on the state

For a nonlinear conservation law $u_t + f(u)_x = 0$, characteristics can intersect and create a discontinuity. Enclose a moving jump in a thin control volume. The net flux imbalance fixes its speed $s = (f(u_R) - f(u_L))/(u_R - u_L)$. For Burgers flux $f = u^2/2$ with left state 2 and right state 0, this gives $s = 1$. The faster characteristics on the left overtake the slower ones to form a shock. Reversing these states produces a spreading fan under the entropy condition even though the algebraic jump formula alone would still supply a speed.

A numerical face flux must be consistent when the two cells agree and must incorporate information from both sides when a nonlinear wave can travel either way. A conservative method that converges can

reproduce the weak jump balance, while selection of a physically admissible solution also requires an entropy criterion. For practical assessment, monitor total inventory, motion of the front, overshoot and the number of cells over which it is smeared. The four measurements address distinct properties, and reducing one defect need not reduce all of them.

Move the control surface instead of the grid

Across a front moving with speed s, flux in the front's own frame is $f(u) - su$. The jump relation enforces equality of this quantity on both sides. For the Burgers jump from 2 to 0 with speed one, the left flux in the moving frame is $2 - 1 \cdot 2 = 0$, and the right flux is also zero. The laboratory flux differs across the front because a moving interface is sweeping conserved material through stationary space.

This distinction helps when checking a simulation by counting material in a region that tracks a front. Differentiating an integral over moving endpoints adds boundary-motion terms. In a periodic fixed window the update should conserve total amount without those extra terms. In a moving measurement window, physical transport through the window and motion of the window both influence the recorded inventory, even under a perfectly conservative scheme.

Distinguish a conservative jump from the physical weak solution

For inviscid Burgers flow $u_t + (u^2/2)_x = 0$, integrating conservation across a moving jump gives the Rankine–Hugoniot speed $s = [f]/[u] = (u_L + u_R)/2$. It uses only balance, so a discontinuity with that speed satisfies the weak conservation equation for either orientation of its left and right states. To select the physically relevant solution one needs an entropy condition. For a compressive shock characteristic speeds enter it from both sides: $u_L > s > u_R$. If $u_L < u_R$, the characteristics separate and a continuous rarefaction fan fills the gap.

For example the entropy $\eta(u) = u^2/2$ has entropy flux $q(u) = u^3/3$, because $q' = \eta' f'$. A jump produces distributional entropy $[q] - s[\eta]$, which must be nonpositive for a dissipative solution. At equal-magnitude opposite states the entropy jump $[\eta]$ is zero, so orientation alone changes the sign of the entropy production. A conservative scheme can preserve every cell total correctly yet converge to an inadmissible expansion shock if it fails this additional condition.

Choose the correct interface flux for two opposite Riemann data

Take $u_L = -1,\ u_R = 1$. Both endpoint fluxes equal one half, and the jump-speed formula gives zero. Holding this discontinuity fixed would conserve the cell totals. Its entropy production, however, is

$$1/3 - (-1/3) = 2/3 > 0.$$

The admissible Riemann solution is the self-similar fan: at coordinate $\xi = x/t$, $u = -1$ for $\xi \le -1$, $u = \xi$ between minus one and one, and $u = 1$ for $\xi \ge 1$. At the interface $\xi = 0$, the flux is therefore

zero. Using an arithmetic average of the two endpoint fluxes would instead maintain the inadmissible jump.

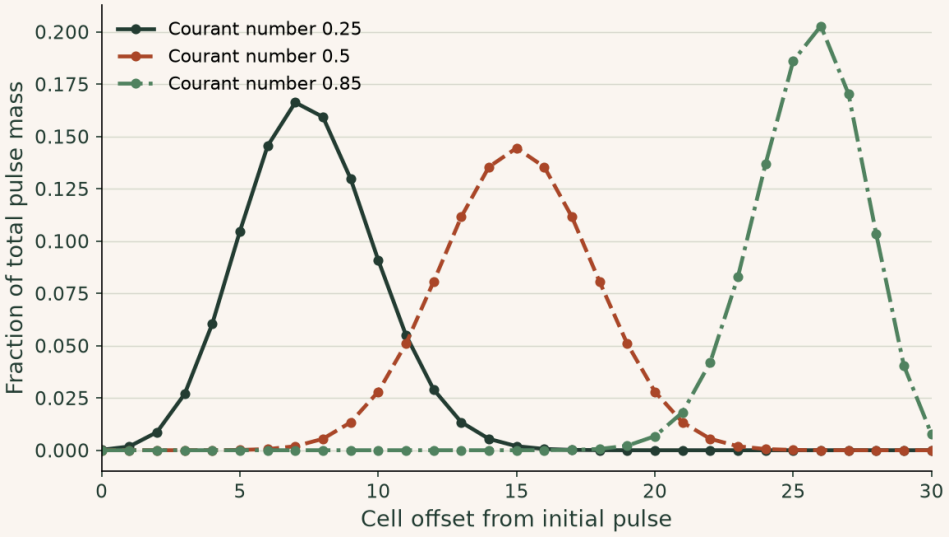
For a finite-volume update with $\Delta t/\Delta x = 0.5$ and initially constant states on either side, the neighbouring left cell changes from -1 to $-1 - 0.5(0 - 0.5) = -0.75$. The right cell changes from one to $1 - 0.5(0.5 - 0) = 0.75$. Their sum remains zero. A direct integration of the fan gives the same cell averages: the disturbance occupies half of each cell and linearly brings that half from its far-state value toward zero. Thus both conservation and the local wave pattern can be verified independently.

Reverse the initial states to $u_L = 1,\ u_R = -1$. The zero-speed jump now has entropy production $-2/3$, and both families of characteristics enter it. Its correct interface flux is one half. The same numerical update then leaves both neighbouring cell averages unchanged, exactly as required for a stationary shock. Testing both orientations is more revealing than testing a single conserved jump: they have identical endpoint flux magnitudes and the same Rankine–Hugoniot speed but different physically admissible interface states.

A further integration test shifts the interface by precisely one cell and reruns the update. On a uniform periodic grid with unchanged physical states, the new solution should be a one-cell translation of the old one. A difference between the two calculations could reveal an indexing error in the face flux even when a global conservation diagnostic remains identically zero.

Measure numerical spreading in a conservative transport step

Start with all mass in one cell. For positive velocity the explicit upwind update keeps fraction $1 - C$ and transfers fraction C to the next cell each step. Thirty updates produce a binomial distribution on an unbounded uniform grid.



Each profile sums to one. Its centre moves 30C cells and its variance is 30C(1−C) cells². Conservation and the correct mean travel distance can coexist with artificial broadening of a pulse that should translate without changing shape.

▶ **Read values from the plot**

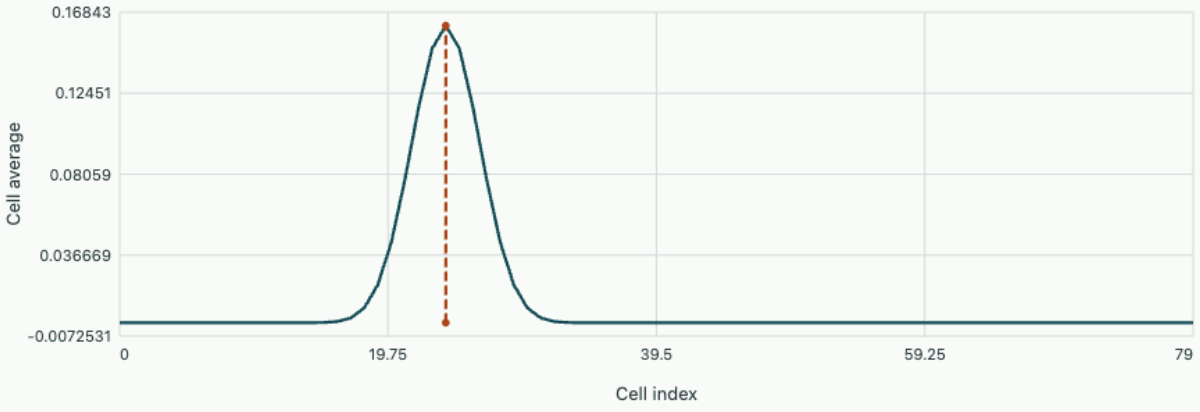
Change parameters · Measure numerical spreading in a conservative transport step

Change one input at a time and compare the recalculated curves with the argument above. The browser evaluates this small model locally; the accompanying numerical values state its assumptions and reference checks. Reset restores the reference inputs, and the data can be downloaded as CSV.

Open this calculation with adjustable inputs

Try the calculation

Courant number C	0.5
Number of steps	24



Total cell sum 1, centroid 24, added variance 6 cells².

Time-dependent diffusion

Numerical partial differential equations

Derive the diffusion time-step restriction from two independent arguments and compare explicit and implicit decay of spatial modes.

FROM FIRST PRINCIPLES

Derive the explicit heat step and its grid-dependent time scale

Take the heat equation $T_t = \alpha T_{xx}$ with constant positive diffusivity on a uniform one-dimensional grid. Fourier conduction through each side of a cell produces a temperature update proportional to the difference of the two neighbour gradients. With forward Euler timing, collect those fluxes to obtain

$$T_i^{n+1} = T_i^n + r(T_{i-1}^n - 2T_i^n + T_{i+1}^n),$$
$$r = \alpha \Delta t / h^2.$$

When $0 \le r \le 1/2$ the new temperature is a convex average of three old ones, preserving bounds on a source-free grid. To check whether a higher step could still be stable for all periodic grid disturbances, substitute $T_j^n = g^n e^{ij\theta}$, where j is the node index. The amplification simplifies to

$$g = 1 + r(e^{-i\theta} - 2 + e^{i\theta}) = 1 - 4r\sin^2(\theta/2).$$

The alternating mode at $\theta = \pi$ gives $g = 1 - 4r$, so $|g| \le 1$ again requires $r \le 1/2$. Halving spatial spacing therefore quarters the largest explicit time step for this discretization. Implicit stepping changes this restriction but still has to resolve any transient relevant to the measurement.

DERIVATION AND APPLICATIONS

Can an energy-stable heat step create an alternating transient?

A diffusion mode should decay rather than oscillate. To distinguish decay of a norm from preservation of sign, examine the finite-element or finite-volume semidiscrete system $M\dot{u} + Ku = 0$, with symmetric positive definite mass matrix M and thermal stiffness K positive semidefinite. An eigenvector satisfying $Kv = \lambda M v$ has scalar amplitude $y(t)$ obeying $y' = -\lambda y$. A generalized θ step applied to this mode gives

$$[1 + \theta z]y^{n+1} = [1 - (1-\theta)z]y^n,$$
$$z = \Delta t\, \lambda.$$

For Crank–Nicolson take $\theta = 1/2$. The factor $G = (1 - z/2)/(1 + z/2)$ has $|G| \le 1$ for all $z \ge 0$ but changes sign when $z > 2$. An initial grid-scale component can therefore alternate in time. To test a remedy, replace the first full step by two half backward-Euler steps. Their combined factor is

$$G_{\text{start}} = (1 + z/2)^{-2}.$$

Thereafter return to the second-order Crank–Nicolson update. After n further full steps the mode amplitudes, with and without the initial damping, are

$$y_{\text{damped}} = G_{\text{start}} G^n y^0,$$
$$y_{\text{plain}} = G^{n+1} y^0.$$

For small z, both startup strategies agree through first order, while for large z the two half steps strongly suppress the unresolved mode. This argument explains a startup treatment for nonsmooth initial data; sign preservation of every physical grid value is a stronger matrix property than sign preservation of one mode.

WORKED EXAMPLE

Check damping of a near-grid-scale eigenmode

Take $\lambda = 2000$ s⁻¹ and $\Delta t = 0.0100$ s. Then $z = 20$. A single Crank–Nicolson step leaves $-9/11 = -0.818$ of the amplitude; two backward-Euler half steps leave $1/121 = 0.00826$. The exact amplitude factor is

$$e^{-20} = 2.06 \times 10^{-9}.$$

Both discrete results are bounded, but the startup treatment removes almost all of a mode that the time grid cannot resolve.

SECOND APPLICATION

Check the cost for a physically slow component

For the same mesh and step, a slow mode with $\lambda = 2.00$ s⁻¹ has $z = 0.0200$. Crank–Nicolson gives $0.99/1.01 = 0.980198$, the startup gives $1/1.01^2 = 0.980296$ and the exact factor is 0.980199. The slow component changes by less than 10^{-4} in this comparison. A timestep must still resolve a transient that is itself part of the desired output.

Discretise space and time separately

A heat pulse in a thin insulated rod smooths over time according to $u_t = D u_{xx}$ for constant positive diffusivity D. Use a uniform grid of spacing h, and initially examine interior points or periodic boundaries so the stencil is unchanged everywhere. Replacing the second derivative by a centred difference gives a system of ordinary differential equations for the nodal temperatures. The time method still has to resolve the fastest rate that this spatial operator can generate.

An explicit Euler step yields $u_i^{n+1} = r u_{i-1}^n + (1-2r) u_i^n + r u_{i+1}^n$ with $r = D\Delta t / h^2$. If $0 \le r \le 1/2$, the update is a convex combination of three temperatures. The maximum cannot rise and the minimum cannot fall in a source-free interior step. Starting with one hot node at 1 and neighbours at zero, $r = 0.4$ gives a centre value 0.2 and neighbours 0.4 after a single step. At $r = 0.6$ the centre would become negative although the continuum heat equation preserves nonnegativity.

Identify the spatial modes that set the limit

Insert $u_j^n = g^n e^{\mathrm{i} j \theta}$, where i is the imaginary unit and j indexes the grid. The two neighbouring values contribute factors $e^{\pm \mathrm{i}\theta}$, yielding $g = 1 - 4r \sin^2(\theta/2)$. Long waves have $\sin^2(\theta/2) \simeq \theta^2/4$, so their decay approaches the continuum rate as the mesh is refined. The alternating grid mode $\theta = \pi$ has the most negative amplification and enforces $r \le 1/2$ for bounded magnitude.

At the limiting value $r = 1/2$ the alternating mode flips sign without losing amplitude. A strict margin damps it. For $D = 10^{-4}$ m²/s and $h = 2$ mm, the maximum explicit step is

$$h^2/(2D) = 0.020\,\mathrm{s}.$$

Refining to 1 mm reduces that limit to 0.005 s. The mesh gives a sharper temperature profile yet can cost four times more steps over the same physical interval, besides the increased number of spatial degrees of freedom.

Compare backward Euler and Crank–Nicolson

Writing the semidiscrete system as $u' = -A_h u$, backward Euler solves $(I + \Delta t A_h) u^{n+1} = u^n$. Crank–Nicolson averages the old and new diffusive fluxes, giving $(I + \Delta t A_h/2) u^{n+1} = (I - \Delta t A_h/2) u^n$. For a Fourier mode put $s = 4r \sin^2(\theta/2)$. Their factors are $1/(1+s)$ and $(1 - s/2)/(1 + s/2)$, respectively. Both remain bounded at arbitrarily large s, but Crank–Nicolson may alternate for under-resolved sharp features.

