Engineering mechanics

Draw a free-body diagram and calculate stress and deformation, then determine whether yielding, instability or a growing crack limits a structure.

Read the subject in depth

Each volume develops its arguments in sequence, with worked problems and links to experiments.

- ### Stress, deformation and structural measurement

  Start from balance, compatibility and material response. Work through structural predictions with alternative load or support conditions and assess the measurements needed to test them.

Advanced study across the subject

Each sequence connects the course to detailed chapters in neighbouring subjects. The complete course PDF includes their derivations, examples and exercises.

1. ### Elastic members assembled into mechanisms

   Choose a load path, propagate its compliance and determine where contact or a bonded connection changes the boundary conditions.

2. ### From stiffness to motion

   Pass from a static deflection to a variational mode estimate and a transient stress wave. Check how choice of interpolation affects the answer.

How the course fits together

1. **Chapters 1–4**

   Draw free bodies and recover support reactions. A cut exposes axial forces, shear and bending moments. A material law and displacement compatibility turn those internal actions into extensions and beam deflections.

2. **Chapters 5–8**

   Stress components combine on an inclined plane. Compressive load can also destabilise a whole member. Torsion and transverse shear explain why the location and direction of the largest stress depend on section geometry.

3. **Chapters 9–12**

   Use compatibility to solve redundant supports and strain measurements to reconstruct the stress state. Plastic redistribution and crack growth introduce two different limits to the elastic load calculation.

Contents

1. Chapter 1 · Free bodies, moments and support reactions
2. Chapter 2 · Axial stress, strain and compatibility
3. Problem sheet 1 · Equilibrium and axial compatibility
4. Chapter 3 · Shear force and bending moment
5. Chapter 4 · Elastic curves and virtual work
6. Problem sheet 2 · Beam actions and deformations
7. Chapter 5 · Stress transformation and combined loading
8. Chapter 6 · Elastic stability of compression members
9. Problem sheet 3 · Strength and stability
10. Chapter 7 · Torsional stiffness of shafts and closed sections
11. Chapter 8 · Transverse shear and connections in built-up beams
12. Problem sheet 4 · Torsion and transferred shear
13. Chapter 9 · Compatibility and redundant supports
14. Chapter 10 · From a strain rosette to principal stresses
15. Problem sheet 5 · Redundants and measured strains
16. Chapter 11 · Yield fronts and plastic bending capacity
17. Chapter 12 · Crack driving force and fracture assessment
18. Problem sheet 6 · Plastic deformation and crack driving force
19. Chapter 13 · From cut forces to local equilibrium
20. Chapter 14 · How a lateral restraint changes a measured modulus
21. Chapter 15 · Radial displacement and hoop stress in a thick ring
22. Chapter 16 · Creep, relaxation and what a holding test measures
23. Chapter 17 · How a continuum balance becomes a finite-element stiffness matrix
24. Chapter 18 · Finding a displacement when no load acts in its direction
25. Chapter 19 · Why a bonded pair of strips bends under uniform heating
26. Chapter 20 · How a beam spreads a point load through an elastic support
27. Problem sheet 7 · Calibrating a compression fixture through three independent readings
28. Problem sheet 8 · What stress does a curved flexure actually carry?
29. Stress, deformation and structural measurement
30. Elastic members assembled into mechanisms
31. From stiffness to motion
32. Applied study
33. Notation and assumptions
34. Further reading

Chapter 1 · Free bodies, moments and support reactions

Replace a distributed load by its force and moment, then determine which reactions a support can supply.

FROM FIRST PRINCIPLES

Derive the force and moment conditions for a free body

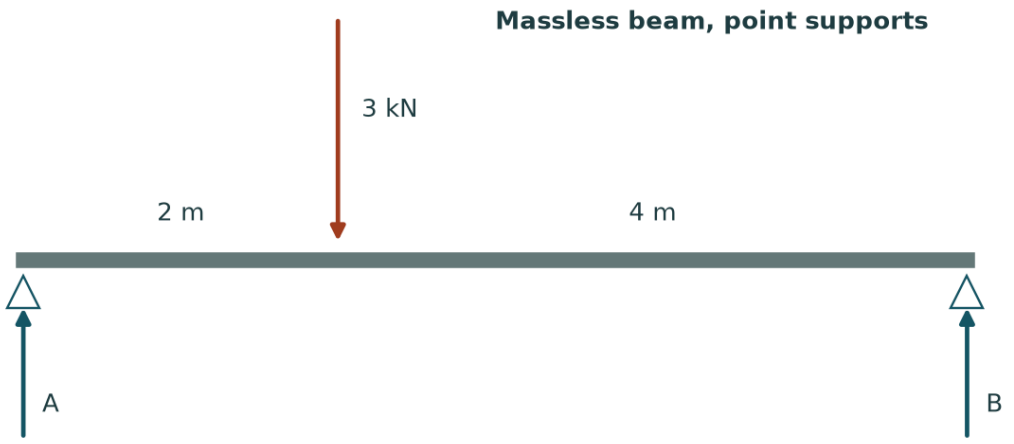
Choose one body and mark the external loads and support reactions crossing its boundary. In an inertial frame, Newton's linear momentum balance for each mass element sums to a balance for the whole body. Internal action–reaction pairs cancel. Taking moments about a fixed point O similarly eliminates internal couples for ordinary continuum forces, leaving

$$\sum \mathbf{F}_{\text{ext}} = \frac{d\mathbf{P}}{dt},$$
$$\sum \mathbf{M}_{O,\text{ext}} = \frac{d\mathbf{H}_O}{dt}.$$

Static equilibrium requires both momenta to remain constant at zero. For planar loading these vector statements reduce to three scalar conditions. A distributed downward load $q(x)$ contributes force and moment by summing its infinitesimal forces with their physical lever arms:

$$\sum F_x = 0,$$
$$\sum F_y = 0,$$
$$\sum M_O = 0,$$
$$W = \int q(x)\,dx,$$
$$W\bar{x} = \int x q(x)\,dx.$$

The equations predict reactions only within the chosen support model. A roller can exert a normal force but no end moment. An upward-only contact cannot supply a negative normal reaction; such a mathematical solution calls for a different contact configuration.



Find the reaction by choosing where to take moments
With the left support as moment centre its own unknown force disappears. The right reaction multiplied by 6 m must balance 3 kN multiplied by 2 m, giving 1 kN. Vertical equilibrium then supplies the remaining 2 kN at the left. Moving the load toward the right support changes both reactions while keeping their sum fixed.

DERIVATION AND APPLICATIONS

Locate the resultant of a linearly varying distributed load

A beam support carries the effect of the complete loading, including where each small portion acts. Consider a rigid horizontal span of length L with a pin at its left end and a roller at the right. Let q_a and q_b be downward line loads at the ends, varying linearly in between. Use upward support forces R_A, R_B. Ignore the beam weight unless already included in q. The aim is to replace the distribution by a single force that reproduces both force and moment.

At position x, a slice of length dx contributes $q(x)dx$ downward. Its moment magnitude about the left end has the additional factor x. Integrating first the force and then the moment gives

$$q(x) = q_a + (q_b - q_a)x/L,$$
$$W = \int_0^L q(x)dx = \tfrac{1}{2}L(q_a + q_b).$$

$$M_A^{\text{load}} = \int_0^L xq(x)dx = L^2\left[\frac{q_a}{2} + \frac{q_b - q_a}{3}\right] = \frac{L^2}{6}(q_a + 2q_b).$$

If $W \neq 0$, its point of application is $\bar{x} = M_A^{\text{load}}/W$. Moment equilibrium about the pin and vertical force equilibrium now give the support readings:

$$R_B = \frac{L}{6}(q_a + 2q_b),$$

$$R_A = \frac{L}{6}(2q_a + q_b).$$

For equal end loads the resultant is at midspan. For a triangle increasing from zero it acts at two thirds of the span from the zero end. These formulas predict external reactions; replacing the distribution by a single force would change the internal shear and moment between the supports.

WORKED EXAMPLE

Predict the support readings of a loaded shelf

A 2.40 m shelf carries $q_a = 100$ N/m on the left increasing to $q_b = 400$ N/m on the right. The resultant is 600 N and its moment about the left end is 864 N m, placing it at 1.44 m. The right support reads 360 N and the left 240 N. Their sum and their moment reproduce both integrals independently.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Reconstruct a linear distribution from the scale readings

The same span now gives $R_A = 320$ N and $R_B = 280$ N. Adding the readings gives

$$q_a + q_b = 2(600)/2.40 = 500\,\mathrm{N/m}.$$

Subtraction gives

$$q_a - q_b = 6(320 - 280)/2.40 = 100\,\mathrm{N/m}.$$

Thus $q_a = 300$ and $q_b = 200$ N/m. If a point load could also be present, these two support readings alone would not uniquely determine the load shape.

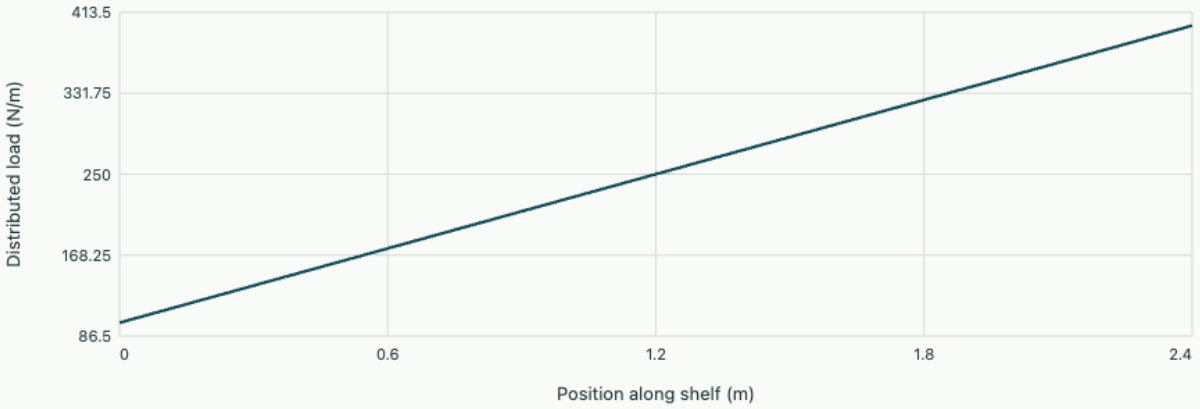
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Left load intensity (N/m)	100
Right load intensity (N/m)	400
Span (m)	2.4



Linear loading

Left reaction 240.000 N; right 360.000 N. Total 600.000 N and resultant moment 864.000 N m.

Start from this chapter. Take a section through the truss and balance moments. Would doubling every bar area change that balance? Compare the predicted chord force and the separately computed joint movement.

OneSimulation laboratory · Distinguish force from deflection in a truss

Choose the body before writing equations

A platform scale under a shelf can read more than the weight placed directly above it. The measurement depends on all the other supports and the locations of all the applied forces. To predict it, imagine removing the shelf from its surroundings. Replace each lost contact by the force or couple that the

surroundings could exert. Internal forces between parts of the isolated body cancel in pairs. The choice of boundary determines which forces are internal.

In a planar model a smooth roller on a horizontal track supplies one vertical reaction. A pin supplies two force components and admits rotation. A clamp supplies both force components and a reaction moment. A cable pulls along its length. A roller simply resting on a track can push, but a negative calculated upward reaction would require a hold-down or a change of contact. These restrictions are part of the physical model, even before a single number is inserted.

Sum forces and moments

A rigid body at rest in a plane obeys three independent scalar equilibrium equations. Choose an arbitrary point O as the moment centre. Taking upward force and counterclockwise moment as positive gives

$$\sum_i F_{x,i} = 0,$$
$$\sum_i F_{y,i} = 0,$$
$$\sum_i M_{O,i} = 0.$$

A force applied at (x, y) relative to O contributes $xF_y - yF_x$ to the last sum. Taking moments about a pin eliminates its two unknown forces at once. A specified couple has the same signed moment about every origin and contributes zero resultant force.

For a vertical load with downward intensity $q(x)$ in N/m, a strip of width dx contributes a downward force of magnitude $q(x)\,dx$. Integrate over the loaded span to obtain the force magnitude and its clockwise moment magnitude about the left end. An equivalent point load must reproduce both.

$$W = \int q(x)\,dx,$$
$$W\bar{x} = \int x\,q(x)\,dx,$$
$$\bar{x} = \frac{\int x\,q(x)\,dx}{\int q(x)\,dx}.$$

This replacement preserves the external reactions. When finding an internal force at a cut, integrate only the load actually lying on the chosen side.

Background · Why can the moment balance be taken about any point?

Keep all forces on the same free body, including support reactions. Let $\mathbf{a}$ point from the first moment centre O to the second P. Every force lever arm changes by the same vector. Applied couples have no lever arm and carry their original signs.

$$\mathbf{M}_P = \sum_i (\mathbf{r}_i - \mathbf{a}) \times \mathbf{F}_i + \sum_j \mathbf{C}_j = \mathbf{M}_O - \mathbf{a} \times \sum_i \mathbf{F}_i.$$

Once force balance holds, the two moment sums coincide. This makes a second choice of moment centre a diagnostic. A missing 100 N vertical load creates a 50 N m difference between sums taken about points 0.5 m apart horizontally. Repeating the moment sum about points displaced parallel to that missing force would hide the discrepancy.

Weigh a platform with a triangular load

A 4.0 m platform is pinned at its left end A and supported by a roller at B at its right end. Bulk material forms a triangular pile, loading the platform with intensity rising linearly from zero at A to 3.0 kN/m at B. A 2.0 kN crate sits 1.0 m from A. Platform self-weight has already been tared off. Write $q(x) = q_B x / L$. Integration gives

$$W = q_B L / 2 = 6.0\,\mathrm{kN},$$

$$\bar{x} = (q_B L^2 / 3) / W = 2L/3 = 2.667\,\mathrm{m}.$$

The resultant is closer to the heavily loaded end.

$$4B_y - 6(2.667) - 2(1) = 0,$$

$$A_y + B_y = 8.0\,\mathrm{kN}.$$

The right reaction is 4.50 kN and the left is 3.50 kN. There is no applied horizontal force, so $A_x = 0$. Check independently about B: the clockwise effect of the left reaction is $3.50(4) = 14.0$ kN·m and the two weights give the opposite moment $6(1.333) + 2(3) = 14.0$ kN·m. Replacing the triangle by a uniform load of equal total weight would pass a force check and fail this moment check.

Predict the next scale reading

Move the crate 0.50 m to the right. Its force is unchanged, but its moment about A increases by 1.0 kN·m. The right reaction therefore rises by $1.0/4.0 = 0.25$ kN and the left falls by the same amount. In practice this relation lets two load cells measure both weight and the position of a moving load. It assumes a stable support arrangement with known moment arms. Three vertical supports under a rigid platform introduce an extra unknown. Their individual readings then require information about deflections and support stiffness, which equilibrium alone cannot supply.

Find the load position from the reactions

An inverse problem uses the same equilibrium equations. A 2.0 m simply supported rail carries a single unknown load and sensors read 900 N at the left and 300 N at the right after taring the empty rail. The total applied downward load is 1200 N. Taking moments about the left support gives its position as $300(2.0)/1200 = 0.50\,\mathrm{m}$. Repeating this reasoning for a distributed load gives the position of its resultant but cannot reconstruct its detailed distribution.

Determine when an assumed contact can exist

A support symbol on paper often hides a condition on the direction of its reaction. A dry foot resting on a horizontal floor can push on the floor, but it cannot pull the floor upwards. A stationary contact can transmit a range of tangential forces up to its friction limit. These statements convert an ordinary equilibrium problem into one with inequalities. First derive the forces required to maintain the proposed configuration. Then ask whether the contact can supply them. Assigning the maximum friction before writing equilibrium would generally give the wrong answer for a stationary body.

Consider a rectangular machine base supported on two narrow feet a distance b apart. Its centre of mass lies midway between them, its weight is W, and a horizontal force P acts to the right at height h. Write upward contact forces N_L, N_R. Neglect the thickness of the contact patches and consider only motion in the vertical plane. Vertical equilibrium requires

$$N_L + N_R = W.$$

Taking moments about the midpoint between feet eliminates both unknown friction forces, which act along the floor. The right foot gains normal load as the left one loses it:

$$(N_R - N_L)b/2 = Ph,$$

$$N_L = W/2 - Ph/b,$$

$$N_R = W/2 + Ph/b.$$

Both feet stay in contact only while $P \le Wb/(2h)$. If both have the same coefficient μ, their combined available friction is

$$\mu(N_L + N_R) = \mu W.$$

Thus impending tip and impending slip are distinct limiting states. The precise sharing of tangential load between the feet depends on their compliance; these three rigid-body equations alone need not fix it.

Decide which motion starts first on a test stand

A movable test stand weighs 1800 N. Its feet are 0.80 m apart and the technician pushes horizontally on a handle 1.20 m above the floor. Suppose the measured static friction coefficient is 0.45. At a push of 400 N, the load shifts by $400(1.20)/0.80 = 600\,\mathrm{N}$ to the right. The left reaction is 300 N and the right

reaction is 1500 N. The available 810 N of total friction exceeds 400 N; a static configuration is possible. A calculation assigning each foot a friction force of 405 N would nevertheless violate the individual 135 N friction limit on the lightly loaded left foot.

Increase the push gradually. The tip threshold is $1800(0.80)/[2(1.20)] = 600\,\mathrm{N}$, below the 810 N sliding threshold. At 600 N the right foot alone carries 1800 N; the required friction ratio there is $600/1800 = 0.333$, within the measured friction limit. Now relocate the handle to 0.60 m. The calculated tip threshold doubles to 1200 N, so the base begins sliding at 810 N while both feet still carry weight. The dividing height follows directly from equating both thresholds,

$$h = b/(2\mu) = 0.889\,\mathrm{m}.$$

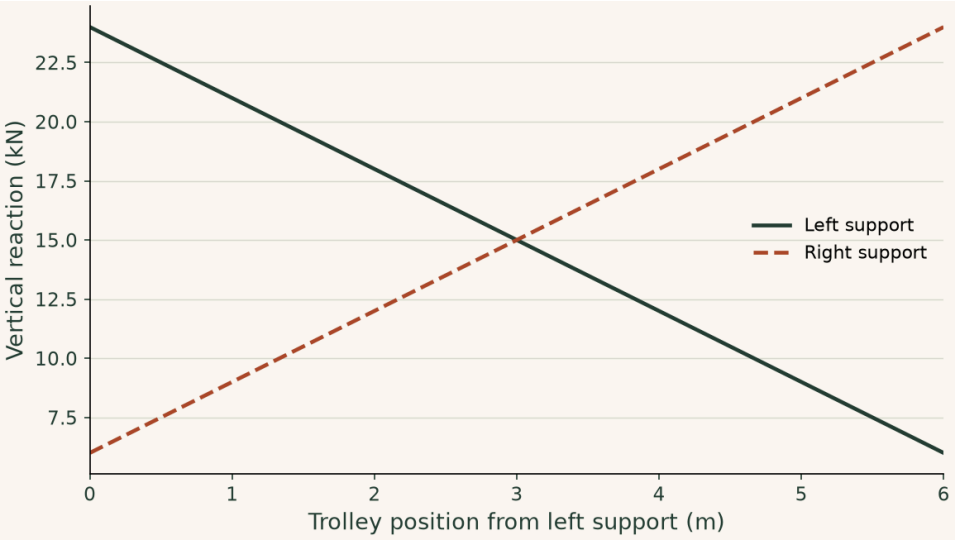
Lowering the handle changes the rotational demand without changing the total horizontal force that floor friction can transmit.

Locate a trolley from two support reactions

A simply supported 6 m runway weighs 12 kN uniformly and carries an 18 kN trolley. Take moments about the left support to find

$$R_B\,6 = 12(3) + 18a.$$

Then close the vertical force balance. Suppose the right load cell reads 18 kN. Where is the trolley?



Subtracting the 6 kN permanent-load reaction leaves 12 kN on the right, placing the trolley at 4 m. The two readings must sum to 30 kN at every position. A constant bias in both sensors would violate this independent check.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

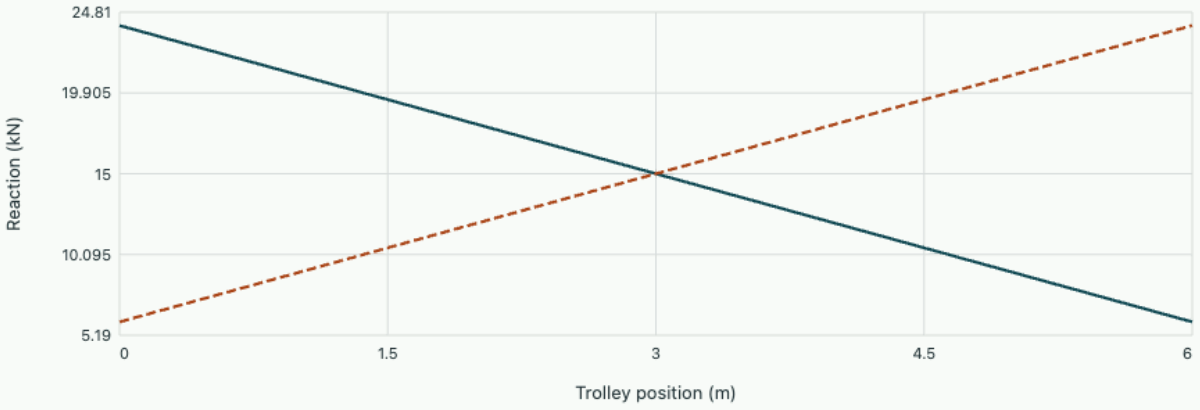
Move the trolley across the span and then increase its weight. Check that both reactions always sum to the total load.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Runway length (m)	6
Runway weight (kN)	12
Trolley load (kN)	18



The support sum is always 30.000 kN. At one third span the right support carries 12.000 kN.

Chapter 2 · Axial stress, strain and compatibility

Determine how much a stepped member stretches and why a restrained temperature change produces force.

FROM FIRST PRINCIPLES

Build the axial displacement from equilibrium, strain and a material law

Take a slender straight member loaded through its centroid, undergoing small axial deformation with no buckling or bending. Let $N(x) > 0$ denote tension, $A(x)$ area and $p(x)$ the axial force per unit length. Balance a slice between x and $x + dx$: its two cut forces enter with opposite signs, while distributed load acts over the slice. Dividing by dx gives

$$N'(x) + p(x) = 0,$$
$$\sigma_x(x) \simeq N(x)/A(x).$$

The stress relation assumes a nearly uniform section stress away from load introduction. The extension of the slice is $u(x + dx) - u(x)$, so strain is du/dx. For a linear elastic material with free thermal strain $\alpha\Delta T$, Hooke's law applies to strain remaining after this stress-free part is removed. Integrating along the actual member gives

$$u(b) - u(a) = \int_a^b \left[\frac{N(x)}{E(x)A(x)} + \alpha(x)\Delta T(x)\right] dx.$$

An imposed end displacement supplies a compatibility equation for unknown reaction forces. A freely heated unloaded bar extends with $N = 0$; setting total extension to zero generally creates thermal stress.

Background · Axial strain

$$\varepsilon = du/dx$$

This is engineering strain for small displacement gradients in a straight bar. A measured extension is an integral over the gauge length.

DERIVATION AND APPLICATIONS

Separate assembly preload from compatible thermal strain

Two axial members attached in parallel to rigid end plates share one extension. Equal extensions need not mean equal elastic strains when their stress-free lengths change differently with temperature. Assume initially unstressed prismatic bars of common length L, small deformation, uniform temperature increments, linear elastic moduli E_i, areas A_i and free thermal strains $\epsilon_i^T = \alpha_i \Delta T_i$. The plates remain parallel; an axial force resultant P acts through a guided fixture that prevents rotation.

