# Thermal processes and transport in engineering

Close energy and entropy balances around real processes. Use measured states and heat flows to separate component performance from modelling assumptions.

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# Thermal processes and transport in engineering · 1. Staged compression with an imperfect intercooler

*A laboratory compressed-air supply has two compressor stages and an intercooler. Equal pressure ratios minimise work only under particular assumptions. Derive the optimum, compare it with a single-stage machine and identify which temperature and power measurements reveal imperfect cooling or unequal efficiency.*

## Work follows the actual enthalpy rise

Model dry air as an ideal gas with constant $c_p = 1.005$ kJ per kilogram kelvin and $\gamma = 1.40$. Assume steady adiabatic compressor stages, negligible changes in kinetic and potential energy, and negligible intercooler pressure drop initially. For a fixed inlet temperature $T_i$ and pressure ratio $r_i$, an isentropic reference ends at $T_{os} = T_i r_i^a$, $a = (\gamma - 1)/\gamma$. Isentropic compressor efficiency compares reference enthalpy rise with actual rise, so actual shaft work per unit mass is $w_i = c_p T_i (r_i^a - 1)/\eta_i$.

An intercooler reduces the second-stage inlet temperature by rejecting heat to a colder stream. It does not undo the pressure work already delivered by the first stage. For overall ratio $R$ and first-stage ratio $r$, assume the measured second-stage inlet $T_2$ can be held fixed across the comparison. Adding both work contributions gives

$$w(r) = c_p \left[ \frac{T_1}{\eta_1}(r^a - 1) + \frac{T_2}{\eta_2}((R/r)^a - 1) \right].$$

Direct differentiation with respect to $\log r$ balances the incremental work costs of transferring pressure ratio between stages. The stationary split is

$$r_{\mathrm{opt}} = \sqrt{R} \left( \frac{T_2 \eta_1}{T_1 \eta_2} \right)^{1/(2a)}.$$

Both exponential-power terms have positive second derivatives in $\log r$, giving a unique interior minimum when the root lies between one and $R$. Otherwise the constrained optimum is a boundary under this simplified model. A hot or inefficient second stage favours assigning more ratio to the first. Perfect intercooling with identical efficiencies gives $r = \sqrt{R}$, the geometric mean rule for intermediate absolute pressure.

**Background · What if the two compressor stages start at different temperatures?**

Consider an ideal gas with constant specific heat. Use equal isentropic efficiencies and a fixed total pressure ratio $R$. Prescribe inlet temperatures $T_1$ and $T_3$ independently of the intermediate pressure. With $a = (\gamma - 1)/\gamma$, the speed-independent part of shaft work is proportional to $T_1(r_1^a - 1) + T_3[(R/r_1)^a - 1]$. Differentiate with respect to $\ln r_1$ and set the derivative to zero.

$$T_1 r_1^a = T_3 r_2^a,$$

$$r_1 = \sqrt{R}\left(\frac{T_3}{T_1}\right)^{1/(2a)},$$

$$r_1 r_2 = R.$$

At the minimum the two ideal discharge temperatures coincide. Incomplete intercooling with $T_3 > T_1$ shifts more compression to the cooler first stage. For very different temperatures the stationary point may leave the allowable interval $1 \le r_1 \le R$; check the boundary then. A cooler with specified effectiveness can make $T_3$ vary with $r_1$, which requires an additional derivative.

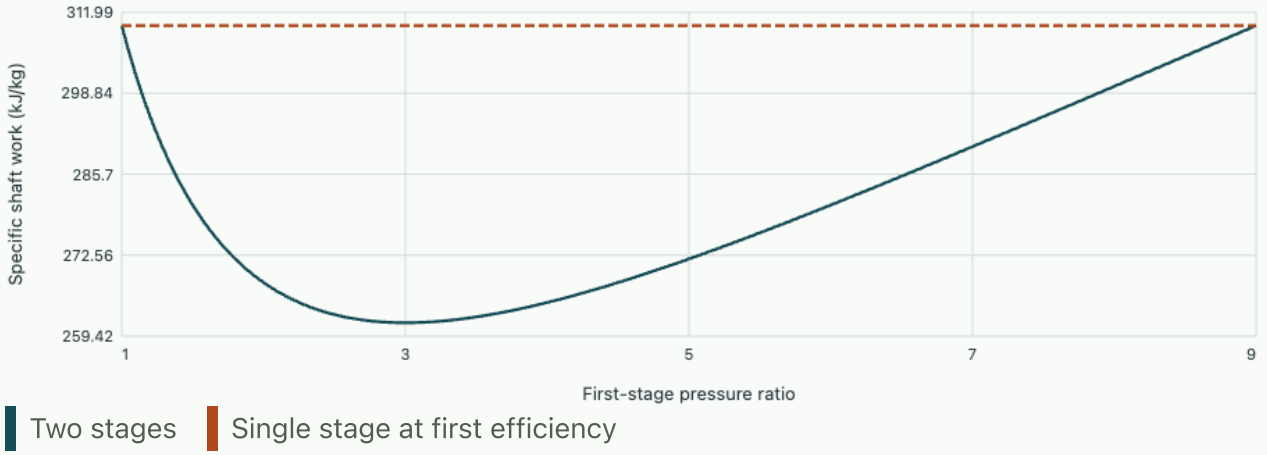
# Use pressure, temperature and electrical power together

For $R = 9, T_1 = T_2 = 300$ K and efficiencies 0.85, each stage ideally uses ratio three. Each consumes 130.79 kJ/kg and the combined work is 261.59 kJ/kg. The otherwise identical single stage uses 309.82 kJ/kg. At 0.020 kg/s, two-stage shaft power is 5.23 kW before motor losses. The first-stage discharge temperature is about 430.1 K; cooling it back to 300 K rejects 130.8 kJ/kg, numerically equal to that first adiabatic stage's work under these property assumptions.

Open the live experiment

### Try the calculation

| | |
|---|---|
| Overall pressure ratio | 9 |
| Second inlet temperature (K) | 300 |
| Second efficiency | 0.85 |




**Minimum work is 261.59 kJ/kg at first ratio 3.000. The equal-ratio split requires 261.59 kJ/kg.**

Raise the second inlet temperature or reduce its efficiency. The minimum now moves toward a higher first pressure ratio. The graph compares total specific work, keeping final absolute pressure fixed. It does not guarantee that a real compressor can operate at every displayed split: stage flow maps, surge, choke, motor limits and heat exchanger capacity may restrict the admissible point. Measure actual mass flow and pressure drop at each tested condition. A wattmeter upstream of a motor reports more than shaft power; use measured or justified motor efficiency when comparing with a gas enthalpy rise.

## Close the work and rejected-heat balances separately

The two-stage minimum is a statement about shaft work per kilogram of ideal gas. An experimenter can test it by weighing delivered gas over a timed interval and measuring electrical energy or shaft torque. Converting an electrical reading to specific shaft work requires the motor efficiency at the measured operating point. A small motor often operates at a different efficiency when the intermediate pressure changes. Dividing the same electrical input by two different mass-flow readings can obscure a compressor change behind a motor change.

For an illustrative mass flow of 0.10 kg/s, a computed 262 kJ/kg corresponds to 26.2 kW at the shaft. At 90 percent motor efficiency the electrical input is about 29.1 kW. If the first compressor delivers at 430 K and the intercooler returns the gas to 300 K, its separate steady heat duty at the stated constant heat capacity is approximately 13.1 kW. A flow of cooling water warming by 10 K at 4.18 kJ per kilogram kelvin would need about 0.31 kg/s. These two measurements test different assumptions. Work

compares both compressor stages; the cooling-water enthalpy rise compares only the intermediate exchanger, including any loss through its casing.

When measuring intake and discharge temperatures, distinguish stagnation from static temperature if velocity changes appreciably in connecting ducts. A thermocouple close to a wall may also respond to conduction along its sheath. Repeat each setting after allowing the intercooler and casing to approach steady temperatures. Heat retained in a warming machine changes the short-time coolant balance while leaving the algebraic optimum calculated for fully steady operation unchanged.

## Tutorial · Find the new optimum for incomplete cooling

Suppose $R = 9, T_1 = 300$ K, $T_2 = 330$ K and both efficiencies remain 0.85. Predict the optimal first ratio and decide whether it is physically sensible. Estimate its work advantage over equal ratios using the graph or direct substitution.

▼ **Balance the two marginal temperature-weighted costs**

Since $1/(2a) = 1.75$, $r_{\text{opt}} = 3(1.10)^{1.75} = 3.545$, so the second ratio is about 2.54. The hotter second stage now receives less compression work. Direct substitution gives approximately 273.5 kJ/kg versus 274.7 kJ/kg at equal ratios, an improvement only around one kJ/kg. Imperfect cooling itself costs much more relative to the original 261.6 kJ/kg. A noisy power sweep can therefore locate a rather broad minimum poorly even when cooling quality clearly matters.

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## Tutorial · Include an intercooler pressure loss

The cooler outlet pressure is a fraction $1 - \delta$ of first-stage discharge pressure. For fixed compressor inlet and final discharge pressures, what happens to the product of the actual compressor-stage ratios? Compute the equal-efficiency, fully cooled optimum at $R = 9$, $\delta = 0.05$.

▼ **Use absolute pressures on both sides of the exchanger**

With $p_2 = (1 - \delta)p_1 r_1$, final pressure $p_f = p_2 r_2$, so

$$r_1 r_2 = R/(1 - \delta) = 9.474.$$

Under identical stage inlet temperatures and efficiencies the optimum actual compressor ratios are equal to $\sqrt{9.474} = 3.078$. Thus the pressures around the cooler straddle the ideal loss-free geometric-mean intermediate pressure. A pressure tap only at first discharge cannot determine the second-stage ratio when exchanger loss is appreciable.

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# Tutorial · A temperature measurement conflicts with the power estimate

A trial records 0.020 kg/s, first-stage inlet 300 K and discharge 430 K. The drive wattmeter reports 3.25 kW for that stage. Estimate gas enthalpy power and explain two possible sources of the difference that have distinguishable signatures.

▼ **Draw separate drive and gas control volumes**

The gas enthalpy gain is $0.020(1.005)(130) = 2.61$ kW. Relative to electrical input this gives an apparent conversion fraction of 0.804. Electrical and mechanical motor losses are candidates, but heat rejected by a nonadiabatic compressor, temperature probe recovery errors, flow calibration and auxiliary equipment can also enter. Compare shaft torque times angular speed with electrical power to isolate drive losses. Measure casing heat rejection or obtain an energy balance around compressor plus coolant if shaft power remains above the gas gain. A single inferred efficiency cannot identify which physical boundary carries the missing 0.64 kW.

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Ohio State thermal sciences · multistage compression with intercooling

# Thermal processes and transport in engineering · 2. How a gravity-drained condensate film limits heat transfer

*Saturated vapour condenses on a cool vertical plate. A liquid film thickens as it drains, adding thermal resistance toward the bottom. Derive the local heat-transfer coefficient from coupled momentum and energy balances, then determine which observations would invalidate the smooth-film model.*

## Find the liquid flow for a specified film thickness

Let $x$ increase downward along a vertical isothermal wall, $y$ normal to it, and $\delta(x)$ be the liquid thickness. Assume a steady laminar film with negligible inertia, constant liquid properties, negligible interfacial shear from slow vapour, saturation temperature at the interface, and a wall uniformly below saturation. Neglect sensible cooling of the condensate relative to latent heat and heat conduction along the plate. The common vapour pressure gradient is approximately hydrostatic, so net downward liquid driving per unit volume is $(\rho_l - \rho_v)g$.