With $s = 20$, backward Euler retains only $1/21$ per step. Crank–Nicolson yields $-9/11$, retaining much more of the fast component and reversing its sign. For a smooth low-frequency mode Crank–Nicolson generally gives more accurate temporal decay because it is second order. Decide whether early sharp transients or later smooth behaviour is the relevant physical quantity. To identify spatial and temporal error separately, refine Δt first on a fixed mesh, then refine space with time error kept well below spatial error. Merely halving both at once cannot reveal which term dominated the original discrepancy.

Apply a prescribed heat input without confusing it with temperature

An insulated periodic rod with uniform heating rate s measured in K/s has an average temperature that must rise by $s\Delta t$ each step. Sum the centred second differences over the periodic indices: each nodal value appears with coefficients $1, -2, 1$, which cancel. The source alone changes the mean. For $s = 3$

K/s and a step of 0.02 s the numerical average must increase by 0.06 K, independently of the diffusivity and initial profile.

A code can pass this global test while still mishandling short-wave damping, since an alternating error has zero spatial mean. Pair a mean-temperature test with a nonconstant mode-decay test. Use a homogeneous periodic run for isolates of diffusion, and a uniform temperature with prescribed source for the storage term. A failure in one test then identifies a different part of the mathematical balance than a failure in the other.

Prove decay of a discrete thermal energy

Let a closed semidiscrete diffusion model be $M\dot{u} + Ku = 0$, with M symmetric positive definite and K symmetric positive semidefinite. Assume $K\mathbf{1} = 0$, expressing zero flux for a uniform temperature. Backward Euler satisfies $M(u^{n+1} - u^n) + \Delta t K u^{n+1} = 0$. Multiplying by the constant vector and using symmetry gives

$$\mathbf{1}^T M u^{n+1} = \mathbf{1}^T M u^n.$$

Thus the capacity-weighted sum, proportional to stored sensible heat for constant capacities and a fixed reference, is conserved for any step size.

Now multiply by u^{n+1}. The identity $a^T M(a-b) = [\|a\|_M^2 - \|b\|_M^2 + \|a-b\|_M^2]/2$ yields the exact balance

$$\tfrac{1}{2}\|u^n\|_M^2 - \tfrac{1}{2}\|u^{n+1}\|_M^2 = \Delta t\,(u^{n+1})^T K u^{n+1} + \tfrac{1}{2}\|u^{n+1} - u^n\|_M^2.$$

Both terms are nonnegative. The quadratic quantity therefore decays although total sensible heat is conserved. Subtracting the conserved mean first yields the same decay statement for temperature variation around equilibrium. The first term represents spatial smoothing measured at the new time, and the second reflects the numerical time update. This derivation gives a stability certificate on nonuniform grids without assuming Fourier modes.

Audit two cells with different heat capacities

Choose capacities 1 and 3 J/K, conductance 2 W/K, and initial temperature deviations $(4, 0)^T$ K. The conserved sensible heat is 4 J, so the final equilibrium in a closed system must be one kelvin above reference. At $\Delta t = 0.5$ s the matrix $M + \Delta t K$ has rows $(2, -1)$ and $(-1, 4)$, and the right-hand side is $(4, 0)^T$. The new temperatures are $(16/7, 4/7)^T$. Checking their weighted sum gives $16/7 + 3(4/7) = 4$, as required. The unweighted average changes, which is physically appropriate for unequal capacities.

The quadratic quantity at the start is $4^2/2 = 8$. After the step it is

$$[(16/7)^2 + 3(4/7)^2]/2 = 152/49.$$

Its drop is $240/49$. The conductive contribution is $(0.5)(2)(12/7)^2 = 144/49$; the time-update contribution is

$$[(-12/7)^2 + 3(4/7)^2]/2 = 96/49.$$

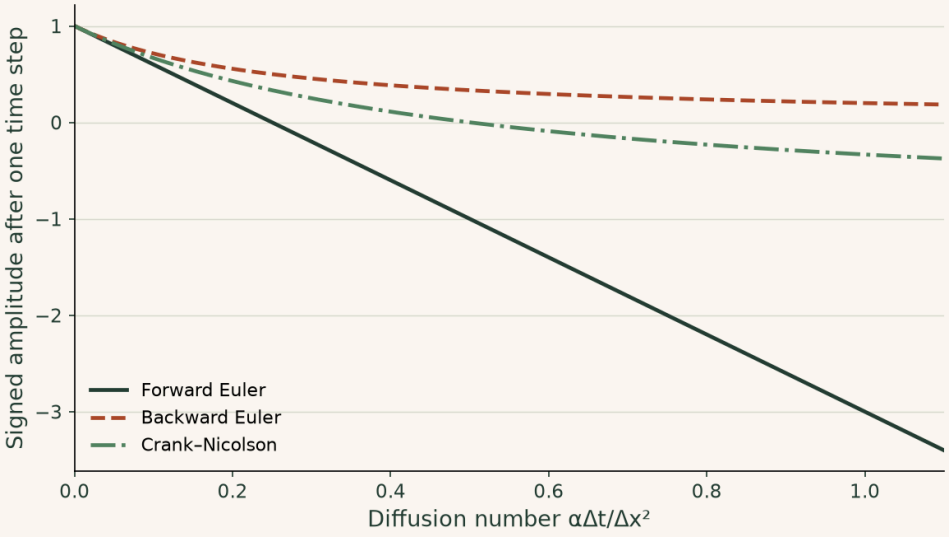
Their sum recovers the measured drop exactly. Neither contribution equals a loss of the 4 J of sensible heat. The quadratic measure depends on squared temperatures and has different physical units.

A positive heat supply or a prescribed exterior temperature introduces a boundary or source work term into these identities. Applying the closed-system decay test to an externally heated experiment would therefore diagnose the wrong physical model. Verify which conserved or dissipated quantity is actually implied by the capacity matrix, boundary conditions and source terms before comparing numerical integrators.

The conserved mean suggests another check after many large steps. The steady state must have zero conductance dissipation, which for two connected cells requires equal temperatures. Together with the weighted heat total this fixes both temperatures at one kelvin above reference. A scheme that instead approaches the unweighted initial average of two would be converging to the wrong conserved quantity despite producing a visually smooth, uniform endpoint. The measured quadratic variation about the correct mean tends to zero; the uncentred squared quantity tends to a positive constant because the remaining uniform state still has temperature above the chosen reference.

What happens to a grid-scale temperature ripple?

The alternating spatial mode on a uniform heat grid has discrete Laplacian eigenvalue $-4/\Delta x^2$. Substitute this mode into each method and compare the amplitude after one step. The plot isolates the shortest resolvable pattern.



Forward Euler reverses the mode above 1/4 and becomes unstable above 1/2. Crank–Nicolson stays bounded for every plotted step but also reverses the mode above 1/2. Smooth initial data can hide this high-frequency diagnostic.

▶ **Read values from the plot**

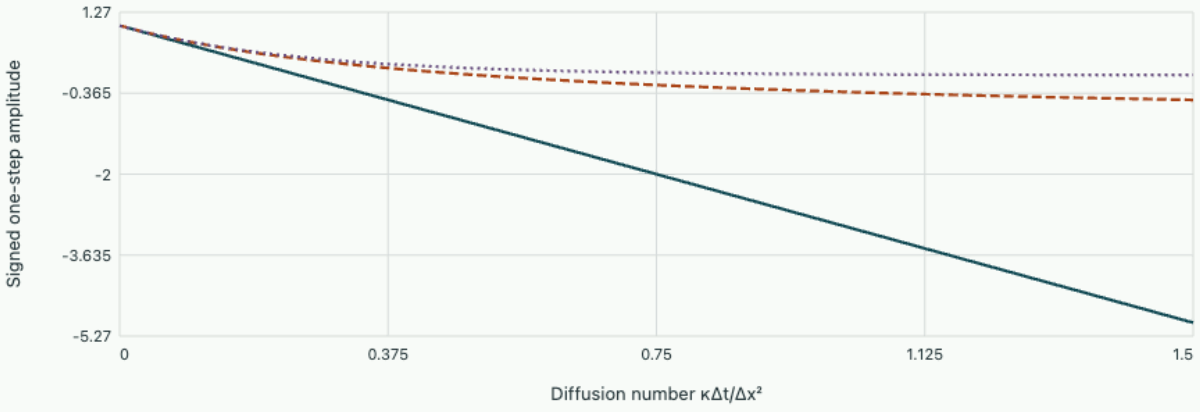
Change parameters · What happens to a grid-scale temperature ripple?

Change one input at a time and compare the recalculated curves with the argument above. The browser evaluates this small model locally; the accompanying numerical values state its assumptions and reference checks. Reset restores the reference inputs, and the data can be downloaded as CSV.

Open this calculation with adjustable inputs

Try the calculation

Highest diffusion number	**1.5**
Mode fraction of Nyquist	**1**



Directional Laplacian magnitude times Δx^2 is 4. Forward Euler is stable up to r = 0.5 and nonnegative only up to r = 0.25.

Interfaces, contact resistance and Robin data

Numerical partial differential equations

Derive a face flux that transports the same heat through two different materials and through a convective boundary.

FROM FIRST PRINCIPLES

Derive an interface flux from two half-cell temperature drops

Two neighbouring cell centres can lie in materials with sharply different conductivities. Consider a planar interface between them, normal to a one-dimensional steady heat flow. Let d_L, d_R be the distances from the centres to the interface, and suppose there is no interface source, storage or contact resistance. Steady conservation requires one heat flux q, positive to the right, and perfect contact requires one interface temperature T_I. Applying Fourier's law to each half gives

$$q = k_L(T_L - T_I)/d_L = k_R(T_I - T_R)/d_R.$$

Solve each expression for its temperature drop, add the two drops and eliminate T_I. The conductance follows from the actual material distances:

$$q = \frac{T_L - T_R}{d_L/k_L + d_R/k_R},$$

$$k_{\text{eff}} = \frac{d_L + d_R}{d_L/k_L + d_R/k_R}.$$

For equal distances this is the harmonic mean; only equal conductivities recover the same result as the arithmetic mean. If one material approaches zero conductivity with positive thickness, the flux approaches zero. A finite thermal contact resistance adds another temperature drop qR_c, while a heat-generating interface alters the flux-continuity condition itself.

DERIVATION AND APPLICATIONS

Enforce the flux jump at a heat-generating interface

Two materials in series carry the same conductive current only if their common interface stores and generates no energy. Consider two homogeneous slabs with an idealized thin heater between them. Let T_L, T_R be prescribed outer surface temperatures, R_L, R_R the positive thermal resistances per unit area in m² K/W, and s the heater release in W/m². Assume perfect thermal contact, steady conduction and negligible thickness of the heater. Define the interface temperature as T_I and flux q positive to the right.

Fourier's law integrated through each slab relates its own constant flux to its temperature drop. The pillbox balance across the heater requires the right flux to exceed the left one by the generated heat:

$$q_L = (T_L - T_I)/R_L,$$
$$q_R = (T_I - T_R)/R_R,$$
$$q_R - q_L = s.$$

Substitute the two face laws into the balance. With $g_j = 1/R_j$ as the two conductances per area, the interface temperature and individual fluxes are

$$T_I = \frac{g_L T_L + g_R T_R + s}{g_L + g_R},$$
$$q_R = g_R(T_I - T_R),$$
$$q_L = g_L(T_L - T_I).$$

Alternatively eliminate T_I to express the fluxes as through-flow plus a split source. The resistances determine the fraction of added heat directed to each boundary:

$$q_L = \frac{T_L - T_R - sR_R}{R_L + R_R},$$
$$q_R = \frac{T_L - T_R + sR_L}{R_L + R_R}.$$

Setting $s = 0$ restores a single common flux. Equal outside temperatures send more generated heat toward the lower resistance. For an interface at a mesh face the two cells therefore need distinct physical face fluxes linked by the stated jump; assigning exactly the same flux while adding no source anywhere violates energy balance.

WORKED EXAMPLE

Determine both heat flows through an instrumented wall

Let $T_L = 320$ K, $T_R = 300$ K, $R_L = 0.100$, $R_R = 0.200$ m² K/W and $s = 150$ W/m². The interface sees conductances 10.0 and 5.00 W/(m² K) and reaches 323.33 K. The left flux is -33.3 W/m² and the right flux 116.7 W/m². Heat leaves on both sides, including toward the hotter external surface. Their oriented difference equals 150 W/m².

SECOND APPLICATION

Select a heater setting that makes one sensor read zero

Keeping the same temperatures and layers, impose $q_L = 0$. Then

$$T_I = T_L = 320\,\mathrm{K},$$
$$s = (T_L - T_R)/R_R = 100\,\mathrm{W/m^2}.$$

All generated heat exits on the right. Reducing s to zero instead gives 66.7 W/m² from left to right through both layers. A zero left-side flux in the heated experiment therefore does not imply an insulating left layer.

Touch two long solids: which temperature does the interface acquire?

Two long blocks, initially at temperatures $T_L > T_R$, are pressed into ideal thermal contact at $x = 0$. Denote conductivity by k, volumetric heat capacity by C, and diffusivity by $\alpha = k/C$. At early times the penetration distance on each side is small compared with its length. Model the blocks as half-spaces with a common interfacial temperature T_*. This transient contact problem has no fixed layer thickness to insert into a pair of steady thermal resistances.

In the right half-space the similarity coordinate is $x/(2\sqrt{\alpha_R t})$. Substituting a function F of this coordinate into the heat equation gives

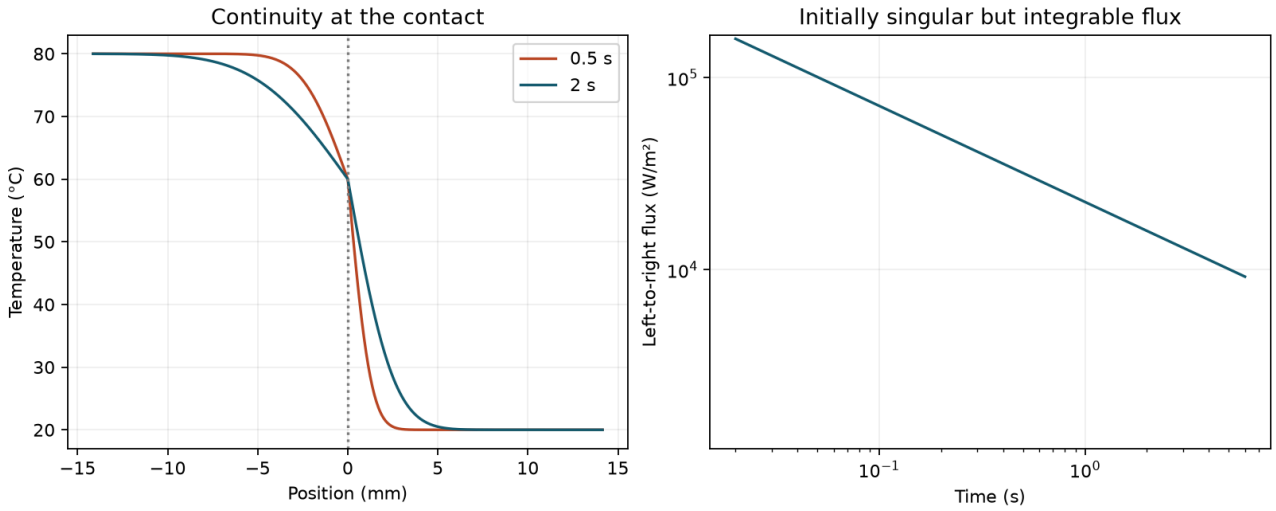
$$F'' + 2\eta F' = 0.$$

Its bounded solution matching a fixed surface and a uniform distant temperature has an error-function shape. Thus $T_R(x,t) = T_* + (T_R - T_*)\,\mathrm{erf}[x/(2\sqrt{\alpha_R t})]$. The left solution uses distance $-x$, its own diffusivity and the initial left temperature. These profiles share their interface temperature by construction; conservation still requires their heat fluxes to match.