Let δ be the common end movement, positive for extension. Subtract thermal strain before applying Hooke's law. The force in each bar then becomes

$$N_i = E_i A_i(\delta/L - \epsilon_i^T) = k_i(\delta - L\epsilon_i^T),$$
$$k_i = E_i A_i / L.$$

Equilibrium of the end plate sets the sum of the two member forces equal to P. Solving before substituting back makes the compatible motion explicit:

$$\delta = \frac{P + k_1 L\epsilon_1^T + k_2 L\epsilon_2^T}{k_1 + k_2}.$$

Separate the force in bar 1 into the usual load-sharing part and an internally balanced thermal part:

$$N_1 = \frac{k_1 P}{k_1 + k_2} + \frac{k_1 k_2}{k_1 + k_2} L(\epsilon_2^T - \epsilon_1^T),$$
$$N_2 = P - N_1.$$

The internal thermal forces cancel at the plates. They vanish under equal free thermal strains even when the elastic moduli differ. This model presumes the members can sustain any predicted compression without buckling or losing contact.

WORKED EXAMPLE

Find stresses in a heated parallel link

A 0.500 m steel bar has $E_1 = 200$ GPa, $A_1 = 100$ mm² and $\alpha_1 = 12 \times 10^{-6}$/K. A second bar has $E_2 = 100$ GPa, $A_2 = 200$ mm² and $\alpha_2 = 20 \times 10^{-6}$/K. Both warm by 50.0 K with $P = 0$. Their axial stiffnesses both equal 40.0 MN/m. Free extensions would be 0.300 and 0.500 mm, so the shared extension is 0.400 mm. Bar 1 stretches elastically by 0.100 mm and carries 4.00 kN tension, or 40.0 MPa. Bar 2 carries 4.00 kN compression, or 20.0 MPa in magnitude.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Choose a load that relieves one member

After the same heating, pull the rigid plates apart with increasing external force. Half the load enters each bar, so $P = 8.00$ kN brings bar 2 from -4.00 kN to zero. The extension then equals its own free extension of 0.500 mm. Bar 1 alone carries 8.00 kN, which follows independently from its 0.200 mm elastic extension.

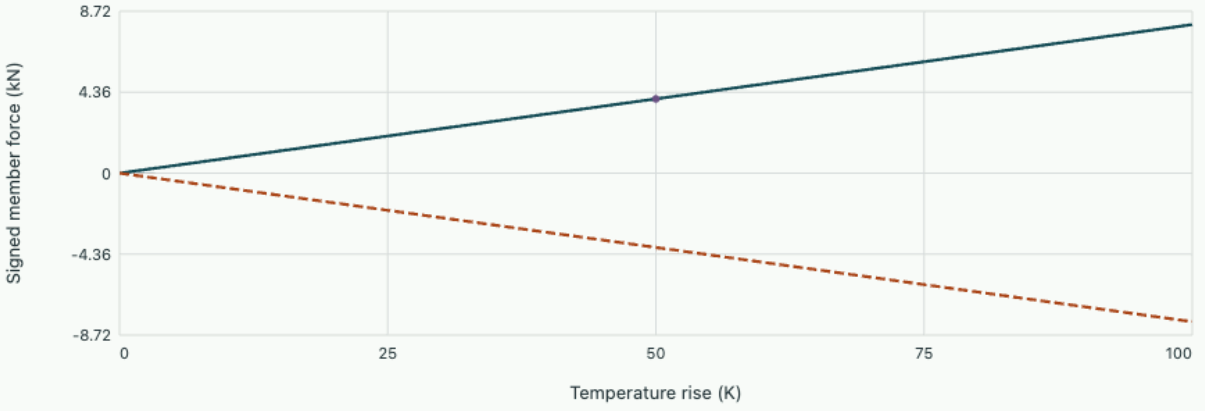
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Common warming (K)	50
External axial force (kN)	0
Second expansion coefficient (µm/m K)	20



At selected temperature N1=4.000 kN; N2=-4.000 kN. Common extension 0.400 mm. Both axial stiffnesses40 MN/m.

Start from this chapter. Use axial extension NL/(EA) with the same load and geometry. Double every area in the model and test the predicted displacement ratio while reading the chord force independently.

OneSimulation laboratory · Distinguish force from deflection in a truss

Find the force carried through a section

Two shafts that support the same suspended mass can stretch by very different amounts. A load cell measures force, while a clearance specification usually limits displacement. Connect them by taking a cut normal to a straight member and replacing the removed material by its axial force N. When the load is centred, changes slowly and is far from grips or abrupt changes in section, the nominal normal stress is

$$\sigma = N/A.$$

Adopt tension as positive. Near holes or grips the mean value remains useful for equilibrium, while local stresses can differ appreciably.

If a small material line of original length dx gains length du, its engineering strain is du/dx. Assume small strain, a uniform temperature across each section and a linear elastic material. Elastic strain is σ/E, where Young's modulus E has units of stress. Free thermal expansion adds the strain $\alpha \Delta T$. Thus an element can change length even if its axial force is zero.

Reassemble the pieces

Multiply the strain by dx and add along the member. The displacement difference between its ends is

$$u(L) - u(0) = \int_0^L \frac{N(x)}{E(x)A(x)}\,dx + \int_0^L \alpha(x)\Delta T(x)\,dx.$$

Every term has units of length. Bars joined end to end carry the same force in a segment without an intervening axial load. Their extensions add, so their compliances $L/(EA)$, measured in m/N, add. Bars between the same two rigid plates instead share the same extension. Their forces add, and their stiffnesses EA/L, measured in N/m, add. Draw the connections before deciding which quantities are equal. The material with the larger modulus does not always carry the larger load because area and length also matter.

Background · What does a short extensometer actually measure?

A sensor spans two material marks with a finite separation L_g. It reports their relative displacement divided by the original separation. When stiffness varies within the gauge length, this reading weights the local strains by their lengths.

$$\bar{\epsilon}_g = \frac{u(x+L_g)-u(x)}{L_g} = \frac{1}{L_g}\int_x^{x+L_g} \frac{N(s)}{E(s)A(s)}\,ds.$$

Take a 20 mm gauge spanning equal lengths of two segments with areas A and $2A$, carrying the same tensile force in the same material. The indicated strain is $0.75N/(EA)$. A gauge located entirely on the thin segment reads $N/(EA)$. Moving the marks changes the inferred modulus if the analysis treats either reading as a point value at the step.

Stretch a two-material tie

A steel section is 0.60 m long with area 120 mm² and $E_s = 200$ GPa. An aluminium section continues another 0.40 m, has area 200 mm² and $E_a = 70$ GPa. Pull on the joined tie with 12 kN at constant temperature. Equilibrium gives 12 kN in both pieces. The nominal stresses are 100 MPa and 60 MPa respectively. Their elongations are

$$12\,000(0.60)/(200\times10^{9}\,120\times10^{-6}) = 0.300\,\mathrm{mm},$$

$$12\,000(0.40)/(70\times10^{9}\,200\times10^{-6}) = 0.343\,\mathrm{mm}.$$

The end displacement is 0.643 mm. The aluminium extends more although its nominal stress is smaller, because its compliance is larger.

Dividing total force by total displacement gives a tie stiffness of 18.7 MN/m. This is smaller than the stiffness of either isolated piece. Checking this limit catches an accidental parallel-spring calculation. The stress calculation still treats each piece separately. Averaging their areas first would conceal the location of the larger nominal stress.

Install the tie between rigid walls

Now let the same tie fit exactly between rigid walls with zero force at its installation temperature. Raise the uniform temperature by 30 K. Take $\alpha_s = 12\times10^{-6}\,\mathrm{K}^{-1}$ and $\alpha_a = 23\times10^{-6}\,\mathrm{K}^{-1}$. If free, the pieces would extend by 0.216 and 0.276 mm. The walls prescribe zero total displacement, so an elastic shortening of 0.492 mm must develop. Multiplying this mismatch by the combined stiffness gives $N = -9.18$ kN. The negative sign denotes compression.

A force balance on the entire tie merely says the two wall reactions are opposite. Their magnitude came from the displacement condition. Real supports deflect, giving part of the expansion back to the tie and

reducing the force. If the tie is long and slender, check whether it can remain straight under this force before interpreting the compressive stress.

Test the difference between member force and extension

In a statically determinate pin-jointed truss the axial forces follow from joint equilibrium before the bar areas enter the displacement calculation. Predict both outputs when every bar area is doubled at unchanged geometry and load. If the compressive top chord initially carries 75 kN and the monitored joint moves down 3.219 mm, which reading changes, and by what factor? Check the two runs and explain the final rounding using $\delta_i = N_i L_i/(E_i A_i)$. Open the truss experiment.

Integrate strain in a tapered member

A weight-saving tie may taper along its length although equilibrium still gives the same force at every cross-section. Its compliance cannot in general be calculated using the arithmetic mean area. Let a straight elastic bar of length L have uniform thickness and linearly varying width, so its area changes linearly from A_0 to A_1. Apply an axial end force P, neglect distributed body force, take constant Young's modulus and exclude local stress disturbances near load introduction. Isolating any segment gives $N = P$; compatibility between adjacent slices gives

$$du/dx = P/[EA(x)].$$

Set $A(x) = A_0 + (A_1 - A_0)x/L$. Changing integration variable from position to area produces a logarithm:

$$\delta = \int_0^L \frac{P}{EA(x)}\,dx = \frac{PL}{E(A_1 - A_0)} \ln\frac{A_1}{A_0}.$$

This expression is unchanged if the member is turned end for end. Its limit as A_1 approaches A_0 follows from $\ln(1+z)/z \to 1$, recovering $PL/(EA_0)$. The effective uniform area for extension is therefore a logarithmic mean. Narrow regions contribute disproportionately to compliance because strain depends on the reciprocal of area. The stress maximum still occurs at the smallest cross-section; a stiffness-equivalent uniform area does not reproduce it.

Size a suspended gauge bar from two different requirements

Take an aluminium tie with $L = 1.20\,\mathrm{m}$, $A_0 = 120\,\mathrm{mm}^2$, $A_1 = 360\,\mathrm{mm}^2$, $E = 70\,\mathrm{GPa}$, and $P = 6.0\,\mathrm{kN}$. Using newtons and millimetres, its elongation is

$$6000(1200)\ln 3/[70000(240)] = 0.471\,\mathrm{mm}.$$

The arithmetic mean area 240 mm² would predict 0.429 mm, about 9% too low. The actual peak nominal stress is $6000/120 = 50.0\,\mathrm{MPa}$, and the far larger end sees 16.7 MPa. A strain gauge near the large end therefore underestimates the average elongation if its measured strain is simply multiplied by the whole length.

A different loading mechanism matters for a very long uniform member. Suspend a uniform steel gauge wire of length L from its top with no end load, and measure x upwards from its free bottom. The force at x carries the weight below, $N(x) = \rho A g x$. The area cancels from the strain $N/(EA) = \rho g x/E$, so its total extension is $\rho g L^2/(2E)$. Taking $L = 30\,\mathrm{m},\ \rho = 7850\,\mathrm{kg/m^3}$ and $E = 200\,\mathrm{GPa}$ gives 0.173 mm. Doubling the wire diameter halves neither the strain nor the elongation under self-weight, since the added area also adds weight. Enlarging the wire does reduce the separate extension caused by a fixed instrument hung from its end.

A practical way to separate the wire's two contributions is to measure extension with two different end weights while leaving the wire itself in place. The difference between readings is $\Delta P L/(EA)$, since self-weight and any fixed initial offset are the same in both tests. A plot of extension against end load therefore has slope $L/(EA)$ and a nonzero intercept if its zero refers to the unstressed wire. If an instrument was instead zeroed after the wire was suspended, the intercept disappears from the record. Stating the reference condition prevents it from being misinterpreted as evidence for or against self-weight deformation.

Creep and relaxation put different constraints on the same bar

A polymer strip may continue to extend after a testing machine reaches its target force. In another test the machine holds the extension fixed and the force decreases. In each case the energy-storing part of the material can exchange load with a time-dependent internal mechanism. Represent the simplest such mechanism by a spring of modulus E_1 in series with a viscous dashpot of viscosity η. This Maxwell branch carries common stress s and its strains add:

$$\dot{\varepsilon} = \dot{s}/E_1 + s/\eta.$$

To represent a solid with finite long-time stiffness, put a second spring E_∞ in parallel. The specimen stress is

$$\sigma = E_\infty \varepsilon + s.$$

Assume small uniaxial strain, uniform temperature and an initially relaxed specimen.

At prescribed step strain ε_0, the dashpot cannot change length instantaneously, so both springs initially contribute. Holding the strain sets $\dot{s} = -sE_1/\eta$. Hence $\sigma(t) = \varepsilon_0(E_\infty + E_1 e^{-t/\tau_r})$ with relaxation time $\tau_r = \eta/E_1$. Under a maintained stress σ_0, substitute $s = \sigma_0 - E_\infty\varepsilon$ into the branch equation. The result is

$$\tau_c \dot{\varepsilon} + \varepsilon = \sigma_0/E_\infty,$$

with $\tau_c = \eta(E_\infty + E_1)/(E_\infty E_1)$. The initial strain is $\sigma_0/(E_\infty + E_1)$ and the creep strain approaches σ_0/E_∞. The two time constants differ even for the same specimen.

A 100 mm strip of area 20 mm² has $E_\infty = 100$ MPa, $E_1 = 300$ MPa and $\eta = 6000$ MPa s. Hold its extension at 0.50 mm. The initial stress is 2.00 MPa, corresponding to 40 N. After 20 s it is $0.005[100 + 300/e] = 1.052$ MPa or 21.0 N. The long-time force is 10 N. Instead hold a previously relaxed strip at 20 N. Its stress stays at 1.00 MPa, giving initial extension 0.25 mm and final extension 1.00 mm. The creep time is 80 s. After that time extension is

$$1.00 - (1.00 - 0.25)/e = 0.724\,\mathrm{mm}.$$

Matching one force reading at one time cannot establish which boundary condition was imposed. A machine compliance comparable to the specimen's compliance moves the test between these ideal limits.

Practise on sheet 1 · Infer a relaxation model from a displacement-hold test. Further reading · MIT · 3.11 Mechanics of Materials · Engineering Viscoelasticity, §4.1 The Maxwell Spring-Dashpot Model and §4.2 The Standard Linear Solid (Maxwell Form).

Can a compliant support relieve thermal stress?

Heat an elastic bar by a uniform temperature increment. One end is fixed, the other pushes a linear spring. Compatibility gives

$$\alpha L \Delta T = FL/(EA) + F/k_s.$$

Normalise the displacement by free expansion and the force by the fully restrained value.



Equal bar and support stiffness split the free expansion equally between bar compression and spring deflection. At a stiffness ratio of nine the bar develops 90% of its fully restrained stress. A nominally fixed end can therefore matter quantitatively.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

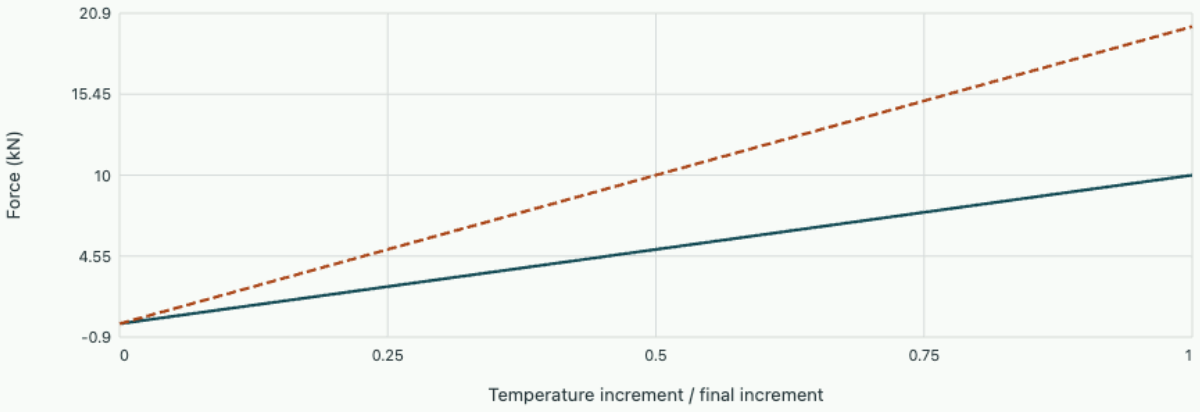
Make the support softer than the bar and compare the resulting force with the perfectly restrained limit. The two elastic shortenings must still close the compatibility equation.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Bar axial stiffness EA/L (MN/m)	20
Support stiffness (MN/m)	20
Free thermal extension (mm)	1



Final force 10.000 kN. Support movement 0.500 mm; bar compression 0.500 mm. Their sum equals the free extension.

Problem sheet 1 · Equilibrium and axial compatibility

Draw each isolated body, state the sign convention and identify the compatibility condition before substituting numbers.

1. Find a load from two readings

A 5 m platform with a pin at the left end and a roller at the right weighs 6 kN uniformly. An added crate makes the reactions 4.2 kN and 5.8 kN respectively. (a) Determine the crate weight. (b) Find its position measured from the left. (c) Slide it 0.50 m left. Predict both new readings and show how their sum and difference respond.

▼ **Work the two balances**

The total vertical load is $4.2 + 5.8 = 10.0$ kN, so the crate weighs 4.0 kN. Taking moments about the left support gives

$$5.8(5) = 6(2.5) + 4x;$$

hence $x = 3.50$ m. Moving left removes $4(0.50) = 2.0$ kN·m from the clockwise load moment. The right reading falls by $2.0/5 = 0.40$ kN to 5.4 kN, while the left rises to 4.6 kN. Total weight remains 10 kN. The right-minus-left difference decreases from 1.6 to 0.8 kN because one reading rises while the other falls.

---

2. Let two bars share one load

Rigid parallel plates are joined by a 0.40 m long steel bar ($E = 200$ GPa, $A = 100$ mm²) and an equal-length aluminium bar ($E = 70$ GPa, $A = 200$ mm²). The plates are guided to remain parallel. A centred 17 kN tensile load separates them. (a) Find each stiffness. (b) Determine the extension and forces. (c) Compare the two strains and two stresses.

▼ **Work the common-displacement condition**

The stiffnesses EA/L are 50 MN/m and 35 MN/m. The common extension is

$$17000/(85 \times 10^{6}) = 0.200\,\mathrm{mm}.$$

Steel takes 10.0 kN and aluminium 7.0 kN, summing to 17 kN. Both strains are $0.0002/0.40 = 0.00050$, but the stresses are 100 and 35 MPa. Equal length and equal displacement enforce equal strain. Different moduli turn equal strain into different stress, and different areas convert those stresses to forces.

---

3. Close a gap before loading a support

A stress-free 1.20 m steel bar points toward a rigid stop with a 0.18 mm gap. Its other end is fixed. Use $A = 150$ mm², $E = 200$ GPa and $\alpha = 12 \times 10^{-6}\,\mathrm{K}^{-1}$. (a) How much warming closes the gap? (b) Find the compressive force and stress after a uniform 30 K rise. (c) Interpret a calculation that uses all 30 K as a restrained rise.

▼ **Separate free and restrained stages**

Free expansion closes the gap at $0.00018/(1.20\alpha) = 12.5$ K. The subsequent 17.5 K would add another 0.252 mm of expansion, which the stop prevents. Compatibility now requires elastic strain $-\alpha(17.5) = -0.000210$. The resulting stress is –42.0 MPa and force –6.30 kN. Treating all 30 K as restrained would give –72 MPa and would incorrectly predict a reaction before contact. This piecewise solution assumes the bar remains straight and the stop carries compression.

Infer a relaxation model from a displacement-hold test

A 40 mm² specimen held at fixed strain 0.004 requires 48 N immediately, 24 N after a long hold and 32.83 N after 30 s. Infer both spring moduli and the branch viscosity. Predict the long-time strain when a fresh specimen instead carries 32 N.

▼ **Show the worked solution**

The instantaneous and long-time stresses are 1.20 and 0.60 MPa. Dividing by strain gives $E_\infty + E_1 = 300$ MPa and $E_\infty = 150$ MPa, hence $E_1 = 150$ MPa. The excess force fraction after 30 s is $(32.83 - 24)/(48 - 24) = 0.368$, so $\tau_r \simeq 30$ s and $\eta = 4500$ MPa s. At 32 N the maintained stress is 0.80 MPa. The strain starts at $0.8/300 = 0.00267$ and tends to $0.8/150 = 0.00533$. Its creep time is 60 s, so transferring the relaxation time directly to the force-hold test would predict the wrong rate.

Revisit chapter 2 · Creep and relaxation put different constraints on the same bar.

Explore the governing equations after solving the sheet: Free bodies, moments and support reactions; Axial stress, strain and compatibility. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Infer a truss area without mistaking it for a height change

On the 2 m high reference truss a 100 kN load produces 3.219 mm of downward displacement and a top-chord force of –75.0 kN. A second build gives 2.146 mm and the same chord force. Estimate the uniform bar area required if the geometry and modulus stayed fixed and the baseline area was 1000 mm². Test the alternative explanation that the height alone increased to 3 m.

▼ **Work through the solution**

With unchanged geometry an internally determinate pin-jointed truss has member forces fixed by equilibrium. Each member's elastic extension is $NL/(EA)$, so uniform area scaling divides the complete displacement by the same factor. Thus $A_2 = 1000(3.219/2.146) = 1500$ mm². Raising height to 3 m gives –50.0 kN and 2.659 mm in the laboratory. The measured force already rules out that specific alternative. A uniform change in modulus remains indistinguishable from the inferred area change using these two readings alone.

Use the OneSimulation laboratory to check your answer

Vary the corresponding laboratory reference interactively. Compare the recalculated observable with the recorded OneSimulation runs and the approximation used in this exercise.

Chapter 3 · Shear force and bending moment

Construct the internal-force diagrams of a simply supported beam and locate the section with the largest nominal bending stress.

FROM FIRST PRINCIPLES

Obtain shear and moment from the equilibrium of a beam slice

Let a planar beam carry a downward line load $q(x)$. Use positive sagging moment M and the shear sign convention that a positive shear raises M with increasing x. Isolate the material between two closely spaced vertical cuts. The shear on the right face differs from the left by its spatial change, while the distributed load has resultant $q(x)dx$ to first order. Vertical force balance gives

$$V(x+dx) - V(x) = -q(x)\,dx,$$
$$dV/dx = -q(x).$$

Moment balance about the left cut includes the shear acting at a lever arm dx. The distributed load on this short interval produces only a term proportional to dx^2. Dividing by dx and taking the limit leaves

$$dM/dx = V,$$
$$M(b) - M(a) = \int_a^b V(x)\,dx.$$

Support reactions set integration constants. A downward concentrated force creates a downward jump in shear, found by integrating across its position; an applied couple causes a jump in moment. Between concentrated actions, a zero of shear identifies a stationary value of moment, which may locate the most highly stressed fibre.

DERIVATION AND APPLICATIONS

Construct a bending-moment diagram directly from load integrals

The maximum bending stress depends on moment, not necessarily on where the distributed loading is largest. Consider a simply supported span L carrying a triangular line load $q(x) = q_0 x/L$ directed down. Define positive sagging moment M and positive shear $V = dM/dx$. The pin and roller transmit no end couples. Find reactions before cutting the span: the resultant load $q_0 L/2$ acts at $2L/3$ from the left. Its moment balance therefore gives

$$R_A = q_0 L/6,$$
$$R_B = q_0 L/3,$$
$$R_A + R_B = q_0 L/2.$$

A cut at x isolates the left support and the load between 0 and x. Vertical equilibrium gives the support reaction minus the integral of all line load up to that cut:

$$V(x) = R_A - \int_0^x q(s)\,ds = \frac{q_0 L}{6} - \frac{q_0 x^2}{2L}.$$

Every small line force contributes a moment arm $x - s$. Equivalently integrate V and fix the integration constant by $M(0) = 0$. Both routes give

$$M(x) = R_A x - \int_0^x (x - s) q(s)\,ds = \frac{q_0 L x}{6} - \frac{q_0 x^3}{6L}.$$

At the far support $M(L) = 0$, a check on the moment boundary condition rather than an additional reaction. Since the shear decreases monotonically across the span, its unique zero locates the maximum bending moment. A point couple, if present, would require a local jump in moment and could invalidate this smooth-maximum argument.