Solve $\mu_l u_{yy} = -(\rho_l - \rho_v)g$ with no slip at $y = 0$ and zero shear at $y = \delta$. The velocity is a downward parabola $u = (\rho_l - \rho_v)g(\delta y - y^2/2)/\mu_l$. Integrating gives mass flow per unit width $\Gamma = \rho_l(\rho_l - \rho_v)g\delta^3/(3\mu_l)$. The velocity increases from the wall to its interface maximum; assigning the interface a second no-slip condition would substantially underpredict gravity drainage.

## A growing mass flow changes the conductive resistance

If transverse heat conduction dominates, approximately linear temperature through the film gives wall heat flux $q'' = k_l(T_{\mathrm{sat}} - T_w)/\delta$. New condensate produced per unit wall length adds $d\Gamma/dx = q''/h_{fg}$. Substitute the cubic film-flow relation and integrate from an ideal leading edge of zero film thickness:

$$\delta(x) = \left[\frac{4\mu_l k_l \Delta T\, x}{\rho_l(\rho_l - \rho_v)g h_{fg}}\right]^{1/4},$$

$$h_x = \frac{k_l}{\delta(x)},$$

$$\bar{h}_L = \frac{4}{3}h_L.$$

The leading coefficient diverges at $x = 0$ in this idealisation, but its integral is finite. A real leading edge, substrate conduction and contact-angle effects regularise the immediate entrance. The factor $4/3$ comes from integrating $x^{-1/4}$, not averaging endpoint values. A plate made twice as long gains heat transfer area but increases film resistance downstream; total heat per unit width rises only as $L^{3/4}$.

As an illustrative water-like fluid take $\rho_l = 960$, $\rho_v = 0.6$ kg/m³, $k_l = 0.68$ W/(m K), $\mu_l = 2.9 \times 10^{-4}$ Pa s, $h_{fg} = 2.26$ MJ/kg, $\Delta T = 5.0$ K and $L = 0.20$ m. The outlet film is roughly 79 micrometres and the outlet coefficient about 8.6 kW per square metre kelvin. The average coefficient is about 11.5 kW per square metre kelvin; a 0.10 m wide plate removes about 1.15 kW and yields roughly 0.51 g/s

condensate. Calculate film Reynolds number with the specified convention before accepting the assumed smooth regime.

Return to the compressor and intercooler calculation to compare this thermal coefficient with an actual heat duty

## Test the quarter-power prediction at several stations

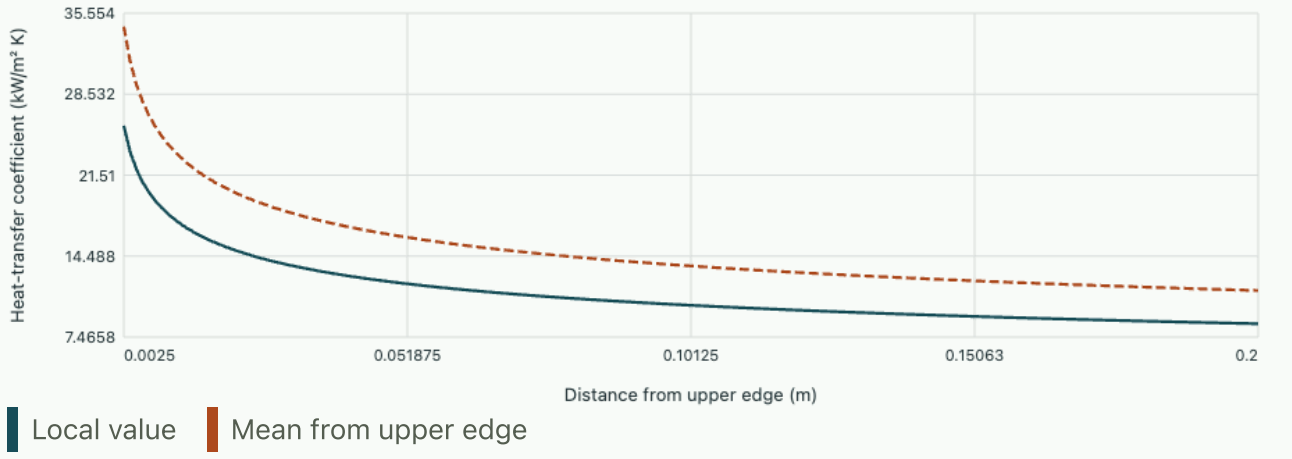
A film-thickness measurement is easiest away from the leading edge. There the ideal formula predicts a thin film and formally unbounded heat transfer coefficient. Real entrance geometry, surface condition, contact-line physics and measurement resolution take over. In the study the curves begin at a small positive fraction of plate length. This preserves the finite integral of the singular ideal heat flux yet avoids assigning a physical measured value at the mathematical edge.

Open the live experiment

### Try the calculation

Wall subcooling (K) **5**

Plate length (m) **0.2**




**The outlet film is 78.8 µm thick. A 0.10 m wide plate rejects 1.150 kW.**

For an independent test place two thermal sensors along the downstream wall and estimate local heat flux by conduction across a thin backing of known thickness and conductivity. At two stations whose distances differ by a factor sixteen, the ideal prediction halves the local coefficient. A common wall heat loss can distort this ratio much more at the downstream station, where the useful heat flux is smaller. Repeat with vapour absent to estimate the backing loss and use an unheated reference section to check temperature gradients in the support.

The liquid mass collected at the bottom integrates all upstream condensation. Doubling plate length increases total condensation by a factor of two raised to three quarters under fixed conditions, even as

the mean coefficient falls. If the collector instead finds exactly twice the condensate and local sensors show small axial variation, test for liquid dripping onto the plate from above, a nonuniform vapour temperature or turbulent mixing. Keep track of condensate retained on the surface when using short timed collection intervals; intermittent release into the collector can dominate the apparent uncertainty.

## Tutorial · Reconstruct a heat balance from collected mass

A plate of the stated size produces 0.48 g/s of condensate in a steady timed collection. Estimate latent heat removal and inferred mean coefficient. Does disagreement with the wall-coolant calorimeter necessarily imply an incorrect latent heat?

▼ **Convert the collected mass into a rate of thermal energy**

Latent duty is

$$0.00048(2.26 \times 10^{6}) = 1085\,\mathrm{W}.$$

The effective mean coefficient is $1085/(0.020 \cdot 5) = 10.85$ kW per square metre kelvin. Disagreement with a coolant energy balance can arise from heat leaks, condensate subcooling, liquid holdup or drainage that the collection misses. Weigh over a sufficiently long stationary interval and measure condensate outlet temperature. A transient wetting front stores liquid and invalidates equating instantaneous formation and collection.

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## Tutorial · Compare two uses of the same plate area

Two plates can be placed vertically in series so liquid feeds from the upper into the lower, or arranged side by side with independently drained leading edges. Each plate has length $L$ and width $b$. Estimate the ideal heat-duty ratio at fixed temperatures.

▼ **Compare fresh versus pre-existing films**

Side-by-side total duty is $2Q(L)$. A continuous vertically stacked film behaves like one length $2L$, giving $Q(2L) = 2^{3/4}Q(L)$. Independently drained plates therefore give a factor $2^{1/4} = 1.189$ more ideal duty for the same area. Additional vapour routing, wall conduction and edge drainage can change real performance. The advantage arises from repeatedly beginning with a thin film, not from more total geometric area.

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## Tutorial · Diagnose a vapour-side resistance

A nominally identical plate shows dramatically smaller heat flux after a small amount of noncondensable gas enters the vapour volume, while liquid film thickness measurements barely

change. Why can changing only liquid conductivity in a fit give a misleading result?

▼ **Identify a missing interface boundary condition**

The model assumed saturated pure vapour at an interface of known temperature. A noncondensable component can accumulate at the interface and require vapour diffusion through a gas layer. Local vapour partial pressure and interface saturation temperature can differ from estimates based on total chamber pressure. Heat and species transport on the vapour side then add resistance. Measure composition and partial pressure, sweep the vapour flow or purge the noncondensable fraction and compare flux with an independent film profile. Fitting a different liquid conductivity can mimic the overall flux but would generally predict the wrong response to those controls.

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Lienhard and Lienhard · A Heat Transfer Textbook, film condensation and phase change

# Thermal processes and transport in engineering · 3. Radiative exchange in a grey enclosure

*A thin radiation shield does not need to absorb all incoming radiation to reduce heat transfer. Multiple reflections connect every surface to every other visible surface. Build the radiosity equations, recover the two-plate result and infer what can be learned from a measured shield temperature.*

## Separate emission, reflection and net heat flux

For opaque diffuse-grey isothermal surface $i$, let $G_i$ be irradiation arriving from the enclosure and $J_i$ be all radiative power leaving per unit area. The outgoing consists of thermal emission plus reflected irradiation. With emissivity $\epsilon_i$, reflectivity $1-\epsilon_i$ and blackbody emissive power $E_{bi}=\sigma T_i^4$, one obtains

$$J_i=\epsilon_i E_{bi}+(1-\epsilon_i)G_i.$$

Net outward heat is $Q_i=A_i(J_i-G_i)$.

Geometry enters through view factors: $G_i=\sum_j F_{ij}J_j$ for a closed nonparticipating enclosure of diffuse surfaces, provided reciprocity $A_iF_{ij}=A_jF_{ji}$ has been used consistently. Each closed-enclosure row sums to one, and summing $Q_i$ over all surfaces gives zero. A concave surface can see itself, so setting every diagonal view factor to zero would break some real enclosure models. For an opening to surroundings include an explicit environmental surface or known irradiance source.

## Eliminate the unknown reflected fields

Two large parallel plates of equal area see only one another. Thus $G_1=J_2, G_2=J_1$. Eliminate both radiosities, or add a surface resistance for each plate and a unit space resistance in the diffuse network. Net flux from hot to cold becomes

$$q''=\frac{\sigma(T_1^4-T_2^4)}{1/\epsilon_1+1/\epsilon_2-1}.$$

The denominator is not simply the reciprocal of the product of emissivities because photons can reflect repeatedly before absorption. In the black limit both emissivities are one and the cold plate's own emission remains in the numerator. At equal temperature flux vanishes even when emissivities differ. Measuring only outgoing hot-side radiance without reflected incoming light cannot determine heat loss.

For $T_1=500$ K, $T_2=300$ K and emissivities 0.80 and 0.60, the blackbody power difference is 3085 W/m² and the denominator 1.9167, giving about 1609 W/m². Place an opaque thermally floating shield between the plates, with both faces of emissivity 0.10 and negligible conduction to its supports. The same steady flux crosses both gaps. The two gap resistances add in series, with total dimensionless denominator $1/\epsilon_1+2/\epsilon_s+1/\epsilon_2-2=20.9167$, leaving about 147.5 W/m².

Check the underlying blackbody emission and view-factor balances in the radiation chapter

## Determine its temperature from the steady flux

For the above example the first gap denominator is $1/0.80 + 1/0.10 - 1 = 10.25$ and the second $10.6667$. Write $E_{bs} = E_{b1} - q''D_1$ or $E_{bs} = E_{b2} + q''D_2$. The shield temperature is about 435 K. It is a weighted average of fourth powers, not the arithmetic average of temperatures. A measured shield temperature without net heat flux generally constrains resistance ratios, while the flux supplies their sum.