Differentiate the error functions at zero and apply Fourier's law. The left flux equals $e_L(T_L - T_*)/\sqrt{\pi t}$, and the right flux is $e_R(T_* - T_R)/\sqrt{\pi t}$, both oriented from left to right. Here $e = \sqrt{kC}$ is thermal effusivity. Equating the fluxes gives

$$T_* = (e_L T_L + e_R T_R)/(e_L + e_R).$$

The interfacial temperature is independent of time within this idealisation, while the penetration layers broaden continuously.



The two temperature gradients differ because flux continuity weights each gradient by its own conductivity. The common temperature depends on heat capacity as well as conductivity.

For the notebook's $k_L = 4,\ k_R = 1\,\mathrm{W/(m\,K)}$ and equal capacities of $10^6\,\mathrm{J/(m^3\,K)}$, the effusivity ratio is two. Between 80 and 20 degrees Celsius the common temperature is 60 degrees Celsius. The initially infinite ideal flux is integrable: the energy transferred by time t equals $2tq(t)$ per unit contact area. The notebook independently integrates temperature loss and gain throughout the two bodies and checks this balance. A real surface resistance, a finite approach time or the far surface of a finite specimen would eventually invalidate different parts of the idealisation.

A finite-volume scheme offers a sensitive check of this benchmark. On a sufficiently long computational domain, sum nodal or cell-average energies on each side at the same time. Internal face fluxes cancel in the total by conservation. A scheme may pass that global check while partitioning energy incorrectly between materials. Compare both individual energy changes, the early flux integrated over the first step and the temperature nearest the interface. This combination tests the material weighting and the time integration independently.

Keep both initial temperatures and conductivities fixed, but multiply the right-hand volumetric heat capacity by four. Predict T_* and the right-hand penetration depth at the same time.

▼ Work through the solution

The right effusivity doubles and now equals the left one. The interface becomes 50 degrees Celsius. At fixed k_R the right diffusivity falls to one quarter, so the penetration distance proportional to $\sqrt{\alpha_R t}$ halves. A larger local capacity can draw more flux while confining the altered temperature to a narrower region.

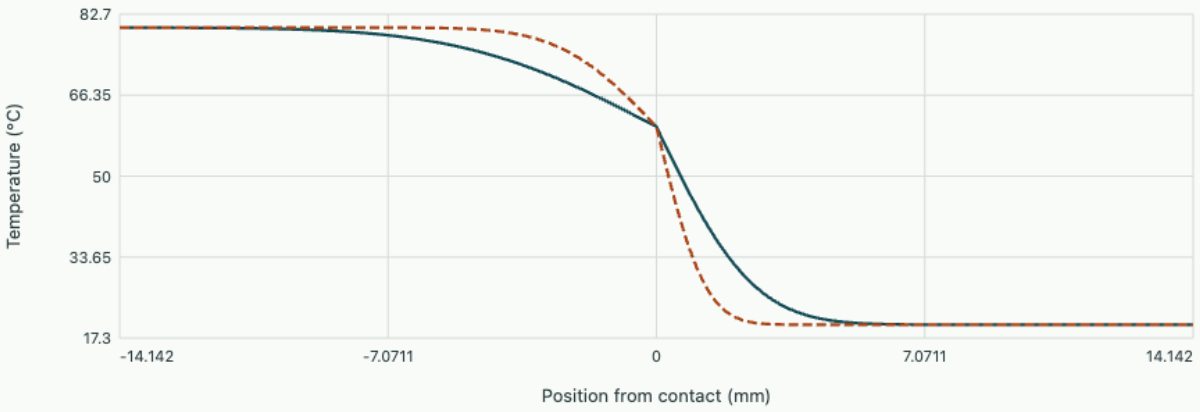
Change parameters · Explore this notebook example

Change one input at a time and compare the recalculated curves with the argument above. The browser evaluates this small model locally; the accompanying numerical values state its assumptions and reference checks. Reset restores the reference inputs, and the data can be downloaded as CSV.

Open this calculation with adjustable inputs

Try the calculation

Left conductivity (W/m/K)	4
Right conductivity (W/m/K)	1
Elapsed time (s)	2



Common interface temperature 60 °C; heat flux 15958 W/m². Both semi-infinite materials have volumetric heat capacity 10⁶ J/m³/K; contact resistance is zero.

Open editable notebook Download .ipynb

One flux through resistances in series

Let $q = -kT'$ be heat flux in the positive coordinate direction, with unit cross-sectional area and no internal source. Between two cell centres separated by an interface, let the left and right centre-to-interface distances be d_L, d_R. Piecewise constant conductivities $k_L, k_R > 0$ make the equilibrium temperature linear in each segment. If the interface has perfect contact, its temperature T_f is continuous. Integrating Fourier's law on each side gives

$$T_L - T_f = q\,d_L/k_L,$$
$$T_f - T_R = q\,d_R/k_R.$$

Adding eliminates the unknown interface temperature.

$$q = \frac{T_L - T_R}{d_L/k_L + d_R/k_R}.$$

For equal distances this amounts to using the harmonic mean conductivity over the full centre spacing. Inserting the same signed q as the outflow of the left cell and the inflow of the right cell produces cancellation when their balances are added. If a material interface stores energy or supplies heat independently, a flux jump belongs in the balance and this steady series calculation must be modified.

Measure the temperature drop

Take $d_L = d_R = 0.01$ m, $k_L = 2,\ k_R = 10$ W/(m K) and $T_L = 30,\ T_R = 18$ °C. The series resistance per unit area is $0.005 + 0.001 = 0.006$ m²K/W. The flux is 2000 W/m² and the interface temperature is $30 - 2000(0.005) = 20$ °C. The lower conductivity carries the larger temperature drop despite carrying the same flux. Using the arithmetic mean 6 with the centre difference over 0.02 m instead predicts 3600 W/m².

Introduce a contact resistance $R_c = 0.002$ m²K/W. Its temperature jump is qR_c; the total resistance rises to 0.008, reducing the heat flux to 1500 W/m². The two sides of the contact now have temperatures 22.5 and 19.5 °C. Imposing a single interface temperature on this model would remove the physically measured 3 K jump.

OneSimulation · Resolve a temperature jump at imperfect contact

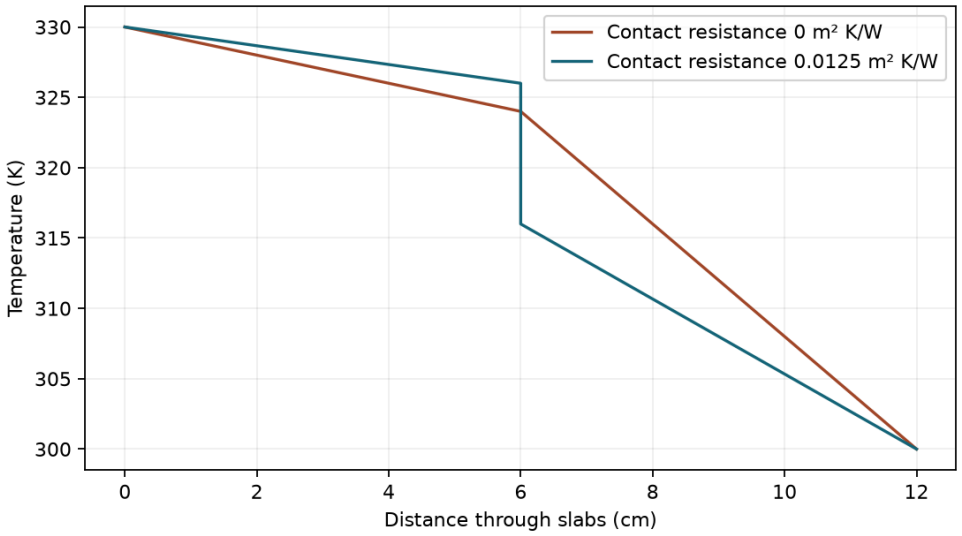
Two slabs are each 0.06 m thick, with conductivities 12 and 3 W/(m K). Their outer surfaces are held 30 K apart. A contact resistance R_c permits two distinct temperatures at the shared physical location. The flux stays continuous. The face law and the exact reference for the whole assembly are

$$T_{f,L} - T_{f,R} = qR_c,$$
$$q = \frac{T_L - T_R}{L_1/k_1 + R_c + L_2/k_2}.$$

The code adds R_c only to the resistance between the two cells adjacent to the contact. OneSimulation solves the resulting balance system. The flux falls from 1200 to 800 W/m² for $R_c = 0.0125$ m² K/W. Reconstructing the two face temperatures gives 326 K and 316 K. Connecting only the cell-centre temperatures would smear this 10 K jump over one mesh cell. Refining an ideal-contact model would converge to the wrong interface law.

```
"""Find the temperature jump at an imperfect contact between two slabs."""
import matplotlib.pyplot as plt
import numpy as np
from onesim.numerics.linear.matrices.dense_linear_algebra import solve_dense

n = 12  # even number of equal, cell-centred control volumes
length = 0.12  # m
dx = length / n
k = np.repeat([12., 3.], n // 2)  # W/(m K)
left, right = 330., 300.  # K
centres = (np.arange(n) + 0.5) * dx
fig, ax = plt.subplots(figsize=(6.8, 3.8), constrained_layout=True)
for contact, colour in [(0.0, "#a4492b"), (0.0125, "#176879")]:  # m² K/W
    resistance = np.r_[dx/(2*k[0]), dx/(2*k[:-1]) + dx/(2*k[1:]), dx/(2*k[-1])]
    resistance[n//2] += contact
    g = 1 / resistance
    matrix = np.diag(g[:-1]+g[1:]) - np.diag(g[1:-1], 1) - np.diag(g[1:-1], -1)
    rhs = np.zeros(n)
    rhs[0], rhs[-1] = g[0]*left, g[-1]*right
    temperature = solve_dense(matrix, rhs)
    fluxes = -np.diff(np.r_[left, temperature, right]) / resistance
    q = fluxes[0]
    assert np.allclose(fluxes, (left-right)/(length/2/k[0]+contact+length/2/k[-1]))
    hot_face = temperature[n//2-1] - q*dx/(2*k[0])
    cold_face = temperature[n//2] + q*dx/(2*k[-1])
    assert np.isclose(hot_face-cold_face, q*contact)
    x = np.r_[0, centres[:n//2], length/2, length/2, centres[n//2:], length]
    y = np.r_[left, temperature[:n//2], hot_face, cold_face, temperature[n//2:], right]
    ax.plot(100*x, y, color=colour, label=f"Contact resistance {contact:g} m² K/W")
    print(f"Flux {q:.0f} W/m²; interface jump {hot_face-cold_face:.1f} K")
ax.set(xlabel="Distance through slabs (cm)", ylabel="Temperature (K)")
ax.legend()
ax.grid(alpha=0.2)
fig.savefig("partial-differential-equations.png", dpi=160)
```



The vertical segment reconstructs both surface temperatures at the same location. All control-volume faces carry the same steady flux, including the imperfect contact.

Download the Python example

Apply a convective boundary

At an exterior face the outward heat flux obeys $q = h_c(T_s - T_\infty)$, a Robin condition. Eliminate the unknown surface temperature as above to obtain

$$q = (T_P - T_\infty)/(d/k + 1/h_c).$$

If h_c tends to zero, the boundary becomes insulating. With very large h_c, the surface approaches ambient temperature while the finite centre-to-face resistance remains. For $d = 0.01$ m, $k = 2$, $h_c = 100$ W/(m² K), $T_P - T_\infty = 15$ K, the predicted flux is 1000 W/m² and the surface is 10 K above ambient. Multiplying by face area converts this flux to the power used in a cell energy balance. Mixing up these two quantities makes the result depend incorrectly on mesh area.

A jump located inside a cell

If the materials change at a known position inside a segment, the fractions of that segment matter. Moving the interface so that 75 percent of the length has conductivity 2 and 25 percent has conductivity 10 produces effective conductivity $1/(0.75/2 + 0.25/10) = 2.5$ W/(m K). Giving each material half the segment instead would give $10/3$ W/(m K). The harmonic rule depends on actual layer thicknesses. A mesh-refinement study that snaps the interface to different positions at every refinement can mix geometric representation error with flux-discretisation error.

Derive effective transport parallel and normal to layers

A fine laminate of isotropic materials can have a direction-dependent macroscopic conductivity. Consider two perfectly bonded planar layers occupying thickness fractions θ and $1 - \theta$, with conductivities k_1, k_2. Assume steady conduction, no internal sources, and boundary conditions that permit a one-dimensional field for the loading orientation being tested. For transport normal to the layers, continuity forces equal flux. Adding their temperature drops across a total thickness L gives

$$k_\perp = [\theta/k_1 + (1 - \theta)/k_2]^{-1}.$$

For transport parallel to the layers the imposed longitudinal gradient is common and the area-weighted fluxes add, giving $k_\parallel = \theta k_1 + (1 - \theta)k_2$.

The arithmetic and harmonic expressions are thus both correct, but apply to different geometries of imposed transport. Their ordering follows without a conductivity table: multiplying the two positive weighted sums $\theta k_1 + (1 - \theta)k_2$ and $\theta/k_1 + (1 - \theta)/k_2$ gives $1 + \theta(1 - \theta)(k_1 - k_2)^2/(k_1 k_2) \geq 1$. Equality requires identical conductivities or a missing layer. Spatial averaging performed before identifying the flux direction would erase precisely the distinction that determines the coarse physics.

Predict two experiments on the same material sample

Suppose a repeated laminate consists of 40 percent material with conductivity 1 and 60 percent with conductivity 10 W/(m K). Then $k_{\parallel} = 0.4 + 6 = 6.4$ W/(m K) and $k_{\perp} = 1/(0.4 + 0.06) = 50/23 \approx 2.174$ W/(m K). A temperature drop of 4 K over 0.02 m imposes gradient magnitude 200 K/m. Perfectly normal loading produces flux magnitude about 434.8 W/m², whereas the same gradient parallel to the layers yields 1280 W/m². The larger flux is consistent with the fast material providing a continuous path in the latter orientation.