WORKED EXAMPLE

Find the location of the largest moment

On a 3.00 m span let $q_0 = 6.00$ kN/m. The end reactions are 3.00 and 6.00 kN. Setting $V = 0$ gives

$$x = L/\sqrt{3} = 1.732\,\mathrm{m}.$$

Substitution yields $M_{\max} = q_0 L^2/(9\sqrt{3}) = 3.46$ kN m. The location lies left of the resultant at 2.00 m. Replacing the triangle by a single force there would give correct reactions but an incorrect peak moment.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Reconstruct a distributed load from section readings

A second unloaded-at-the-left triangular span has $L = 2.00$ m and a measured midpoint moment $M(L/2) = 0.750$ kN m. Substituting $x = L/2$ into the exact diagram gives

$$M = q_0 L^2/16.$$

Thus $q_0 = 3.00$ kN/m. The midpoint shear should be $q_0 L/24 = 0.250$ kN. Moment and shear measured independently test whether the assumed linear load profile is plausible.

Interactive calculation · Change the inputs of this example

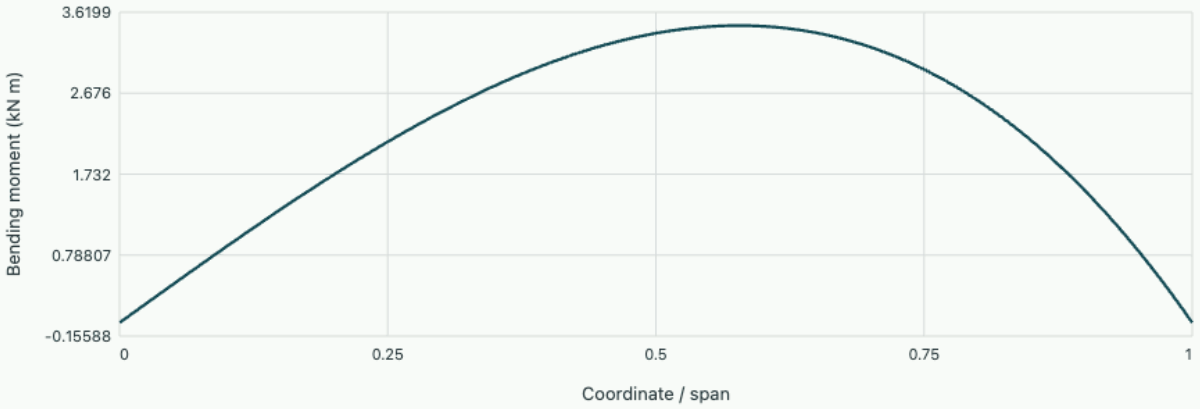
Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Peak triangular load (kN/m) 6

Span (m) 3



Triangular-load bending moment

Maximum at x/L=1/√3, moment 3.464 kN m. Reactions 3.000 and 6.000 kN.

Read a section cut as an equilibrium problem

A transfer rail carries a trolley and the distributed weight of its own structure. Its supports may be strong enough while the rail still bends excessively at an interior section. Start with reactions for the entire beam. Next isolate the part from its left end to a cut at coordinate x. The portion discarded from the drawing exerts an axial force, a shear force and a bending moment on the cut. For a horizontal beam with vertical loading the axial force is zero.

Use downward distributed load $q(x) > 0$, take $M > 0$ for sagging and define positive shear V by the resultant upward external load on the left-hand part. Thus a positive V corresponds to a downward internal shear on its right cut face. Choose the signs once and test a cut just to the right of the left support. With a positive left reaction the first value of the shear diagram must be positive.

Build a diagram from the loading

Take an element between x and $x + dx$. Its external load increases the downward resultant by $q\,dx$, so $V(x + dx) - V(x) = -q\,dx$. Moment balance retains the term $V\,dx$; the distributed load times its half-element arm is second order in dx. Passing to the limit gives

$$\frac{dV}{dx} = -q(x),$$

$$\frac{dM}{dx} = V(x).$$

A point load produces a finite jump in V and a change of slope in M. The bending moment is continuous there unless an applied concentrated couple acts. On an unloaded interval shear is constant and moment is a straight line. Under uniform loading the moment is parabolic. At a simple end support without an applied end couple the bending moment is zero. These four facts are often enough to find a sign mistake before calculating stress.

Combine a point load and distributed weight

Let a simply supported 6.0 m rail carry uniform weight 2.0 kN/m over its full span and a 6.0 kN trolley 2.0 m from the left end. The total downward load is 18 kN. Moment balance about the left gives $6R_B = 12(3) + 6(2) = 48$ kN·m, so $R_B = 8$ kN and $R_A = 10$ kN. To the left of the trolley, $V = 10 - 2x$ kN and $M = 10x - x^2$ kN·m for x in metres. At $x = 2$, $M = 16$ kN·m. Shear falls from 6 kN to zero on crossing the trolley.

To the right of the trolley, $V = 4 - 2x$ kN and $M = 10x - x^2 - 6(x - 2)$ kN·m. Shear is negative throughout $2 < x < 6$; the moment therefore falls to zero at the right support. The largest bending moment is 16 kN·m under the trolley. At the right end $V(6^-) = -8$ kN. The upward reaction returns it to zero, another check on the construction.

Translate the moment into a fibre stress

For a symmetric homogeneous section undergoing small elastic curvature, strain varies linearly through depth. Integrating the moments of the Hookean normal stresses about the neutral axis gives

$$M = EI\kappa.$$

A fibre a distance c from that axis has stress magnitude $Mc/I = M/Z$, where $Z = I/c$ is the section modulus. Take a rectangular section 80 mm wide and 160 mm deep. Then $Z = bh^2/6 = 3.41 \times 10^{-4}\,\mathrm{m}^3$, giving a nominal bending stress of 46.9 MPa under the trolley. At a positive sagging moment the bottom is in tension. A local wheel-contact stress or weld stress requires its own length scale and model.

What jumps under an applied couple?

A concentrated transverse force produces a jump in shear. A concentrated moment produces a jump in the bending-moment diagram. For example, apply only a clockwise couple of 4 kN m midway along a 2 m simply supported beam. Global moment equilibrium gives a right reaction of 2 kN upwards and a left reaction of 2 kN downwards. The shear stays at −2 kN between supports. The moment thus falls from zero to −2 kN m on the left half, jumps to +2 kN m at the couple and falls to zero at the right support. The two zero end moments provide a check on the jump direction.

Derive an influence line at a fixed section

A bending-moment diagram follows a fixed loading pattern along a beam. A travelling wheel poses the converse question: how does the moment at one strain gauge vary as the load crosses the span? Take a simply supported beam of span L and a downward point load P located at z from the left support. Equilibrium gives

$$R_A = P(L - z)/L.$$

Keep the section coordinate a fixed while changing z. The moment at a must be evaluated from a cut that contains the wheel only if the wheel lies to its left.

When $z < a$, the moment is $R_A a - P(a - z)$; when $z > a$, it is $R_A a$. Divide through by P to obtain an influence ordinate i_M, whose units are length. The two linear expressions agree when the wheel crosses the gauge:

$$i_M(z) = \begin{cases} z(L - a)/L, & 0 \le z \le a, \\ a(L - z)/L, & a \le z \le L. \end{cases}$$

The ordinate is zero when the force is directly over a support and reaches $a(L - a)/L$ at the gauge. For a distributed load $w(z)$ the moment is the integral of $w i_M$ because a short loaded length contributes $w(z)\,dz$ of force. This is an application of superposition in a beam with fixed geometry and fixed support conditions. A support that lifts off would invalidate the same influence line.

Place a pair of unequal wheels to maximise one reading

Consider a 6 m span with a gauge at $a = 2.0\,\mathrm{m}$. A carriage has a front wheel carrying 8 kN and a rear wheel carrying 4 kN, a fixed 1.5 m behind it. It travels towards increasing z. To maximise the moment, move both loads along the triangular influence line. The weighted sum is piecewise linear in carriage position, so its maximum occurs at a change of slope or over a level interval. The changes happen when a wheel enters the span, passes the gauge or exits. There is no need to sample many intermediate positions.

With the front wheel over the gauge the rear wheel is at 0.50 m. The ordinates are $2(4)/6 = 1.333$ m and $0.5(4)/6 = 0.333$ m, giving $M = 8(1.333) + 4(0.333) = 12.0\,\mathrm{kN\,m}$. With the rear over the

gauge the front is at 3.5 m. Its ordinate is $2(6-3.5)/6 = 0.833$ m, and the moment is $4(1.333) + 8(0.833) = 12.0\,\mathrm{kN\,m}$ again. Between these positions the front contribution falls at $8(2/6)$ kN for each metre moved, exactly balancing the rear contribution rising at $4(4/6)$ kN per metre. The peak is a plateau, so several carriage positions give exactly the same design moment at this gauge.

For a narrow strain gauge on the bottom fibre of a symmetric elastic cross-section, the measured strain follows this moment history through $\varepsilon = My/(EI)$. Once the two wheels straddle the gauge, a flat measured strain interval is therefore physically plausible even though both wheels continue moving. A gauge near another section has a different influence line and need not show the same plateau.

Find curvature and stress in a bonded two-material beam

A homogeneous beam has its elastic neutral axis at the geometric centroid. A timber strip bonded to an aluminium cap can carry much larger normal stress on one side of their interface even though both materials have identical longitudinal strain there. Assume the interface prevents slip, each material is linear elastic, the beam is slender and no net axial force acts. With depth y measured from a common datum, compatible strain is $\epsilon_x = -\kappa(y - y_n)$ and stress in layer i is $E_i \epsilon_x$. The unknown neutral height y_n follows from axial force equilibrium:

$$\sum_i E_i A_i (\bar{y}_i - y_n) = 0,$$

$$y_n = \frac{\sum_i E_i A_i \bar{y}_i}{\sum_i E_i A_i}.$$

Now sum moments of the layer stresses about this axis. The effective bending rigidity is $D = \sum_i E_i [I_{i,c} + A_i (\bar{y}_i - y_n)^2]$, giving curvature magnitude M/D. Dividing all moduli by one reference modulus gives the equivalent transformed-section construction. Taking an arithmetic mean modulus and the geometric centroid separately generally gives the wrong neutral position as well as the wrong rigidity.

As an independent worked section, bond two strips each 20 mm wide and 5 mm thick. Let the lower strip have $E = 70$ GPa and the upper strip $E = 210$ GPa. Measured from the bottom, their centroids are at 2.5 and 7.5 mm and each area is 100 mm². Thus

$$y_n = (2.5 + 3 \times 7.5)/4 = 6.25\,\mathrm{mm}.$$

Each layer has $I_c = 208.33\,\mathrm{mm}^4$. Their distances from the neutral axis are 3.75 and 1.25 mm. The resulting rigidity is $1.896 \times 10^{8}\,\mathrm{N\,mm}^2$. A sagging 20 N m moment gives

$$\kappa = 1.055 \times 10^{-4}\,\mathrm{mm}^{-1}.$$

The lower surface carries 46.2 MPa tension and the upper carries 83.1 MPa compression. At the interface $y = 5$ mm the strain is continuous at about 132 microstrain in tension, but the two normal stresses are 9.23 and 27.7 MPa. Such a longitudinal stress jump is compatible with a sound bond. Delamination would instead invalidate the assumed strain compatibility.

Practise on sheet 2 · Size a symmetric sandwich beam from its bending rigidity. Further reading · MIT · 1.050 Solid Mechanics · Chapter 9, §9.5 Stresses within a Composite Beam.

Study the full derivation

Why an open channel twists under a transverse load · Stress, deformation and structural measurement

A channel can twist even when a force passes through its centroid. Trace the longitudinal shear flow around its flanges, find the load line that produces bending without torsion, and design a loading experiment that separates the stiffness in bending from eccentricity relative to this shear centre.

Chapter 4 · Elastic curves and virtual work

Predict a specified displacement from the bending-moment field and distinguish a stiffness limit from a stress limit.

FROM FIRST PRINCIPLES

Why cantilever stiffness scales as the inverse cube of span

Take a straight slender beam clamped horizontally, with a downward end force P. The empirical inputs are small-strain linear elasticity and the beam assumption that plane sections remain normal to the deflected centreline. Denote its centroidal bending rigidity by EI. With constant section and modulus the stress moment of the stretched and compressed fibres is proportional to curvature. With downward deflection positive, equilibrium of a cut at x from the clamp supplies

$$M(x) = P(L - x),$$
$$w''(x) = M(x)/(EI).$$

Curvature is the rate at which slope changes. Integrating it once gives the slope, and integrating slope gives deflection. Combine both integrations into a weighted curvature integral: curvature occurring near the clamp rotates a longer remaining segment than curvature near the tip. The clamp fixes initial slope and displacement to zero, hence

$$w(L) = \int_0^L (L - x) w''(x)\, dx = \frac{P}{EI} \int_0^L (L - x)^2 dx = \frac{PL^3}{3EI}.$$

This is a displacement prediction under small slopes, not just a stress limit. Its input $M = EIw''$ is established from individual fibre strains and the neutral-axis condition in the detailed derivation. Short beams can also require shear deflection.

Background · Second moment of area

$$I = \int_A y^2\, dA$$

Measure the transverse coordinate from the stated bending axis. This section property has units of length to the fourth power and enters the flexural rigidity.

DERIVATION AND APPLICATIONS

From the extension of a fibre to cantilever curvature and

displacement

Why does section depth enter a bending calculation through a second moment of area? Take a straight prismatic cantilever with small slopes, uniform Young's modulus E, linear elastic response and slender dimensions. Euler–Bernoulli kinematics assumes that cross-sections remain plane and normal to the deflected centreline. Measure x from the clamp and transverse deflection w downward. Measure cross-section coordinate y downward from an initially arbitrary reference line. Sections separated by dx rotate relative to one another by $d\theta$; a fibre at y shortens relative to the reference line by $y\,d\theta$.

$$\epsilon_x(y) = \epsilon_0 - y\,d\theta/dx \simeq \epsilon_0 - y w''(x),$$
$$\sigma_x = E\epsilon_x.$$

With no imposed axial resultant, the integral of this normal stress must vanish. Writing $\bar{y} = A^{-1}\int_A y\,dA$, we obtain the strain of the reference line, and thereby identify the zero-strain fibre:

$$0 = N = \int_A E(\epsilon_0 - y w'')\,dA,$$
$$\epsilon_0 = \bar{y} w'',$$
$$\sigma_x = -E(y - \bar{y})w''.$$

Define bending moment positive for this downward curvature. Each fibre contributes its force times its lever arm about the neutral axis. Integrating introduces the centroidal second moment I:

$$M = -\int_A (y - \bar{y})\sigma_x\,dA = EIw'',$$
$$I = \int_A (y - \bar{y})^2\,dA.$$

A downward tip load P gives $M(x) = P(L - x)$ by equilibrium of the portion beyond the cut. For constant EI, integrate twice and enforce the actual clamp conditions $w(0) = w'(0) = 0$:

$$EIw' = P(Lx - x^2/2),$$
$$EIw = P(Lx^2/2 - x^3/6),$$
$$w(L) = PL^3/(3EI).$$

The slope and displacement constants follow from compatibility at the support. The zero moment at the free tip follows independently from the load on the right-hand portion. Plane sections can also remain plane under pure bending of a short specimen, but shear deformation near loads and supports can matter in short cantilevers.

WORKED EXAMPLE

Calculate a tip reading and maximum fibre strain

A rectangular steel strip has $L = 0.200$ m, $b = 20.0$ mm, $h = 8.00$ mm and $E = 200$ GPa. Depth h lies along the deflection direction, so $I = bh^3/12 = 8.53 \times 10^{-10}$ m⁴. A 10.0 N tip load produces $w(L) = 0.156$ mm. The clamp moment is 2.00 N m; its curvature 0.0117 m⁻¹ gives outer-fibre strain magnitude $(0.00400)(0.0117) = 46.9$ microstrain. The top is tensile and the bottom compressive for the stated load.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Vary the span to distinguish two sources of compliance

On the same strip double the free span to 0.400 m. Bending predicts eight times the displacement, or 1.25 mm; outer-fibre strain doubles to 93.8 microstrain. A nearly span-independent displacement offset would instead suggest motion of the clamp fixture, and a shear-displacement component at fixed load and section scales only linearly with span. The different scalings let two displacement readings test the assumptions behind a bending-only interpretation.

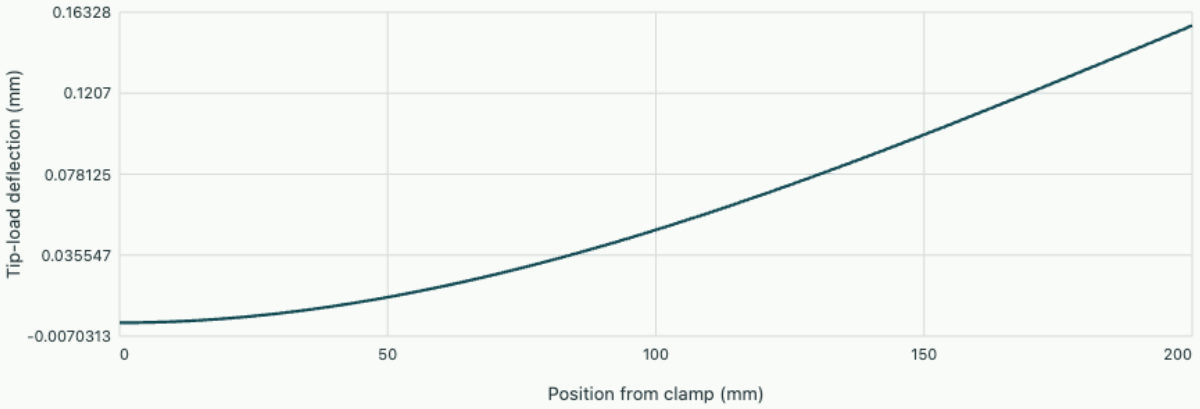
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Length (mm)	200
Section depth (mm)	8
Tip force (N)	10



Integrated curvature

Tip reading 0.1563 mm; clamp outer-fibre strain 46.87 microstrain. Width20 mm and E200 GPa.

ONESIMULATION LABORATORY

Where does a three-dimensional beam depart from bending theory?

A 120 mm aluminium beam is clamped over its entire left end face. The free face carries a downward resultant of 30 N distributed uniformly. Keep the width at 18 mm and Young's modulus at 70 GPa. Change only the height. The experiment asks how far a one-dimensional bending law can predict a three-dimensional displacement.

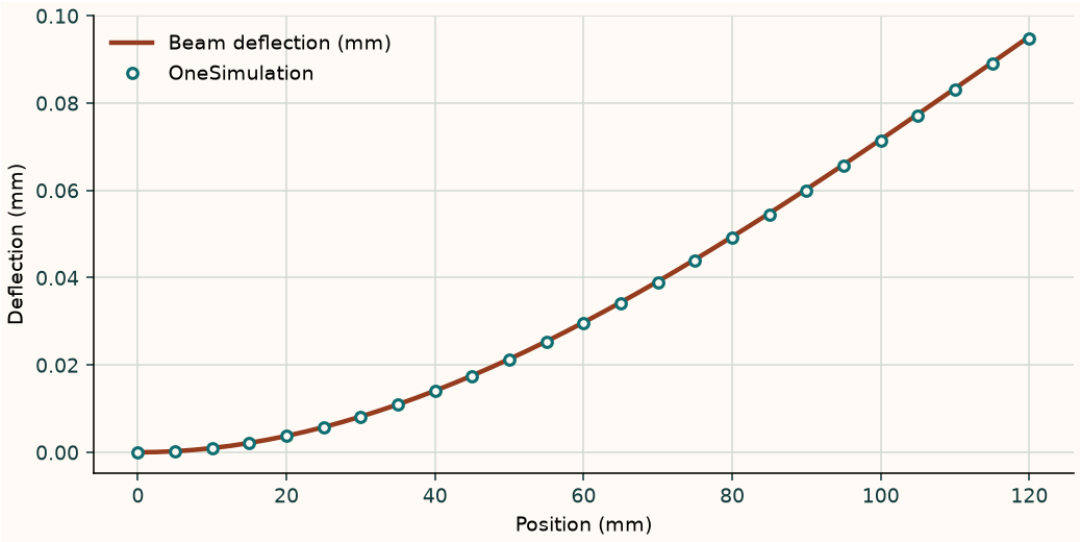
Make the paper prediction. Take downward deflection as positive. A cut at distance x from the clamp has bending-moment magnitude $F(L - x)$. Using the rectangular section's $I = bh^3/12$ and integrating from the clamp gives

$$EIw'(x) = F(Lx - x^2/2),$$

$$EIw(x) = F(Lx^2/2 - x^3/6),$$

$$w(L) = \frac{4FL^3}{Ebh^3}.$$

The integration constants vanish because both displacement and rotation are fixed at the clamp. Doubling the height from 6 to 12 mm should cut the tip deflection to one eighth. Calculate the 12 mm prediction, then select the two heights in the figure. Move the position control to the loaded end to read the measured ratio.



Downward centreline displacement against distance from the clamp, comparing the solid model with beam theory. Beam height (mm) = **12**.

Download this run (CSV)

Explain the discrepancy. The 12 mm beam gives 0.09491 mm in the solid model and 0.09524 mm from bending. The 36 mm beam gives 0.003717 and 0.003527 mm, a difference of about 5.4% relative to the beam prediction. The solid model permits transverse shear and constrains the whole end face. Both assumptions differ from beam theory. Changing axial mesh density tests one numerical error source. It cannot by itself identify the physical origin of every difference.

Try a fourth section. Keep the force fixed at 30 N. Predict the result for an 18 mm height, run the model, then repeat with `--cells 36` to check the effect of axial refinement on the centreline. The CSV reports downward displacements in millimetres.

Further reading · David Roylance, Beam displacements.

Download the OneSimulation model (Python). Set up the local runtime, then run the downloaded file:

```
python cantilever.py --height 18
```

Test a beam reduction against a solid in OneSimulation

A 120 mm cantilever of width 18 mm and modulus 70 GPa carries 30 N uniformly distributed over its end face. Make a prediction before using the slider. If the height changes from 12 to 24 mm, which changes more, the section area or its bending stiffness?

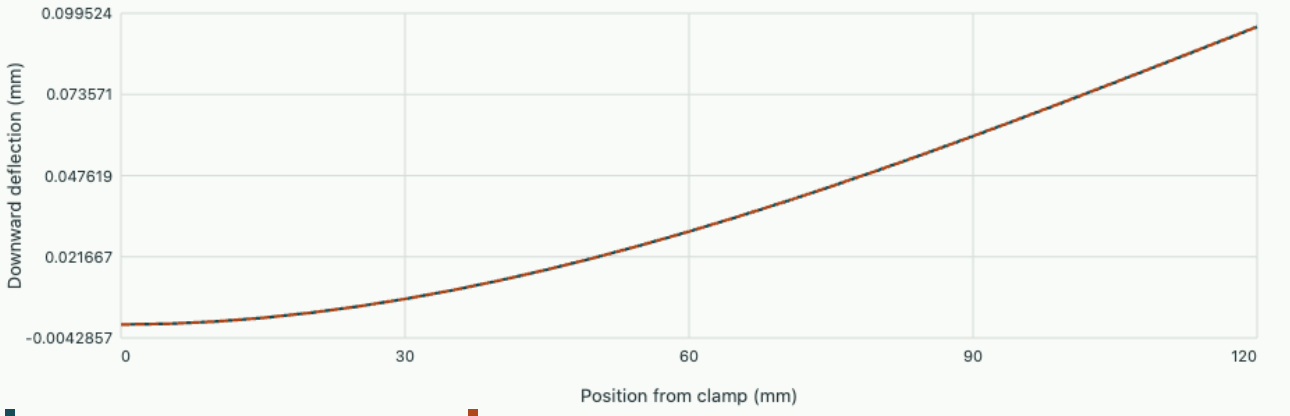
The end moment about a section a distance x from the clamp has magnitude $F(L - x)$. The small-slope beam model first converts it into curvature $w'' = F(L - x)/(EI)$, then integrates twice with $w(0) = w'(0) = 0$. The browser performs both integrations on the selected grid by trapezoidal sums. The first integral is exact for linear curvature. The second integrates a quadratic slope and retains a grid error. Its reference is $w(x) = Fx^2(3L - x)/(6EI)$.