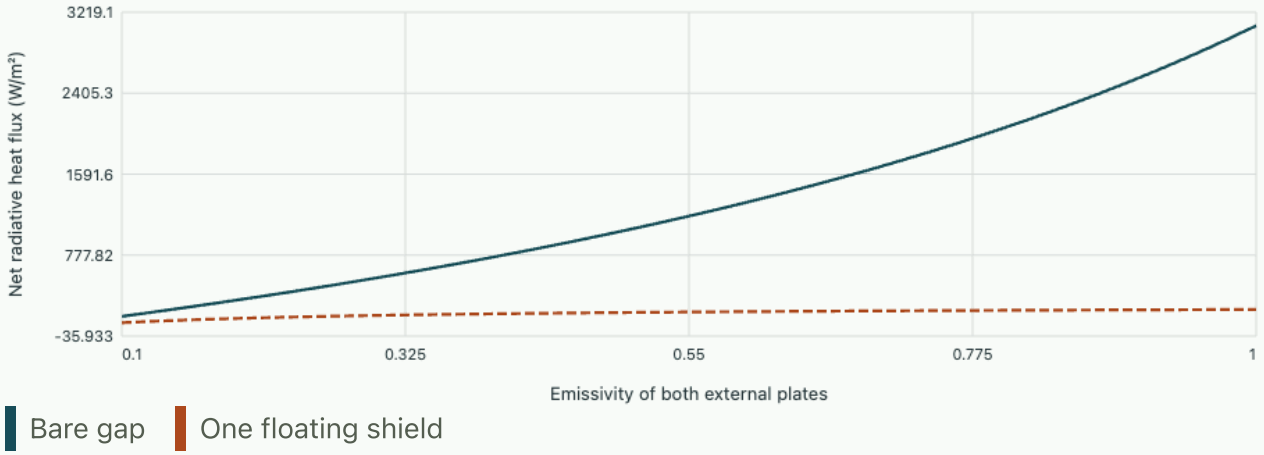
## Use a floating shield as a two-sided calorimeter

Insert an opaque isothermal sheet between the two parallel walls. At steady state its net accumulation vanishes, so energy transferred across the hot gap equals energy leaving the cold gap. For identical external surface emissivities and identical shield-face emissivities, the two radiative resistances of the gaps are the same. Equality of the fourth-power temperature drops then places the shield temperature at the fourth-power mean of the two plate temperatures. It is generally above their arithmetic mean. Lower shield emissivity changes the size of the common heat flow while leaving the intermediate temperature fixed in this symmetric problem.

Open the live experiment

### Try the calculation

| | |
|---|---|
| Hot surface temperature (K) | 500 |
| Cold surface temperature (K) | 300 |
| Shield emissivity, both faces | 0.15 |




**At plate emissivity 0.8 the bare and shielded fluxes are 2056.5 and 223.0 W/m². Both faces of the isothermal shield share temperature 433.5 K.**

Make a paired test using a matte shield and a polished shield of the same dimensions. A heat-flux transducer should respond strongly, while an ideal shield-temperature measurement would indicate the

same equilibrium value. That contrast provides a cleaner test than relying on a single fitted emissivity. A thermocouple lead can conduct heat away from the shield and change its temperature, particularly for the polished sheet when radiative conductance is small. A radiation thermometer aimed at that sheet can receive reflected hot-wall radiation. Use a conducting contact sensor with estimated lead loss and a separately characterised radiance measurement to investigate discrepancies.

If the shield faces have different coatings, reverse it without changing either plate. The series resistance and net heat flow stay the same under the simple diffuse-grey model because the two face resistances enter as a sum. The equilibrium shield temperature changes, however, because the allocation of resistance to the hot and cold gaps changes. This distinguishes an orientation-dependent temperature from an orientation-dependent heat flux.

## Tutorial · Can two identical black plates benefit from a black shield?

Keep the same temperatures but make both plates and both shield faces black. Calculate flux with and without one floating shield. Would replacing one shield by two double the benefit again?

▼ **Count separate spaces and solve their fluxes**

Without a shield the net flux is about 3085 W/m². With one black shield two identical gap resistances divide it in half, about 1542 W/m². With two black shields there are three equal resistances and flux is one third of the unshielded value, about 1028 W/m². Each plate emits and absorbs on its own faces even when black. Equal changes in the count of shields add equal resistances in this ideal case rather than multiplying the previous flux by one half each time.

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## Tutorial · Test a view-factor proposal by reciprocity

Two surfaces have areas 0.020 and 0.050 square metres. A ray calculation gives $F_{12} = 0.70$. What should $F_{21}$ be and why cannot the uncovered row sum automatically be assigned to the same neighbouring surface?

▼ **Use the area-weighted symmetry**

Reciprocity gives

$$F_{21} = 0.020(0.70)/0.050 = 0.28.$$

Row remainders are respectively 0.30 and 0.72. They can describe additional walls, an opening or a surface's self-view if geometry permits. Two convex finite unequal plates alone do not form a closed enclosure with unit rows. Give their surrounding environment a temperature and surface properties or prescribe incoming irradiation before computing the net powers.

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# Tutorial · Infer an omitted conductive heat leak

An instrument with the example shield and plate area 0.10 square metres reports net heat input of 19 W at the hot wall after other losses are removed. Assume the ideal radiation estimate holds. Estimate the parallel conductance directly connecting hot and cold walls and propose a second test.

**▼ Compare absolute powers and their different temperature scaling**

Predicted radiation is about 14.75 W, leaving 4.25 W not explained. At a 200 K temperature difference a direct linear bypass would require $0.0213$ W/K. This could represent support conduction but is not unique: emissivity errors, finite plate-edge view to surroundings or residual gas conduction can also contribute. Repeat at a different mean absolute temperature while holding the same wall-temperature difference. Linear solid conduction with weak property variation remains similar, whereas radiative transfer grows strongly with mean temperature. Monitor the floating shield temperature too; direct bypass need not heat it, whereas a support attached to the shield changes its equilibrium.

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Lienhard and Lienhard · A Heat Transfer Textbook, radiation exchange in enclosures

# Thermal processes and transport in engineering · 4. Periodic heating as a measurement of thermal diffusivity

*Two buried thermometers see the same applied surface temperature oscillation with different amplitudes and phases. Derive how a solid filters that oscillation, recover diffusivity independently from both observations and identify when a rear boundary or a poor sensor contact changes the inference.*

## Choose the decaying branch of a complex thermal wave

A homogeneous solid fills $x > 0$ and has constant conductivity $k$, density $\rho$ and heat capacity $c_p$. For small one-dimensional disturbances its temperature excess obeys $\partial_t \theta = \alpha \partial_x^2 \theta$, $\alpha = k/(\rho c_p)$. Impose sinusoidal surface temperature with amplitude $A$ and angular frequency $\omega$, and wait until the initial transient has decayed. With a positive-exponential time convention look for $\theta = \mathrm{Re}[A e^{-sx + i\omega t}]$. Then $s^2 = i\omega/\alpha$, and boundedness far from the wall selects

$$s = (1 + i)/\delta,$$

$$\delta = \sqrt{2\alpha/\omega},$$

$$\theta(x, t) = A e^{-x/\delta} \cos(\omega t - x/\delta).$$

One penetration depth produces both an amplitude factor $e^{-1}$ and a phase lag of one radian. This is diffusive propagation: it is not an undamped travelling wave and its phase velocity depends on frequency. Temperature is continuous at the surface condition, while required input heat flux has complex amplitude $ksA$. Its magnitude is $kA\sqrt{\omega/\alpha}$ and its phase leads surface temperature by 45 degrees. A known heat flux and surface amplitude can therefore supply information about thermal effusivity $\sqrt{k\rho c_p}$, distinct from diffusivity.

## Extract a complex slope from two thermometer depths

For two sensors separated by known $\Delta x$, the amplitude ratio gives $\log(A_1/A_2) = \Delta x/\delta$ and unwrapped phase difference gives

$$\Delta\phi = \Delta x/\delta.$$

Either yields $\alpha = \omega(\Delta x)^2/(2\Delta\phi^2)$ when amplitude log is substituted for phase if desired. A common unknown drive amplitude and common timing phase cancel. Different sensor gains or unequal sensor time constants do not cancel automatically.

For a solid with $\alpha = 1.0 \times 10^{-5}$ square metres per second heated at 0.10 Hz, $\delta = 5.64$ mm. Sensors 3.0 mm apart should display amplitude ratio $A_2/A_1 = 0.588$ and lag 0.532 rad, about 30.5 degrees and 0.847 s. Increasing frequency by a factor four halves penetration depth and doubles the phase lag at the same separation. The signal at depth falls more strongly and may eventually become noise-dominated although the theoretical phase difference grows.

Compare time-dependent wall-temperature inference with a steady-stage energy balance

## A finite slab creates a returning thermal disturbance

The half-space solution is valid only if the far boundary is several penetration depths beyond the region of interest and if lateral heat losses or heat spreading remain small. In a slab of thickness $L$ with its back face held at zero temperature excess, an exact periodic solution satisfying both faces is proportional to $\sinh[s(L-x)]/\sinh(sL)$. An insulating back instead gives $\cosh[s(L-x)]/\cosh(sL)$. Both reduce to the decaying exponential far from the back at sufficiently high frequency. Fitting the half-space law at very low frequency in a thin specimen can bias diffusivity even with perfect temperature readings.

## Use the time traces to separate a material delay from a sensor delay

In the experiment the two displayed signals share a clock and a known drive amplitude. Read off the time difference between successive maxima, multiply it by angular frequency, and compare with the logarithm of the two amplitudes. Both dimensionless readings estimate depth divided by penetration depth. Increase drive frequency while keeping all geometry fixed. Thermal propagation predicts the same square-root frequency scaling for both quantities. An electronic timing offset instead contributes a phase proportional to frequency and leaves amplitude unchanged.

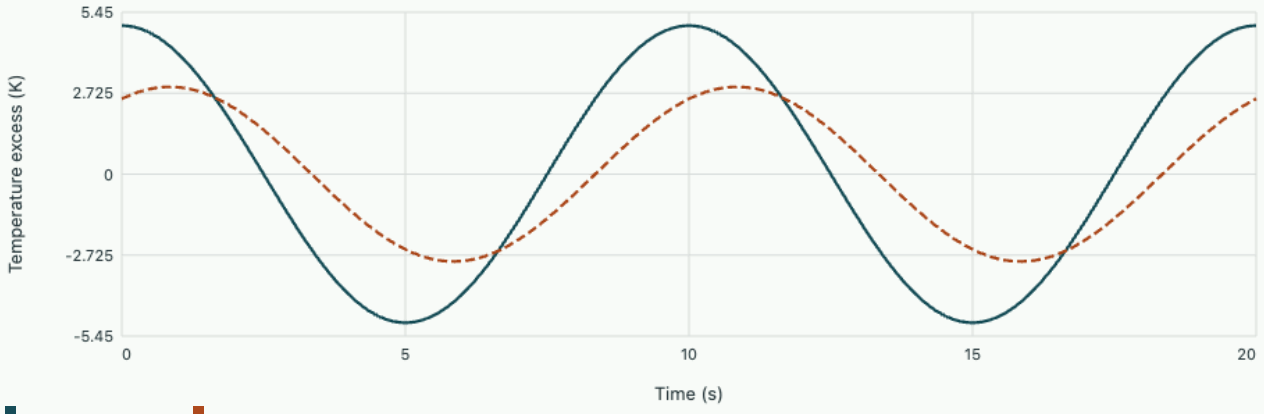
Open the live experiment

### Try the calculation

Drive frequency (Hz) **0.1**

Thermal diffusivity (mm²/s) **10**

Sensor separation (mm) **3**




**Penetration depth is 5.64 mm. At the selected depth the amplitude ratio is 0.588 and the phase lag is 30.5°.**

A thermistor embedded in a drilled hole may exchange heat with adhesive and its own leads. A simple first-order sensor with time constant one second has gain magnitude about 0.847 and phase lag about 32 degrees at one tenth hertz. That is already comparable with the nominal three-millimetre material phase in the example. Exchange the two sensors at the same holes after a baseline run. A phase shift which follows the sensor points to unequal time response; one that stays attached to the location is more consistent with diffusion or a location-dependent contact layer.