For the normal experiment the low-conductivity phase occupies 0.008 m and accounts for a drop $434.8(0.008)/1 \approx 3.478$ K. The high-conductivity phase occupies 0.012 m and drops only $434.8(0.012)/10 \approx 0.522$ K. Adding recovers the imposed four kelvin. For the parallel experiment individual flux densities are 200 and 2000 W/m². Averaging by their area fractions gives

$$0.4(200) + 0.6(2000) = 1280\,\mathrm{W/m^2}.$$

These local checks use the same two material properties and explain the difference in the observed coarse responses.

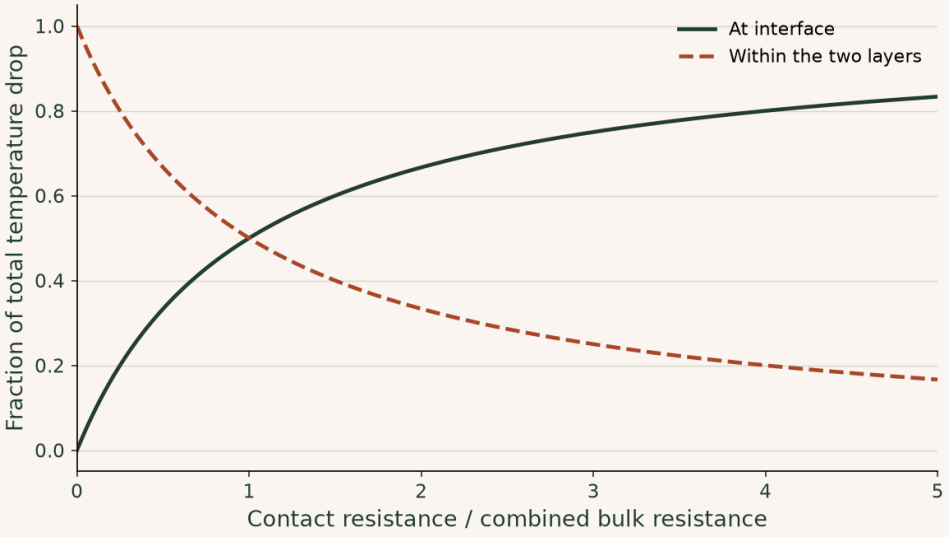
Using the effective tensor assumes the observation scale spans representative layers. A sensor within one layer may respond to its local temperature slope instead. Finite contact resistance adds a temperature jump in the normal direction; if the number of interfaces per macroscopic length changes, the effective normal resistance changes even at identical material fractions. Measuring both orientations can distinguish geometric or contact effects that a single scalar fitted conductivity would conceal.

Rotating a laminate specimen through ninety degrees interchanges the roles of these two effective values without changing its volume fractions. This makes an orientation experiment particularly discriminating: a fit that happened to reproduce the first orientation by adjusting a scalar conductivity predicts the same flux for the rotated specimen, whereas the fine-layer balance predicts a factor of about $6.4/(50/23) = 2.94$ between them.

Repeating the test over multiple layer spacings also probes the assumption of negligible contact resistance.

Separate the temperature jump from the bulk gradient

A steady one-dimensional stack has two positive bulk conduction resistances in series with a contact resistance R_c. Flux is common to all three. Scale R_c by the sum of bulk resistances and predict where probes would measure the imposed end-to-end temperature difference.



When contact resistance equals the bulk sum, half the end-to-end drop occurs at a geometrically thin interface. A numerical scheme that enforces temperature continuity there can converge smoothly to the wrong interface model.

▶ **Read values from the plot**

Change parameters · Separate the temperature jump from the bulk gradient

Change one input at a time and compare the recalculated curves with the argument above. The browser evaluates this small model locally; the accompanying numerical values state its assumptions and reference checks. Reset restores the reference inputs, and the data can be downloaded as CSV.

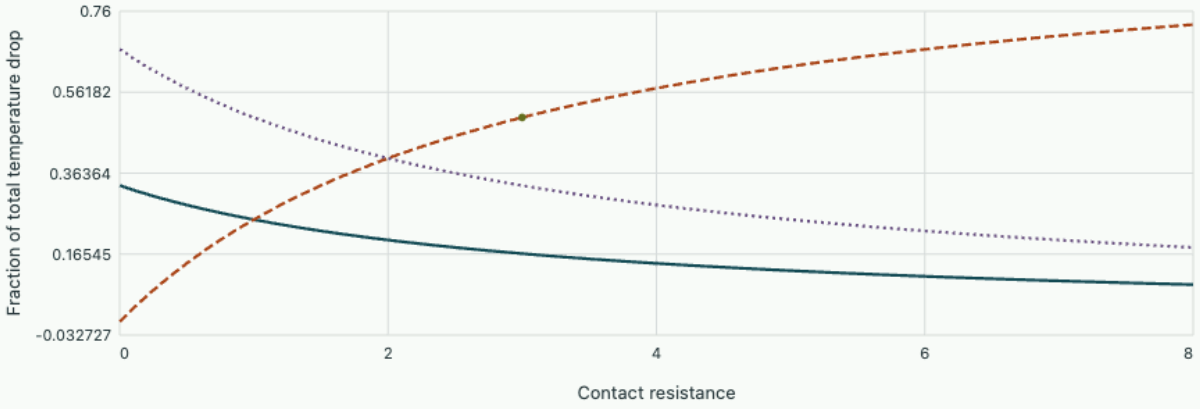
Open this calculation with adjustable inputs

Try the calculation

First bulk resistance **1**

Second bulk resistance **2**

Selected contact resistance **3**



First bulk layer · Contact jump · Second bulk layer · Selected contact fraction

At selected contact resistance, the jump carries 50% of the imposed drop. The three fractions always sum to one.

Cell Péclet number and monotonicity

Numerical partial differential equations

Derive the sign restriction for centred transport, identify a numerical boundary layer and quantify the diffusion introduced by upwinding.

FROM FIRST PRINCIPLES

Derive the Péclet restriction for a centred steady stencil

Take steady one-dimensional transport with constant speed $a > 0$, diffusivity $D > 0$ and no interior source. Conservation requires constant total flux $J = au - Du'$. On a uniform mesh of spacing h, approximate the advective face value by the arithmetic mean and the face gradient by a difference. The right face flux becomes

$$J_{i+1/2} = a(u_i + u_{i+1})/2 - D(u_{i+1} - u_i)/h.$$

Equating the two neighbouring face fluxes and collecting the centre coefficient gives a relation among the three consecutive values:

$$2u_i = (1 + \mathrm{Pe}_h/2)u_{i-1} + (1 - \mathrm{Pe}_h/2)u_{i+1},$$
$$\mathrm{Pe}_h = ah/D.$$

For $|\mathrm{Pe}_h| \le 2$ the centre value is a positive weighted average of its neighbours. Above that threshold one weight becomes negative and the discrete solution can overshoot even when the continuum solution is monotone. This restriction is on the local cell Péclet number, not on aL/D for the whole physical domain. Smaller cells, an upwind flux or an exponentially fitted flux can change the discrete behaviour.

DERIVATION AND APPLICATIONS

Distinguish an imposed inlet flux from an imposed inlet concentration

With diffusion, a concentration measurement alone does not specify the total material entering a channel. Consider steady one-dimensional advection at constant speed $a > 0$ and diffusion coefficient $D > 0$, constant area, no reaction, and coordinate $0 < x < L$. Concentration is $u(x)$. The net species flux is the sum of advection and diffusive backflow. Conservation makes it constant throughout the interval:

$$J = au - Du',$$
$$J' = 0.$$

Suppose the upstream supply delivers a specified total flux $J = au_f$, where u_f is the incoming feed concentration used to normalize the supply rate. Prescribe an outlet concentration $u(L) = u_L$ maintained by a reservoir. Subtract the uniform particular solution u_f. The resulting first-order homogeneous equation for the excess has a convenient outlet-referenced solution:

$$(u - u_f)' = (a/D)(u - u_f),$$
$$u(x) = u_f + (u_L - u_f)e^{-a(L-x)/D}.$$

At the inlet the interior concentration can differ from the feed because diffusion also contributes to the prescribed mass rate. Define $\mathrm{Pe} = aL/D$ and evaluate both terms at $x = 0$:

$$u(0) = u_f + (u_L - u_f)e^{-\mathrm{Pe}},$$
$$-Du'(0) = -a[u(0) - u_f].$$

For large Péclet number the outlet affects only a layer of thickness D/a. For small Péclet number the outlet reservoir can affect the whole channel. Imposing both $u(0) = u_f$ and the specified total flux at the inlet would impose more conditions than this problem needs. A numerical method should record which physical boundary condition it approximates.

WORKED EXAMPLE

Compute a moderate-Péclet-number channel profile

Take $L = 1.00$ m, $a = 0.0200$ m/s, $D = 0.0100$ m²/s, $u_f = 1.00$ and $u_L = 0$ kg/m³. The inlet concentration is $1 - e^{-2} = 0.8647$ kg/m³ and the midpoint is $1 - e^{-1} = 0.6321$ kg/m³. Inlet advection carries 0.01729 kg/(m² s) and diffusion supplies 0.00271 kg/(m² s), giving the prescribed total 0.0200.

SECOND APPLICATION

Quantify a boundary-condition error

If a code instead fixes $u(0) = 1.00$ and $u(L) = 0$, its exact constant flux for those Dirichlet data satisfies

$$u(0) = (J/a)(1 - e^{-\mathrm{Pe}}).$$

Hence $J = 0.0200/(1 - e^{-2}) = 0.0231$ kg/(m² s), about 15.7% too large for the original feed problem. Grid refinement can converge cleanly to that incorrect-for-purpose value because the error lies in the boundary model.

Which length sets the balance?

Model a steady scalar on $[0, L]$ with $au' - \kappa u'' = 0$, constant speed $a > 0$ and diffusivity $\kappa > 0$. Advection and diffusion balance over length $\ell = \kappa/a$. On a mesh with spacing h, the cell Péclet number

$P = ah/\kappa$ measures how many layer lengths fit into one cell. This definition uses the full cell spacing; some conventions divide it by two. Stating the convention matters when quoting a threshold.

Write the centred interior equation in flux form. Its coefficients multiplying u_{j-1}, u_j, u_{j+1} are $-\kappa/h^2 - a/(2h), 2\kappa/h^2, -\kappa/h^2 + a/(2h)$. For $P \le 2$, the neighbour coefficients are nonpositive and the row sum is zero, so each interior value is a weighted average of its neighbours. For $P > 2$, one weight changes sign. The discrete maximum principle can fail even though the centred derivative is formally second-order accurate on smooth resolved fields.

Watch a node cross below the boundary range

Take $L = 1, a = 1, \kappa = 0.1$, $u(0) = 0, u(1) = 1$ and two equal cells. Then

$$h = 0.5, P = 5.$$

At the only interior node the coefficients are $-1.4, 0.8, 0.6$. The numerical value is -0.75, outside the prescribed boundary range. The exact solution is $(e^{ax/\kappa} - 1)/(e^{aL/\kappa} - 1)$, increasing and close to zero except near the outlet. At $x = 0.5$ it is $1/(e^5 + 1) \approx 0.00669$. The coarse grid does not resolve a physical layer of width 0.1.

Using the upstream difference $(u_j - u_{j-1})/h$ changes the coefficients to $-2.4, 2.8, -0.4$; the computed midpoint is $1/7$. The value lies in range but the layer is excessively broad. A monotone method can replace an oscillation with a large smoothing error. Plot the magnitude of the profile at fixed physical locations while refining to distinguish those outcomes.

Explain the extra diffusion

Expand $u_{j-1} = u_j - hu'_j + h^2 u''_j/2 + \cdots$. The upstream derivative is $u'_j - hu''_j/2 + O(h^2)$. The leading modified equation therefore contains effective diffusivity $\kappa + ah/2$. In the example the artificial addition is 0.25, larger than the physical diffusivity 0.1. Reducing h below 0.2 restores the coefficient-sign requirement for the centred formula, while resolving the layer shape accurately generally needs several cells across ℓ. A convergence study should report both bounds compliance and the actual layer width or flux.

Fit the face flux to a constant-coefficient cell

For smooth constant coefficients the total flux $J = au - \kappa u'$ is constant in a source-free cell. Solve this first-order equation between endpoints separated by h. The exact relation is $u_R = e^P u_L + (J/a)(1 - e^P)$, hence $J = a(e^P u_L - u_R)/(e^P - 1)$. At large positive P, the flux approaches au_L, the value carried from upstream. As P tends to zero, expansion gives a diffusive difference $-\kappa(u_R - u_L)/h$ plus the centred advective contribution $a(u_L + u_R)/2$. This exponential fitting uses additional local knowledge of the differential equation. It reproduces the constant-coefficient cell relation exactly, but variable material properties and sources require new local approximations.

Allocate grid points to a singularly perturbed outflow layer

For steady positive advection and small diffusivity the outflow layer shape is approximately $e^{-(L-x)/\ell}$, with physical thickness scale $\ell = \kappa/a$. A uniform grid that resolves that scale at a fixed number of points may waste most of its cells where the solution is almost flat. A simple layer-adapted strategy divides the domain into a coarse interval and a fine interval ending at the outflow boundary. Choose the fine-region width τ by specifying how small the layer should already be at their interface.

If $e^{-\tau/\ell} = N^{-2}$ for a grid with N total cells, then $\tau = 2\ell \log N$. Capping this width at $L/2$ prevents a separate-layer construction when the physical layer is no longer thin. Place half the cells on each subinterval. This produces a piecewise-uniform grid, often called a Shishkin mesh for this boundary-layer setting. The amplitude threshold motivates its construction; it is not itself a proof of a particular numerical convergence rate. Such a proof must also control truncation, stability and matching across the grid transition.

Compare the information captured by sixty-four cells

Consider $L = 1$ m, $a = 1$ m/s and $\kappa = 0.002$ m²/s. Then $\ell = 0.002$ m. A uniform mesh of 64 cells has spacing 0.015625 m, almost eight physical layer lengths per cell. The first interior point back from the outflow has layer amplitude about $e^{-7.8125} \approx 0.000405$. Most of the transition from nearly zero to one takes place in a single cell. A method may fit an acceptable boundary value while giving a poor near-wall shape or slope.