Open this experiment online

Try the calculation

Section height (mm) **12**

Axial cells **24**



Integrated curvature on the grid Euler–Bernoulli solution

Tip deflection 0.095197 mm on this 1D grid, 0.095238 mm analytically. The OneSimulation model solves the same load on a 3D solid.

Download the OneSimulation model

In your OneSimulation environment, run the model with these inputs.

```
python cantilever.py --height 12 --cells 24
```

Download the model for the same dimensions and change the parameters shown with the graph. OneSimulation discretises the three-dimensional solid with quadratic hexahedral elements and applies

distributed end traction. Its CSV compares centreline deflection with the analytical curve. First fix height and refine axially. Then increase height at a converged mesh. The first experiment addresses numerical error within a model, the second probes the slender-beam assumptions. Under those assumptions doubling height reduces deflection eightfold because $I = bh^3/12$.

When a beam is strong enough but too flexible

A tool rail may stay below its elastic limit while its cutting head moves outside a position tolerance. The maximum bending stress depends on the largest moment at one section. Deflection accumulates the effect of curvature over a length. A displacement sensor near an unloaded end can therefore move even if the nearby bending moment is zero. To predict it, identify both the point and direction of the required displacement.

Consider a straight, slender Euler–Bernoulli beam. Cross-sections remain plane and perpendicular to the deformed centreline. Take w positive upward and M positive in sagging. For small slopes the signed curvature is w'', with $\kappa = M/(EI)$. Integrate once to find the slope and once more to find the displacement. Each constant of integration has physical meaning. A clamp fixes displacement and rotation. A simple support fixes its vertical displacement while permitting rotation. At an ordinary join without a hinge, both displacement and slope are continuous.

Work through a tip reading

Clamp a beam at $x = 0$. Apply a downward point load P at $x = a$ on a beam of total length $L > a$. The signed moment in $0 < x < a$ is $-P(a - x)$. For $x > a$ the free segment has no external force and no moment. Taking constant stiffness and using the clamp conditions gives

$$EIw'(x) = -P(ax - x^2/2),$$

$$EIw(x) = -P(ax^2/2 - x^3/6), \quad 0 \le x \le a.$$

At the load point the displacement is $-Pa^3/(3EI)$ and the slope is $-Pa^2/(2EI)$. The free extension beyond it is straight but tilted. Use those values as initial conditions for the unloaded segment:

$$w(L) = w(a) + (L - a)w'(a) = -Pa^2(3L - a)/(6EI).$$

This expression correctly vanishes when the force is applied at the clamp and reduces to the standard end-load expression when $a = L$.

Calculate both position and slope

Let $EI = 8400\,\mathrm{N\,m^2}$, $L = 1.20$ m, $a = 0.80$ m and $P = 100$ N. Under the load the downward displacement is 2.03 mm and the rotation magnitude is

$$100(0.80)^2/[2(8400)] = 0.00381\,\mathrm{rad}.$$

The 0.40 m overhang contributes another 1.52 mm at the tip, for a total downward reading of 3.56 mm. The slope is small compared with a radian, consistent with the linear curvature assumption. A gauge at the load and one at the tip would give different stiffness estimates if their positions were omitted.

Obtain one displacement directly

There is another way to compute a single sensor reading. Introduce a fictitious force Q at the sensor in its desired positive direction. The moment becomes $M_P + Qm$, where m is the moment per unit sensor force. The bending energy is

$$U = \int (M_P + Qm)^2/(2EI)\, dx.$$

Differentiating with respect to Q gives the displacement conjugate to that force. Set $Q = 0$ afterwards to recover the original beam.

$$\delta = \int_0^L \frac{M_P(x) m(x)}{E(x) I(x)}\, dx.$$

For a downward unit force at the free tip, $m = -(L - x)$. Only the portion up to a contributes because $M_P = 0$ further along. The integral gives $Pa^2(3L - a)/(6EI)$, the positive downward reading obtained above. This form is especially useful for stepped sections: split the integral where either stiffness or loading changes. Its positive self-flexibility for a load and a sensor pointing the same way is a useful sign check.

Exchange the load and the gauge

The unit-load formula supplies a test that can be made on the same beam without knowing its exact stiffness distribution. Put the same 100 N force at the tip and move the displacement sensor to $x = 0.80$ m. The new integral has exactly the previous pair of moment fields with their roles exchanged. Since their product is unchanged, the new reading is also 3.56 mm downwards. This Maxwell–Betti reciprocity applies to these work-conjugate forces and displacements in a linear elastic structure with unchanged restraints. A slipping clamp, a support that lifts off or motion large enough to change the load geometry can cause discrepancies between reciprocal tests. Swapping load and sensor positions is therefore also a practical check of the assumed support behaviour.

Quantify the deformation omitted by a bending-only model

The straight-normal assumption of Euler–Bernoulli theory omits transverse shear distortion. For a slender beam this is often small, but for a deep beam or a material with a small effective shear modulus it can dominate a displacement measurement. In a simple Timoshenko description let $\kappa_s G A$ be the effective shear stiffness. The dimensionless shear correction factor κ_s accounts for the nonuniform

cross-sectional shear field when a single transverse shear resultant V is used. For the illustration use $\kappa_s = 5/6$ for a homogeneous rectangular section.

For an end-loaded cantilever the bending moment magnitude is $P(L - x)$ and the shear force magnitude is P throughout the span. Add the elastic shear energy to the bending energy. Differentiating with respect to the end force yields the total deflection under that force:

$$U = \int_0^L \left[\frac{P^2 (L - x)^2}{2EI} + \frac{P^2}{2\kappa_s G A} \right] dx,$$

$$\delta = \frac{PL^3}{3EI} + \frac{PL}{\kappa_s G A}.$$

The shear contribution grows linearly with span while the bending contribution grows cubically. Their ratio is $3EI/(\kappa_s G A L^2)$. Writing $I/A = h^2/12$ for a rectangle gives $[E/(4\kappa_s G)](h/L)^2$. This form identifies both the geometric slenderness and material stiffness ratio required for a sensible model choice. A fixed rule based on span-to-depth alone can be poor for a sandwich section with a compliant core.

Interpret a short cantilever stiffness test

An aluminium fixture projects 120 mm from a rigid clamp. Its cross-section is 20 mm wide and 40 mm deep in the bending direction. Take $E = 70{,}000\,\mathrm{MPa}$, $G = 26{,}000\,\mathrm{MPa}$, and an end load of 500 N. The area is 800 mm² and the bending second moment is

$$20(40)^3/12 = 106667\,\mathrm{mm}^4.$$

In consistent N–mm units the bending deflection is

$$500(120)^3/[3(70000)(106667)] = 0.0386\,\mathrm{mm}.$$

The shear deflection is

$$500(120)/[(5/6)(26000)(800)] = 0.00346\,\mathrm{mm}.$$

A dial indicator would measure about 0.0420 mm under these ideal assumptions.

Fitting a bending-only formula to this reading suggests an apparent modulus of about $70(0.0386/0.0420) = 64.2\,\mathrm{GPa}$. The discrepancy arises from omitted shear deformation despite a perfectly elastic material. Repeat the test at twice the overhang while keeping the force unchanged: bending deflection rises by eight, shear deflection by two. The fractional shear error falls to one quarter of its original value. Measuring several spans can therefore separate an incorrect modulus from a shear correction. Clamp compliance is another candidate in a real experiment. A clamp rotation produces a term with a different span dependence, and a gauge mounted close to the fixture can reveal that motion independently.

For equal applied loads at spans L and $2L$, express measured compliances as $C_1 = B + S$ and $C_2 = 8B + 2S$, where B and S are the original bending and shear contributions per unit load. Then $B = (C_2 - 2C_1)/6$ and $S = C_1 - B$. A negative inferred S signals measurement uncertainty or an omitted boundary deformation, since the assumed elastic shear energy must be positive. This inversion makes the two-span proposal a quantitative model test.

Taking three spans instead of two permits a residual check, because the third measured displacement is then predicted without adjusting either fitted contribution.

Problem sheet 2 · Beam actions and deformations

Follow each load from the support diagram to the internal moment and the sensor reading.

1. Move a wheel along a beam

A single 8 kN wheel loads a simply supported 4.0 m span 1.0 m from A. (a) Find the reactions and largest moment. (b) Place the wheel 2.0 m from A and repeat. (c) A section has $Z = 160\,000$ mm³. Compare the largest nominal stresses.

▼ Work from reactions to stress

The first moment balance about A gives $R_B = 8(1)/4 = 2$ kN and $R_A = 6$ kN. Shear changes sign at the wheel, which is where $M_{\max} = 6(1) = 6$ kN·m. At midspan the reactions are both 4 kN and $M_{\max} = 8$ kN·m. Convert moments to N·mm and divide by Z. The extreme stresses are 37.5 and 50.0 MPa. Symmetry maximises the moment from one mobile wheel at midspan for this support arrangement.

---

2. Locate the maximum moment under a tapered load

A 3.0 m simple span carries downward load rising from zero at A to 6.0 kN/m at B. (a) Determine the reactions. (b) Obtain $V(x)$ and $M(x)$ on the open span. (c) Find the maximum and check the other end condition.

▼ Integrate the triangular load

The resultant is 9 kN at 2 m from A, so $R_B = 6$ kN and $R_A = 3$ kN. In metres the intensity is $2x$ kN/m. Integrating $V' = -2x$ gives $V = 3 - x^2$ kN. With $M(0) = 0$, $M = 3x - x^3/3$ kN·m. The positive shear vanishes at $x = \sqrt{3} = 1.732$ m, where $M = 2\sqrt{3} = 3.46$ kN·m. At B, $M(3) = 9 - 9 = 0$ and the 6 kN support reaction exactly cancels the remaining negative shear.

---

3. Compare two displacements on a cantilever

A 0.90 m cantilever with $EI = 6000\,\mathrm{N\,m^2}$ carries a 240 N downward tip force. (a) Derive the end deflection by the unit-load method. (b) Compute the deflection at the halfway station using a fictitious load there. (c) Explain why linear interpolation from the tip is inaccurate.

▼ Work the two virtual-load diagrams

For the tip reading multiply $M = -P(L - x)$ by $m = -(L - x)$ and integrate from 0 to L. The displacement is

$$PL^3/(3EI) = 9.72\,\mathrm{mm}.$$

For a unit load at $a = L/2$, $m = -(a - x)$ up to a and zero beyond. Thus

$$\delta(a) = P \int_0^a (L - x)(a - x)\, dx/(EI) = Pa^2(3L - a)/(6EI) = 3.04\,\mathrm{mm}.$$

Half the tip reading would be 4.86 mm. The zero slope at the clamp and bending curvature throughout the beam give a nonlinear deflection shape.

Size a symmetric sandwich beam from its bending rigidity

A 30 mm wide beam has a 16 mm core of modulus 2 GPa and two 2 mm faces of modulus 70 GPa. For a 15 N m moment, locate the neutral axis, calculate curvature and maximum face stress, and determine the fraction of bending rigidity supplied by the core.

▼ **Show the worked solution**

Symmetry of materials and geometry fixes the neutral axis halfway through the 20 mm total depth. Core $I = 30(16)^3/12 = 10240\,\mathrm{mm}^4$. Each face centroid is 9 mm from mid-depth and contributes $30(2)^3/12 + 60(9)^2 = 4880\,\mathrm{mm}^4$. Rigidity is

$$2000(10240) + 2(70000)(4880) = 7.0368 \times 10^8\,\mathrm{N\,mm}^2.$$

Curvature is

$$15000/D = 2.132 \times 10^{-5}\,\mathrm{mm}^{-1}.$$

Outer face stress magnitude is $70000(10)\kappa = 14.9\,\mathrm{MPa}$. The core provides only $20.48/703.68 = 2.91\%$ of bending rigidity. It still keeps the faces separated and can carry substantial transverse shear.

Revisit chapter 3 · Find curvature and stress in a bonded two-material beam.

Explore the governing equations after solving the sheet: Shear force and bending moment; Elastic curves and virtual work. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Chapter 5 · Stress transformation and combined loading

Find principal stresses and elastic yield measures at a shaft surface carrying bending and torsion.

FROM FIRST PRINCIPLES

Recover principal normal stresses from traction balance

For a small material tetrahedron, equilibrium of the tractions on its faces gives Cauchy's relation $\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma}\mathbf{n}$. In a classical continuum without body couples, local moment balance makes $\boldsymbol{\sigma}$ symmetric. In plane stress select a section normal $\mathbf{n} = (n_x, n_y)$ in the sheet. A principal section carries traction parallel to its normal, so it solves $\boldsymbol{\sigma}\mathbf{n} = \lambda\mathbf{n}$. A nonzero normal exists only if

$$(\sigma_x - \lambda)(\sigma_y - \lambda) - \tau_{xy}^2 = 0.$$

Solve the quadratic by completing the square about the mean normal stress. This gives both principal values:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}.$$

The associated real eigenvectors are orthogonal when the values differ. For a pure shear, the principal stresses are equal and opposite even though normal stresses on the original coordinate cuts are zero. For a traction-free sheet the third principal value is zero; this third direction must be included when applying a three-dimensional yield criterion.

DERIVATION AND APPLICATIONS

Find the plane carrying the largest shear traction

A tensile test measures a normal stress on one specified cut. In a machine part another cut through the same point can carry shear even when the original axes coincide with the loading. Assume plane stress: the free faces normal to z have zero traction and the nonzero components are $\sigma_x, \sigma_y, \tau_{xy}$. Choose an in-plane unit normal $\mathbf{n} = (\cos\theta, \sin\theta)$ and tangent $\mathbf{s} = (-\sin\theta, \cos\theta)$. Cauchy's traction relation gives the force per actual area on that cut:

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = (\sigma_x \cos\theta + \tau_{xy}\sin\theta,\ \tau_{xy}\cos\theta + \sigma_y \sin\theta).$$

Project onto the normal and tangent. Introduce $m = (\sigma_x + \sigma_y)/2$ and $d = (\sigma_x - \sigma_y)/2$. The double-angle identities rearrange the two projections into a rotation:

$$\sigma_n - m = d\cos 2\theta + \tau_{xy}\sin 2\theta,$$
$$\tau_{ns} = -d\sin 2\theta + \tau_{xy}\cos 2\theta.$$

Squaring and adding cancels the cross terms. This supplies both the in-plane principal stresses and the range of shear traction:

$$(\sigma_n - m)^2 + \tau_{ns}^2 = d^2 + \tau_{xy}^2 = R^2,$$
$$\sigma_{1,2} = m \pm R,$$
$$|\tau_{ns}|_{\max} = R.$$

Planes of largest in-plane shear are rotated 45° from principal planes. The three-dimensional maximum also includes planes involving the out-of-plane principal stress, which equals zero under plane stress. That distinction matters whenever both in-plane principal stresses have the same sign.

WORKED EXAMPLE

Resolve the stress on a particular cut

Take $(\sigma_x, \sigma_y, \tau_{xy}) = (90, 30, 40)$ MPa. The centre and radius are 60 and 50 MPa, giving in-plane principal stresses 110 and 10 MPa. On a cut with normal at 45°, the projections give $\sigma_n = 100$ MPa and $\tau_{ns} = -30$ MPa. Their point lies on the circle because $40^2 + 30^2 = 50^2$. Including the zero out-of-plane stress gives absolute maximum shear 55 MPa.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Test a reported shear measurement

For equal biaxial tensile stresses 70 MPa with $\tau_{xy} = 0$, both d and R vanish. Every in-plane cut has 70 MPa normal stress and zero shear. A gauge claiming 20 MPa shear on a plane normal entirely in the sheet contradicts this state. A plane tilted 45° out of the sheet can carry 35 MPa, since it mixes principal stresses of 70 and zero MPa.

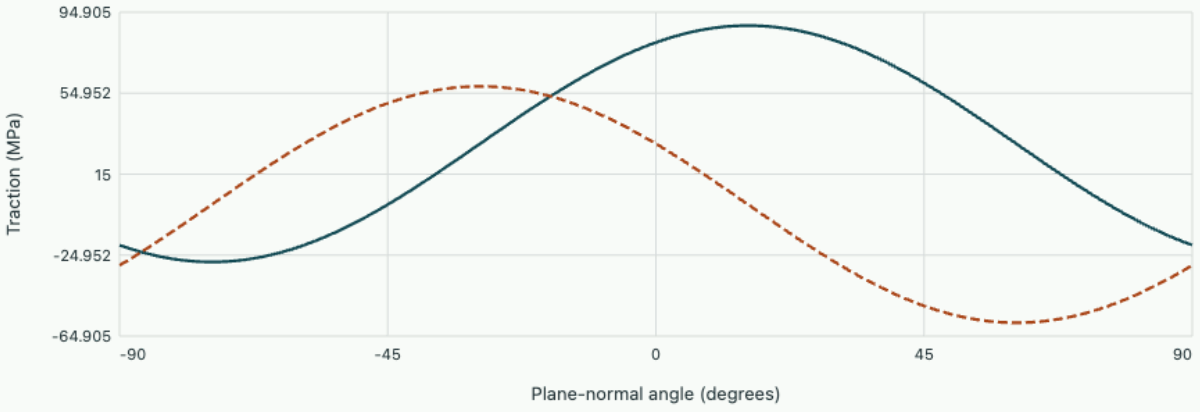
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Stress σx (MPa)	80
Stress σy (MPa)	-20
Shear τxy (MPa)	30



Principal stresses 88.310 and -28.310 MPa; principal direction 15.482°. In-plane maximum shear 58.310 MPa.

A section resultant does not specify a failure plane

The free-body diagram of a shaft gives bending moments and torque. A potential crack, however, can cut the shaft on any plane. To compare the local stress with a material test, first evaluate the stress components at the same material point. At the outer fibre of a circular shaft the bending normal stress may be large. At the neutral axis the bending stress vanishes. Values from different points cannot be combined into one stress state.

For a circular shaft of diameter d bending moment M gives surface stress $32M/(\pi d^3)$, and torque T gives surface shear $16T/(\pi d^3)$. The torsion relation uses $J = \pi d^4/32$. The combination assumes elastic small deformation in a smooth prismatic portion, sufficiently far from shoulders, load introduction

and keyways. At a traction-free outer surface the radial traction is zero. Use a local basis with longitudinal and circumferential axes for the remaining plane-stress components.

Change the orientation of a cut

In two dimensions write the stress tensor with normal components σ_x, σ_y and shear τ_{xy}. The traction vector on a cut with unit normal $\mathbf{n}$ is $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$. Project that vector along and perpendicular to $\mathbf{n} = (\cos\theta, \sin\theta)$. The normal stress becomes a constant average plus a sinusoidal term; the shear varies with twice the same angle:

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta.$$

The largest and smallest in-plane normal stresses differ from the mean by $R = \sqrt{[(\sigma_x - \sigma_y)/2]^2 + \tau_{xy}^2}$. The shear on those principal planes vanishes. Taking the third principal stress as zero allows a three-dimensional maximum-shear check: evaluate half the difference of the largest and smallest of all three principal stresses. In a biaxial tensile state, ignoring the zero third stress can underestimate this difference.

Compare principal stresses on a machine shaft

Let a 40 mm diameter steel shaft carry $M = 300$ N·m and $T = 200$ N·m. At its tensile bending surface, $\sigma_x = 47.75$ MPa, $\sigma_y = 0$ and $\tau_{xy} = 15.92$ MPa. The in-plane mean is 23.87 MPa and the radius is 28.69 MPa. The principal stresses are about 52.57 and −4.82 MPa. The plane of greatest tension is rotated through half of $\operatorname{atan2}(2\tau_{xy}, \sigma_x - \sigma_y)$, here 16.8° from the original plane normal. The maximum shear magnitude is 28.7 MPa.

A useful check is the trace: the principal stresses sum to 47.75 MPa, the same sum as the original normal components. A second check is the determinant: the two principal values have opposite signs since the determinant of this particular matrix is $-\tau_{xy}^2$. A picture with the two directions marked prevents confusing the angle of a plane normal with that of the line drawn for the plane itself.

Interpret an equivalent stress

For isotropic ductile materials a common onset-of-yield model uses von Mises equivalent stress. Under plane stress it is $\sqrt{\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2}$, giving 55.1 MPa here. Compare this scalar with the uniaxial yield strength from a material test under a comparable temperature and loading rate. A given yield strength of 220 MPa corresponds to a nominal static factor of about 4.0. That number answers a defined static question. A shaft rotating under a stationary transverse force cycles the tensile and compressive stress of a given material point once per revolution. Evaluating its fatigue resistance requires the stress range and a separate material model.

Change the mean stress without changing the distortion

Apply a uniform external hydrostatic pressure of 50 MPa to a material point whose original principal stresses are 100, 0 and 0 MPa. The new principal stresses are 50, −50 and −50 MPa, since pressure acts on every plane. Their pairwise differences remain 100, 0 and −100 MPa, so the von Mises stress remains 100 MPa. This illustrates why a distortion-based yield condition must include all three normal directions under confinement. Applying a plane-stress formula after adding pressure only to the in-plane directions would represent a different three-dimensional loading.

Derive the two membrane stresses of a thin cylinder

A hydraulic accumulator behaves differently in its axial and circumferential directions even if it is made of isotropic steel. Consider a long cylindrical shell of mean radius r, wall thickness $t \ll r$, closed ends and internal gauge pressure p. Away from nozzles and end transitions, symmetry permits a membrane model with tensile stresses uniform through the thickness. The radial stress changes from approximately $-p$ at the inner wall to zero outside. Its magnitude is small compared with pr/t if the shell is thin.

Cut the cylinder longitudinally into two halves of length L. The net force from pressure acts on the projected rectangle of area $2rL$, not on the curved wetted area. The two cut edges carry hoop tension. A different cut across the full circular end isolates the cap thrust and the single axial membrane stress. The resulting balances are

$$2\sigma_\theta t L = p(2rL),$$

$$2\pi r t \sigma_z = p\pi r^2,$$

$$\sigma_\theta = pr/t,$$

$$\sigma_z = pr/(2t).$$

The factor of two follows from geometry and end load transmission. If the ends are open and pressure thrust returns through a separate frame, the cylinder may carry little axial membrane force. Always follow the cap force into the structure before assigning axial stress. A cap anchored externally and a welded pressure shell have different load paths even at the same gauge pressure.

Infer shell strain and the effect of axial restraint

For a closed vessel with $r = 0.20\,\mathrm{m}$, $t = 4.0\,\mathrm{mm}$ and $p = 1.2\,\mathrm{MPa}$, the membrane stresses are 60 MPa circumferentially and 30 MPa axially. These directions are principal at an unperturbed surface. Under plane stress the von Mises stress is

$$\sqrt{60^2 + 30^2 - 60(30)} = 52.0\,\mathrm{MPa}.$$

With $E = 200\,\mathrm{GPa}$ and $\nu = 0.30$, an outer hoop gauge reads $(60 - 0.30 \times 30)/200000 = 255$ microstrain and an axial gauge reads only $(30 - 0.30 \times 60)/200000 = 60$ microstrain. The radius

increases by about $0.20(255\times10^{-6})=51\,\mu\text{m}$. The smaller axial signal is particularly sensitive to temperature correction.

Now imagine an axial restraint strong enough to hold the length constant without affecting the radius directly. The condition $\varepsilon_z=0$ gives

$$\sigma_z=\nu\sigma_\theta=18\,\text{MPa}.$$

The cap thrust has not disappeared. An external axial reaction must remove the difference of 12 MPa integrated over the shell area $2\pi rt$, about 60.3 kN. A gauge-based interpretation using the unrestrained 30 MPa estimate would miss this load. Conversely, where the restraint attaches, local bending and stress concentration may dominate the smooth membrane solution. Locate comparison gauges several characteristic wall dimensions away from such load introduction.