To compare a finite solid with the semi-infinite model, estimate the reflected thermal contribution from the far boundary. A disturbance travelling a distance equal to five penetration depths decays in amplitude by a factor near 0.0067. A boundary much nearer than that may noticeably alter phase and attenuation, particularly for a low drive frequency. The relevant distance for a reflected disturbance includes both the outward and return journey. Repeating the measurement at two thicknesses can reveal this effect while preserving material properties and sensor spacing.

## Tutorial · Check two independent diffusivity estimates

A run at 0.20 Hz with 2.0 mm separation gives amplitude ratio 0.600 and lag 0.500 rad. Infer diffusivity separately from the two measurements and decide whether a simple phase-wrap error is likely.

▼ **Invert the attenuation and lag laws**

The amplitude logarithm is 0.5108. Its estimate is $\alpha = 2\pi(0.20)(0.002)^2/[2(0.5108)^2] = 9.63 \times 10^{-6}$ square metres per second. The phase estimate is $1.01 \times 10^{-5}$ in the same units. The small difference is plausible with roughly percent-scale amplitude or phase uncertainty. A full missed wrap would add $2\pi$ and change the inferred diffusivity dramatically. Repeat at adjacent frequencies to establish a continuous unwrapped phase rather than selecting a branch from a single fit.

---

## Tutorial · Separate conductivity and volumetric heat capacity

A second measurement of imposed surface heat-flux amplitude and temperature identifies effusivity $e = 1.50 \times 10^4$ in SI units. Diffusivity is $1.0 \times 10^{-5}$ square metres per second. Recover conductivity and volumetric heat capacity.

▼ **Combine the product and ratio of material properties**

Write $e = \sqrt{kC}$, $\alpha = k/C$, where $C = \rho c_p$. Then $k = e\sqrt{\alpha} = 47.4$ W/(m K) and $C = e/\sqrt{\alpha} = 4.74 \times 10^6$ J/(m³ K). Neither a temperature attenuation nor a phase slope alone identifies both quantities. Surface convection, heater thermal mass and contact resistance can distort the inferred heat actually entering the specimen, so an electrical heater input must be corrected before assigning the effusivity.

---

## Tutorial · Design a frequency window for a finite coupon

A 20 mm specimen has its second sensor at depth 5 mm. A designer wants at least three penetration depths from that sensor to the back wall and a sensor-depth amplitude exceeding one tenth of the surface amplitude. Find an approximate interval of allowable penetration depths, then convert it to frequencies for the example diffusivity.

▼ **Satisfy attenuation and boundary constraints simultaneously**

The back-wall criterion gives

$$\delta \le (20 - 5)/3 = 5.0\,\mathrm{mm}.$$

The signal criterion gives $e^{-5\,\mathrm{mm}/\delta} > 0.10$, or $\delta > 2.17$ mm. Since $f = \alpha/(\pi\delta^2)$, the interval is approximately 0.127 to 0.675 Hz. These are design heuristics for the stated geometry and signal requirement, not universal boundaries of diffusion theory. A finite-slab calculation at both endpoints can quantify actual back-boundary error before committing to the measurement.

---

Lienhard and Lienhard · A Heat Transfer Textbook, transient conduction and penetration depth

# Thermal processes and transport in engineering · 5. What temperature does a probe in a hot gas measure?

*A small bead in a duct exchanges energy with both the gas and the duct walls. Calculate the indicated temperature from an energy balance, infer the gas temperature from the reading, and test how emissivity changes the measurement.*

## Draw the control surface around the bead

Let the gas have temperature $T_g$ and the bead a uniform temperature $T_s$. A large, isothermal enclosure at $T_w$ fills the bead's field of view. Use kelvins throughout. Convection supplies the bead with heat at $hA(T_g - T_s)$. A diffuse grey bead emits $\epsilon\sigma A T_s^4$ and absorbs $\epsilon\sigma A T_w^4$ from the enclosure. The absorptivity and emissivity are equal for the grey surface. The common area $A$ cancels from the steady energy balance when axial conduction through the support is negligible.

$$F(T_s) = h(T_g - T_s) - \epsilon\sigma(T_s^4 - T_w^4) = 0.$$

For a hotter gas, $F(T_w)$ is positive, $F(T_g)$ is nonpositive, and $F'(T_s) = -h - 4\epsilon\sigma T_s^3$ is strictly negative. There is exactly one equilibrium between the two reservoir temperatures. A colder gas reverses their ordering and keeps the same argument. This physical bracket gives a bisection solver two trustworthy endpoints. With positive heat capacity, a small rise in bead temperature makes the net heat input negative, which restores equilibrium.

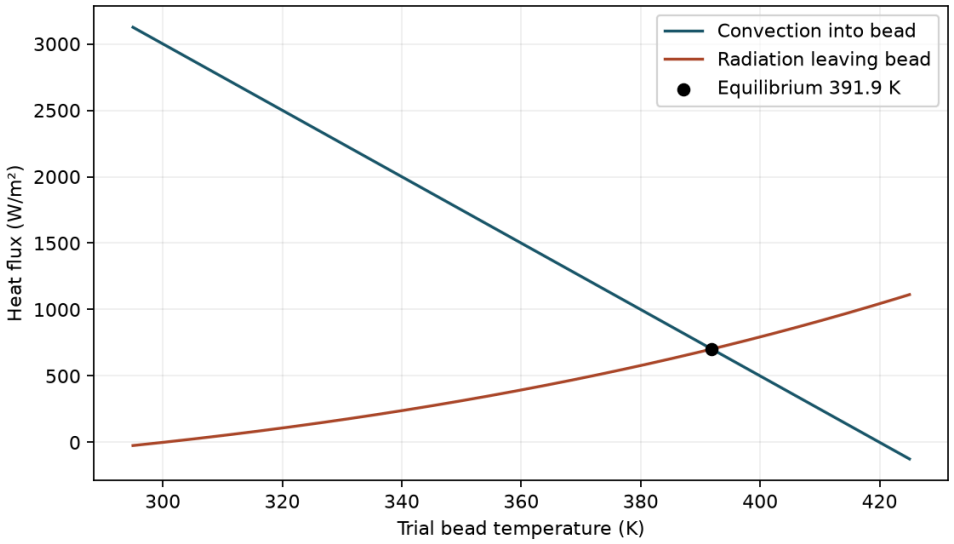
The factorization $T_s^4 - T_w^4 = (T_s - T_w)(T_s + T_w)(T_s^2 + T_w^2)$ gives an exact coefficient $h_r = \epsilon\sigma(T_s + T_w)(T_s^2 + T_w^2)$. At the root,

$$T_g - T_s = h_r(T_s - T_w)/h.$$

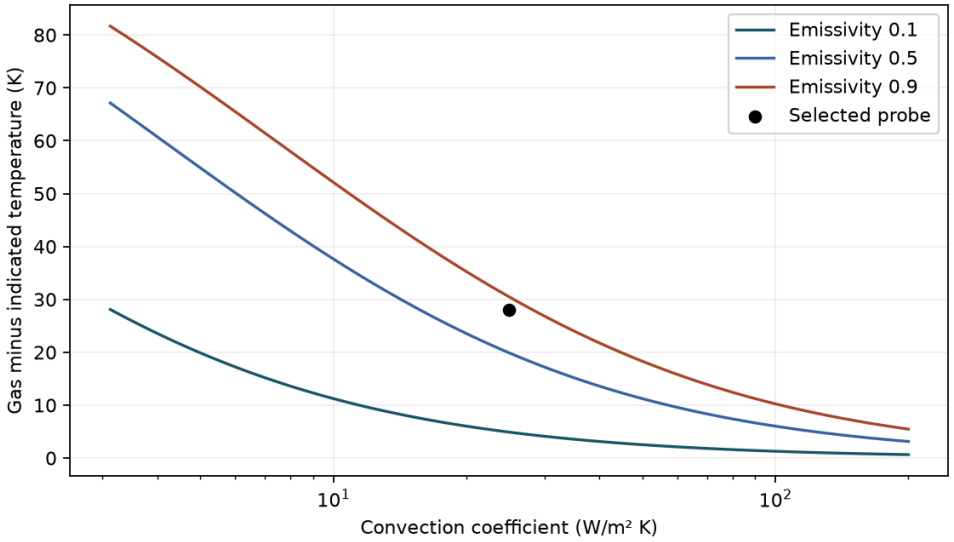
A shiny surface and a fast gas stream therefore act on different terms of the same equation. A larger bead changes the transient response through its mass-to-area ratio, and may also change $h$. Area alone cannot correct the steady reading at fixed $h$.

## Predict the indicated temperature

Take $T_g = 420$ K, $T_w = 300$ K, $h = 25$ W m$^{-2}$ K$^{-1}$ and $\epsilon = 0.8$. At 390 K, convection supplies 750 W m$^{-2}$ and the radiative loss is approximately 682 W m$^{-2}$. The bead thus warms. At 400 K the convective input has fallen to 500 W m$^{-2}$ while radiative loss reaches 794 W m$^{-2}$. The intersection gives $T_s = 391.90$ K, an under-reading of 28.10 K. The two independently evaluated fluxes at equilibrium agree to round-off. Doubling $h$ gives 403.34 K for the same gas and wall. Metal lead wires connected to a cold mount could supply a third heat path, which the two-flux residual cannot independently diagnose.




Follow the curves to the same bead temperature. Their vertical separation is the energy available to warm or cool the bead.




Track one surface along its curve to change the flow, or hold the heat-transfer coefficient fixed and compare surface finishes.

# Problem 1 · Correct a recorded temperature

The probe reads 400 K. The walls remain at 300 K, $h = 25$ W m$^{-2}$ K$^{-1}$ and $\epsilon = 0.8$. Recover the gas temperature without iteration. How would a hotter enclosure affect the sign of this correction?

**▼ Work through the solution**

Rearranging the balance gives

$$T_g = T_s + \epsilon\sigma(T_s^4 - T_w^4)/h.$$

The correction is $793.85/25 = 31.75$ K, giving 431.75 K. If the walls were hotter than the bead, absorbed radiation would exceed emitted radiation and the correction to the reading would be

negative. A Celsius temperature can be reported at the end after the fourth powers have been evaluated in kelvins.

---

## Problem 2 · Specify a minimum convection coefficient

For the original gas at 420 K, find $h$ required for the under-reading to be at most 5 K. Then reduce the emissivity to 0.2 and repeat. Explain the ratio.

▼ **Work through the solution**

At the limiting reading $T_s = 415$ K the required coefficient is $\epsilon\sigma(415^4 - 300^4)/5$. For $\epsilon = 0.8$ this is approximately 196 W m$^{-2}$ K$^{-1}$. The required coefficient scales directly with emissivity at this prescribed reading, so the lower-emissivity bead needs about 49 W m$^{-2}$ K$^{-1}$. Above those coefficients the gas transfers more heat at the limiting temperature and drives the equilibrium closer to 420 K.