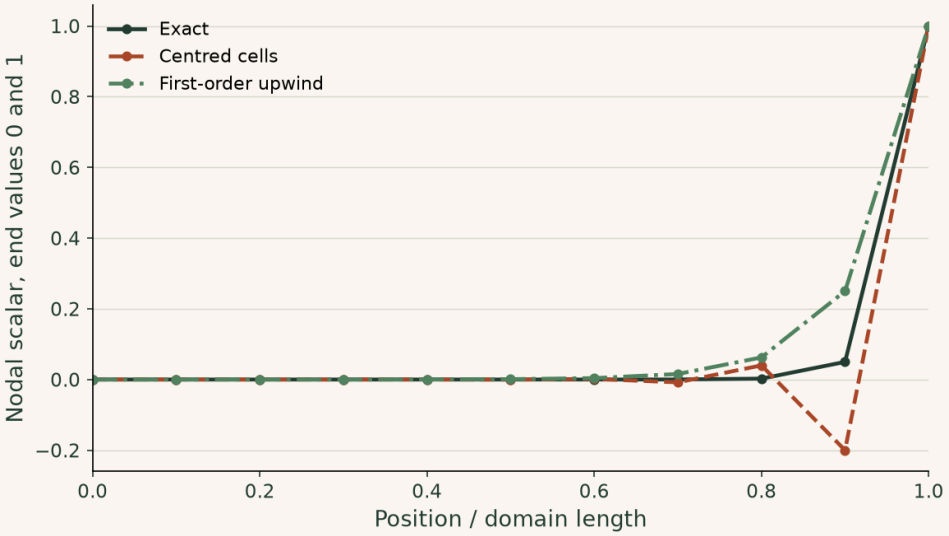
The two-region design instead places its transition $\tau = 0.004 \log 64 \approx 0.01664$ m before the end. With 32 fine cells, their width is approximately 0.000520 m, so the fine cell Péclet number is about 0.260. At the last interior point the layer amplitude is now $e^{-0.260} \approx 0.771$. Several distinct intermediate values are available near the wall. At the coarse-to-fine transition the amplitude is $64^{-2} \approx 0.000244$, just as designed.

The coarse cells on the remaining 0.98336 m have width about 0.03073 m and cell Péclet number about 15.4. Their small expected layer amplitude does not license an unstable or nonmonotone coarse discretisation. Use a spatial scheme whose behaviour is controlled there, and check flux continuity through the changing cell sizes. If advection reverses, the thin layer moves to the opposite boundary; retaining the old mesh would concentrate cells in the wrong region. In time-dependent transport, moving layers may similarly call for adaptive tracking instead of a fixed partition.

The wall derivative supplies a more demanding check than a prescribed boundary value. For the exponential layer its exact slope at outflow has magnitude $1/\ell = 500$ per metre. A backward difference across the final fine cell gives approximately $(1 - 0.771)/0.000520 = 440$ per metre. Even this substantially improved mesh still underestimates the wall slope by about twelve percent, which matters if wall flux is the reported observable.

Expose a monotonicity failure on ten cells

For constant positive advection and diffusion solve $u'' - \mathrm{Pe}\, u' = 0$ with end values zero and one. At cell Péclet number three, the centred recurrence has ratio $(1 + \mathrm{Pe}_h/2)/(1 - \mathrm{Pe}_h/2) = -5$; the upwind ratio is four.



The centred solution overshoots the allowed interval by oscillating below zero near the outflow. Upwinding gives a monotone but overly broad layer. Neither the small algebraic residual nor a monotone curve alone establishes spatial accuracy.

▶ **Read values from the plot**

Change parameters · Expose a monotonicity failure on ten cells

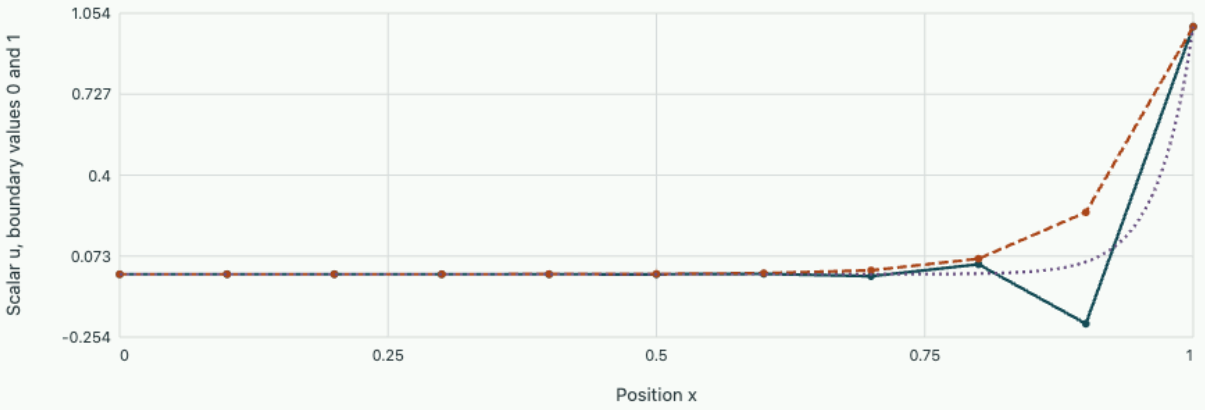
Change one input at a time and compare the recalculated curves with the argument above. The browser evaluates this small model locally; the accompanying numerical values state its assumptions and reference checks. Reset restores the reference inputs, and the data can be downloaded as CSV.

Open this calculation with adjustable inputs

Try the calculation

Cell Péclet number **3**

Mesh cells **10**



Total Péclet number 30. The centred minimum is -0.2. The monotonicity threshold is cell Pe = 2.

What a periodic grid does to a decaying Fourier mode

Numerical partial differential equations

A periodic diffusion solver predicts the shape of a sampled wave while its amplitude decays too slowly. Fourier analysis separates spatial differentiation error, an unresolved physical mode and errors caused by the time integrator.

Diagonalise the differential operator on the periodic interval

Use a dimensionless coordinate $0 \le x < 2\pi$ and solve $u_t = \kappa u_{xx}$ with periodic values and periodic derivatives. The complex exponentials e^{imx}, with integer m, satisfy both conditions and are eigenfunctions of the second derivative. Put $u(x,t) = A_m(t)e^{imx}$ and cancel the common exponential. The scalar amplitude equation is

$$A_m' = -\kappa m^2 A_m.$$

Thus every cosine or sine mode decays by $\exp(-\kappa m^2 t)$, with no phase shift, while the m=0 mean is conserved.

On a uniform grid $x_j = 2\pi j/N$, the samples of m and $m + \ell N$ agree for every integer ℓ. A discrete Fourier transform returns coefficients in one representative band. For the real cosine test with even N we can fold a positive physical mode into $k = \min(r, N - r)$, $r = m \bmod N$. A Fourier spectral Laplacian multiplies that coefficient by $-k^2$. For a resolved mode it differentiates the trigonometric interpolant exactly, up to arithmetic. For unresolved data it differentiates the aliased representative; no operation using only those samples can identify the original m.

Apply the centred difference stencil to a single wave

The standard second difference is $(u_{j+1} - 2u_j + u_{j-1})/h^2$, $h = 2\pi/N$. Put the grid wave $u_j = e^{ikjh}$ into the stencil. Factor out the common phase and simplify with $e^{i\xi} + e^{-i\xi} = 2\cos\xi$. This gives the symbol and its small-h expansion:

$$\frac{e^{ikh} - 2 + e^{-ikh}}{h^2} = -\frac{4}{h^2}\sin^2(kh/2) = -k^2 + \frac{k^4 h^2}{12} + O(k^6 h^4).$$

The effective squared wavenumber is smaller than k^2 for every nonconstant mode in the resolved band, so this spatial approximation decays too slowly if the time equation is integrated exactly. The error is modest when $kh \ll 1$, but at the Nyquist mode $kh = \pi$ the ratio is

$$4/\pi^2 = 0.4053.$$

A stable time integrator cannot repair this spatial attenuation error without additional modeling. The experiment exponentiates both discretizations' symbols exactly, isolating space and sampling here.

Measure separate errors with the same sampled initial data

Choose $\kappa = 0.5,\ m = 3,\ N = 16$. The true squared wavenumber is 9 and the difference stencil gives 8.006. Starting at unit amplitude, at $t = 0.20$ exact and spectral amplitudes are 0.40657; finite differences give 0.44906. Doubling N to 32 changes the difference result to 0.41716. The absolute amplitude error falls from 0.04249 to 0.01059, a ratio of 4.01 consistent with the second-order expansion. This is a spatial-convergence study because time evolution introduces no truncation error.

Now take $m = 12,\ N = 16$. Every sampled cosine equals the low mode $k = 4$ at $t = 0$. At $t = 0.02$, the continuum amplitude for the true physical mode is $e^{-1.44} = 0.237$, whereas the spectral grid predicts

$$e^{-0.16} = 0.852.$$

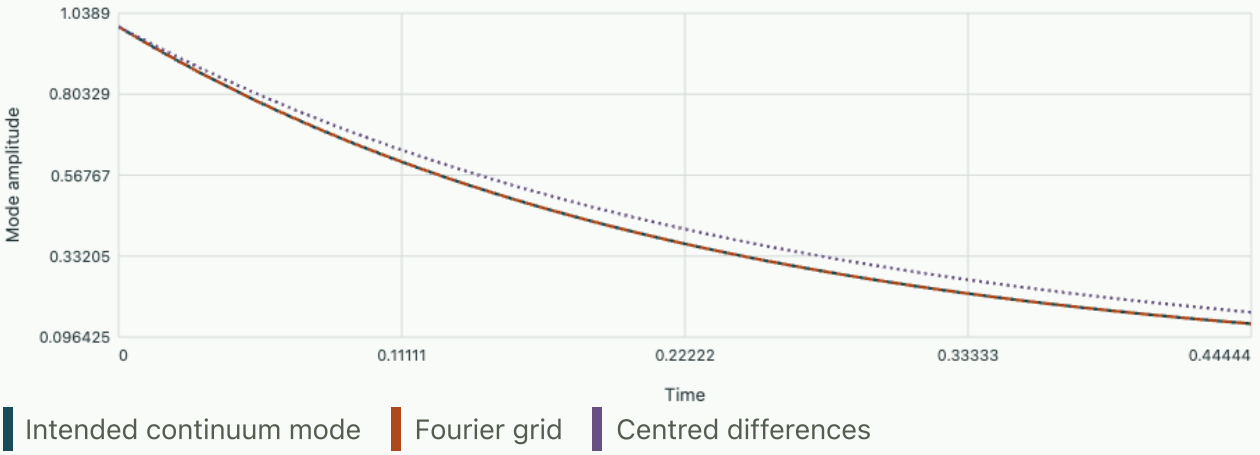
A spectral computation can therefore agree exactly with its trigonometric interpolant and disagree strongly with the evolution of the intended signal. Sampling on a finer grid or obtaining prior bandwidth information resolves that ambiguity. The mean mode is particularly revealing when a nonconstant physical cosine aliases to a constant.

Start at mode 3 and successively double N from 8 to 32. Compute two amplitude-error ratios at the same time. Then use mode 12 on 16 points to observe an alias, and increase N until the Fourier grid represents the intended mode.

Open this experiment online

Try the calculation

Periodic grid points	16
Physical cosine mode	3
Diffusivity	0.5



The grid represents mode 3 as 3. At t = 0.20, amplitudes are 0.40657 for the physical mode, 0.40657 for the Fourier grid and 0.44906 for the difference stencil.

Tutorial · resolve a damping rate to a target tolerance

For a mode well inside the grid bandwidth, estimate the number of samples per wavelength needed for a relative error below 1% in the decay rate with centred differences. Test the estimate for $m = 2$. How could the time integrator contaminate a measurement of this spatial error?

▼ **Show the resolution estimate**

Writing p for samples per wavelength gives

$$kh = 2\pi/p.$$

The relative rate error is approximately $(kh)^2/12$. Requiring at most 0.01 gives $p \gtrsim 18.14$; try 19 samples per wavelength, or $N = 38$ for $m = 2$. The exact ratio $[\sin(\pi/19)/(\pi/19)]^2$ is about 0.991. A forward-Euler step would replace exponential propagation by $1 - \kappa k_{\mathrm{eff}}^2 \Delta t$. Its repeated-step error could partly mask the underdamping of the spatial stencil. Compare with the exponential of the discrete symbol before attributing a measured rate to space alone.

Further reading · Lloyd N. Trefethen, Spectral Methods in MATLAB · periodic grids, Fourier differentiation and downloadable programs

Check the discrete eigenvalue with OneSimulation

A spatial sine also diagonalises the five-point Dirichlet Laplacian on a square. Use the same modified wave-number argument in both coordinate directions. Run `python poisson-mode.py --mode 2 --cells 8 16 32` and compare the observed nodal error with the discrete eigenvalue prediction in the CSV. This is a steady forced problem whose spatial operator can be checked independently of any time integrator. Download the OneSimulation verification model.

Moving a cell average without inventing or losing material

Numerical partial differential equations

Derive a conservative transport step from cell overlap and measure how a monotone scheme spreads a pulse despite conserving its total mass exactly.

Transport is a translation

For constant $a > 0$, the solution to $u_t + a u_x = 0$ is $u(x,t) = u_0(x - at)$. Consider periodic cells of width h. Over one step Δt, a cell translated back through distance $a\Delta t$ overlaps part of its original cell and part of its left neighbour. If the initial reconstruction is constant in each cell and $C = a\Delta t/h$ lies between zero and one, the new average is

$$u_j^{n+1} = (1 - C)u_j^n + C u_{j-1}^n.$$

Summing all cells telescopes on a periodic grid, preserving total material $h\sum_j u_j$. Each new value is a convex combination, so the maximum cannot increase and the minimum cannot fall. For $C > 1$ this particular two-cell overlap ceases to describe the translated interval.

Derive the spreading instead of reading it off a plot

Start from one unit of material in one cell and follow its distribution on an effectively unbounded grid. After one step its index increment behaves like a Bernoulli choice: zero with weight $1 - C$, one with weight C. Repeated convolution gives a binomial profile. Its mean index after n steps is nC and variance $nC(1 - C)$. Multiplying by h^2 and using $n = t/\Delta t$ gives a physical variance $ah(1 - C)t$. Equating this with $2D_{\mathrm{num}}t$ gives

$$D_{\mathrm{num}} = ah(1 - C)/2.$$

This growth describes numerical spreading of a translated shape even though the scheme conserves mass.

Two ways to reach the same physical time

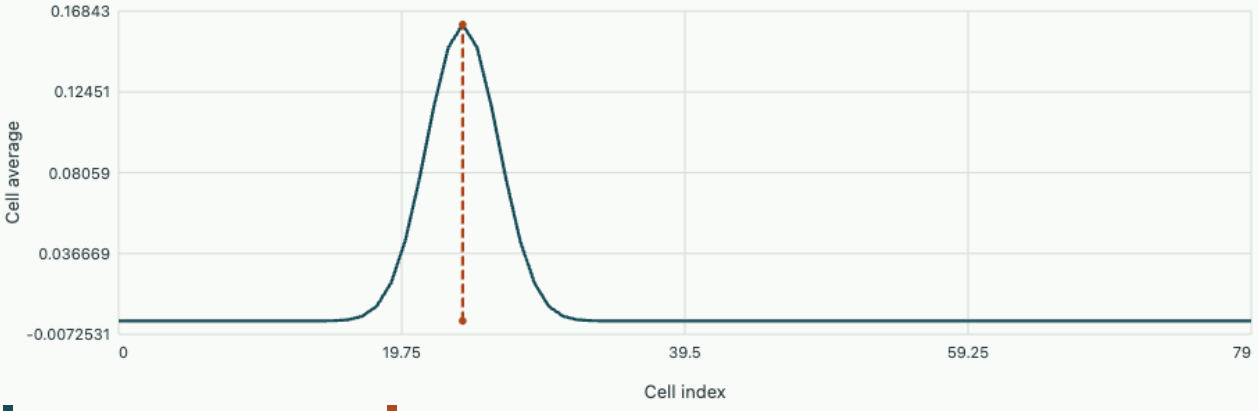
A pulse should move 12 cells. Taking 12 steps with $C = 1$ shifts data by exactly 12 indices. Taking 24 steps with $C = 0.5$ has the same mean displacement but an added variance of six cell widths squared. In a source-free transport test, peak height and width reveal an error hidden from the total-mass budget.