The force calculation can be audited directly. Pressure thrust on the end cap is $1.2\pi(200)^2=151\,\text{kN}$ in MPa and millimetres. An 18 MPa shell stress carries

$$18[2\pi(200)(4)]=90.5\,\text{kN}.$$

Their difference, 60.3 kN, is precisely the required external load path. A restraint instrumented with a load cell would test the assumed axial boundary condition independently of the strain readings.

Extend the traction balance to three dimensions

In three dimensions a cut normal has three components. Cauchy’s relation remains $\mathbf{t}=\boldsymbol{\sigma}\mathbf{n}$. Project onto the unit normal to obtain signed normal stress, then remove that projection to obtain the shear traction. Rotating a diagram does not rotate the tensor physically; changing its components does.

$$\sigma_n=\mathbf{n}^T\boldsymbol{\sigma}\mathbf{n},$$
$$\mathbf{t}_s=\boldsymbol{\sigma}\mathbf{n}-\sigma_n\mathbf{n}.$$

The eigenvectors give principal cut directions with zero shear traction. The following notebook evaluates the symmetric eigensystem and independently verifies both the tensor reconstruction and the orthogonality of each shear traction to its cut normal.

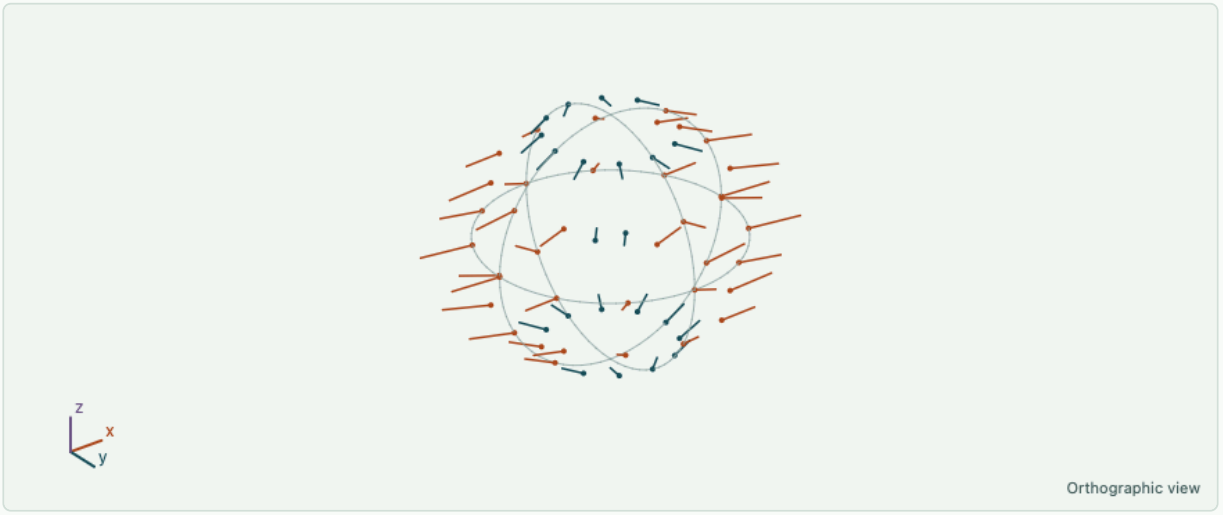
Interactive calculation · Inspect three-dimensional traction on every cut

Change the out-of-plane shear to leave the original plane-stress example. Rotate the sphere of cut normals, then tilt the plotted slice through it. The force segments use a common symbolic scale; they show traction directions, not displacements.

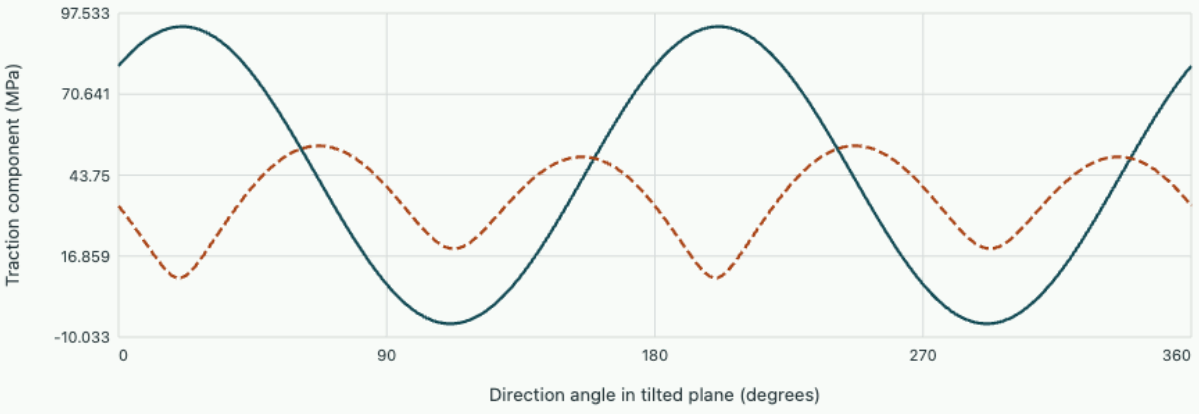
Open this live calculation

Try the calculation

Normal stress σx (MPa)	80
Shear τxy (MPa)	30
Shear τxz (MPa)	15
Slice-plane tilt (degrees)	30



Unit-normal sphere with three-dimensional traction vectors. Each segment starts at a cut normal and ends in the traction direction, using one common symbolic force scale. Orange indicates tensile normal traction, teal compressive. The sphere is a directional diagram, not a deformed body. The OneSimulation notebook also computes and verifies the principal directions. Axes: x = normal x; y = normal y; z = normal z.



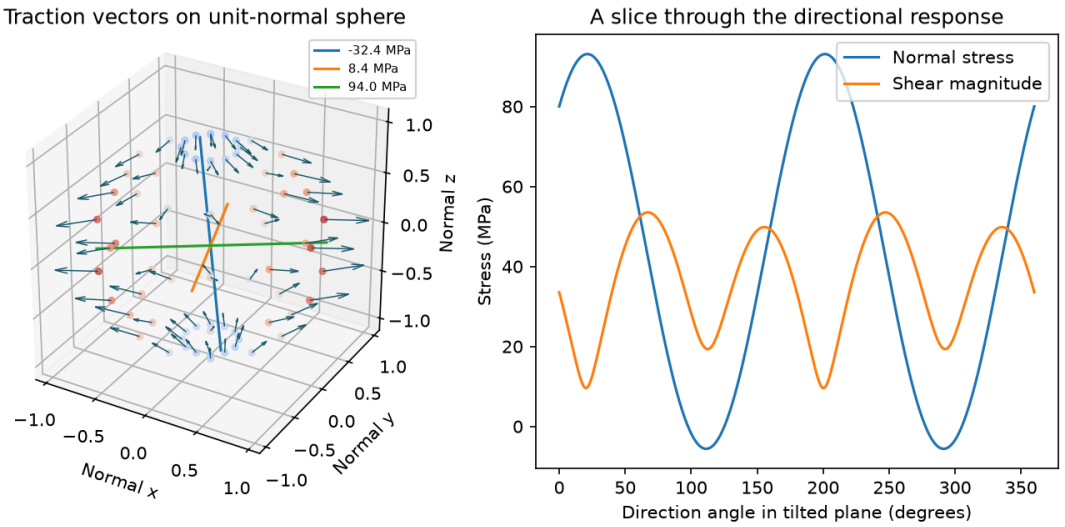
Cauchy traction t=σn is resolved into a normal part and a tangential part for every cut. Fixed σy=20 MPa and σz=−30 MPa; τyz=0. At slice angle45°, normal=77.231 MPa and shear=39.925 MPa.

OneSimulation notebook · solved spatial stresses, code and output

The code solves the illustrated stress model with OneSimulation and checks its equilibrium or eigensystem. The figure is the output of that code.

```
"""Compute principal stresses with OneSimulation and show Cauchy traction in 3D."""
import numpy as np
import matplotlib.pyplot as plt
from onesim.numerics.linear.matrices.dense_linear_algebra import symmetric_eigensystem

# Symmetric Cauchy tensor in MPa; edit the entries to reproduce the browser controls.
sigma=np.array([[80.,30.,15.],[30.,20.,0.],[15.,0.,-30.]])
principal, axes = symmetric_eigensystem(sigma)
assert np.allclose(axes@np.diag(principal)@axes.T,sigma)
assert np.allclose(axes.T@axes,np.eye(3))
print('Principal stresses [MPa]:',np.round(principal,5))
print('Absolute maximum shear [MPa]:',round((principal[-1]-principal[0])/2,5))
phi=np.linspace(0,2*np.pi,12,endpoint=False)
theta=np.linspace(.25,np.pi-.25,6)
n=np.array([[np.sin(t)*np.cos(p),np.sin(t)*np.sin(p),np.cos(t)] for t in theta for p in phi])
traction=n@sigma.T
normal=np.sum(n*traction,axis=1)
shear=traction-normal[:,None]*n
assert np.max(abs(np.sum(n*shear,axis=1)))<1e-12
assert normal.min()>=principal[0]-1e-10 and normal.max()<=principal[-1]+1e-10
fig=plt.figure(figsize=(9,4.2),layout='constrained');ax=fig.add_subplot(121,projection='3d')
ax.quiver(*n.T,*(traction/120).T,length=.5,normalize=False,color='#19596a',linewidth=.7)
ax.scatter(*n.T,c=normal,cmap='coolwarm',s=12,vmin=-100,vmax=100)
for j in range(3):ax.plot(*np.array([-axes[:,j],axes[:,j]]).T,label=f'{principal[j]:.1f} MPa')
ax.set(xlabel='Normal x',ylabel='Normal y',zlabel='Normal z',title='Traction vectors on unit-normal
sphere')
ax.set_box_aspect((1,1,1));ax.legend(fontsize=7)
a=np.linspace(0,2*np.pi,361);incline=np.deg2rad(30)
directions=np.column_stack((np.cos(a),np.sin(a)*np.cos(incline),np.sin(a)*np.sin(incline)))
t=directions@sigma.T;sn=np.sum(t*directions,axis=1);st=np.linalg.norm(t-
sn[:,None]*directions,axis=1)
ax2=fig.add_subplot(122);ax2.plot(np.rad2deg(a),sn,label='Normal
stress');ax2.plot(np.rad2deg(a),st,label='Shear magnitude')
ax2.set(xlabel='Direction angle in tilted plane (degrees)',ylabel='Stress (MPa)',title='A slice
through the directional response');ax2.legend()
fig.savefig('complete-engineering-traction3d.png',dpi=170);plt.show()
```

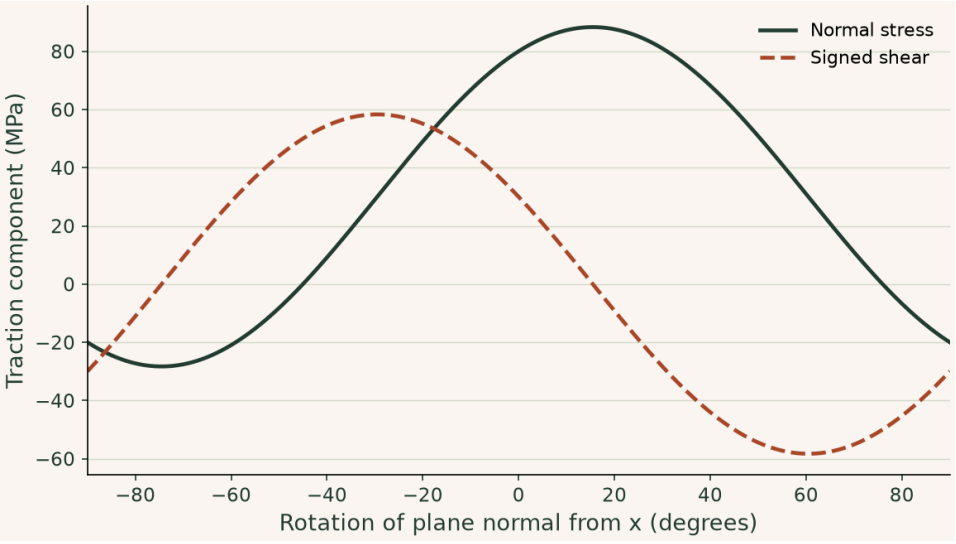


Recorded output from the executed OneSimulation notebook; change its inputs to reproduce the browser study.

Download the executed editable notebook · Download the Python model

Choose a cut plane for a combined stress state

A material point has $\sigma_x = 80,\ \sigma_y = -20,\ \tau_{xy} = 30$ MPa. Resolve the traction on a rotated plane into normal and tangential components. The stress transformation contains twice the rotation angle because both the traction vector and the basis change.



The major principal stress is $30 + \sqrt{3400} \approx 88.3$ MPa at about 15.5°, where the shear vanishes. The maximum absolute in-plane shear is 58.3 MPa and occurs on planes 45° from the principal planes.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

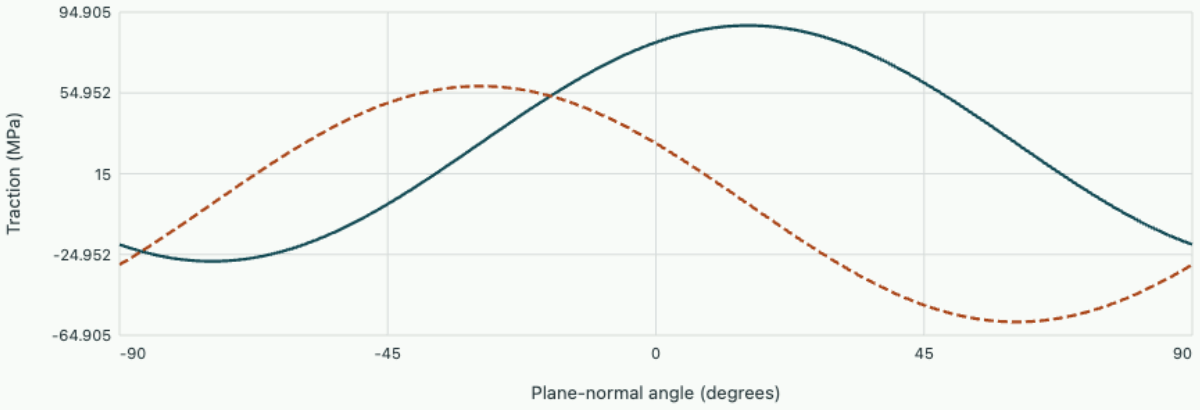
Vary each tensor component and locate the plane on which shear vanishes. Compare that angle with the principal-direction readout.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Stress σx (MPa)	80
Stress σy (MPa)	-20
Shear τxy (MPa)	30



Principal stresses 88.310 and -28.310 MPa; principal direction 15.482°. In-plane maximum shear 58.310 MPa.

Study the full derivation

Contact pressure between elastic cylinders · Stress, deformation and structural measurement

Two polished cylindrical rollers touch along a narrow strip. Dividing load by a nominal full face area gives little indication of the actual pressure. Use the local shape of the gap and elastic half-space compliance to obtain the pressure distribution, and decide whether a measured contact width can determine material stiffness.

Chapter 6 · Elastic stability of compression members

Derive the ideal buckling load, check its governing axis and explain how an imperfection changes the measured response.

FROM FIRST PRINCIPLES

Derive Euler's pinned-column load from transverse equilibrium

Take a straight slender column of length L with pins at both ends, constant rigidity EI and centred compressive load $P > 0$. Assume a small lateral displacement $w(x)$, linear elastic material and negligible shear deformation. With no external transverse load, the axial force acting through the end line has a moment arm equal to the lateral displacement. With a consistent bending sign convention this gives the homogeneous equilibrium equation

$$EIw'' + Pw = 0,$$
$$w(0) = w(L) = 0.$$

A perfectly straight shape solves this equation for every load. To find when a neighbouring bent equilibrium becomes possible, solve for a nonzero amplitude. The first pin removes the cosine term; the far pin then selects discrete wave numbers:

$$w = A\sin(\beta x),$$
$$\beta^2 = P/(EI),$$
$$\sin(\beta L) = 0 \quad \Longrightarrow \quad P_n = n^2\pi^2 EI/L^2.$$

The lowest eigenvalue is the first elastic buckling load within this ideal model. Buckling concerns loss of stiffness to lateral perturbations. Material yielding, crookedness and other end conditions can change a measured threshold even when the axial stress is still nominally uniform.

Background · Euler load

$$P_{\mathrm{cr}} = \pi^2 EI/L_{\mathrm{eff}}^2$$

The ideal slender elastic column buckles in its lowest admissible mode. Support conditions determine effective length and change the result quadratically.

DERIVATION AND APPLICATIONS

When does a continuous lateral support change the buckling

mode?

A slender compressed member may rest on an elastic bed rather than being unsupported between pins. The axial compression lowers the energy of lateral deflection; bending stiffness and the support both raise it. Take a straight, prismatic Euler–Bernoulli column with span L, bending rigidity EI and compressive dead load $P > 0$. Let a distributed support supply restoring force per length $-k_f w$, proportional to transverse displacement w. Neglect imperfections, shear deflection and nonlinearities beyond second-order potential energy.

For small slopes the ends approach by $\int w'^2 dx/2$. The change of potential energy therefore contains three contributions:

$$\Delta\Pi = \tfrac{1}{2}\int_0^L \left[EI(w'')^2 + k_f w^2 - P(w')^2\right] dx.$$

With pins at both ends take $w = A\sin(n\pi x/L)$, which satisfies zero deflection and zero moment there. For uniform properties these are independent eigenfunctions. With $\beta_n = n\pi/L$, each squared sine or cosine integrates to $L/2$, giving

$$\Delta\Pi_n = \tfrac{1}{4}LA^2(EI\beta_n^4 + k_f - P\beta_n^2).$$

At the threshold the coefficient of A^2 first vanishes. Dividing by β_n^2 gives a candidate compression for each mode; the physically relevant first threshold is their minimum:

$$P_n = EI\beta_n^2 + k_f/\beta_n^2,$$
$$P_{\mathrm{cr}} = \min_{n\geq 1} P_n.$$

A long-wavelength deflection costs little bending energy but displaces the elastic bed strongly for a prescribed end shortening. Short wavelengths avoid that displacement at the expense of large curvature. For $k_f = 0$ the first sine mode recovers Euler's pinned-column result. A sufficiently stiff bed can instead favor a mode having more than one half-wave.

WORKED EXAMPLE

Find the first threshold for a supported strip

Take $L = \pi$ m, $EI = 100$ N m² and $k_f = 900$ N/m². Then $\beta_n = n$ m⁻¹ and $P_1 = 1000$ N, $P_2 = 625$ N, $P_3 = 1000$ N. For larger n the bending term alone exceeds 625 N. The first instability is therefore the two-half-wave shape at 625 N, even though the ends are still ordinary pins.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Specify a bed stiffness at the mode exchange

For this same strip increase k_f from zero. Equating P_1 and P_2 gives $EI\beta_1^2 + k_f/\beta_1^2 = 4EI\beta_1^2 + k_f/(4\beta_1^2)$, hence $k_f = 4EI\beta_1^4 = 400$ N/m². Both modes then have $P = 500$ N. Raising the support stiffness increases both candidate loads but changes which one is lowest. A calculation checking only the first shape would overpredict the threshold beyond this exchange.

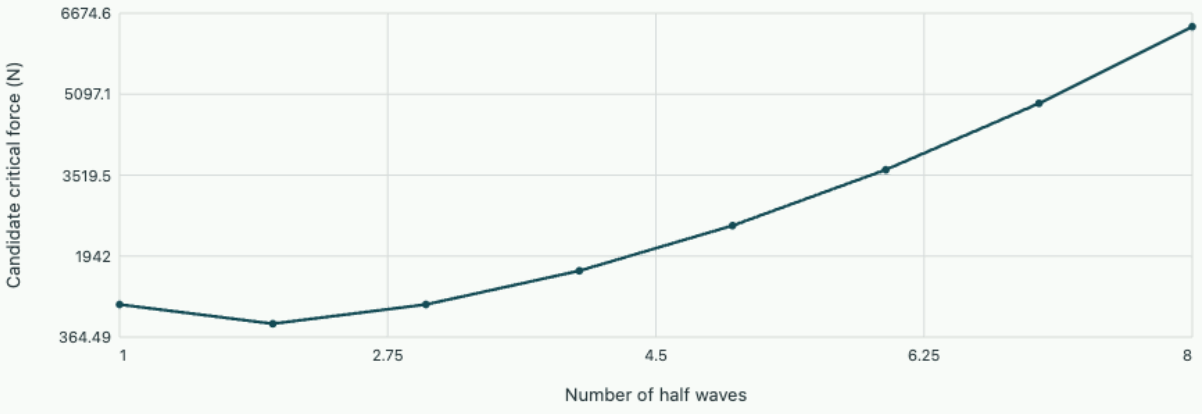
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Bending rigidity (N m²)	100
Bed stiffness (N/m²)	900
Length / π (m)	1



Pinned-end admissible modes

Among these modes the first threshold is 625.000 N at mode 2. Bed stiffness can make a shorter wavelength govern.

Equilibrium can persist while stability changes

Press gently on the ends of a thin steel ruler. The straight configuration satisfies force equilibrium over a range of compressive loads. Eventually a small lateral disturbance produces a large deflection. The issue concerns the restoring response to a disturbance, even when the nominal compressive stress is well within the material's elastic range.

Model a slender, initially straight, uniform elastic column with ideal pins at both ends. Apply a conservative compressive load P along the initial end-to-end line, with small transverse slope and negligible shear deformation. A lateral excursion $w(x)$ creates a bending moment of magnitude $P|w|$. Combining its restoring sign with the small-curvature law gives

$$EIw'' + Pw = 0.$$

The difference from a laterally loaded beam is that the load itself creates a moment through the unknown displacement.

Apply two pin conditions

Write $k^2 = P/(EI)$ and integrate:

$$w = A\sin(kx) + B\cos(kx).$$

The first pin imposes $w(0) = 0$, so $B = 0$. The second imposes $A\sin(kL) = 0$. The zero-amplitude straight solution always exists. A nonzero amplitude is possible within this linearised model only when $kL = n\pi$, for positive integer n. The smallest compressive load admitting a bent mode is

$$P_{\mathrm{cr}} = \frac{\pi^2 EI}{L^2},$$

$$w_1(x) \propto \sin(\pi x/L).$$

The proportionality leaves the amplitude undetermined. It locates a loss of stiffness, not a post-buckled deflection. The same result follows from energy: for $w = A\sin(\pi x/L)$ the quadratic elastic energy is $EI\pi^4 A^2/(4L^3)$, while end shortening releases $P\pi^2 A^2/(4L)$. Their difference changes sign at the critical load.

Find which direction buckles first

A rectangular steel strut is 1.50 m long with section 40 mm by 12 mm and $E = 200$ GPa. Bending across the 12 mm dimension gives

$$I_{\min} = 40(12)^3/12 = 5760\,\mathrm{mm}^4.$$

Across the 40 mm dimension,

$$I = 12(40)^3/12 = 64\,000\,\mathrm{mm}^4.$$

Free to buckle either way, the column chooses the smaller bending stiffness. Converting $5760\,\mathrm{mm}^4$ to $5.76 \times 10^{-9}\,\mathrm{m}^4$ gives $P_{\mathrm{cr}} = 5.05$ kN.