---

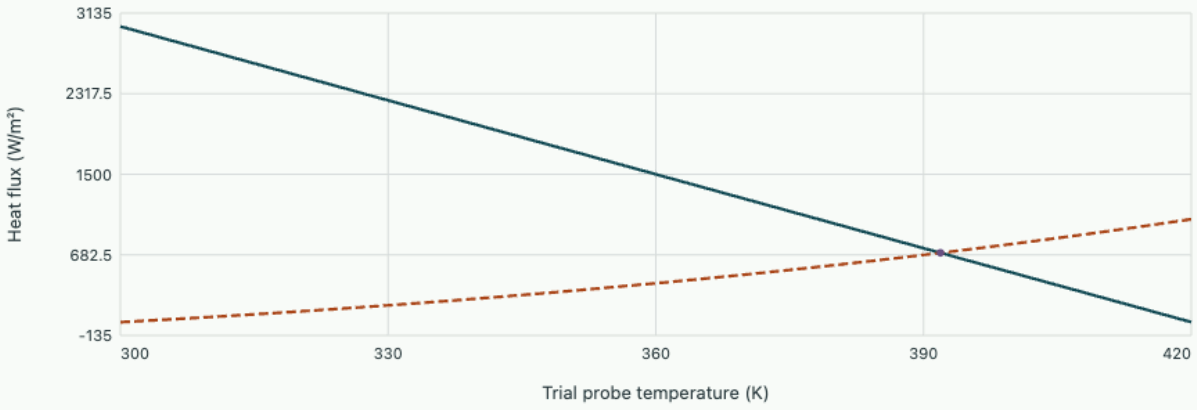
**Interactive calculation · Probe balance**

Change the physical and numerical inputs of the notebook model. Compare the updated curves and balance checks with the recorded notebook figures. Each plot uses the same equations and assumptions documented in that notebook; its readout names any analytical or reduced-model reference.

Open this live calculation

## Try the calculation

| | |
|---|---|
| Gas temperature (K) | **420** |
| Wall temperature (K) | **300** |
| Film coefficient (W/m² K) | **25** |
| Probe emissivity | **0.8** |




**Indicated temperature 391.90 K; bias 28.10 K. The equilibrium balances convection and grey-surface radiation.**

Download this example's editable notebook

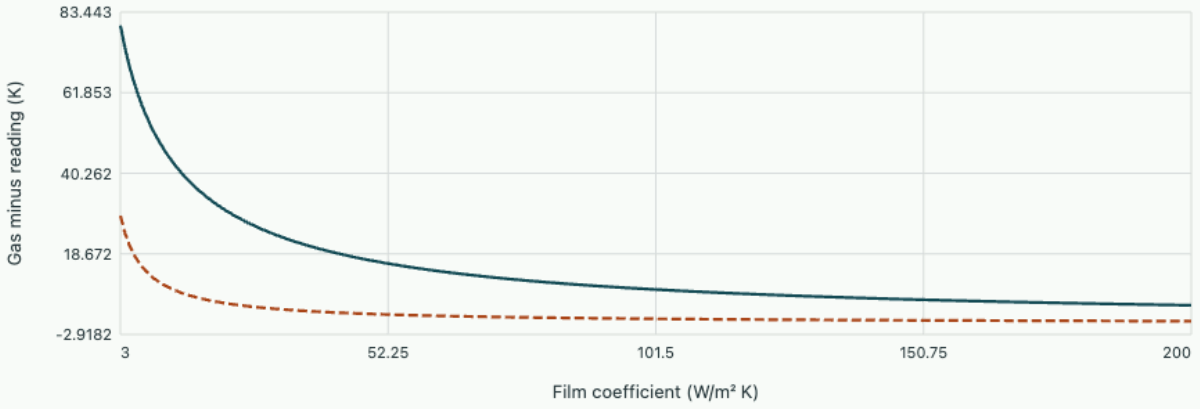
**Interactive calculation · Probe bias**

Change the physical and numerical inputs of the notebook model. Compare the updated curves and balance checks with the recorded notebook figures. Each plot uses the same equations and assumptions documented in that notebook; its readout names any analytical or reduced-model reference.

Open this live calculation

## Try the calculation

| | |
|---|---|
| Gas temperature (K) | **420** |
| Wall temperature (K) | **300** |
| Emissivity | **0.8** |




**At25 W/(m² K), bias 28.10 K. More convective coupling or less radiative coupling moves the reading toward the gas.**

Download this example's editable notebook

**Open editable notebook** Download .ipynb

# Thermal processes and transport in engineering · 6. Where does the temperature of a reacting gas come from?

*A heater in an ideal cycle can supply a prescribed amount of heat. A steady combustor converts chemical enthalpy into sensible enthalpy. Compute a state from atomic balances and show where the complete-conversion model begins to overpredict its temperature.*

## Element balances determine the complete-conversion products

Choose one mole of methane as the feed basis, ideal dry air represented by oxygen plus 3.76 nitrogen molecules per oxygen molecule, and an air ratio $\lambda$ relative to stoichiometric. For $\lambda \geq 1$ and complete conversion to carbon dioxide and water vapour, balance carbon, hydrogen, oxygen and nitrogen separately:

$$\mathrm{CH_4} + 2\lambda(\mathrm{O_2} + 3.76\,\mathrm{N_2}) \longrightarrow \mathrm{CO_2} + 2\mathrm{H_2O} + 2(\lambda - 1)\mathrm{O_2} + 7.52\lambda\mathrm{N_2}.$$

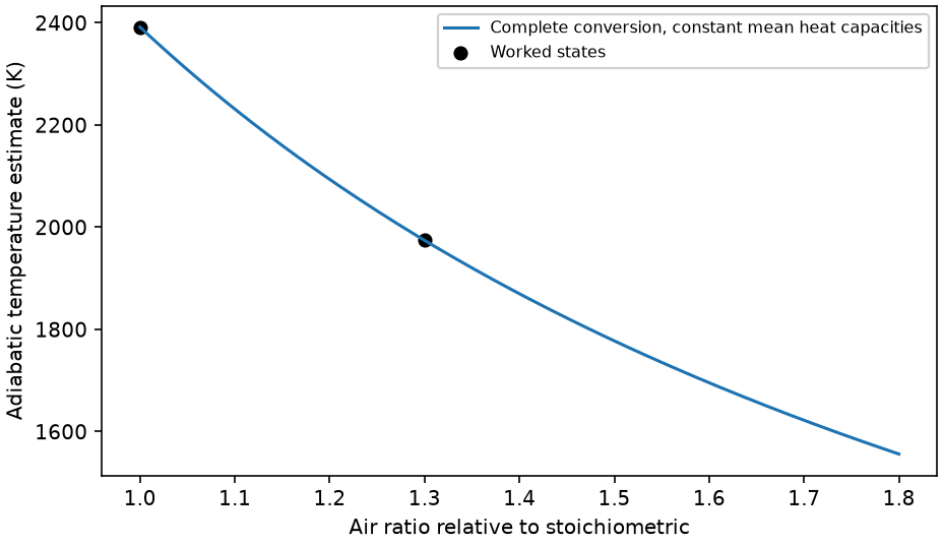
This balances atoms even though individual chemical species are created and consumed. At $\lambda = 1.3$ the products include 0.6 mol oxygen and 9.776 mol nitrogen per mole of fuel. The equivalence ratio is $1/\lambda$ for this feed. If oxygen is deficient, the assumed set of products cannot satisfy these balances with nonnegative amounts.

## Use a common reference for chemical and sensible energy

In steady adiabatic operation without shaft work or appreciable kinetic-energy change the inlet and outlet enthalpy flows are equal. For an ideal gas write $h_i(T) = \Delta h^\circ_{f,i}(T_r) + \int_{T_r}^{T} c_{p,i}(u)du$. Elements in their reference states have zero standard formation enthalpy. Water vapour and liquid water have different formation enthalpies. Use the product phase appearing in the balance.

$$\sum_{p} n_i \int_{T_r}^{T_{ad}} c_{p,i}(u)\,du = -\Delta H^\circ_{rxn}(T_r) + \sum_{r} n_i \int_{T_r}^{T_{in,i}} c_{p,i}(u)\,du.$$

Use feeds at 298 K and an illustrative heat release of 802.3 kJ/mol of methane for vapour water. Take constant mean product heat capacities 51, 42, 35 and 33 J/(mol K) for carbon dioxide, water, oxygen and nitrogen, respectively. For stoichiometric air the total is $51 + 2(42) + 7.52(33) = 383.16$ J/K per mole of fuel. The resulting estimate is 2392 K. With 30% excess air the sum rises to 478.61 J/K and the estimate falls to 1974 K. Both are deliberately conditional on constant averaged heat capacities and the assumed product set.




Additional air raises the heat capacity of the products. Both plotted states assume complete conversion with water vapour and the mean heat capacities given above.

## Close energy and composition together

At high temperature allow the product reaction $\mathrm{CO_2} \rightleftharpoons \mathrm{CO} + \tfrac{1}{2}\mathrm{O_2}$ to proceed by extent $\xi$. Replace carbon dioxide by $1-\xi$, add $\xi$ carbon monoxide, and add $\xi/2$ oxygen. For ideal gases define $a_i = y_i p/p^\circ$. Chemical equilibrium and the energy balance are now two equations for $T$ and $\xi$:

$$\frac{a_{CO}\,a_{O_2}^{1/2}}{a_{CO_2}} = K_p(T),$$

$$\sum_i n_i(\xi)\,h_i(T) = H_{in}.$$

Dissociation stores energy in chemical form, altering temperature and oxygen content. With one illustrative reaction the calculation isolates that effect. A quantitative high-temperature property model requires all important species, pressure and temperature-dependent thermochemical data. The physical root must have nonnegative species amounts.

> **Is there heat crossing the wall?**
>
> In this adiabatic model wall heat transfer is zero. Formation enthalpies carry the chemical energy with the flowing matter. Adding a separate fuel heating value to a balance that already includes formation enthalpies counts the same conversion twice.

# Solve two independent reactions and the energy equation numerically

Use the preceding methane basis with stoichiometric air, at 298.15 K and initially 1 bar. Permit the carbon reaction above with extent $x$ and the water dissociation $\mathrm{H_2O} \rightleftharpoons \mathrm{H_2} + \tfrac{1}{2}\mathrm{O_2}$ with extent $y$. The product amounts in the order carbon dioxide, carbon monoxide, steam, hydrogen, oxygen and nitrogen are

$$n = (1 - x,\; x,\; 2 - y,\; y,\; (x + y)/2,\; 7.52).$$

These amounts preserve all four elemental inventories for any admissible pair. At an air ratio above one, add $2(\lambda - 1)$ mol oxygen and use $7.52\lambda$ mol nitrogen. Both reactions share the same oxygen activity. Solving them as independent single-reaction mixtures would give two incompatible oxygen partial pressures.

In the stated species order use standard formation enthalpies $(-393.52, -110.53, -241.83, 0, 0, 0)$ kJ/mol and standard entropies $(213.79, 197.66, 188.84, 130.68, 205.15, 191.61)$ J/(mol K) at $T_r = 298.15$ K. For this pedagogical approximation use constant mean heat capacities $(51, 33, 42, 31, 35, 33)$ J/(mol K) over the whole interval. The inlet formation enthalpy is −74.87 kJ per mole of methane. The common pressure reference is 1 bar. For each species integrate $c_p$ for enthalpy and $c_p/T$ for entropy. With reaction stoichiometric vectors $\nu_j$, compute

$$h_i = h^\circ_{f,i} + c_{p,i}(T - T_r),$$

$$s^\circ_i = s^\circ_{i,r} + c_{p,i}\ln(T/T_r),$$

$$\ln K_j = -\frac{\sum_i \nu_{ji}(h_i - T s^\circ_i)}{RT}.$$

Enforce $\sum_i \nu_{ji}\ln[(n_i/\sum n)(p/p^\circ)] - \ln K_j = 0$ for both reactions and $\sum n_i h_i - H_{in} = 0$ for energy. The notebook bounds $x$ between zero and one and $y$ between zero and two. It scales the energy residual by 100 kJ and verifies both equilibrium residuals and the atomic balances independently. At 1 bar the solution has $x = 0.11117$, $y = 0.03649$ mol and $T = 2286.01$ K, about 106 K below the corresponding complete-conversion estimate. Products contain 0.07383 mol oxygen per mole of methane even though the feed was stoichiometric.