Open this experiment online

Try the calculation

Courant number C **0.5**

Number of steps **24**



Total cell sum 1, centroid 24, added variance 6 cells².

The experiment starts with one full cell on a ring of 80 cells. Up to 40 steps avoid wraparound across the displayed pulse for the chosen starting index. The blackboard binomial calculation predicts centroid and spread independently of the plotted recurrence.

Tutorial on interpreting artificial diffusion

If $C = 0.25$ and the pulse has moved ten cells on average, what is its added variance?

▼ **Worked solution**

Mean travel requires $n = 40$ steps. Variance in cell units is $40(0.25)(0.75) = 7.5$. Multiplying by h^2 gives the physical variance.

Is a larger time step always the cure for pulse spreading?

▼ **Worked solution**

For constant transport on this uniform grid $C = 1$ happens to align the translation with cell boundaries. Multiple speeds, nonuniform meshes and source physics generally preclude such alignment everywhere. A higher-order bounded reconstruction can reduce spreading while preserving conservation.

Read the university treatment

Use the advection examples to compare convergence in a norm with conservation of an integral and propagation of an extremum. Randall LeVeque · Finite Difference Methods for Ordinary and Partial Differential Equations

Splitting a stiff relaxation from a slowly varying drive

Numerical ordinary differential equations

A sensor tries to follow a changing reference while a fast relaxation damps departures. An explicit method may oscillate wildly even though the underlying transient decays monotonically. Treating the restoring term implicitly can resolve the slow motion using steps longer than the transient time. Whether those steps measure the transient accurately is a different question.

Construct an exact benchmark with two time scales

Let $s(t) = \sin t$ and choose a relaxation time $\varepsilon > 0$. Consider $y' = s'(t) + (s(t) - y)/\varepsilon$, with $y(0) = 1$. Set $e = y - s$. Subtraction of s' gives $e' = -e/\varepsilon$, hence the reference solution $y(t) = \sin t + e^{-t/\varepsilon}$. The slow reference changes over times of order one, while the initial mismatch decays significantly over ε. We know both pieces exactly and can diagnose errors even when a stable result looks visually smooth.

Explicit Euler applied to a perturbation of the state multiplies that perturbation by $1 - r$, with $r = h/\varepsilon$. It damps monotonically only for $0 < r \le 1$, changes sign while damping for $1 < r < 2$, and grows for $r > 2$. These statements compare nearby solutions at equal drive. The full signal also contains the time variation of s. At the boundary $r = 2$ a state perturbation alternates without decaying. Refining the step to follow a slow input alone can therefore give the wrong choice of h.

Solve only the fast part implicitly

Split $f_{\mathrm{slow}}(t) = \cos t$ from $f_{\mathrm{fast}}(t, y) = (\sin t - y)/\varepsilon$. Evaluate the slow part at t_n and the fast part at t_{n+1}, y_{n+1}. This first-order additive implicit-explicit Euler step needs one scalar linear solve:

$$(1 + r)y_{n+1} = y_n + h\cos t_n + r\sin t_{n+1}.$$

The state perturbation now multiplies by $1/(1 + r)$. It is damped at every positive step and tends to zero as r grows. Subtract $\sin t_{n+1}$ from both sides after solving. The mismatch obeys $e_{n+1} = [e_n + h\cos t_n - (\sin t_{n+1} - \sin t_n)]/(1 + r)$. The bracketed forcing defect is $O(h^2)$ for smooth input. In the stiff limit at fixed h, the endpoint approaches the current reference. Moving the reference in the fast term to the old time would instead drag it one sample behind in that limit. The choice of what to treat implicitly includes the time at which a fast equilibrium is imposed.

Test early-time accuracy and late-time stability

For $\varepsilon = 0.02$, $h = 0.05$, $r = 2.5$. At the first step the exact value is

$$\sin(0.05) + e^{-2.5} = 0.1321.$$

Explicit Euler gives

$$1 + 0.05(1 - 50) = -1.45.$$

The implicit-explicit value is

$$[1 + 0.05 + 2.5\sin(0.05)]/3.5 = 0.3357.$$

The latter is stable but visibly overestimates the surviving transient. Its damping factor $1/3.5 = 0.286$ differs substantially from $e^{-2.5} = 0.0821$. A question about peak rate or initial settling requires shorter steps even if the late-time reference is recovered.

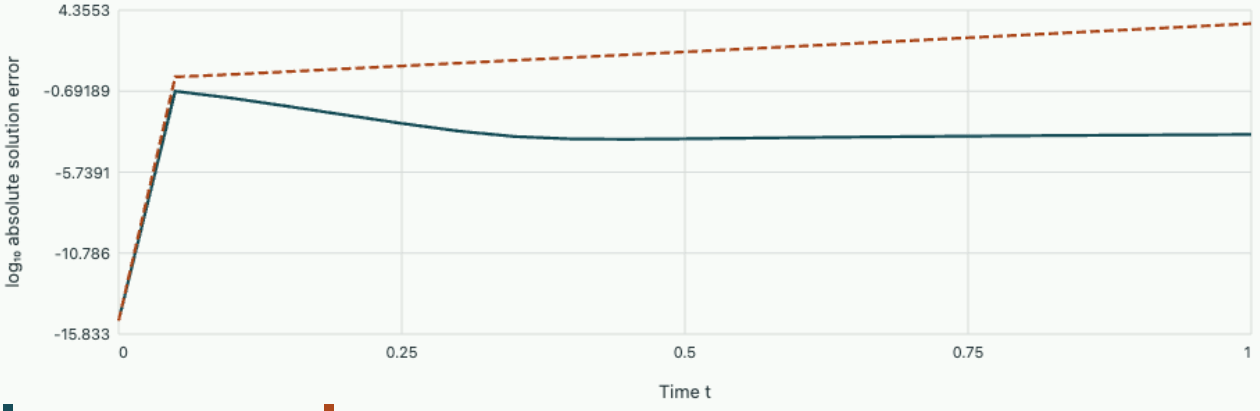
The experiment holds the final time at one while the number of steps changes. The vertical coordinate is the base-ten logarithm of absolute error, with values below 10^{-15} drawn at that floor. Negative heights therefore correspond to errors less than one. Track the maximum error separately from the final error as the transient dies out. Try an initially unstable explicit run, then double the number of steps. The stiffness criterion depends on the ratio h/ε, while accuracy on the reference also depends on h itself.

Open this experiment online

Try the calculation

Relaxation time ε **0.02**

Steps to final time 1 **20**



h/ε = 2.5. Explicit perturbation factor -1.5, implicit factor 0.2857. IMEX maximum error 0.2036, final error 0.0004054.

Tutorial on identifying which component is stiff

Keep $h = 0.04$ and reduce ε from 0.04 to 0.004. For each case compare exact, explicit and implicit amplification of a state perturbation. What quantity would force a reduction of h even with the implicit-explicit method?

▼ **Work through the solution**

At $r = 1$ the three factors are $e^{-1} = 0.368$, zero and $1/2$. At $r = 10$ they are 4.54×10^{-5}, -9 and $1/11 = 0.0909$. For a measurement of the initial decay rate the implicit discretisation substantially misrepresents both exact factors. Bounding error during that transient requires smaller steps relative to ε, even though its numerical amplification is stable.

Further reading · Ascher, Ruuth and Wetton, implicit-explicit time discretisations.

Transport coefficients from equilibrium fluctuations

Statistical mechanics from microscopic counting to material response

How can a fluid in equilibrium reveal the diffusion constant that would govern a spreading drop, or the viscosity that would resist an applied shear? Derive an exact connection between displacements and velocity correlations, generalise it to stress and heat currents and test how a finite record can bias each number. The assumptions behind convergence matter as much as the prefactor.

Integrate the velocity correlation before taking a long-time limit

Observe one Cartesian coordinate of a tagged particle in a homogeneous equilibrium fluid with zero mean drift. The displacement over time t is the integral of its velocity. Square that integral, average over the equilibrium ensemble and use stationarity to replace the two-time expectation by $C_v(t_1 - t_2)$. For a scalar component in a time-reversal-symmetric equilibrium state this correlation is even. Integrate over the two triangles of the time square:

$$\langle [x(t) - x(0)]^2 \rangle = 2 \int_0^t (t - s) C_v(s)\, ds,$$

$$\frac{1}{2} \frac{d}{dt} \langle \Delta x^2(t) \rangle = \int_0^t C_v(s)\, ds.$$

If the correlation is integrable and boundary effects remain absent, the long-time slope yields a diffusion coefficient $D = \int_0^\infty C_v(s) ds$. In d isotropic dimensions one may use $D = (1/d) \int \langle \mathbf{v}(s) \cdot \mathbf{v}(0) \rangle ds$ or the mean-square-displacement slope divided by $2d$. Using the full vector dot product without division by dimension gives an overestimate. Directional diffusion in an anisotropic material requires tensor components rather than a forced scalar average.

The derivation is kinematic and does not assume a Markovian motion. A fluid with velocity memory or an oscillating solid still obeys the finite-time relation. Those systems need not exhibit the same long-time proportionality to time. A bound oscillator can have persistent or alternating velocity correlations and a bounded displacement. An observation window in which the mean square grows over a fraction of an oscillation is insufficient to claim diffusion.

A solvable benchmark from ballistic to diffusive motion

For one coordinate take the underdamped Langevin equation $m\dot{v} = -\zeta v + \xi$, with zero-mean white noise satisfying $\langle \xi(t) \xi(t') \rangle = 2 k_B T \zeta\, \delta(t - t')$. In equilibrium $\langle v^2 \rangle = k_B T / m$. Condition on $v(0)$ and integrate for positive time. The later random force has no correlation with the initial velocity, giving $C_v = (k_B T / m) e^{-t/\tau_v}$, where $\tau_v = m/\zeta$. Its area and the resulting displacement are

$$D = \frac{k_B T}{\zeta},$$

$$\langle \Delta x^2(t) \rangle = 2D[t - \tau_v (1 - e^{-t/\tau_v})].$$

At $t \ll \tau_v$, the constant and linear terms inside the brackets cancel and displacement is ballistic:

$$\langle \Delta x^2 \rangle \simeq (k_B T/m) t^2.$$

At $t \gg \tau_v$ the slope approaches $2D$. Fitting a line through the origin at times only a few relaxation times overestimates the influence of the negative intercept in this long-time expression and underestimates D. Comparisons at multiple lags distinguish crossover curvature from a true anomalous power law.

For an inert sphere of radius 0.50 micrometres and density 2000 kg per cubic metre in water with $\eta = 1.00$ mPa s at 298 K, its bare mass is 1.05×10^{-15} kg and Stokes drag 9.42×10^{-9} kg/s. The estimate $\tau_v = 0.111$ microseconds is far faster than ordinary video tracking. $D = 0.437$ square micrometres per second and the one-coordinate rms displacement over 10 ms is 0.0935 micrometres. Hydrodynamic added mass and memory affect very short-time behaviour; the simple exponential is an analytical reference, not a claim that a water suspension remains Markovian down to arbitrary times.

Relate a weak imposed force to equilibrium motion

In the same soluble model a small constant force F turns the mean velocity into $F(1 - e^{-t/\tau_v})/\zeta$, so the mobility $b = \lim \langle v \rangle / F = 1/\zeta$. Together with equilibrium diffusion this gives the Einstein relation $D = k_B T b$. A broader equilibrium argument considers a dilute suspension in a weak conservative potential $U(x)$. Its current is

$$j = -D\,\partial_x n + bFn,$$

with $F = -\partial_x U$. The Boltzmann distribution $n \propto e^{-U/k_B T}$ must carry zero current; substitution gives the same relation without assuming exponential velocity relaxation.

Linear response connects the full weak-force impulse response to an equilibrium correlation. For a perturbation coupled as $-F(t)x$, classical equilibrium response of velocity has causal kernel $C_v(t)/(k_B T)$ for $t > 0$ under the usual microreversibility assumptions. Integrating the response kernel for a constant weak force gives

$$b = (1/k_B T) \int_0^\infty C_v(t)\,dt,$$

which recovers the kinematic diffusion relation. Strong forcing, an active bath or a nonequilibrium steady state need not obey this simple equilibrium identity.

A direct check compares independently measured weak sedimentation or calibrated drift with freely diffusing trajectories. For the previous bead a force 10 fN would produce a mean drift about 1.06 micrometres per second in the ideal Stokes regime. During a 10 ms frame its mean drift is about 0.011 micrometres, much smaller than rms Brownian motion, so many frames or many particles are needed. A single short jagged record can appear to move against the force. Use equal positive and negative applied forces to cancel constant stage drift in the estimated mobility.

Work through diffusion and mobility in the random-walk chapter

Stress and heat current carry their own normalisations

In a homogeneous isotropic laboratory fluid, define P_{xy} as a spatially averaged off-diagonal microscopic stress, with units Pa. Its equilibrium mean is zero in an isotropic state without imposed shear. A weak velocity gradient couples to momentum transport. The Green–Kubo expression for dynamic shear viscosity is

$$\eta = \frac{V}{k_B T} \int_0^\infty \langle P_{xy}(t) P_{xy}(0) \rangle \, dt.$$

If a program outputs an extensive stress moment $J_{xy} = V P_{xy}$, the prefactor is $1/(V k_B T)$. Multiplying an already extensive current by volume again is a common orders-of-magnitude error. Check dimensions: pressure squared times time times volume divided by energy gives pascal seconds. Kinetic and interaction contributions to microscopic stress must both be included according to the model; a calculation with ideal kinetic flux alone need not reproduce the viscosity of a dense fluid.

Similarly for a component of total heat current $J_{Q,x}$ with units energy times length per time, the thermal conductivity in an isotropic equilibrium single-component system is

$$\kappa = \frac{1}{V k_B T^2} \int_0^\infty \langle J_{Q,x}(t) J_{Q,x}(0) \rangle \, dt.$$

Here $k_B T^2$ means k_B times T^2. If a heat-flux density is used instead, the volume prefactor moves as for stress. Heat current must exclude energy carried by a net mass flow as appropriate to the chosen thermodynamic constraint. In mixtures several conserved species currents can couple, and the measured conductivity at zero mass flux can differ from one unprojected current coefficient. The meaning of the measured flux and the imposed condition must be specified before comparing coefficients.