The corresponding mean stress is only $5050/480 = 10.5$ MPa. The radius of gyration about the weak axis is

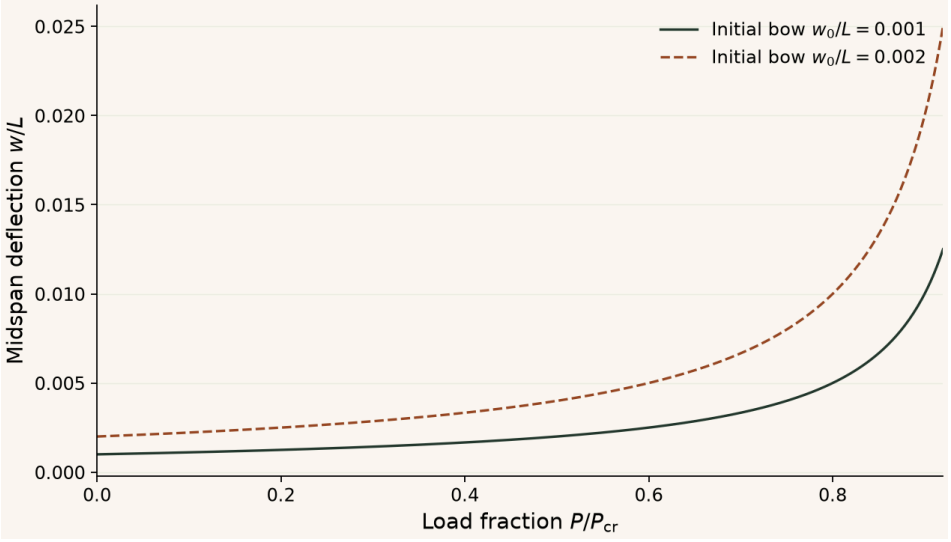
$$r_g = \sqrt{I/A} = 3.46\,\mathrm{mm}.$$

Writing $P_{\mathrm{cr}}/A = \pi^2 E/(L/r_g)^2$ makes the sensitivity to slenderness explicit. If a real brace prevents weak-axis deflection at midspan, a pin-like brace divides that buckling length by two. Check other unbraced axes and the brace stiffness before crediting the fourfold ideal increase.

Interpret a pre-curved strut

Suppose the unloaded column already bows with amplitude w_0 in the first sine shape. Elastic bending resists the curvature change from that initial shape. Writing its total amplitude as w, the mode equation now gives $P_{\mathrm{cr}}(w - w_0) = Pw$, hence $w = w_0/(1 - P/P_{\mathrm{cr}})$. If $w_0 = 1.0$ mm, the predicted amplitude is 2.0 mm at half the ideal load and 5.0 mm at 80%. The steep increase can show up in a test well before a distinct peak load.

This model assumes small slopes and a material still in its elastic range. Large deflection changes the geometry, and local yielding can reduce the tangent stiffness. End restraint also changes the eigenvalue. The often-used effective-length notation replaces L by the pin-ended length giving the same ideal load. It must reflect the actual translations and rotations permitted by the supports.



First-mode imperfection model below the Euler load. Doubling the initial bow doubles the predicted displacement at each load within the linearised theory.

Use two readings to estimate an elastic critical load

For an imperfect elastic column dominated by its first buckling shape, write total offset $e = e_0/(1 - P/P_{\mathrm{cr}})$. Let $y = e - e_0$ be the additional deflection measured from its unloaded shape. Rearrangement gives

$$y/P = (y + e_0)/P_{\mathrm{cr}}.$$

A graph of y/P against y should therefore be linear with slope $1/P_{\mathrm{cr}}$. As a consistency test, measured pairs $P = 20$ kN, $y = 0.50$ mm and $P = 30$ kN, $y = 1.00$ mm give slope $(1/30 - 0.5/20)/(1.0 - 0.5) = 1/60\,\mathrm{kN}^{-1}$. The implied critical load is 60 kN and the intercept implies a 1 mm initial offset. Clearance or nonlinear material response can spoil this inference before the theoretical critical load is reached.

Estimate a buckling load from an assumed lateral shape

The differential buckling equation is especially simple for uniform pinned columns. An energy method extends the calculation to cases where stiffness varies or extra bracing is present. Take a conservative compressive dead load P acting along the original axis. A small lateral displacement $w(x)$ stores bending energy $\frac{1}{2}\int EI(w'')^2 dx$. To the same quadratic order in slope the ends move towards one another by $\frac{1}{2}\int (w')^2 dx$, allowing the axial load to do positive work. The straight configuration loses stiffness against a given shape when these quadratic contributions balance.

$$P_R = \frac{\int_0^L EI(w'')^2 dx}{\int_0^L (w')^2 dx}.$$

The trial shape must satisfy the geometric support conditions and must possess finite bending energy. Its amplitude cancels, since numerator and denominator are both quadratic in the same displacement. For this conservative model the smallest possible quotient over all admissible shapes equals the fundamental buckling load. A particular admissible shape generally yields an upper estimate. This is a potential hazard in assessment: a plausible trial shape can overestimate the true capacity and refinement should make the estimate decrease.

For the pinned uniform column choose $w = A\sin(\pi x/L)$. Squaring and integrating the derivatives recovers $P_R = EI\pi^2/L^2$. A deliberately less accurate trial $w = Ax(L - x)$ has constant curvature and gives

$$P_R = 12EI/L^2.$$

It satisfies zero displacement at both pins, hence is admissible in this energy calculation, but it misses the exact sinusoidal curvature. Its predicted load is about 22% high.

Decide whether a central brace can suppress the lowest mode

Install a lateral spring of stiffness k at the midpoint of a 2.0 m pinned member with $EI = 40\,\mathrm{kN\,m^2}$. The spring adds $\tfrac{1}{2}kw(L/2)^2$ to stored energy. Using the first sine shape gives

$$P_R = \pi^2 EI/L^2 + 2kL/\pi^2.$$

Its unbraced value is 98.7 kN. For $k = 100\,\mathrm{kN/m}$, the quotient rises by $2(100)(2)/\pi^2 = 40.5$ kN, giving 139.2 kN. The value estimates the improved symmetric mode; the spring generally changes that mode's shape.

The second sine shape has a node at midspan, so its frequency-independent elastic buckling load remains $4\pi^2 EI/L^2 = 394.8\,\mathrm{kN}$. Equating the two trial values would suggest a transition near $k = 731\,\mathrm{kN/m}$. This crossing identifies when a one-shape description merits scrutiny. It cannot prove the actual transition because the symmetric trial result is an upper bound. A brace added at midspan can suppress a large single bow and still leave a two-half-wave buckling mode essentially unaffected. Checking the other bending direction and the brace's own connection stiffness is essential before applying either idealised improvement to a real member.

A midspan dial indicator responds strongly to the symmetric shape and sees zero displacement for the ideal two-half-wave shape. An experiment observing only that indicator might report successful suppression even as the second mode grows. A second indicator at quarter span has nonzero response to both shapes and reveals the change of governing mechanism.

Interactive calculation · Change the inputs of this example

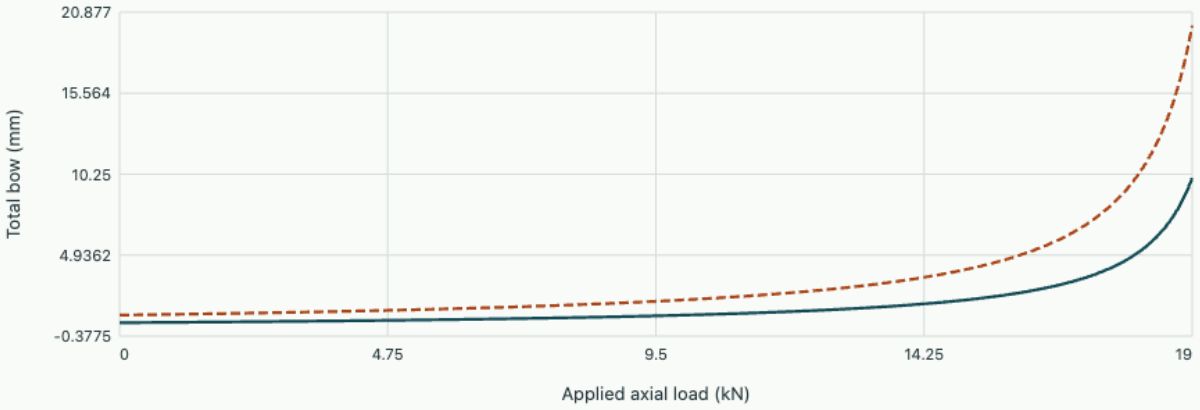
Change the parameters of this reference plot and compare its limiting cases. The fixed illustration remains beside the derivation; the adjustable calculation exposes the same dependence on geometry, material properties or loading.

Open this live calculation

Try the calculation

Initial bow (mm) **0.5**

Euler load (kN) **20**



First-mode imperfection amplification Twice initial bow

At80% of the Euler load, total bow is 2.500 mm. The linear geometric amplification is not reliable near material yielding or its pole.

Problem sheet 3 · Strength and stability

Use point stress for a material check and structural stiffness for a buckling check.

1. Change shaft speed at constant power

A smooth solid shaft of diameter 30 mm transmits 6 kW at 1200 rpm and carries a fixed bending moment of 80 N·m. (a) Find the torque and surface stresses. (b) Compute von Mises stress. (c) Repeat at 600 rpm at the same power and explain which stress changes.

▼ **Combine bending and torsion**

At 1200 rpm the angular speed is 40π rad/s and $T = 6000/(40\pi) = 47.75$ N·m. At the tensile surface $\sigma = 32(80)/(\pi d^3) = 30.18$ MPa and $\tau = 16(47.75)/(\pi d^3) = 9.01$ MPa. Thus

$$\sigma_{\mathrm{vm}} = \sqrt{\sigma^2 + 3\tau^2} = 33.98\,\mathrm{MPa}.$$

Halving the speed doubles the torque and the shear to 18.01 MPa while bending stays fixed. The equivalent stress becomes about 43.4 MPa. Keeping power constant does not keep the torque demand constant.

---

2. Decide which plane controls a strut

A straight pin-ended column is 2.0 m long with $E = 200$ GPa, $I_y = 8.0 \times 10^{-8}\,\mathrm{m}^4$ and $I_z = 2.0 \times 10^{-7}\,\mathrm{m}^4$. (a) Calculate both mode loads. (b) Add an ideal midspan lateral brace that halves only the effective length for bending about y. Determine the controlling load. (c) Compare your result with a prediction multiplying the original capacity by four.

▼ **Compare both elastic modes**

Using $P = \pi^2 EI/L^2$ gives 39.5 kN about y and 98.7 kN about z; initially the first governs. After bracing, the ideal y load is 158 kN. The z load is still 98.7 kN and now governs. The ideal capacity therefore rises by a factor 2.5, because the weak mode switches. The brace must restrain displacement in the direction in which the original weak mode bends.

---

3. Interpret a lateral-displacement reading

A test column has initial sinusoidal bow 0.40 mm. At a compressive load of 12 kN the total lateral amplitude is 1.00 mm. (a) Infer its first-mode elastic critical load. (b) Estimate amplitude at 16 kN. (c) If the area is 300 mm² and $Z = 1200$ mm³, estimate the most compressive nominal stress at 16 kN.

▼ **Infer the stiffness loss and add the bending stress**

From $w_0/w = 1 - P/P_{\mathrm{cr}}$, the 12 kN reading gives

$$P_{\mathrm{cr}} = 12/(1 - 0.4) = 20\,\mathrm{kN}.$$

At 16 kN the predicted amplitude is

$$0.40/(1 - 16/20) = 2.0\,\mathrm{mm}.$$

The direct stress is

$$16\,000/300 = 53.3\,\mathrm{MPa}.$$

The additional moment is $16\,000(2.0) = 32\,000$ N·mm, giving $M/Z = 26.7$ MPa. At the more compressed fibre the two effects add, reaching about 80.0 MPa. The inference assumes an elastic member, a first-mode imperfection and small slopes.

Explore the governing equations after solving the sheet: Stress transformation and combined loading; Elastic stability of compression members. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Chapter 7 · Torsional stiffness of shafts and closed sections

Derive torque–twist relations and explain the large difference in stiffness between an intact tube and an open strip.

FROM FIRST PRINCIPLES

Derive the torque–twist relation from local shear deformation

Consider a straight circular shaft in small elastic torsion. In Saint-Venant torsion of a circular section, each cross-section can rotate as a rigid plane relative to the next. Let its rotation be $\phi(x)$ and take two sections separated by dx. At radius r their relative tangential displacement is $r\,d\phi$. Dividing by the axial spacing gives engineering shear strain and, through the measured shear modulus G, shear stress:

$$\gamma_{x\theta} = r\,d\phi/dx,$$
$$\tau_{x\theta} = Gr\,d\phi/dx.$$

Each patch of area dA carries tangential force $\tau\,dA$ with moment arm r. Sum these moments over the section and integrate the resulting twist rate along the shaft:

$$T = \int_A r\tau\,dA = GJ\phi',$$
$$J = \int_A r^2\,dA,$$
$$\Delta\phi = \int \frac{T(x)}{G(x)J(x)}\,dx.$$

For a homogeneous solid circular shaft $J = \pi R^4/2$, so doubling radius strongly increases torsional stiffness. Noncircular sections generally warp and their torsion constant need not equal this polar area moment. The circular-shaft kinematics should not be used unchanged for an open thin-walled section.

DERIVATION AND APPLICATIONS

How two bonded shaft materials share one applied torque

Two shafts in series carry the same torque; a bonded core and sleeve instead undergo the same twist per length. Consider a circular core of radius a and shear modulus G_c, bonded to a concentric sleeve with outer radius b and modulus G_s. Use a long prismatic region away from load introduction, small elastic deformations, perfect bond and no interfacial slip. Let $\phi(x)$ describe section rotation. Two cross-sections separated by dx acquire a relative tangential motion $r\,d\phi$ at radius r:

$$\gamma_{x\theta} = r\phi',$$
$$\tau_i(r) = G_i r\phi'.$$

Compatibility makes ϕ' common. The shear stress may change at the material boundary because the moduli differ. The torque contribution from a thin annulus is its area $2\pi r\,dr$ times shear stress times radius. Integrating in each material gives

$$T_c = G_c J_c \phi',$$
$$T_s = G_s J_s \phi',$$
$$J_c = \frac{\pi a^4}{2},$$
$$J_s = \frac{\pi (b^4 - a^4)}{2}.$$

The applied torque T is their sum. Solving first for the common twist and then for the load split yields

$$\phi' = \frac{T}{G_c J_c + G_s J_s},$$
$$\frac{T_c}{T} = \frac{G_c J_c}{G_c J_c + G_s J_s}.$$

The denominator has units of N m². Setting both shear moduli equal recovers the homogeneous solid shaft independently of the arbitrary division at a. If the sleeve is unbonded and torque is applied only to the core, equal twist is no longer a valid compatibility condition.

WORKED EXAMPLE

Predict the twist of a reinforced shaft

Take $a = 10.0$ mm, $b = 15.0$ mm, $G_c = 80.0$ GPa and $G_s = 30.0$ GPa. The rigidities are 1257 and 1914 N m². Under $T = 60.0$ N m the twist rate is 0.0189 rad/m; over 0.500 m the end rotates 0.00946 rad. The core carries 23.8 N m and the sleeve 36.2 N m. Their stresses at a are 15.1 and 5.68 MPa respectively despite equal shear strain there.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Infer a sleeve modulus from a rotation test

A second shaft with the same dimensions and core rotates through 0.0100 rad over 0.500 m under 50.0 N m. The measured total rigidity is $TL/\phi = 2500$ N m². Subtracting the known core leaves 1243 N m². Since $J_s = 6.38 \times 10^{-8}$ m⁴, $G_s = 19.5$ GPa. The test would yield an apparent modulus if compliance of the grips were included in the measured rotation.

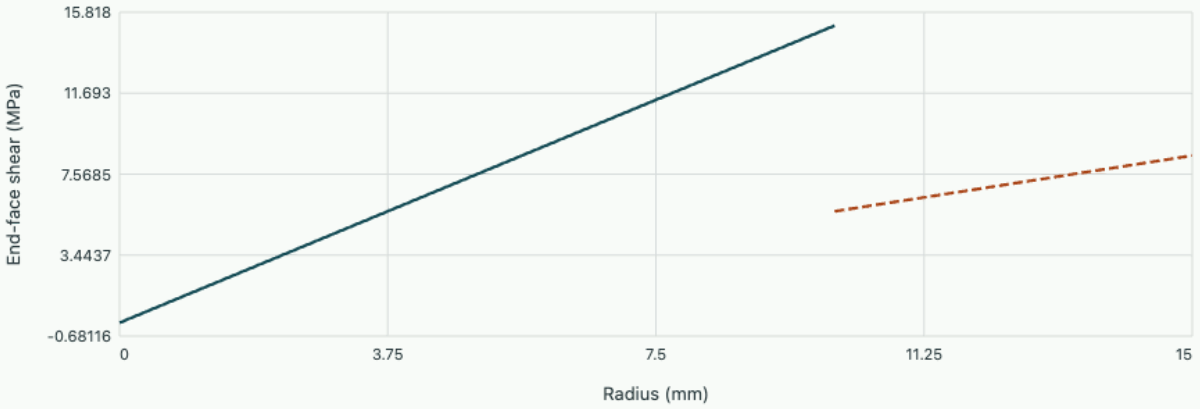
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Core radius (mm)	10
Outer radius (mm)	15
Torque (N m)	60



Twist rate 0.01892 rad/m; core torque 23.777 N m; sleeve 36.223 N m. Compatibility sets equal twist, not equal interface stress.

How much torque does the sleeve actually carry?

A steel core and an aluminium sleeve extend the full length of a circular shaft. They are bonded along their cylindrical interface, and the end fittings transmit the resultant torque to the complete section. A student dividing the torque in proportion to cross-sectional area obtains a plausible answer that misses both lever arm and material stiffness. We can reconstruct the division from the displacement at the interface before writing any torque balance.

Let a denote the core radius and b the outside radius. A cross section rotating by angle $\varphi(x)$ moves a point at radius r tangentially by $r\varphi$. Over a short axial interval the resulting engineering shear strain is $\gamma_{x\theta} = r\,d\varphi/dx$. Bonding requires a common twist per length φ' for the two materials. The shear on a cross-sectional end face is $Gr\varphi'$, so it can jump at $r = a$ even though the strain there is continuous. It

is the traction on a radial-normal interface that must equilibrate on opposite sides of the bond. Confusing it with the traction on the shaft end face would wrongly force the plotted shear stresses to coincide.

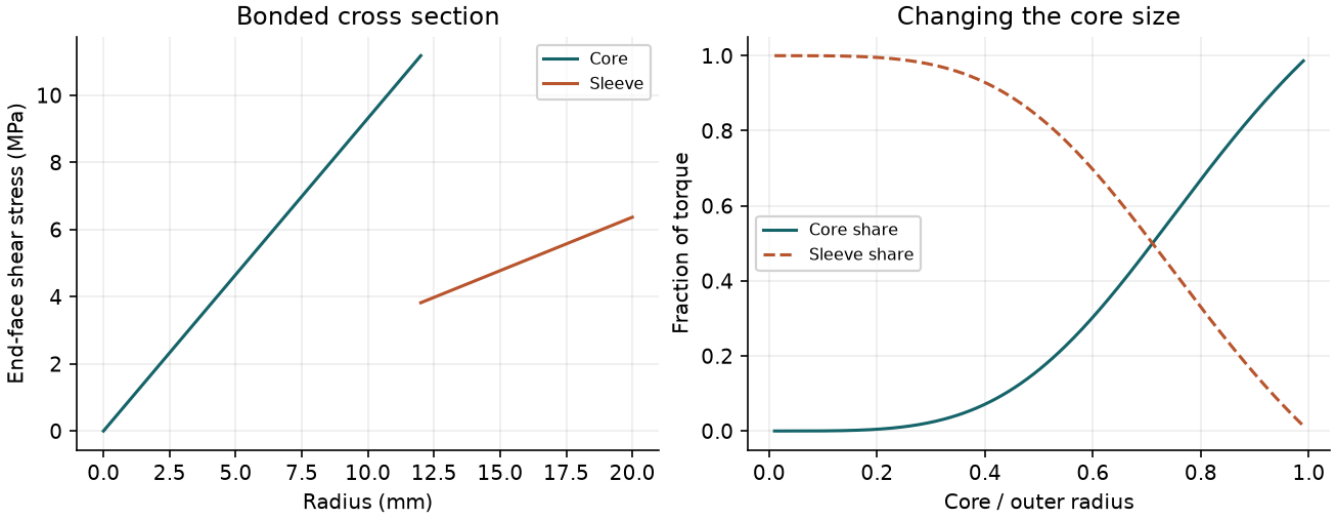
The torque on an annular element is the shear times its area times its lever arm. Integration of $r(Gr\varphi')(2\pi r\,dr)$ gives

$$T = [G_c J_c + G_s J_s]\varphi',$$

$$J_c = \pi a^4/2,$$

$$J_s = \pi(b^4 - a^4)/2.$$

Use $a = 12$ mm, $b = 20$ mm, $G_c = 79$ GPa and $G_s = 27$ GPa. Under 100 N m the core carries 30.35 N m and the sleeve 69.65 N m. A 600 mm length rotates by 0.405 degrees. Immediately inside the interface the end-face shear is 11.18 MPa. On the other side of that same radius the lower modulus reduces it to 3.82 MPa; continuing outward increases it again. An accurate shaft twist cannot by itself identify the local peak stress unless the section and material distribution are known.



The radial stress is local, while each torque share is an integral weighted by the lever arm. A thin outer ring may contribute disproportionately to the total rigidity.

In the notebook, integrate the stress directly over the core and sleeve and compare the total with the imposed torque. Evaluate the two integrals separately so a numerical trapezoid does not smear the material jump. Then make the two moduli equal: the end-face shear curve becomes straight across the entire section, while the fraction of torque in the core equals $(a/b)^4$. This limit offers an independent check against a uniform solid shaft.

A separate way to check the torque split uses stored energy. In a unit length of shaft the strain energy density is $\tau\gamma/2$. Integrating it over each annulus gives $GJ(\varphi')^2/2$ for that material. Because the twist is common, the energy fractions equal the torque fractions. For the example the total strain energy is $T\varphi/2$, about 0.35 J over 600 mm. If the two materials were joined end to end, the applied torque would be common and the angles of twist would add. The continuous cylindrical bond is the feature that selects the parallel arrangement used here.

At what radius fraction will the default materials divide torque exactly in half?

▼ **Work through the solution**

Set $G_c a^4 = G_s(b^4 - a^4)$. Rearranging gives

$$a/b = [G_s/(G_c + G_s)]^{1/4} = 0.7106.$$

At $b = 20$ mm this corresponds to a core radius of about 14.2 mm. Run the notebook with the revised radius and read both shares. This calculation assumes a bonded circular shaft away from the ends; it cannot predict the local loading of a short end fitting.

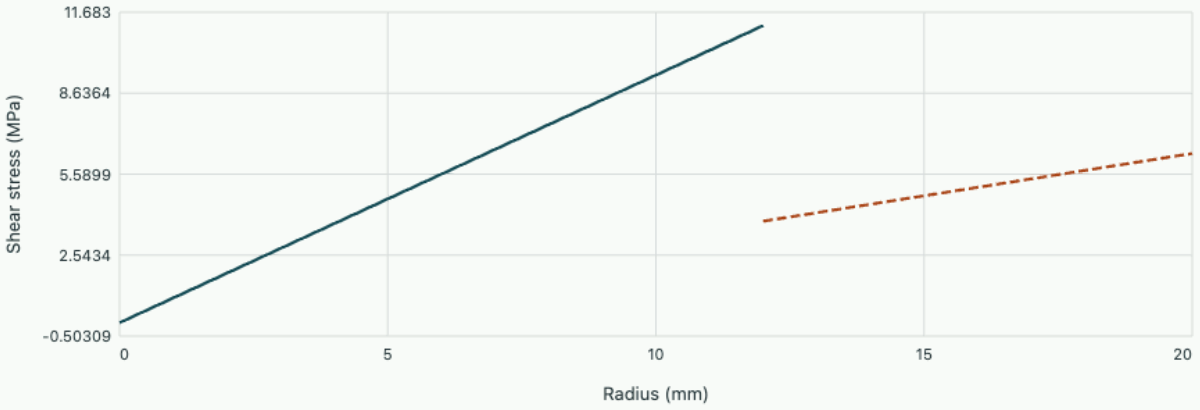
Interactive calculation · Bonded torsion

Vary the notebook inputs and compare both plotted observables. The same equations generate the browser curves and the notebook reference; fixed material, geometry and model assumptions are stated in the readout.