**Interactive experiment · Change the pressure and air ratio of an adiabatic mixture**

Move the pressure slider and compare the recomputed equilibrium temperature with the complete-conversion curve. The air-ratio slider places an operating point on that curve and reports the two reaction extents. Every update solves the same two chemical-equilibrium equations and enthalpy balance specified above, using damped Newton steps in the browser. A bounded composition parametrisation keeps all species amounts positive.

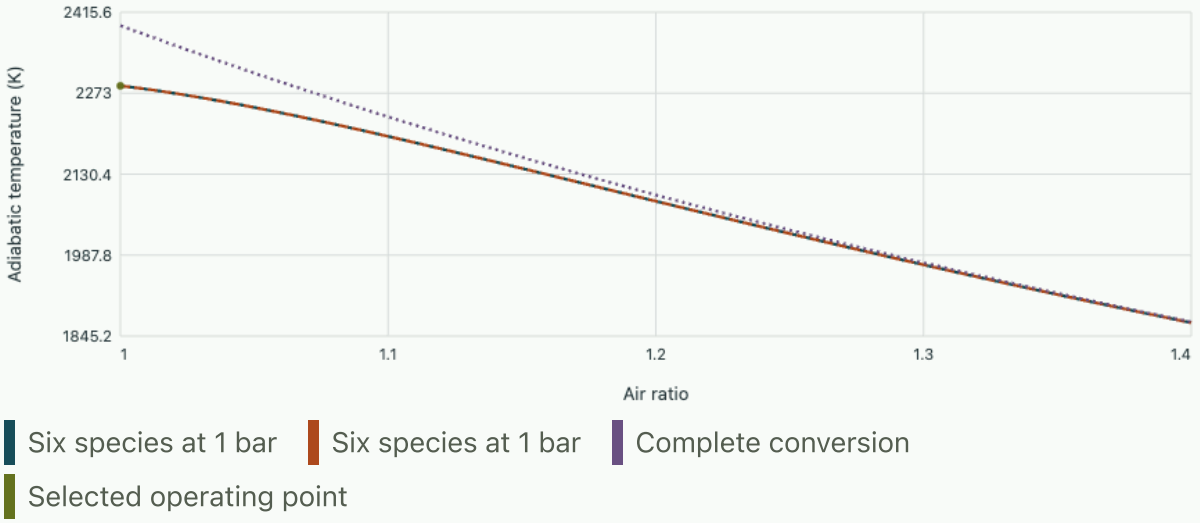
At stoichiometric feed, raise pressure from 1 to 5 bar and explain why both dissociation reactions are suppressed: each increases the total number of gas molecules. Now add excess air and identify where dilution dominates the temperature change. The readout includes a residual check on the solved equations. The fixed mean heat capacities and six-species product set are deliberately limited thermochemical assumptions, shared with the downloadable OneSimulation model.

Open this experiment online

## Try the calculation

Air ratio **1**

Total pressure (bar) **1**




**Selected state: 2286.01 K; CO extent 0.11117 mol; $H_2$ extent 0.03649 mol per mole of methane. Dissociation lowers the complete-conversion estimate by 106.07 K. Scaled equilibrium/energy residual 1.8e-13. Constant heat capacities and six ideal-gas species, as in the notebook.**

Continue in the OneSimulation notebook · Download its Python model

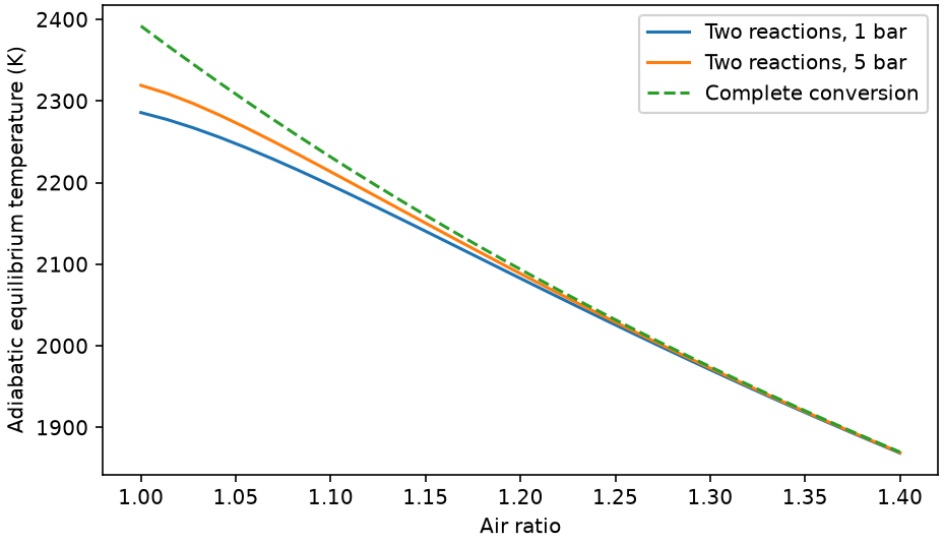
### OneSimulation experiment · Air ratio and total pressure change the temperature of the six-species adiabatic equilibrium mixture

Run the notebook with OneSimulation 0.2.0, NumPy, SciPy and Matplotlib. Change the input data and compare your run with the recorded result.

```
"""Solve a two-reaction, six-product adiabatic equilibrium model for a methane-air stream."""
import numpy as np
import matplotlib.pyplot as plt
from onesim.numerics.nonlinear import LeastSquares

# Species order: CO2, CO, H2O(g), H2, O2, N2. Reference state 298.15 K and 1 bar.
reference = 298.15
gas_constant = 8.314462618  # J/(mol K)
hf = 1000*np.array([-393.52, -110.53, -241.83, 0.0, 0.0, 0.0])  # J/mol
entropy_ref = np.array([213.79,197.66,188.84,130.68,205.15,191.61])  # J/(mol K)
cp = np.array([51.,33.,42.,31.,35.,33.])  # constant teaching-model mean heat capacities
nu = np.array([[-1.,1.,0.,0.,0.5,0.], [0.,0.,-1.,1.,0.5,0.]])
element_counts = np.array([[1,1,0,0,0,0], [0,0,2,2,0,0],
                           [2,1,1,0,2,0], [0,0,0,0,0,2]])
inlet_enthalpy = -74870.0  # J per mole methane, with all reactants at 298.15 K

def equilibrate(air_ratio, pressure_bar):
    complete = np.array([1.,0.,2.,0.,2*(air_ratio-1),7.52*air_ratio])
    def residual(state):
        extents, temperature = state[:2],state[2]
        moles = complete + extents @ nu
        enthalpies = hf + cp*(temperature-reference)
        entropies = entropy_ref + cp*np.log(temperature/reference)
        log_k = -(nu @ (enthalpies-temperature*entropies))/(gas_constant*temperature)
        log_q = nu @ np.log(pressure_bar*moles/moles.sum())
        return np.r_[log_q-log_k, (moles @ enthalpies-inlet_enthalpy)/1e5]
    optimum = LeastSquares(options={"gtol":1e-13,"xtol":1e-13,"ftol":1e-13}).solve(
        residual,[0.06,0.04,2200.0], bounds=([1e-12,1e-12,500.],[1-1e-12,2-1e-12,3500.]))
    assert np.max(np.abs(optimum.fun)) < 1e-8
    moles = complete + optimum.x[:2] @ nu
    assert np.allclose(element_counts @ moles, element_counts @ complete)
    return optimum.x, moles

for pressure in [1.,5.]:
    result, moles = equilibrate(1.0, pressure)
    print(f"{pressure:.0f} bar: T = {result[2]:.2f} K, extents = {result[0]:.5f}, {result[1]:.5f}
mol")
    print("product moles",np.round(moles,5))
ratios = np.linspace(1.0, 1.4, 30)
fig,ax = plt.subplots(figsize=(6.4,3.7),layout="constrained")
for pressure in [1.,5.]:
    temperature = [equilibrate(r,pressure)[0][2] for r in ratios]
    ax.plot(ratios,temperature,label=f"Two reactions, {pressure:g} bar")
complete_temp = reference + 802310/(51+2*42+2*(ratios-1)*35+7.52*ratios*33)
ax.plot(ratios,complete_temp,"--",label="Complete conversion")
ax.set(xlabel="Air ratio",ylabel="Adiabatic equilibrium temperature (K)")
ax.legend()
plt.show()
```

Reference run

```
1 bar: T = 2286.01 K, extents = 0.11117, 0.03649 mol
product moles [0.88883 0.11117 1.96351 0.03649 0.07383 7.52   ]
5 bar: T = 2319.34 K, extents = 0.07711, 0.02403 mol
product moles [0.92289 0.07711 1.97597 0.02403 0.05057 7.52   ]
```




Air ratio and total pressure change the temperature of the six-species adiabatic equilibrium mixture

Download editable notebook · Download Python script

At 5 bar the computed extents are 0.07711 and 0.02403 mol. Reconstruct the oxygen amount and explain why the new temperature, 2319.34 K, is higher.

▼ **Worked solution**

The released oxygen is $(0.07711 + 0.02403)/2 = 0.05057$ mol. Both reactions increase the mole count. At higher pressure their equilibrium extents shrink, leaving more energy as sensible enthalpy. The difference between 2286 and 2319 K follows from the model, not a direct pressure dependence of ideal-gas species enthalpy.

---

**How could this model approach a thermochemical-table calculation?**

Replace the constant heat capacities by temperature-dependent species functions with a common reference, then enlarge the allowed product set to include radical species when required. The elemental constraints and energy residual still apply. The present six-species answer is reproducible for its named property law.

Reference gas data · NIST carbon dioxide · carbon monoxide · water · hydrogen · oxygen · nitrogen · methane.

# Exercises on model closure

At $\lambda = 1.3$ preheat all feed gases so their combined sensible inlet enthalpy rises by 20 kJ per mole of methane. Estimate the temperature change.

▼ **Worked solution**

Holding the product composition and heat capacities fixed gives $20000 / 478.61 = 41.8$ K of additional temperature rise. At equilibrium the reaction extent changes and this simple ratio is only an initial estimate.

---

In an ideal-gas equilibrium calculation, increase total pressure while holding $T$ and composition momentarily fixed. Which way does the dissociation quotient move?

▼ **Worked solution**

The net mole change is +1/2, so the quotient scales as $p^{1/2}$. It rises above its original equilibrium value, favouring recombination. The coupled adiabatic temperature must then be solved again.

---

Further reading · Çengel, Boles and Kanoğlu, Thermodynamics: An Engineering Approach, 10th ed., Chapters 16–17.

# Thermal processes and transport in engineering · 7. How can a flow measurement determine a wall temperature?

*Move from a heat-transfer coefficient supplied as data to one inferred from geometry and flow. Compare a turbulent pipe, a laminar external boundary layer and a plate cooled by buoyancy.*

## Distinguish a defining relation from a predictive law

At an impermeable wall heat crosses the fluid immediately adjacent to the solid by molecular conduction. Define $h$ by equating that flux to the observable temperature difference. In a tube use the bulk mean temperature from the enthalpy flux. On an external plate use the free-stream temperature.

$$q'' = -k\,\partial_n T|_w = h(T_w - T_b),$$

$$Nu_L = hL/k,$$

$$Re_L = UL/\nu,$$

$$Pr = \nu/\alpha.$$

Nondimensionalising the advective and diffusive terms of the thermal energy equation produces $Re\,Pr$. Geometry, boundary conditions, momentum transport and entrance development decide the actual dependence of Nusselt number. The exponents fitted to turbulent data do not follow from dimensional analysis alone.