For an illustrative volume 10^{-24} cubic metres at 300 K, assume one stress autocorrelation $C_P(t) = (2.0 \times 10^6 \ \mathrm{Pa})^2 e^{-t/(2.0 \ \mathrm{ps})}$. Its area is 8.0 square pascal seconds. Multiplication by $V/(k_B T)$ gives viscosity 1.93×10^{-3} Pa s. Doubling the simulated volume of an otherwise similar homogeneous fluid generally reduces the variance of its volume-averaged stress, keeping the physical transport coefficient intensive. Holding the same numerical stress variance while merely changing the prefactor would describe a different equilibrium fluctuation model.

Find a correlation integral's useful time window

An estimate at lag s averages over pairs of samples separated by that lag. In a record of duration T_{rec}, the number of available pairs decreases with lag. Overlapping time origins strongly correlate the estimates, so plotting one error bar per raw pair exaggerates precision. Integrating an estimated correlation to the maximum possible lag frequently accumulates late random noise rather than useful signal. Conversely stopping too early omits a physical tail.

For a positive single-exponential reference, integrating to three correlation times retains $1 - e^{-3} = 95.0$ percent of the area. Five times retain 99.3 percent. A correlation containing a small, much slower

component can look essentially flat on the main decay scale while contributing appreciably to the area. Divide a long run into blocks, compare running integrals and check several window lengths. Report the variation between independent blocks and any explicit tail model used beyond the measured interval.

At short lags, finite exposure averages correlated positions or velocities. Uncorrelated localisation errors in position increase nonzero-lag displacement variance by a constant $2s_x^2$ per coordinate. Numerically differentiating those same noisy positions creates large artificial velocity variance and negative correlations between adjacent increments. An apparent negative velocity lobe can therefore be physical backscattering or a measurement artefact. Comparing a raw displacement fit with the velocity integral after independently characterising detector noise is more informative than differentiating first and interpreting every feature as fluid memory.

A forced shear wave gives a second viscosity measurement

An equilibrium stress correlation estimates viscosity without imposing any flow. A separate driven measurement can test that coefficient against a directly observed penetration depth. Take a Newtonian fluid filling $y > 0$, driven by a laterally oscillating plate at $y = 0$ with speed $U_0 \cos \omega t$. Neglect boundaries at large y, compressibility and nonlinear convective effects in this spatially uniform tangential shear. The momentum equation reduces to

$$\partial_t u = \nu \partial_y^2 u,$$

with kinematic viscosity $\nu = \eta/\rho$.

Seek $u = \mathrm{Re}[U_0 e^{-\lambda y - i\omega t}]$ and choose the square root with positive real part. Then $\lambda = (1 - i)/\delta$,

$$\delta = \sqrt{2\nu/\omega}.$$

Both amplitude decay and phase lag carry information: at distance one penetration depth the amplitude is e^{-1} of the wall speed and the lag magnitude is one radian. A single complex velocity map can determine viscosity from either slope and use their agreement to test the simple model.

For a fluid of density 1000 kg per cubic metre and measured penetration depth 0.126 mm at 20 Hz, $\nu = \omega \delta^2/2 \simeq 1.00 \times 10^{-6}$ square metres per second and $\eta \simeq 1.00$ mPa s. A stress-fluctuation integral yielding a value twice as large would predict a penetration depth larger by $\sqrt{2}$, approximately 0.178 mm. Such a comparison can discriminate a missing component or volume factor in the equilibrium calculation if density, temperature and fluid composition match.

At high enough frequency the fluid may have a complex frequency-dependent viscosity rather than the zero-frequency coefficient from a Green–Kubo area. In linear response that generalisation is the one-sided Fourier transform of the equilibrium stress correlation with appropriate prefactors. A single-exponential stress memory gives a frequency-dependent response with a characteristic roll-off and phase lag. Comparing a high-frequency penetration measurement directly to a zero-frequency area without checking the timescale can therefore produce a genuine difference, not a statistical failure.

Remove a centre-of-mass mode only when the observable requires it

In a finite periodic particle simulation total momentum may be exactly conserved. If the initial centre of mass has a random velocity and no thermostat removes it, every particle's measured laboratory coordinate includes the common drift $V_{\mathrm{CM}}t$. Its mean-squared displacement gains a ballistic term $\langle V_{\mathrm{CM}}^2\rangle t^2$. In a canonical ensemble the centre-of-mass velocity variance in one direction is k_BT/M_{tot}. Though it falls with system size, at sufficiently long times its quadratic contribution exceeds the linear diffusive part.

For 1000 equal particles of mass m, an equilibrium centre-of-mass mode contributes one thousandth of the single-particle instantaneous velocity variance. In a long-time autocorrelation this small constant never integrates to a finite diffusion coefficient. Subtract centre-of-mass displacement or work in a zero-total-momentum ensemble before taking the long-time integral of an internal tracer's motion. In a real sample with externally imposed drift, decide whether the physical quantity sought is diffusion about that drift or absolute laboratory displacement; the subtraction has a physical meaning rather than merely a cosmetic improvement.

Conversely, indiscriminately subtracting the mean velocity of a tiny observation subregion separately at every time can remove genuine hydrodynamic long-wavelength motion. A strongly applied single-particle thermostat can alter current correlations and suppress transport even while maintaining the correct static temperature histogram. Check the thermostat strength and system size; compare an equilibrium-prepared weakly thermostatted trajectory with an energy-conserving production segment where appropriate. Maintaining the correct one-time distribution is not sufficient to preserve all dynamical correlation functions.

The distinction also matters when removing an estimated mean stress before integrating its fluctuations. The true shear mean may vanish by symmetry, but a short noisy record has a nonzero sample mean. Subtracting a slowly varying trend on the same timescale as a genuine stress tail can erase physical viscosity. Use independent samples, separately measured drifts and explicit sensitivity to the subtraction window. A persistent nonzero off-diagonal stress under nominal isotropic conditions may indicate incomplete equilibration, a deformed box or an unrelaxed solid rather than a fluid transport coefficient.

Heat and particle currents can share fluctuations

Consider two thermodynamic fluxes J_1, J_2 driven weakly by conjugate forces X_1, X_2, chosen so entropy production density is $J_1X_1 + J_2X_2$. Linear response writes $J_i = \sum_j L_{ij}X_j$. Near time-reversal-symmetric equilibrium, with flux parities chosen appropriately and no magnetic-field reversal complications, reciprocal relations give

$$L_{12} = L_{21}.$$

The symmetric part must be positive semidefinite so that entropy production is nonnegative for every small force. Thus $L_{11}, L_{22} \geq 0$ and $L_{12}^2 \leq L_{11}L_{22}$.

Suppose a thermal measurement enforces $J_1 = 0$ by allowing a compensating force X_1 to develop, as when a concentration or electrochemical gradient cancels a coupled particle flux. Solving the first row gives

$$X_1 = -L_{12}X_2/L_{11}.$$

Substitution into the second yields response coefficient $L_{22} - L_{21}L_{12}/L_{11}$. The measured constrained heat response can therefore be smaller than the coefficient at fixed $X_1 = 0$. The difference has a simple operational origin in the boundary condition enforced during the experiment.

Take consistent normalised coefficients $L_{11} = 4, L_{22} = 3, L_{12} = L_{21} = 2$. At zero first force the heat flux slope is three. At zero first flux it is $3 - 4/4 = 2$. Both are positive and consistent with the same equilibrium correlation matrix. Comparing a simulated heat-current autocorrelation corresponding to the unconstrained coefficient against a zero-particle-flux experiment requires the mixed-current correlation too. Apparent disagreement by a factor of one and a half can arise entirely from this distinction.

Green–Kubo cross-correlations must use properly defined conjugate fluxes and units. Redefining heat current by adding a constant multiple of particle current changes the matrix entries but must leave a physically defined zero-particle-flux heat measurement invariant when the forces are transformed consistently. This reference invariance is analogous to subtracting a per-particle energy reference in grand-canonical covariance. It provides a powerful test of a multicomponent transport implementation without relying on any empirical transport value.

A lattice trajectory provides an exactly known correlation integral

For a one-dimensional nearest-neighbour walk with steps of length a every interval Δt, suppose successive step directions have correlation $\langle s_{j+k}s_j\rangle = r^k$ with $|r| < 1$, and zero mean. Squaring the sum of n steps gives

$$a^2[n + 2\sum_{k=1}^{n-1}(n-k)r^k].$$

In the large-n limit divide by $2n\Delta t$ to obtain

$$D = a^2(1+r)/[2\Delta t(1-r)].$$

The uncorrelated random-walk value is recovered at $r = 0$.

With $a = 1$ micrometre and $\Delta t = 0.10$ s, uncorrelated diffusion is five square micrometres per second. Persistence $r = 0.5$ raises it to fifteen, while anticorrelation $r = -0.5$ lowers it to $5/3$. All three walks have the same step-size variance, so an estimate based only on one-frame displacement misses the entire correlation effect. The finite-lag autocorrelation sum and a long-time displacement fit should recover the same value on a long simulated record.

The discrete formula also clarifies the special zero-lag weight in numerical integration. An integral approximated from sampled symmetric correlations gives half weight to the zero lag on a one-sided

trapezoidal rule. A careless double count of the instantaneous self term predicts twice the uncorrelated diffusion. Benchmark a numerical estimator with this exact walk before applying it to molecular data with unknown late tails and noisy velocity derivatives.

Measure an off-diagonal diffusion coefficient after rotating the sample

In an anisotropic sheet a tracer's two-dimensional displacement covariance need not align with the camera axes. Let its long-time principal diffusion coefficients be D_1, D_2 and the first principal axis make angle θ with laboratory x. In matrix form the laboratory tensor is $R\,\mathrm{diag}(D_1, D_2)R^{\mathsf{T}}$, with R the planar rotation. At sufficiently large lag in an effectively unbounded drift-free region, the displacement covariance matrix tends to $2Dt$. An off-diagonal covariance is a measurable correlation between displacement components even if the noise is independent in the material principal directions.

Take $D_1 = 2.0$ and $D_2 = 0.50$ square micrometres per second with $\theta = 30^\circ$. Laboratory components are $D_{xx} = 1.625$, $D_{yy} = 0.875$ and $D_{xy} = 0.650$, in the same units. At 0.10 s the cross covariance is 0.130 square micrometres. Rotating the physical sheet by ninety degrees interchanges the two diagonal values and reverses the sign of the off-diagonal one. An instrument-fixed camera readout artefact would not generally follow this material rotation.

To infer the principal axes from data, fit each covariance component versus lag using a common drift subtraction and independent position calibration. Diagonalise the symmetric fitted diffusion matrix. Positive semidefiniteness requires

$$D_{xy}^2 \le D_{xx} D_{yy}.$$

A strong violation indicates incompatible estimates, unit conversion errors or statistical uncertainty, not a physically permissible strongly anisotropic diffusion tensor. Near-degenerate principal values make the orientation poorly determined even when the mean diffusion coefficient is precise. Report uncertainties on eigenvalues and orientation separately.

A finite field of view introduces a further bias. Fast particles along the high-diffusion direction leave the image sooner, and a tracking pipeline retaining only long trajectories can favour slow or fortuitously confined paths. Compare different fields of view and minimum track durations, or fit a measurement model incorporating censoring. A physical material anisotropy should persist after correcting these effects; apparent anisotropy aligned with the short direction of a rectangular observation window demands particular scrutiny.

A weak calibrated force along different axes supplies an independent check through the tensor Einstein relation under equilibrium conditions. If the diffusion tensor predicts off-axis mean drift, the mobility experiment should observe its sign and magnitude after reversing the force. A discrepancy could reveal active forcing, nonstationarity, an incorrect temperature or a mismatch between the mobility and displacement observables. It cannot be diagnosed from a scalar mean square alone.

Another rotational invariant is the trace of the diffusion tensor. In the example its value stays 2.50 square micrometres per second for every sample orientation, while the determinant stays one square

micrometre to the fourth per second squared. A comparison of two camera orientations therefore provides numerical checks independent of the fitted angle. A drift mistakenly left in the displacement covariance can violate the expected linear dependence on lag, making the inferred effective eigenvalues grow with the selected fitting window. Subtract the ensemble mean displacement at each lag and record that mean separately as a flow measurement. Conversely do not subtract an individual best-fit line from every short trajectory without modelling the resulting loss of long-time fluctuations; that procedure can bias both diffusion coefficients downward.

Choose an integration window for a transport coefficient

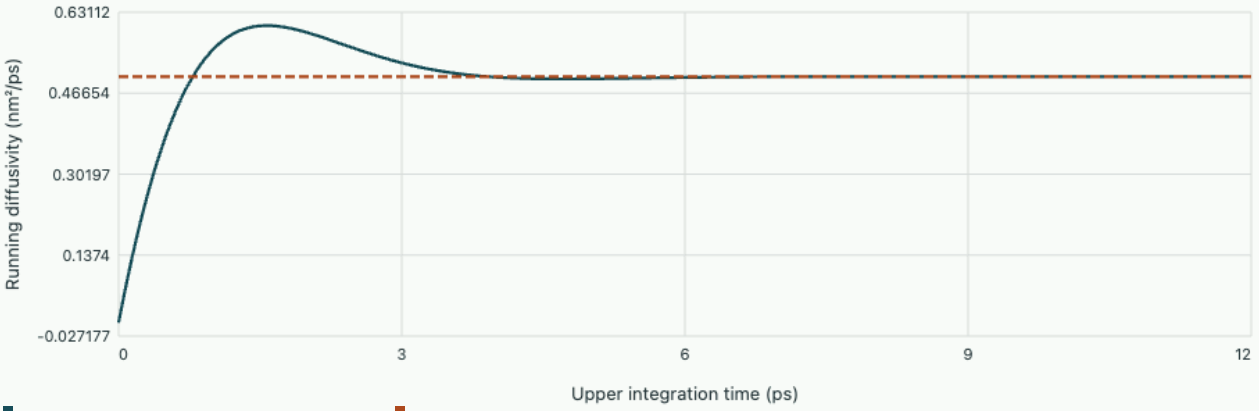
Integrate an oscillatory velocity correlation over a finite record. Compare a short apparent plateau with the independently computed infinite-time result.

Open this experiment online

Try the calculation

Correlation decay time τ (ps) 1

Oscillation frequency Ω (rad/ps) 1



Finite Green–Kubo integral Infinite-time transport coefficient

With C(0) = 1 nm²/ps² the exact 1D diffusion coefficient is 0.5 nm²/ps. At 2τ the integral gives 0.58969 nm²/ps.

Tutorial · Invert diffusion measurements at two lags

In a one-dimensional diffusive window a measured displacement variance is 0.024 square micrometres at 10 ms and 0.064 at 50 ms. Assume long-time Brownian displacements plus independent frame-localisation variance per position measurement. Estimate D and localisation precision.

▼ **Fit slope and intercept separately**

The variance slope is $(0.064 - 0.024)/(0.050 - 0.010) = 1.00$ square micrometre per second, so $D = 0.500$ square micrometres per second. The intercept is 0.014 square micrometres, equal to twice the single-frame localisation variance. The implied rms localisation error is $\sqrt{0.007} = 0.0837$ micrometres. Forcing the fit through the origin at the shortest lag would give $D = 1.20$, more than twice the inferred value.