Open this live calculation

Try the calculation

Core radius (mm)	12
Core shear modulus (GPa)	79
Sleeve shear modulus (GPa)	27
Applied torque (N m)	100



At length0.60 m the rotation is 0.405°. Core torque 30.346 N m. Shear strain is continuous while stress jumps when the moduli differ.

Download the editable notebook for this example

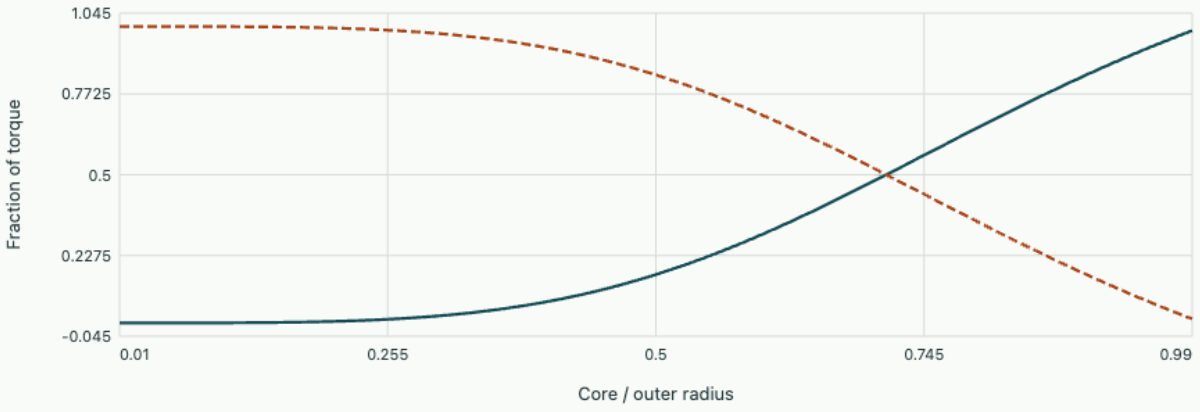
Interactive calculation · Bonded torque share

Vary the notebook inputs and compare both plotted observables. The same equations generate the browser curves and the notebook reference; fixed material, geometry and model assumptions are stated in the readout.

Open this live calculation

Try the calculation

Core radius (mm)	12
Core shear modulus (GPa)	79
Sleeve shear modulus (GPa)	27
Applied torque (N m)	100



At length0.60 m the rotation is 0.405°. Core torque 30.346 N m. Shear strain is continuous while stress jumps when the moduli differ.

Download the editable notebook for this example

Open editable notebook Download .ipynb

Rotate two adjacent sections

A torque meter measures twist between two stations on a shaft. To infer torque, first ask how material between the stations deforms. A straight circular shaft in uniform, small-deformation torsion has cross-sections that rotate as rigid discs. Two discs separated by dx rotate relative to one another by $d\theta$. A fibre at radius r moves sideways by $r\,d\theta$, giving shear strain $\gamma = r\,d\theta/dx$. With elastic shear modulus G, the shear stress increases linearly from the centre to the surface.

Each area element transmits a torque $r\tau\,dA$. Integrating produces the polar second moment of area J, whose dimensions are length to the fourth power. With constant torque and cross-section, integration along the shaft then converts twist per length into total relative angle.

$$T = \int_A r\tau\,dA = G\theta' \int_A r^2 dA,$$

$$J = \frac{\pi}{2}(r_o^4 - r_i^4),$$

$$\theta = \frac{TL}{GJ}.$$

Follow a torque through a stepped shaft

A 0.75 m aluminium tube of outer radius 15 mm and inner radius 12 mm transmits 45 N m. For $G = 26\,\mathrm{GPa}$, $J = 4.695 \times 10^{-8}\,\mathrm{m}^4$. Its twist is

$$45(0.75)/(26 \times 10^9 J) = 0.02765\,\mathrm{rad} = 1.58^\circ.$$

Surface shear is

$$45(0.015)/J = 14.4\,\mathrm{MPa}.$$

The inside surface still carries 80% of this stress. Drilling a central hole removes the least highly stressed material first.

Twist can also identify an unknown joint compliance. If an instrument measures 2.00° across this tube plus its end couplings, and all parts carry the same torque, the couplings contribute about 0.42°. Their combined torsional compliance is approximately $0.42\pi/[180(45)] = 1.6 \times 10^{-4}\,\mathrm{rad/(N\,m)}$. Such a comparison assumes that the gauges do not include a second shaft segment and that the torque is transmitted without slip.

Carry torque around a single closed cell

For a long, thin-walled closed tube with a single cell, use the shear flow $q = \tau t$, the force per unit length of wall. Longitudinal equilibrium makes this flow approximately constant around the perimeter if no intermediate longitudinal force is introduced. A length ds carries tangential force $q\,ds$. Its moment about an arbitrary origin is q times twice the triangular area swept by the radius vector. Around the median wall contour this gives

$$T = 2A_m q.$$

For a rectangular cell with median dimensions 80 by 40 mm, a torque of 320 N m gives

$$q = 320000/[2(80)(40)] = 50\,\mathrm{N/mm}.$$

A 2 mm wall carries 25 MPa while a 4 mm wall carries 12.5 MPa. A slender longitudinal slit interrupts the closed path. The constant circulating flow solution then ceases to apply. A seam that reconnects the two lips must transmit the longitudinal shear flow if the component is to recover its closed-section behaviour.

Distinguish two section properties

For a noncircular closed tube, the polar area moment and torsion constant generally differ. Compatibility around a single thin-walled cell gives

$$\theta' = T \oint ds/(Gt)/(4A_m^2).$$

Making only a short wall segment very thin substantially increases this integral. A wall segment can therefore control angle of twist even when equilibrium fixes the same shear flow everywhere around the cell.

Determine torque sharing between two closed cells

A single closed cell supports one constant circulating shear flow. In an aircraft-style box or a partitioned machine boom, adjacent cells share a web. Torque balance alone cannot tell how much flow each cell carries. The missing equation is twist compatibility: because the cells are bonded along the common web, their twist per unit length must agree. Assume a long, thin-walled prismatic box of one material carrying Saint-Venant torsion with negligible distortion. Use median-line areas and length-to-thickness ratios throughout.

Assign positive clockwise circulations q_1, q_2 to both cells. They oppose each other on the common web, where physical flow becomes $q_1 - q_2$ measured in the first cell's direction. Let S_1, S_2 denote $\int ds/t$ over the exterior walls of each cell and S_w that quantity along their common web. The torque and the two contour compatibility relations become

$$T = 2A_1 q_1 + 2A_2 q_2,$$
$$2A_1 G\theta' = (S_1 + S_w)q_1 - S_w q_2,$$
$$2A_2 G\theta' = (S_2 + S_w)q_2 - S_w q_1.$$

These equations express the same shear-strain compatibility that leads to $J_t = 4A_m^2 / \oint ds/t$ for one cell. If both cells and exterior wall thicknesses are mirror images, symmetry requires equal circulations and zero common-web flow under pure torque. The common web may still carry transverse shear from a separate load case. A missing web flow under pure torque is therefore a property of that loading and symmetry.

Compute the stress in an internal dividing web

Take adjacent median rectangles 40 mm high, 60 and 30 mm wide, with every wall 2 mm thick. Then

$$A_1 = 2400,\ A_2 = 1200\,\mathrm{mm}^2.$$

Their three exterior walls give $S_1 = (60+40+60)/2 = 80$ and $S_2 = (30+40+30)/2 = 50$, while $S_w = 20$. Equal twist requires $(100q_1 - 20q_2)/2400 = (70q_2 - 20q_1)/1200$, giving $140q_1 = 160q_2$. Thus $q_2 = 0.875q_1$, larger than one might expect from the smaller area alone.

Applying $T = 345\,\mathrm{N\,m} = 345000\,\mathrm{N\,mm}$ gives $345000 = 4800q_1 + 2400(0.875)q_1$, so $q_1 = 50.0$ and $q_2 = 43.75\,\mathrm{N/mm}$. The dividing web carries only 6.25 N/mm, or 3.13 MPa. The primary exterior walls carry 25.0 and 21.9 MPa. Using $G = 27000\,\mathrm{MPa}$, either contour gives $\theta' = (100\times 50 - 20\times 43.75)/(2\times 2400\times 27000) = 3.18\times 10^{-5}\,\mathrm{rad/mm}$, about 1.82° per metre. Checking the other contour independently catches a reversed sign on the shared wall. If a weld along that wall fails, equal twist and the closed paths that led to these flows can no longer be assumed.

Individual cell torque contributions are informative when interpreting a section cut. The larger cell contributes $2(2400)(50) = 240\,\mathrm{N\,m}$, and the smaller contributes $2(1200)(43.75) = 105\,\mathrm{N\,m}$. Their 345 N m sum is independent of the coordinate origin. These are bookkeeping circulations rather than two torques applied independently to separate tubes. Cutting through the common web changes the compatibility and available flow paths, so using the original partition after a cut is not justified. Surface strain measurements on the upper exterior walls and the shared web would distinguish the predicted 50, 43.75 and 6.25 N/mm flows more effectively than measuring a single outer-wall strain alone.

A comparison with a thin unpartitioned box is possible without changing outer dimensions. For the outer median rectangle 90 by 40 mm, $A_m = 3600\,\mathrm{mm}^2$ and $\oint ds/t = 2(90+40)/2 = 130$. Its single-cell torsion constant is

$$4A_m^2/130 = 398769\,\mathrm{mm}^4.$$

The partitioned solution implies

$$J_t = T/(G\theta') = 345000/[27000(3.183\times 10^{-5})] \simeq 401500\,\mathrm{mm}^4.$$

Under this specific pure torque the additional web therefore raises torsional stiffness by less than one percent. It experiences little shear because the neighbouring circulations are similar. Its contribution to transverse load sharing, panel buckling resistance or local attachment stiffness can nevertheless be important and needs separate calculations.

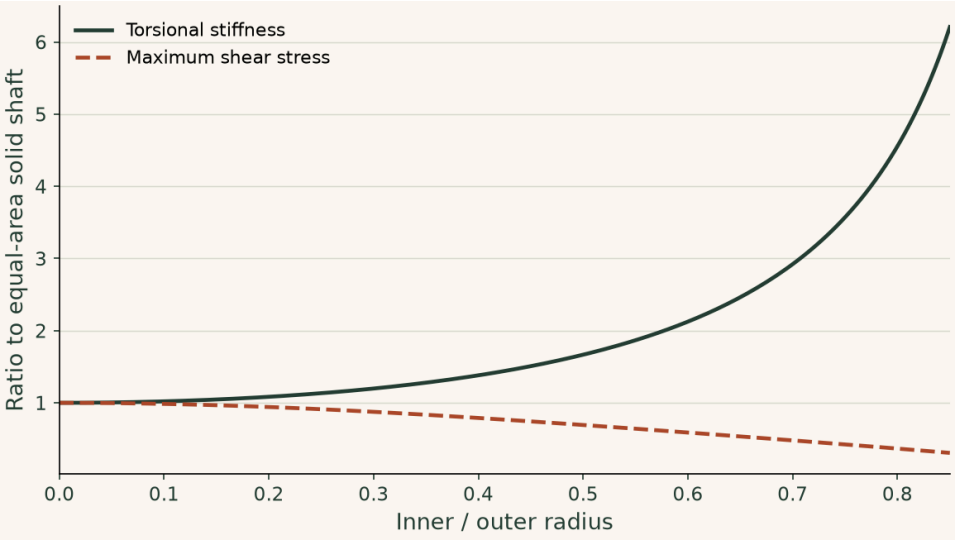
Move the web to the centre, producing identical cells of median width 45 mm. Pure torque then gives equal circulations, zero shear flow on the common web and precisely the single outer-cell twist in this idealisation. This limiting case is a demanding model check. A numerical section calculation that predicts double the torsional stiffness simply because there are now two cells has likely treated the common web as two independent exterior walls.

Compare a tube and a shaft at equal material area

Hold length, material, torque and cross-sectional area fixed while hollowing a circular shaft and increasing its outer radius. For radius ratio r, area conservation gives

$$R/R_s = 1/\sqrt{1-r^2}.$$

Substitute in $J = \pi R^4(1-r^4)/2$ and $\tau_{\max} = TR/J$.



At inner/outer radius 0.6 the torsional stiffness is 2.125 times the solid-shaft value, while peak shear is about 0.588 times as large. This comparison requires 25% greater outer radius; under a fixed envelope it would answer a different design question.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

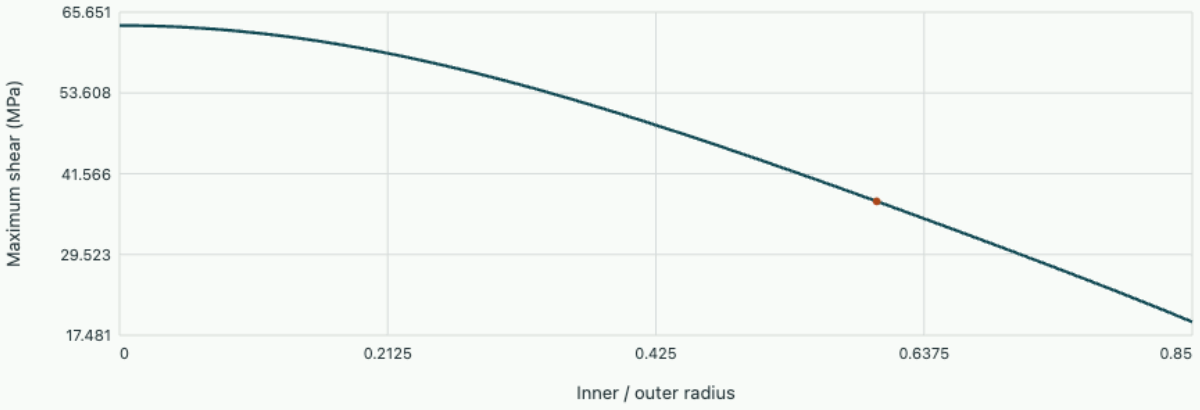
Hollow the shaft while keeping material area fixed. Watch both the required outer envelope and the peak shear change.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Equivalent solid radius (mm)	10
Torque (N m)	100
Hollow radius ratio	0.6



Equal material area | Selected tube

Selected outer radius 12.500 mm; stiffness / solid 2.125; peak shear 37.448 MPa. The outer envelope increases.

Study the full derivation

Torsion in a thin-walled two-cell section · Stress, deformation and structural measurement

Two closed cells share one slender web. The applied couple gives only their combined torque, yet the common web can carry much less shear than either exterior wall. Derive the missing compatibility equation, calculate a twist and test which measurements distinguish the two circulations.

Chapter 8 · Transverse shear and connections in built-up beams

Derive shear flow from changing bending stress and determine the force transferred between parts of a built-up section.

FROM FIRST PRINCIPLES

Derive transverse shear flow by balancing a longitudinal strip

Consider a slender prismatic beam undergoing elastic bending with moment $M(x)$ and shear $V = dM/dx$. Select a horizontal cut through the cross-section and keep the cap of material above it. Let y be measured from the neutral axis. The longitudinal bending stress $-My/I$ has a resultant on the end of this cap proportional to its first area moment $Q = \int_{\text{cap}} y\, dA$:

$$N_{\text{cap}}(x) = -M(x)Q/I.$$

Over distance dx the two end resultants differ. Longitudinal traction along the cut must balance this difference. Dividing its force by the beam length gives interfacial shear flow q_s; spreading the flow over the local contact breadth b gives the elementary shear stress:

$$|q_s| = |dN_{\text{cap}}/dx| = |VQ/I|,$$
$$|\tau| = |q_s|/b.$$

This is also the longitudinal load per length to be transferred by continuous glue or by discrete fasteners in a built-up beam. At a free outer surface the cap has zero area and flow vanishes. A small breadth at fixed Q can produce high local stress even when the section-average V/A is modest.

Background · Shear flow

$$q = VQ/I$$

In elementary beam theory the first moment Q belongs to the cut portion of the section. Shear flow is force per length. Divide by the local width to obtain shear stress.

DERIVATION AND APPLICATIONS

Recover the entire shear resultant from stresses across a rectangular section

The average value V/A from force balance cannot locate the most heavily sheared fibres. Consider a slender rectangular beam of breadth b and depth h, made from a homogeneous linear elastic material.

Use a coordinate y upward from the neutral axis. At a section where bending moment is $M(x)$, a strip carries longitudinal normal stress $-My/I$. Changes in that stress along x must be balanced by longitudinal shear on the strip boundary. Take the entire cap of material above a horizontal cut at y.

The resultant longitudinal force in this cap depends on its first area moment Q:

$$N_{\text{cap}} = -\frac{M}{I} Q(y),$$

$$Q(y) = \int_{y}^{h/2} y' b \, dy' = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right).$$

Across length dx the cap force changes as the bending moment changes. With $dM/dx = V$, equilibrium requires a shear flow of magnitude $|VQ/I|$ per unit length. Dividing by the local breadth converts flow to shear stress:

$$\tau(y) = \frac{VQ(y)}{Ib},$$

$$I = bh^3/12.$$

This expression uses the conventional signed direction for positive section shear. Substituting the rectangular first and second moments produces

$$\tau(y) = \frac{3V}{2bh}\left(1 - \frac{4y^2}{h^2}\right),$$

$$\int_{-h/2}^{h/2} \tau(y) b \, dy = V.$$

The integral provides a direct equilibrium check, and the profile vanishes at the free surfaces. Maximum shear occurs at the neutral axis, precisely where bending normal stress is zero. The elementary profile is a beam-theory approximation away from concentrated loads or end details.

WORKED EXAMPLE

Determine the force carried by the middle half-depth

Take $b = 30.0$ mm, $h = 60.0$ mm and $V = 4.80$ kN. The average is 2.67 MPa, but the centre reaches 4.00 MPa. Integrating the parabola only from $-h/4$ to $h/4$ gives the fraction $11/16 = 0.6875$. The central half of the material therefore transmits 3.30 kN of the shear resultant; the outer two quarters together carry the remaining 1.50 kN.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Set a bond-line force requirement away from the neutral axis

Suppose the same beam is made from bonded strips and a glue line lies at $y = h/4$. The local shear stress is $(3/4)(4.00) = 3.00$ MPa and the flow across the 30.0 mm breadth is 90.0 kN/m. A 20.0 mm length of bond must transfer 1.80 kN under nearly constant local V. Treating the bond stress as the section average would give 1.60 kN and miss the location dependence.

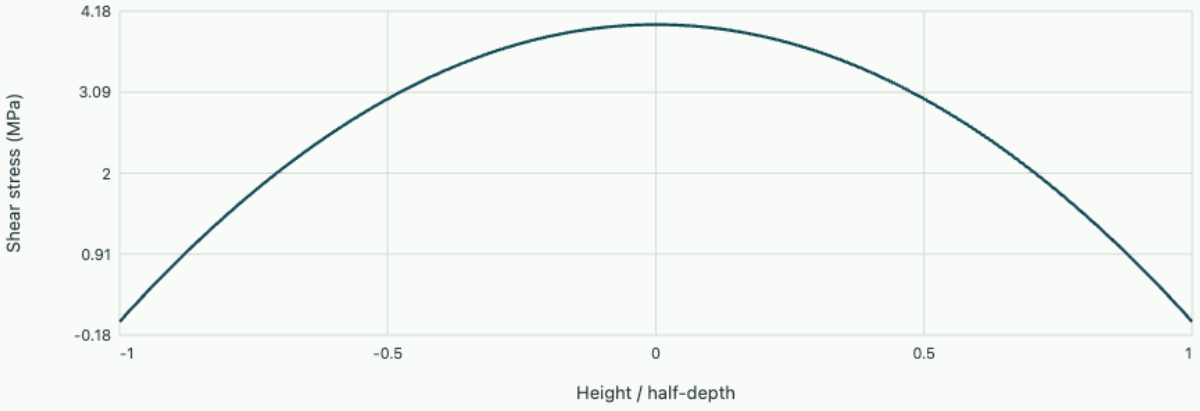
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Shear resultant (kN)	4.8
Width (mm)	30
Depth (mm)	60



Rectangular-section shear

Peak stress 4.000 MPa; central half-depth transmits 3.300 kN, exactly11/16 of the total.

Why bending generates an interface force

Stack two boards and support them as a beam. Under a transverse load their neighbouring surfaces tend to slide. A bond or row of fasteners connects the boards so they can act as one deeper section. To

calculate the demand on that connection, isolate only the material above the interface over a short beam length. Even when the external load points vertically, horizontal equilibrium of this isolated piece requires longitudinal shear at its lower face.

Under elastic bending about a centroidal principal axis, longitudinal stress is $\sigma_x = My/I$ up to the chosen sign. Its resultant on the isolated partial area A' has magnitude $N = MQ/I$, where $Q = \int_{A'} y\,dA$ is its first moment about the neutral axis of the complete section. For constant cross-section, the difference between adjacent resultants is $dN = (Q/I)\,dM$. The balancing interface force is $q\,dx$.

$$q = \frac{dN}{dx} = \frac{VQ}{I},$$

$$\tau = \frac{q}{b_i}.$$

Here b_i is the local contacting width, and $dM/dx = V$ for the sign convention used. Shear flow has units N/mm and stress has units N/mm². Evaluate Q with the portion on one side of the cut and I with the complete section. Using the second moment of an individual board defeats the assumption that the connection makes them share a curvature.

Select fastener spacing for an assembled beam

Two boards each 60 mm wide and 40 mm deep form an 80 mm deep rectangular beam. At a section carrying $V = 2.4\,\mathrm{kN}$, the complete centroidal second moment is

$$60(80)^3/12 = 2.56 \times 10^6\,\mathrm{mm}^4.$$

The upper board has area 2400 mm² and its centroid lies 20 mm above the interface, which coincides with the full neutral axis. Hence $Q = 48000\,\mathrm{mm}^3$.

The required flow is

$$2400(48000)/(2.56 \times 10^6) = 45\,\mathrm{N/mm}.$$

An adhesive over the full contact width would carry an average longitudinal shear stress of $45/60 = 0.75\,\mathrm{MPa}$. A single row of connectors rated for 900 N each in the idealised shear mode would require a spacing no greater than $900/45 = 20\,\mathrm{mm}$. For a load varying significantly over a connector spacing, integrate the flow over its tributary length instead.

Recover the shear profile in a rectangle

At any elevation y in that rectangle the partial area extends from y to $h/2$. Its first moment is

$$Q = b[(h/2)^2 - y^2]/2.$$

Substituting $I = bh^3/12$ gives

$$\tau(y) = [3V/(2bh)](1 - 4y^2/h^2).$$

The stress vanishes at the top and bottom free surfaces, reaching one and a half times the section-average transverse shear at mid-depth. The elastic bending stress, in contrast, vanishes at that mid-depth. A joint placed near the neutral axis can therefore carry substantial shear even when its local normal bending stress is small.

Where should fasteners become more closely spaced?

Under a uniform transverse load a simply supported beam has largest shear magnitude near the supports and zero shear at its symmetric midspan. With the same cross-section along the span, the connector force per unit length follows that shear diagram. Midspan may govern bending strength while the regions near the supports govern connector spacing.

Locate a transverse load that produces bending without twist

For a cross-section with one axis of symmetry, the shear centre lies somewhere on that axis. It can fall outside the section itself. To see why, consider a thin open channel with horizontal flanges of median length b, a vertical web of median height h and uniform wall thickness t. Both flanges extend to the right of the web. Apply a vertical shear V; seek the transverse location of its line of action for zero torsional moment. Assume an undeformed cross-section and neglect terms of order t/h and t/b.

Measure s from each free flange edge towards the web. The partial flange area for the longitudinal shear-flow formula is ts, located about $h/2$ from the horizontal neutral axis. Thus $q_f(s) = Vtsh/(2I)$ in magnitude. Integrating along one flange gives horizontal resultant $F_f = Vthb^2/(4I)$. Upper and lower flange forces point in opposite directions and form a couple of magnitude $F_f h$. The web shear acts along the vertical web median and contributes no moment about that line. The vertical resultant must therefore be offset from the web by

$$e = \frac{F_f h}{V} = \frac{t b^2 h^2}{4I},$$

$$I \simeq \frac{t h^3}{12} + 2bt(h/2)^2.$$

Following the directions of the shear flows places the shear centre on the side of the web opposite the flanges. In a symmetric I-section, left and right flange couples cancel and the shear centre instead lies on the intersecting symmetry axes.