## A heated smooth tube with developed turbulent flow

For a smooth, sufficiently long, fully developed turbulent tube with moderate property changes, a common first estimate for heating the fluid is $Nu_D = 0.023 Re_D^{0.8} Pr^{0.4}$. Use it here within its usual range, roughly $Re_D \gtrsim 10000$ and $0.7 \lesssim Pr \lesssim 160$. Check that the physical setting and temperature convention match before comparing correlations. For laminar fully developed circular flow the exact constant-wall-temperature and constant-wall-flux limits instead give 3.66 and 4.36, respectively.

At $Re_D = 30000$, $Pr = 5$, $D = 0.020$ m, $k = 0.60$ W/(m K) and imposed $q'' = 9000$ W/m², the turbulent estimate gives $Nu = 167.1$, $h = 5013$ W/(m² K) and $T_w - T_b = 1.80$ K. That temperature rise is small enough that sensor offsets matter. The following notebook fits one prefactor to deliberately perturbed synthetic readings and tests whether the residual size is compatible with thermometer precision.

**OneSimulation experiment · Measured wall superheat decreases with flow rate in a heated tube**

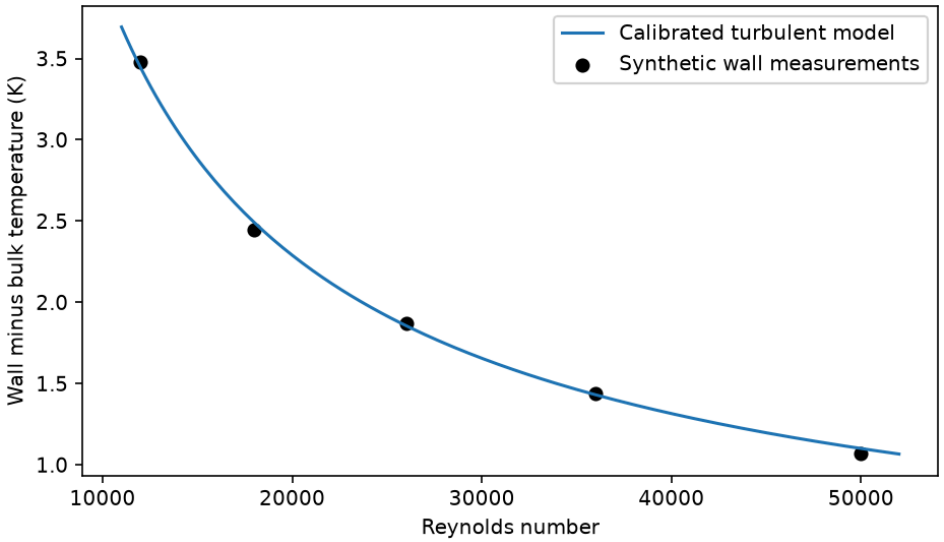
Run the notebook with OneSimulation 0.2.0, NumPy, SciPy and Matplotlib. Change the input data and compare your run with the recorded result.

```
"""Estimate a turbulent tube coefficient using synthetic wall temperature readings."""
import numpy as np
import matplotlib.pyplot as plt
```

```
from onesim.numerics.nonlinear import LeastSquares

reynolds = np.array([12000., 18000., 26000., 36000., 50000.])
q, diameter, conductivity, prandtl = 9000.0, 0.02, 0.60, 5.0
def superheat(coefficient, re):
    nusselt = coefficient*re**0.8*prandtl**0.4
    return q*diameter/(conductivity*nusselt)

readings = superheat(0.025, reynolds)+np.array([0.04,-0.04,0.02,0.01,-0.03])
fit = LeastSquares().solve(lambda c: superheat(c[0], reynolds)-readings,
                           [0.023], bounds=([0.005],[0.08]))
assert fit.success
coefficient = fit.x[0]
print(f"fitted prefactor = {coefficient:.5f}")
print(f"RMS temperature residual = {np.sqrt(np.mean(fit.fun**2)):.4f} K")
print(f"predicted rise at Re=30000 = {superheat(coefficient, 30000):.3f} K")
```

## Reference run

```
fitted prefactor = 0.02494
RMS temperature residual = 0.0299 K
predicted rise at Re=30000 = 1.655 K
```




Measured wall superheat decreases with flow rate in a heated tube

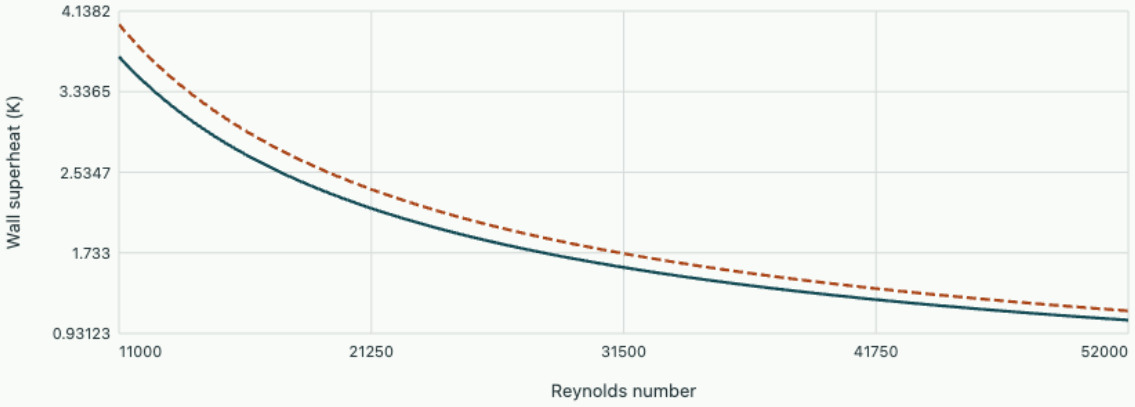
**Interactive calculation · Change a turbulent heat-transfer closure**

Vary heat flux, Prandtl number and correlation prefactor. The fixed synthetic calibration measurements in the notebook determine one prefactor; here you can inspect its predictive consequences.

Open this live calculation

### Try the calculation

| | |
|---|---|
| Heat flux (W/m²) | 9000 |
| Correlation prefactor | 0.025 |
| Prandtl number | 5 |




**At Re30000, predicted wall superheat 1.652 K. Tube diameter20 mm and conductivity0.60 W/(m K) are fixed; use the notebook to fit the prefactor from temperature data.**

Download editable notebook · Download Python script

## Changing the flow geometry changes the correlation

For a smooth isothermal plate with a laminar external boundary layer starting at the leading edge and moderate Prandtl number, use the area-mean value $\overline{Nu}_L = 0.664 Re_L^{1/2} Pr^{1/3}$. A local correlation at the trailing edge has half the prefactor: local and average heat transfer answer different measurement questions. For air with $\nu = 1.6 \times 10^{-5}$ m²/s, $Pr = 0.71$, $k = 0.026$ W/(m K), $L = 0.20$ m and $U = 2$ m/s, the mean result is $h = 12.18$ W/(m² K). At 4 m/s it becomes 17.22 W/(m² K).

For a still fluid by a vertical plate the motion instead follows from thermal expansion. Buoyant acceleration is of order $g\beta\Delta T$. Balancing it against thermal and momentum diffusion supplies $Ra_L = g\beta|T_w - T_\infty|L^3/(\nu\alpha)$. A vertical isothermal plate correlation across laminar and turbulent regimes is

$$\overline{Nu}_L = \left[0.825 + \frac{0.387 Ra_L^{1/6}}{\left[1 + (0.492/Pr)^{9/16}\right]^{8/27}}\right]^2.$$

With the same length, $\Delta T = 20$ K, $\beta = 1/300$ K⁻¹ and $\alpha = 2.25 \times 10^{-5}$ m²/s, $Ra_L = 1.45 \times 10^7$ and $h = 4.51$ W/(m² K). Evaluate properties consistently, usually at a representative film temperature. If both buoyancy and forced motion matter, their orientation can aid or oppose the flow.

**What changes when the wall temperature is unknown?**

For imposed heat flux in free convection, the temperature difference appears both in $h$ and in $q'' = h\Delta T$. Solve the scalar nonlinear balance with updated properties as warranted. Treating $h$ as independent of wall temperature suppresses the coupling driving the motion.

## Exercises on correlation selection

Double the pipe velocity with constant properties inside the turbulent range and unchanged imposed heat flux. How far does the wall rise above the fluid?

▼ **Worked solution**

The coefficient rises by $2^{0.8} = 1.741$. The 1.80 K rise falls to about 1.03 K. A fixed wall-temperature experiment would observe the opposite dependent variable, an increased heat rate.

---

A stationary vertical plate in air has a known 20 K excess temperature. Its height is doubled. May the original coefficient simply be reused?

▼ **Worked solution**

Rayleigh number grows eightfold. Reevaluate the stated correlation at that Rayleigh number and divide the new $Nu$ by the doubled length when recovering $h$. The heat transfer area also changes if width stays fixed.

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Further reading · Bergman and Lavine, Fundamentals of Heat and Mass Transfer, 8th ed., Chapters 6–9.

# Thermal processes and transport in engineering · 8. Why can a heated surface suddenly become much hotter?

*Trace the pool-boiling curve of a saturated liquid. Compare changing a heater's temperature with changing its electrical power and estimate the hydrodynamic scale at which nucleate boiling becomes unstable.*

## Read a boiling curve as a family of states

Plot heat flux $q''$ against excess wall temperature $\Delta T_e = T_w - T_{sat}(p)$ for a saturated liquid at fixed pressure. Natural convection carries small heating rates. When active cavities support vapour embryos, rising bubbles renew liquid at the wall. Nucleate boiling heats the liquid much more effectively and the slope is usually steep. At the critical heat flux vapour structures begin to impede liquid return. The transition branch can have negative slope. Farther along, a continuous vapour film separates liquid from the surface and heat crosses a poorly conducting layer.

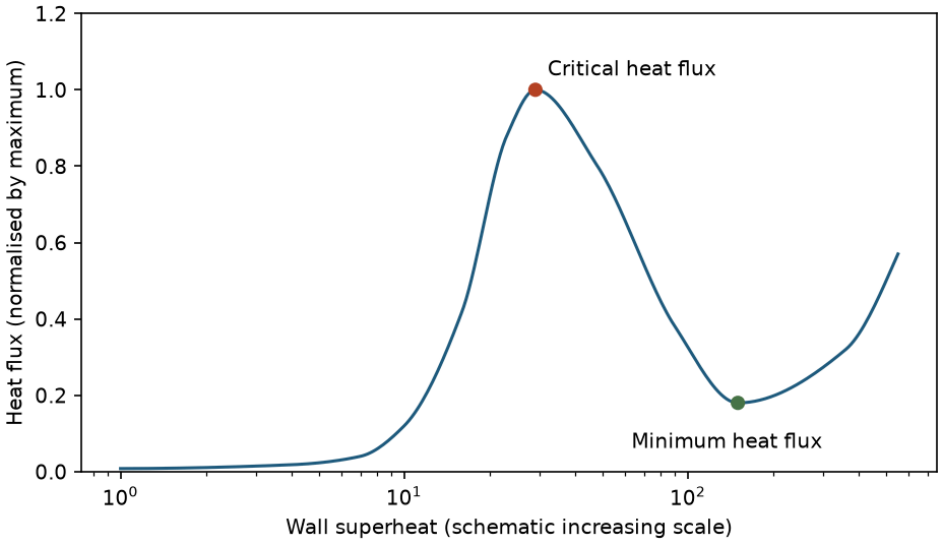
A boiling curve depends on wetting, surface finish, pressure, orientation and heating history. Saturation temperature gives the equilibrium temperature at a given pressure. It is insufficient to determine the local wall superheat or boiling branch. A condensing film and a boiling boundary involve the same latent heat but different interfacial dynamics.