Acquire more lags to test the linear model. A static drift contributes a term growing as t^2 if its mean displacement is not removed, while finite exposure can lower the physically measured short-lag variance. Two points always define a line and cannot independently validate either noise model.

Tutorial · Estimate a hidden slow tail in viscosity

A stress correlation is modelled as $A[e^{-t/\tau} + 0.02e^{-t/(40\tau)}]$. What fraction of its true area comes from the low-amplitude tail? How much does a cutoff at 5τ capture?

▼ **Weight a correlation by its duration**

The fast and slow areas are $A\tau$ and $0.80A\tau$. The slow contribution is 44.4 percent of total. Integration to five fast times captures $A\tau[(1 - e^{-5}) + 0.80(1 - e^{-5/40})] \simeq 1.087A\tau$, or about 60.4 percent. Visual disappearance of the dominant peak therefore does not establish convergence of the transport coefficient.

The slow contribution must be supported by independent long blocks or a physically motivated model. A fitted tiny constant baseline can generate arbitrarily large apparent area if extrapolated indefinitely. Subtracting a sample mean estimated from the same short record can conversely suppress a genuine slow mode. Compare records much longer than the proposed tail time before trusting the extrapolation.

Tutorial · Is an algebraic tail compatible with ordinary diffusion?

Suppose a measured long-time velocity correlation remains positive and scales as $t^{-\alpha}$. Determine when its Green–Kubo area exists and infer the displacement growth for $0 < \alpha < 1$ and for $\alpha = 1$. Why would a finite periodic simulation require care?

▼ **Integrate the time-dependent slope before naming a diffusion coefficient**

For $\alpha > 1$ the tail integral converges. For $0 < \alpha < 1$, the running area grows as $t^{1-\alpha}$, so displacement variance grows as $t^{2-\alpha}$ under the stated persistent positive tail. At $\alpha = 1$, the running area grows logarithmically and the displacement variance is of order $t \log t$. Sign oscillations, finite-size termination or another late-time crossover can alter these conclusions.

A periodic box admits only discrete long-wavelength modes and may cut off a hydrodynamic tail. Repeating at larger sizes and comparing the range before signals traverse the box helps identify a bulk regime. Unwrap particle coordinates crossing periodic boundaries before calculating a displacement. A

wrapped coordinate can falsely saturate and imitate a confined particle even in a freely diffusing periodic fluid.

Cambridge Kinetic Theory · correlations, response and transport

Applied study

Connect a growing boundary layer to wall shear, momentum loss and drag.

How does a surface slow down a moving fluid?

Upstream of a smooth plate, every parcel in a uniform stream moves at speed U. At the plate the no-slip condition brings the fluid to rest. Momentum diffuses across the flow while the stream carries it downstream, so the region slowed by the wall thickens with distance. The wall shear determines how much power is needed to move a plate through the fluid.

Consider steady two-dimensional incompressible laminar flow at constant density ρ, dynamic viscosity μ and zero streamwise pressure gradient. Measure x from the leading edge and y normal to the plate. When the viscous layer is thin relative to x, the streamwise momentum and continuity equations reduce to

$$u u_x + v u_y = \nu u_{yy},$$

$$u_x + v_y = 0,$$

$$\nu = \mu / \rho.$$

At the wall $u = v = 0$, while u approaches U far above it. Balancing U^2/x with $\nu U/\delta^2$ gives the thickness scale $\delta \sim \sqrt{\nu x / U}$. The thickness increases downstream even though δ/x falls. Doubling the speed gives the fluid less time to diffuse momentum before it reaches the observation point.

Relate a measured velocity slope to drag

Profiles at different stations have nearly the same shape when plotted against $\eta = y\sqrt{U/(\nu x)}$. The laminar Blasius solution gives a wall slope $\partial_y u|_0 = 0.332\, U\sqrt{U/(\nu x)}$. Therefore, with $Re_x = Ux/\nu$,

$$\tau_w = \mu\, \partial_y u|_0,$$

$$c_f(x) = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{Re_x}},$$

$$\delta_{99} \simeq \frac{5.0 x}{\sqrt{Re_x}}.$$

The last thickness is the height at which $u = 0.99U$. Integrating τ_w from the leading edge to length L gives the drag on one face per unit width,

$$D/b = 0.664 \rho U^2 \sqrt{\nu L / U}.$$

The mean friction coefficient over the plate is twice the local coefficient at its trailing edge. Surface roughness, transition to turbulence and strong pressure gradients change this result.

Worked example · Sample a plate halfway along

Use $U = 1.0$ m/s, $\rho = 1.0\,\mathrm{kg/m^3}$, $\mu = 10^{-4}\,\mathrm{Pa\,s}$ and $x = 0.50$ m. Then $\nu = 10^{-4}\,\mathrm{m^2/s}$ and $Re_x = 5000$. The 99% thickness is 35.4 mm, the friction coefficient is 0.00939, and the shear stress is

$$0.5(1)(1)^2(0.00939) = 0.00470\,\mathrm{Pa}.$$

At $x = 1.00$ m the layer has grown to 50 mm and the wall shear has fallen to 0.00332 Pa. The amount of fluid affected has increased while the near-wall velocity gradient has weakened.

Transfer problem · Use a pressure drop to infer pipe flow

In a sufficiently long straight pipe of radius R, the viscous regions growing from the walls merge and the velocity profile stops changing downstream. For a fluid cylinder of radius r and length L, the pressure force $\Delta p\,\pi r^2$ balances the resisting shear force $2\pi r L|\tau|$. Hence $|\tau| = \Delta p\, r/(2L)$. Since $du/dr = -|\tau|/\mu$, integration with $u(R) = 0$ gives

$$u(r) = \Delta p(R^2 - r^2)/(4\mu L).$$

Integrating over circular rings yields

$$Q = \pi R^4 \Delta p/(8\mu L).$$

A water line has $R = 5.0$ mm, $L = 2.0$ m, $\mu = 0.0010\,\mathrm{Pa\,s}$ and $\rho = 1000\,\mathrm{kg/m^3}$. Delivering $Q = 1.0 \times 10^{-6}\,\mathrm{m^3/s}$ requires $\Delta p = 8.15$ Pa over the developed portion. The mean speed is $Q/(\pi R^2) = 0.0127$ m/s and the diameter-based Reynolds number is 127, consistent with laminar flow. Halving the pipe radius at fixed flow multiplies the required pressure by sixteen. Compare the derivation for a parallel-plate channel.

Exercises

1. Choose an appropriate force measurement

A 0.20 m wide plate is exposed on one face from its leading edge to $L = 1.00$ m under the conditions above. What force should a balance measure? What bias would follow from using the trailing-edge wall shear across the entire plate?

▼ Show the drag estimate

The integrated force is

$$0.20(0.664)(1)(1)^2\sqrt{10^{-4}(1)/1} = 0.00133\,\mathrm{N}.$$

Using the final local shear gives only $0.00332(1)(0.20) = 0.000664$ N. Wall shear varies as $x^{-1/2}$, so the leading part contributes more drag per unit length.

2. Distinguish dimensionless and dimensional changes

At the same station, double the free-stream speed without changing the fluid. Predict the changes in δ_{99}, c_f, τ_w and power lost to skin friction at fixed plate length.

▼ **Show the scaling argument**

Both thickness and friction coefficient scale as $U^{-1/2}$, giving a factor 0.707. Dimensional shear includes U^2, hence grows by $2^{3/2} = 2.83$. Drag has the same speed exponent and required power is DU, which grows by $2^{5/2} = 5.66$.

ONESIMULATION LABORATORY

Measure boundary-layer growth from a velocity profile

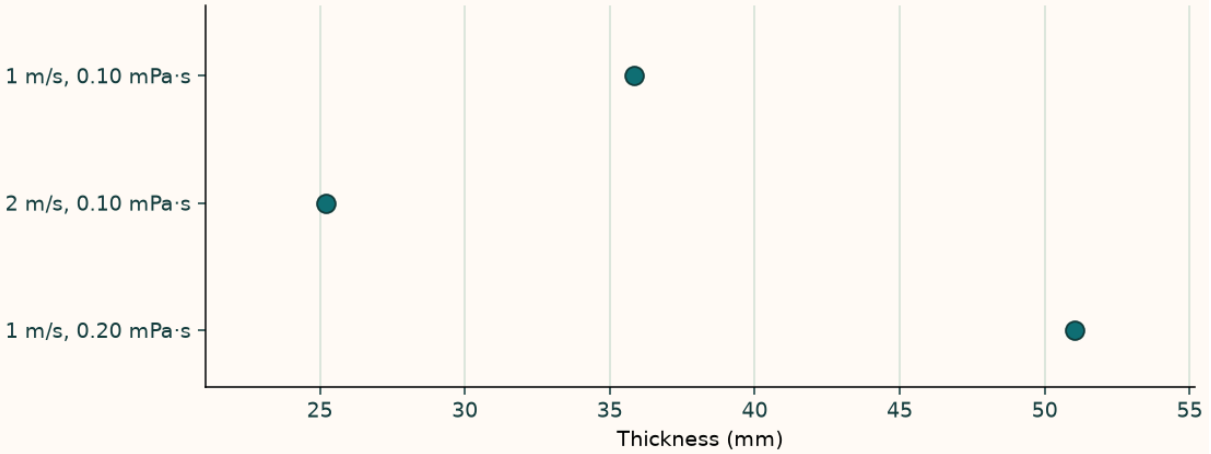
Compare recorded OneSimulation runs on this page. To test your own inputs, download the notebook and model files and run the model in an activated OneSimulation environment.

The OneSimulation domain begins 0.2 m downstream of the leading edge. Use the row labelled $x = 0.50$ m, which already includes that offset. The baseline uses `stream=1.0` m/s, `density=1.0` kg/m³ and `dynamic_viscosity=1.0e-4` Pa·s.

1. Double `stream` to `2.0` m/s. Rebuild the model so the imposed inlet profile also changes. Expect a thickness near 25 mm and smaller c_f, even though the dimensional wall stress increases.
2. Restore `stream=1.0` and double `dynamic_viscosity` to `2.0e-4` Pa·s. At unchanged density, expect a thickness near 50 mm and c_f near 0.0133. Compare the departures from the similarity approximation with the baseline.

Computed boundary layer at 0.50 m from the leading edge

Free stream and viscosity	Thickness (mm)	Friction coefficient (×10⁻³)
1 m/s, 0.10 mPa·s	35.84	9.188
2 m/s, 0.10 mPa·s	25.19	6.559
1 m/s, 0.20 mPa·s	51.03	12.878



At the same downstream station, doubling the speed thins the layer from 35.84 to 25.19 mm. Doubling viscosity thickens it to 51.03 mm.

View the OneSimulation flat-plate model and its velocity profiles.

Download experiment files (ZIP)

Extract the files together and open `blasius_flat_plate.ipynb` in a local OneSimulation environment.

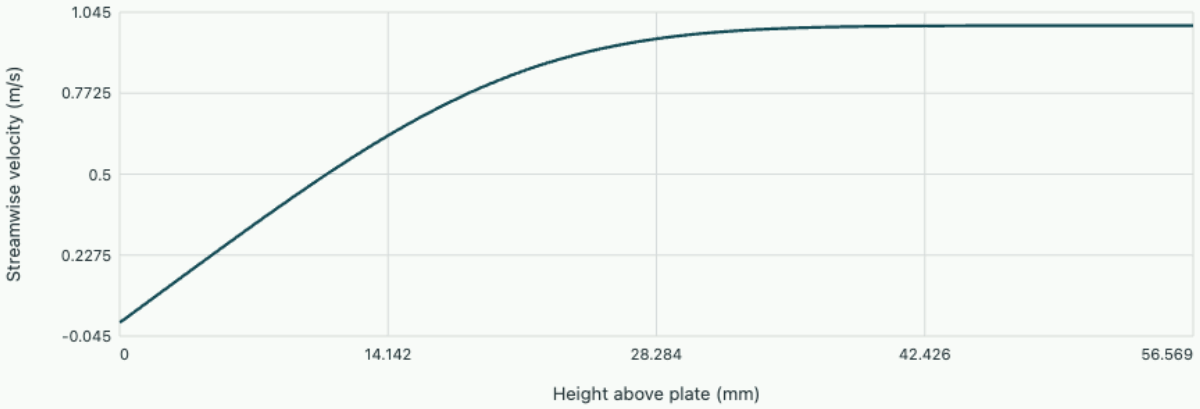
Interactive reference for the OneSimulation laboratory

Reintegrate the Blasius similarity profile and vary its dimensional scales. Compare this laminar1D similarity reference with the spatial OneSimulation run.

Open the live laboratory reference

Try the calculation

Free-stream speed (m/s)	1
Dynamic viscosity (mPa s)	0.1
Station x (m)	0.5



Integrated Blasius similarity profile

At density1 kg/m³, local Re=5000, cf=0.009392, δ99≈34.719 mm. This laminar similarity reference neglects pressure gradient; compare with the linked spatial OneSimulation calculation.

Problem sheet 4 · Infer viscosity from the edge of a boundary layer

Notation and assumptions

Read each defining equation with its assumptions and units, then follow the link to the argument that gives the equation its meaning.

Capillary pressure

$$\Delta p = \gamma(1/R_1 + 1/R_2)$$

Principal curvatures carry signs set by the chosen normal. A cylindrical interface contributes only one finite principal curvature.

Derivation · Curvature pressure, wetting and capillary rise

Froude number

$$\mathrm{Fr} = U/\sqrt{gh}$$

For shallow rectangular flow with gravity waves this compares material speed to wave speed. Critical depth occurs when the two are equal.

Derivation · Critical depth and the hydraulic jump

Head loss

$$h_L = \Delta p_{\mathrm{loss}}/(\rho g)$$

This is an irreversible loss expressed as a length. A static pressure change can also include reversible changes of speed or elevation.

Derivation · Bernoulli head, losses and pump duty

Material acceleration

$$D\mathbf{u}/Dt = \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}$$

A fluid parcel can accelerate as it follows a steady but spatially varying velocity. A probe fixed in space measures only the local time history.

Derivation · Velocity fields, flow rates and conservation of mass

Reynolds number

$$\mathrm{Re} = \rho U L / \mu$$

Choose speed and length scales appropriate to the geometry. Matching this ratio compares inertial and viscous terms under the stated nondimensionalisation.

Derivation · Dynamic similarity, boundary layers and model tests

Vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

Its units are reciprocal seconds. Circulation around a closed loop integrates the corresponding vorticity flux when the interior is regular.

Derivation · Vorticity, circulation and swirling flow

Further reading

- Look for the distinction between an integral momentum balance and a local velocity field. Rework the pipe and boundary-layer cases after changing the control surface. MIT · Fluid Dynamics
- Use the advanced problems to practise identifying the dimensionless groups and boundary conditions that survive a limiting process. MIT · Advanced Fluid Mechanics

Compare specific topics with the university sources