## Balance capillary and buoyant lengths

At a gravity-disturbed interface a pressure difference of order $\sigma/\ell$ competes with a hydrostatic difference $(\rho_l - \rho_v)g\ell$. Their equality gives capillary length $\ell_c = [\sigma/(g(\rho_l - \rho_v))]^{1/2}$. A vapour inertial stress of order $\rho_v u_v^2$ matching the same pressure sets a velocity scale. Multiplication by $\rho_v h_{fg}$ converts that speed to a latent-heat flux scale. A hydrodynamic model for saturated pool boiling on a large horizontal upward-facing plate gives the empirical/geometric coefficient $C \simeq 0.131$:

$$q''_{CHF} = C h_{fg} \sqrt{\rho_v}\, [\sigma g(\rho_l - \rho_v)]^{1/4}.$$

For saturated water near atmospheric pressure take $h_{fg} = 2.257$ MJ/kg, $\rho_l = 958$, $\rho_v = 0.598$ kg/m³ and $\sigma = 0.0589$ N/m. The scale predicts about 1.11 MW/m². This order-of-magnitude estimate pertains to a specific large-surface model, not a universal limit for small heaters or forced flow.




Schematic saturated pool-boiling curve. The marked extrema bound the negative-slope branch. Its coordinates illustrate the topology and are not experimental data.

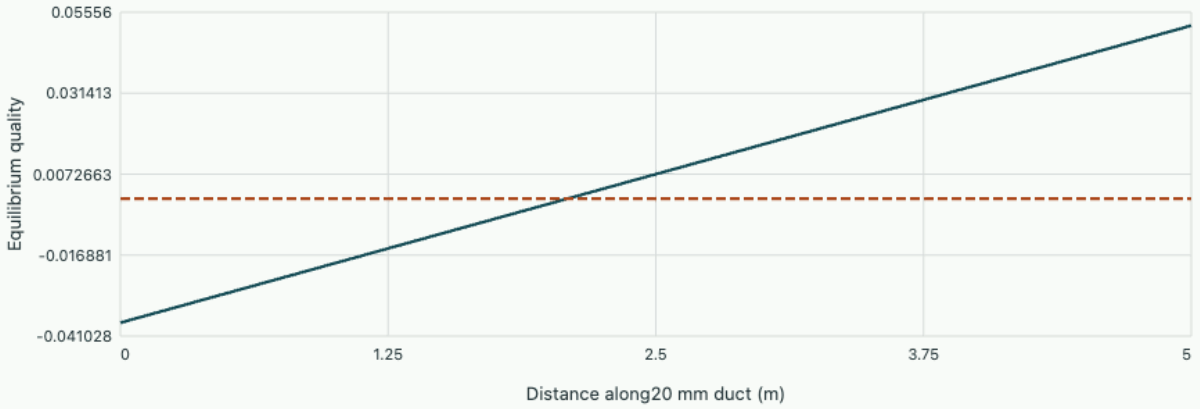
**Interactive calculation · Locate bulk saturation in a heated tube**

Increase heating or decrease mass flux and find where equilibrium quality first reaches zero. This energy balance does not predict the wall-boiling onset or dryout threshold.

Open this live calculation

## Try the calculation

| | |
|---|---|
| Heat flux (kW/m²) | 100 |
| Mass flux (kg/m² s) | 500 |
| Inlet subcooling (K) | 20 |




Equilibrium quality  Bulk saturation

**Bulk saturation begins at 2.090 m. Negative quality denotes sensible subcooling, not a negative vapour fraction. This energy balance alone predicts neither flow pattern nor dryout.**

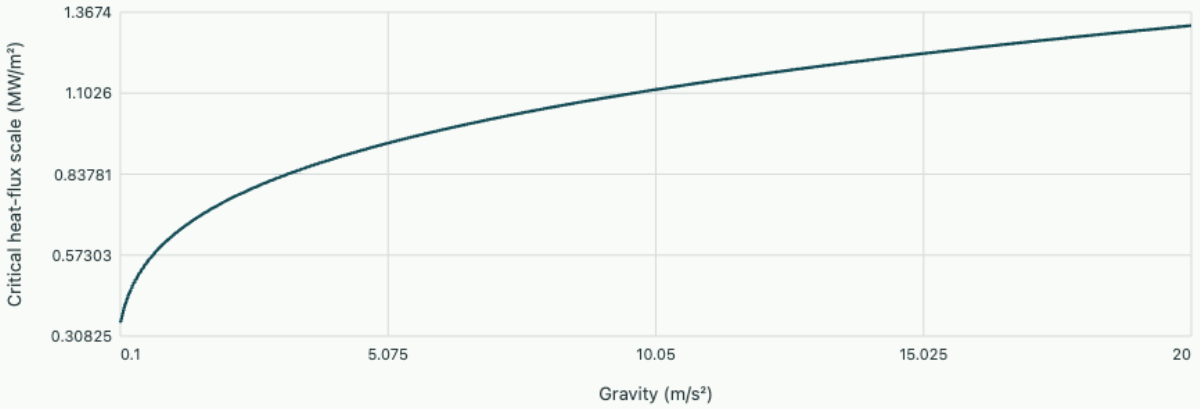
**Interactive calculation · Explore the critical-flux scaling**

Vary one property at a time and compare the gravity dependence. The property inputs are independent scaling experiments; they are not a thermodynamically coupled fluid-property table.

Open this live calculation

### Try the calculation

| | |
|---|---|
| Latent heat (MJ/kg) | **2.25** |
| Surface tension (N/m) | **0.059** |
| Vapour density (kg/m³) | **0.6** |




Large horizontal surface model

**At terrestrial gravity the scale is 1.108 MW/m². Liquid density958 kg/m³ is held fixed; property sliders explore scaling, not a consistent pressure-dependent fluid table.**

## Explain the jump using the heater energy balance

Write $C_A dT_w/dt = q''_{in} - q''_{out}(T_w)$ for heat capacity per area $C_A$. Perturb a steady point by $\delta T$ at fixed input. Then $C_A\, d\delta T/dt = -q'_{out}\delta T$. A positive local slope suppresses a perturbation and a negative slope amplifies it. Consequently an imposed-power sweep can jump past much of the transition branch. A temperature-controlled experiment can in principle visit states that constant-power control cannot stabilize.

**Does a target heat flux determine a unique wall temperature?**

A horizontal line on a nonmonotone boiling curve can intersect more than one branch. Surface state, control mode and the direction of the preceding sweep decide which intersections can be reached.

## Follow a flowing liquid from subcooling into a two-phase tube

For a uniformly heated circular duct, wall area per axial length is $\pi D$ and the mass flow is $G\pi D^2/4$ with mass flux $G$. Neglect pressure change, kinetic-energy change, loss to surroundings and inlet vapour. A differential steady energy balance yields $dh/dz = 4q''/(GD)$. Define the equilibrium quality $x_{eq} = (h - h_f)/h_{fg}$ at the local pressure. Negative values measure subcooling. A positive value describes the mass vapour fraction when both phases are in local thermal equilibrium.

$$z_{sat} = \frac{GDc_{p,l}(T_{sat} - T_{in})}{4q''},$$

$$x_{eq}(z) = \frac{4q''(z - z_{sat})}{GDh_{fg}}.$$

Wall bubbles may form before the bulk reaches saturation and condense in the colder core. Downstream, bubbly or slug flow can give way to an annular liquid film with droplets in a fast vapour core. Dryout of that film and departure from nucleate boiling in a subcooled region are distinct routes to a sharp rise in wall temperature. Orientation, mass flux, pressure, quality and heating history influence the transitions.

## Use boiling and momentum scales before choosing a closure

The boiling number $Bo_q = q''/(Gh_{fg})$ measures vapour generation relative to mass transport. The liquid-only Froude number $Fr_{lo} = G^2/(\rho_l^2 gD)$ compares inertia and gravity. The liquid-only Weber number $We_{lo} = G^2 D/(\rho_l \sigma)$ compares inertia and capillarity. Confinement number $Co = D^{-1}\sqrt{\sigma/[g(\rho_l - \rho_v)]}$ measures capillary length relative to diameter. These definitions specify their reference density. A regime map may instead use superficial velocities $j_g = Gx/\rho_v$ and $j_l = G(1-x)/\rho_l$. Enter that map with its own orientation, geometry and definitions.

For a vertical water tube take $D = 16$ mm, $G = 300$ kg/(m² s), $q'' = 42$ kW/m², 8 K inlet subcooling and representative 1 MPa saturation data $\rho_l = 887$, $\rho_v = 5.15$ kg/m³, $h_{fg} = 2.015$ MJ/kg, $c_{p,l} = 4.4$ kJ/(kg K), $\sigma = 0.042$ N/m. Bulk saturation occurs at 1.006 m. At 3 m the predicted quality is only 0.03464, yet $j_g = 2.018$ m/s and $j_l = 0.327$ m/s. The much smaller vapour density explains why a small mass fraction can dominate volumetric transport. The reference groups are $Bo_q = 6.95 \times 10^{-5}$, $Fr_{lo} = 0.729$, $We_{lo} = 38.7$ and $Co = 0.138$.

For a local illustrative closure suppose a validated superposition fit at this station supplies $h_{lo} = 4500$ W/(m² K), convective enhancement $F = 1.35$, pool reference $h_{nb} = 6500$ W/(m² K) and nucleate suppression $S = 0.60$. Then $h_{tp} = Fh_{lo} + Sh_{nb} = 9975$ W/(m² K) and the required superheat is $42000/9975 = 4.21$ K. Those coefficients are supplied station data. Extending them through flow-regime change would require a validated local closure. A single-phase fit cannot predict film dryout.

At what distance does the simple uniform-pressure model first predict quality 0.10? What additional evidence would justify calling that station a dryout location?

▼ **Worked solution**

The quality slope is $4q''/(GDh_{fg}) = 0.01737$ per metre. Adding $0.10/0.01737$ m to the saturation length gives 6.76 m. Dryout needs evidence of film depletion, measured wall-temperature excursion or a correlation validated for this pressure, mass flux, tube and heated length. A quality of 0.10 alone provides none of these.

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## Exercises on energy and stability

A saturated pool removes 200 kW/m² solely by evaporation. What is the associated mass flux of water vapour?

▼ **Worked solution**

The evaporation rate is $q''/h_{fg} = 200000/2257000 = 0.0886$ kg/(m² s). The fraction of area covered by vapour cannot be inferred from energy alone.

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Near a transition point $q'_{out} = -4000$ W/(m² K) and $C_A = 2000$ J/(m² K). Predict the initial growth time of a small temperature error at fixed power.

▼ **Worked solution**

The linear growth exponent is $-q'_{out}/C_A = 2$ s⁻¹. An initial deviation grows by a factor $e$ in 0.5 s until nonlinear changes of the heat-removal law become significant.

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Further reading · Bergman and Lavine, Fundamentals of Heat and Mass Transfer, 8th ed., Chapter 10, especially sections 10.2–10.5.