Compare centroidal and twist-free load paths

A thin channel has $h = 80$ mm, $b = 30$ mm and $t = 2.0$ mm. Its horizontal-axis second moment is

$$2(80)^3/12 + 2(30)(2)(40)^2 = 277333\,\mathrm{mm}^4.$$

The shear centre offset is $2(30)^2(80)^2/[4(277333)] = 10.38\,\mathrm{mm}$ to the left of the web. A 1.50 kN vertical force through the web therefore imposes a torque of magnitude $1500(10.38) = 15570\,\mathrm{N\,mm} = 15.6\,\mathrm{N\,m}$ as well as bending. That additional torque can matter because a thin open section is relatively compliant in torsion.

The material centroid lies on the flange side at $x_c = 2(bt)(b/2)/(th + 2bt) = 6.43\,\mathrm{mm}$ to the right of the web. Loading through it increases the lever arm relative to the shear centre to 16.81 mm, giving 25.2 N m of torque. A bracket that transfers the load to the left of the web instead can reduce twist without enlarging the section. For a real channel with finite corner radii or unequal flange thicknesses, use its measured section geometry. The result also describes a free member; bracing against twist can replace a visible rotation by torsional reactions at the restraints.

One way to locate the shear centre experimentally is to apply the same small transverse load at several lateral offsets and measure twist per unit length in a region free of local load introduction. Linear elastic torsion predicts an approximately linear dependence of twist on offset. Its zero crossing locates the twist-free load line without requiring the torsion constant beforehand. A fixture restraining warping near the gauges can distort this test.

Study the full derivation

Why an open channel twists under a transverse load · Stress, deformation and structural measurement

A channel can twist even when a force passes through its centroid. Trace the longitudinal shear flow around its flanges, find the load line that produces bending without torsion, and design a loading experiment that separates the stiffness in bending from eccentricity relative to this shear centre.

Problem sheet 4 · Torsion and transferred shear

Trace torque and longitudinal shear through solid, tubular and assembled sections.

A hollow drive shaft

A 1.2 m long hollow shaft has outer and inner diameters 40 and 30 mm and carries 160 N m. Use $G = 78\,\mathrm{GPa}$. Find the peak shear stress and twist. If its length doubles, which result changes?

▼ **Reveal the worked solution**

The polar moment is

$$\pi(0.040^4 - 0.030^4)/32 = 1.718 \times 10^{-7}\,\mathrm{m}^4.$$

The peak shear is $160(0.020)/J = 18.6\,\mathrm{MPa}$ and twist is

$$160(1.2)/(78 \times 10^9 J) = 0.0143\,\mathrm{rad} = 0.821^\circ.$$

Doubling length doubles accumulated twist at unchanged torque, but the stress at each section is unchanged.

A box-section seam

A single-cell tube with median dimensions 100 by 60 mm and thickness 1.5 mm transmits 420 N m. Find its circulating shear flow, wall shear and force per fastener for one longitudinal seam at 25 mm pitch. What changes if thickness doubles without changing the median dimensions?

▼ **Reveal the worked solution**

Using N and mm gives

$$q = 420000/(2 \times 100 \times 60) = 35\,\mathrm{N/mm}.$$

The wall stress is $35/1.5 = 23.3\,\mathrm{MPa}$, and one fastener transmits $35(25) = 875\,\mathrm{N}$. Doubling thickness halves wall stress but leaves the equilibrium shear flow and nominal fastener force unchanged.

Connect a cap plate

A built-up beam carries 8 kN shear. Its second moment is $12.0 \times 10^6\,\mathrm{mm}^4$. The cap plate area is 800 mm² and its centroid is 60 mm above the neutral axis. Two equivalent rows of fasteners share longitudinal shear. Find the force per fastener at 40 mm pitch. Determine the largest pitch for a 1.0 kN per-fastener limit.

▼ **Reveal the worked solution**

The cap first moment is $800(60) = 48000\,\mathrm{mm}^3$, and $q = 8000(48000)/(12 \times 10^6) = 32\,\mathrm{N/mm}$. Each row takes 16 N/mm under the equal-sharing assumption. At the specified pitch that is 640 N per fastener. The theoretical limiting pitch is $1000/16 = 62.5\,\mathrm{mm}$; the practical arrangement must also supply the assumed composite action.

Explore the governing equations after solving the sheet: Torsional stiffness of shafts and closed sections; Transverse shear and connections in built-up beams. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Notebook practice · How much torque does the sleeve actually carry?

Chapter 9 · Compatibility and redundant supports

Use elastic work to determine a support reaction that cannot be obtained from force and moment balance alone.

FROM FIRST PRINCIPLES

Derive a redundant reaction by releasing one compatibility condition

In a statically indeterminate beam, force and moment balances leave more unknown reactions than independent equations. Choose one support force R as the redundant, release that support and confirm that the remaining structure is stable. Use positive displacement in the direction of R. Under small linear elastic bending, the released structure moves at the former support by δ_L under prescribed loads and by fR under the redundant. Restoring a support whose measured movement is δ_s gives

$$\delta_L + fR = \delta_s,$$
$$R = (\delta_s - \delta_L)/f.$$

To calculate the flexibility instead of measuring it, apply a unit virtual force at the released coordinate. Its bending-moment diagram is $m(x)$. The actual load-only diagram M_L produces curvature $M_L/(EI)$. Equal virtual external and internal work then gives

$$\delta_L = \int \frac{mM_L}{EI}dx,$$
$$f = \int \frac{m^2}{EI}dx.$$

Positive rigidity makes $f > 0$ for an effective released coordinate. A negative computed reaction is meaningful for a bilateral support, but a compression-only contact requires checking whether the assumed support remains engaged.

DERIVATION AND APPLICATIONS

How the stiffness of a prop sets a redundant reaction

Replacing a rigid roller by a calibrated spring provides a continuous path between a cantilever and a propped cantilever. Take a prismatic beam of length L clamped at its left end, a uniform downward load q, and an initially just-touching vertical spring of stiffness k at its right tip. Choose downward deflection positive and denote the upward spring reaction by R. Assume small bending, constant EI, negligible shear deformation and no lift-off. Equilibrium alone cannot give the reaction because the clamp can transmit both force and moment.

Remove the spring temporarily. The tip deflection from the distributed load follows by integrating the free-beam moments or by virtual work with a unit downward tip force. The corresponding flexibility for

an additional concentrated tip load is obtained using the same unit moment field:

$$w_q = \int_0^L \frac{[q(L-x)^2/2](L-x)}{EI}\,dx = \frac{qL^4}{8EI},$$
$$f = \int_0^L \frac{(L-x)^2}{EI}\,dx = \frac{L^3}{3EI}.$$

Restore the upward reaction. Superposition gives a smaller downward deflection $w = w_q - fR$. At the same time the compressed spring carries $R = kw$. These two descriptions must agree:

$$R = k(w_q - fR),$$
$$w = \frac{w_q}{1+kf}.$$

Introducing the dimensionless stiffness ratio $\eta = kf$ makes both limiting support conditions transparent:

$$R = \frac{3qL}{8}\frac{\eta}{1+\eta}.$$

The free end is recovered as $k \to 0$. An infinitely stiff prop carries $3qL/8$; even then the clamp still carries the rest of the distributed load. If a downward support settlement exceeds the free deflection, a compression-only spring loses contact and the bilateral force equation no longer applies.

WORKED EXAMPLE

Predict the reaction at finite stiffness

Take $L = 2.00$ m, $EI = 24.0$ kN m², $q = 1.20$ kN/m and $k = 9.00$ kN/m. The free tip drops 0.100 m, while $f = 0.111$ m/kN and $\eta = 1.00$. The spring cuts the drop to 0.0500 m and supplies 0.450 kN upward. The clamp balances the remaining 1.95 kN vertical force.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Select a spring for a reaction target

The rigid-support limit for the same loading is 0.900 kN. To achieve 80.0% of that, solve $\eta/(1+\eta) = 0.800$ to obtain $\eta = 4.00$ and $k = 36.0$ kN/m. The predicted spring compression is

$$0.720/36.0 = 0.0200\,\mathrm{m}.$$

This agrees with $w_q/(1+\eta)$ and distinguishes reaction fraction from displacement fraction.

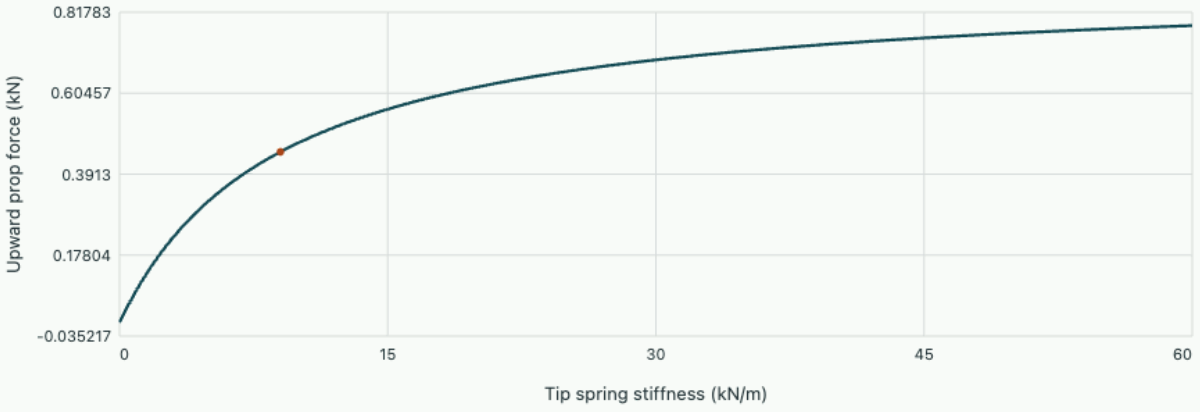
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Uniform load (kN/m)	1.2
Tip spring (kN/m)	9
Bending rigidity (kN m²)	24



With span2 m, tip drop 50.000 mm and prop force 0.450 kN. Rigid-prop limiting force 0.900 kN.

Choose a redundant force

A cantilever with a vertical prop at its far end has one more reaction than the independent planar equilibrium equations can determine. A change of support elevation changes those reactions even though the applied loads stay the same. This sensitivity provides the additional equation: the displacement at the prop equals its specified settlement. Remove the prop temporarily and replace it by an unknown upward force R. The released cantilever must be stable, and its displacement at that location must be calculable.

For small elastic deformation the effects of a known load and of the unknown force may be superposed. Write downward displacements as positive. Under a distributed downward load the released tip

displaces by δ_w. An upward reaction displaces it by $-fR$, where f is tip flexibility measured in metres per newton. If the prop settles downwards by s, compatibility becomes

$$\delta_w - fR = s.$$

Find the coefficients by virtual work

For constant EI, impose a unit downward load at the free tip. Its unit-load bending moment has magnitude $m(x) = L - x$, measured from the fixed end. With a uniformly distributed load of intensity w, the moment magnitude is $M_w = w(L-x)^2/2$. Multiplying the actual and virtual moments and integrating gives

$$\delta_w = \int_0^L \frac{M_w m}{EI}\,dx = \frac{wL^4}{8EI},$$

$$f = \int_0^L \frac{m^2}{EI}\,dx = \frac{L^3}{3EI}.$$

For a rigid prop at its initial elevation $s = 0$, the reaction becomes

$$R = 3wL/8.$$

The remaining vertical force at the fixed end is $5wL/8$. The external support moment there has magnitude $wL^2/2 - RL = wL^2/8$. Recovering equilibrium at the end provides an independent check after using flexibility to close the calculation.

Predict the effect of a small support movement

Take $L = 3.0\,\mathrm{m}$, $w = 4.0\,\mathrm{kN/m}$ and $EI = 1.8\,\mathrm{MN\,m^2}$. Without the prop, $\delta_w = 0.0225\,\mathrm{m}$, and $f = 5.00\,\mathrm{mm/kN}$. Without settlement the prop takes 4.50 kN. With a measured downward settlement of 6 mm, it takes $(22.5 - 6)/5 = 3.30\,\mathrm{kN}$. The fixed-end vertical reaction rises to 8.70 kN and its moment magnitude rises from 4.50 to $18.0 - 3.3(3) = 8.10\,\mathrm{kN\,m}$.

A unilateral prop can push upwards but cannot pull the beam down. At settlements exceeding 22.5 mm the linear compatibility solution would give a negative prop reaction. In that case the beam hangs clear of the prop and behaves as the released cantilever. A bolted hold-down could sustain the negative reaction and would require a different contact model. Replacing the rigid prop by a spring gives another direct extension: its reaction moves its top by R/k, so use $\delta_w - fR = s + R/k$ and obtain $R = (\delta_w - s)/(f + 1/k)$. An increasingly compliant prop takes an increasingly small fraction of the load.

Check a reaction against a stiffness change

If all beam cross-sections gain twice their original bending stiffness, both the load-induced tip deflection and the tip flexibility halve. For a rigid prop with zero settlement their ratio stays constant, so reactions are unchanged. For a prescribed settlement, however, the term s/f doubles. A stiff structure can therefore develop large additional reactions from a small imposed support movement.

Use an interior reaction to join two spans

A rail carried over three supports provides a second use of the force method. Remove its central support to form a simple beam of total length $2L$, retaining the central reaction R_B as an upward force to be found. Let both spans carry uniform downward load w with constant EI. For downward displacement at the centre use a downward unit force there. On the left half its moment function is $m = x/2$, while the corresponding uniform-load moment is

$$M = wx(2L - x)/2.$$

Symmetry provides the other half of the displacement integral.

$$\delta_w = \frac{2}{EI}\int_0^L M m\,dx = \frac{5wL^4}{24EI},$$

$$f_{BB} = \frac{2}{EI}\int_0^L (x/2)^2 dx = \frac{L^3}{6EI}.$$

If the centre support is at its original elevation, setting $\delta_w - R_B f_{BB} = 0$ gives

$$R_B = 5wL/4.$$

Each exterior reaction is then $3wL/8$ from overall equilibrium. Taking moments at an internal section directly over the middle support gives

$$M_B = (3wL/8)L - wL^2/2 = -wL^2/8.$$

Its negative sign describes hogging. An interior vertical support has not created a hinge in the beam; the continuous beam can transmit bending moment across that location even though the support itself supplies only a vertical force.

Set a settlement tolerance from a support load

For two 2.0 m spans under $w = 3.0\,\mathrm{kN/m}$ with $EI = 500\,\mathrm{kN\,m^2}$, the zero-settlement central reaction is

$$5(3)(2)/4 = 7.50\,\mathrm{kN}.$$

The two end reactions are 2.25 kN each and the middle moment is –1.50 kN m. The flexibility at the central support is

$$2^3/[6(500)] = 0.002667\,\mathrm{m/kN}.$$

A 4.0 mm downward support settlement therefore reduces the central reaction by $0.004/0.002667 = 1.50\,\mathrm{kN}$. Rebalancing the beam gives 3.00 kN at each end and zero moment over the middle support. The bar is still geometrically continuous; this particular combination of loading and support elevation happens to eliminate that moment.

If the central load cell must read at least 5.0 kN, setting $R_B = 7.50 - s/f_{BB}$ gives a limiting downward settlement of 6.67 mm. Conversely, a 2 mm rise in the support increases its reading by 0.75 kN. The sign changes because the constraint prescribes a displacement and the reaction adapts. Loss of central contact would occur only at $s = 7.5(2.667) = 20.0\,\mathrm{mm}$ in this one-sided arrangement. Temperature gradients that curve the rail create another imposed displacement in the same compatibility equation; measurements of elevation and temperature are needed to distinguish the two influences.

The numerical values illustrate a sign check for an after-installation alignment survey. At 4 mm descent the central reaction has fallen by 1.50 kN and each outer reaction must grow by 0.75 kN. Their three changes add to zero because the applied rail loading did not change. If all three load cells instead show smaller forces after settlement, some load has moved to another contact or one measurement reference has shifted. Compatibility selects the internal redistribution, but it never replaces the external balance of the complete rail.

A cross-level survey and a repeated load-cell tare can distinguish a genuine reaction change from a changed instrument zero.

Assemble a displacement model of a redundant braced joint

An entry in a truss stiffness matrix can be derived from the work of a single member. For a straight pin-ended bar with direction $n = (c, s)$, length L and end translations u_i, u_j, the first-order extension is $n^T(u_j - u_i)$. Its force in tension is $N = (EA/L)n^T(u_j - u_i)$, and its stored energy is $EA[n^T(u_j - u_i)]^2/(2L)$. Differentiating with respect to the four translations gives the element stiffness $K_e = (EA/L)\begin{pmatrix} nn^T & -nn^T \\ -nn^T & nn^T \end{pmatrix}$. Each member adds its coefficients to the rows and columns of the actual end joints. Shared translations enforce compatibility. Summing member contributions at a joint enforces equilibrium. Prescribed displacements are imposed before solving the remaining equations.

Consider one moving joint connected to three fixed anchors. Give each bar axial stiffness $k = 10\,\mathrm{MN/m}$ and take the directions from each anchor to the moving joint as $(1,0)$, $(0,1)$ and $(1,1)/\sqrt{2}$. The first two members supply independent horizontal and vertical stiffness. The diagonal adds $k/2$ to every entry. After the anchors are constrained, the equation under load $F = (4, 0)$ kN becomes

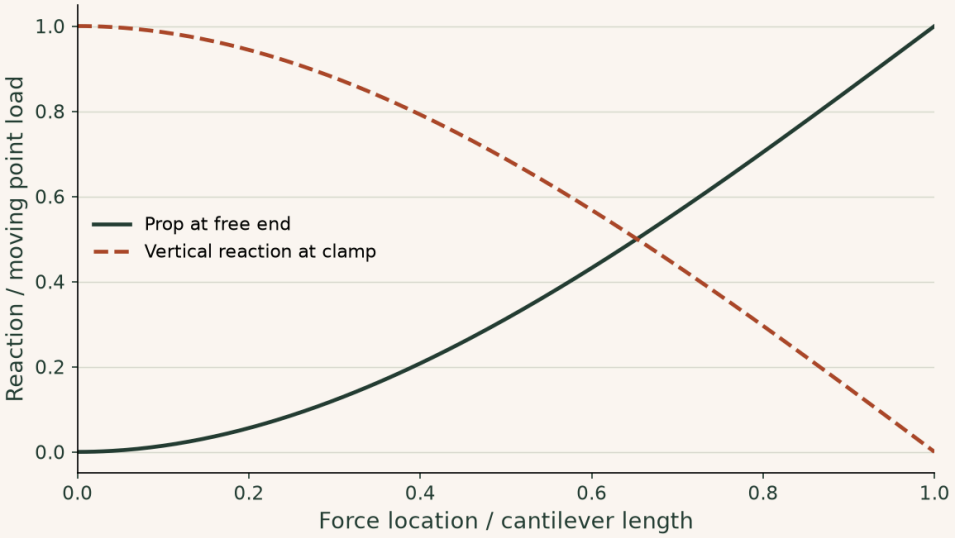
$$k\begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 4000 \\ 0 \end{pmatrix}.$$

Solving gives $u_x = 0.300$ mm, $u_y = -0.100$ mm. The recovered forces $k\,n^T u$ are 3.00 kN tension, 1.00 kN compression and 1.414 kN tension. The diagonal projects 1.00 kN onto each axis, so the resisting components sum to 4.00 kN horizontally and zero vertically. As an energy check, $\tfrac{1}{2} F^T u = 0.600$ J and $\sum N_i^2/(2k) = 0.600$ J. Two joint equilibrium equations alone cannot determine three member forces. If a brace softens, its contribution to K changes and the forces redistribute. This is the compatibility effect hidden by a purely static count.

Practise on sheet 5 · Find a mechanism before adding a diagonal spring. Further reading · MIT · 3.11 Mechanics of Materials · Trusses, “Matrix analysis of trusses”.

Trace a moving load onto an added prop

A cantilever of length L has a vertical prop at its free end. Under a point load at a, the released tip displacement is $Pa^2(3L - a)/(6EI)$. Choose the prop force to cancel it using the tip flexibility $L^3/(3EI)$.



At midspan the prop carries 5/16 of the moving load. An ordinary simply supported span would send one half to the right. The clamped-end moment and the compatibility condition explain the change in load path.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

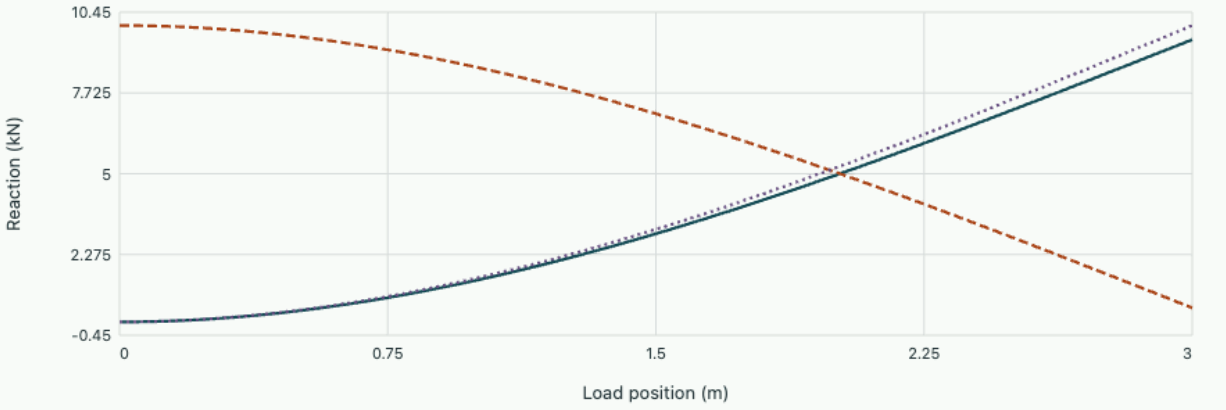
Compare a compliant prop with the rigid-prop limit. Moving the load changes its moment and the compatibility displacement together.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Point load (kN)	10
Cantilever length (m)	3
Prop / beam tip stiffness	20



At half span the prop carries 2.976 kN. The rigid limit is 3.125 kN. Prop stiffness is normalised by $3EI/L^3$.

Chapter 10 · From a strain rosette to principal stresses

Recover a complete plane-strain measurement at a surface from three gauges and infer the corresponding plane-stress state.

FROM FIRST PRINCIPLES

Calculate the extension recorded by an angled strain gauge

A bonded straight gauge records fractional change of length along its own axis. Let adjacent material points be separated by $d\mathbf{x} = \mathbf{n}\, ds$, where $\mathbf{n} = (\cos\theta, \sin\theta)$ is a unit direction. For a smooth small displacement field $\mathbf{u}$, their relative displacement is approximately $(\nabla \mathbf{u}) d\mathbf{x}$. To first order only the projection along the original separation changes its length:

$$\frac{\delta ds}{ds} = \mathbf{n}^{\mathsf{T}} (\nabla \mathbf{u}) \mathbf{n} = \mathbf{n}^{\mathsf{T}} \boldsymbol{\epsilon}\, \mathbf{n},$$

$$\boldsymbol{\epsilon} = (\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathsf{T}})/2.$$

The antisymmetric part represents infinitesimal rotation and has zero quadratic form. Writing engineering shear as $\gamma_{xy} = 2\epsilon_{xy}$ yields the observation equation

$$\epsilon_\theta = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta \cos\theta.$$

Three independently oriented grids provide three scalar observations of a uniform in-plane strain tensor. For directions $0, 45, 90$ degrees, their combination $2\epsilon_{45} - \epsilon_0 - \epsilon_{90}$ recovers engineering shear. Converting this strain to stress additionally requires elastic properties and a local plane-stress or plane-strain model.

DERIVATION AND APPLICATIONS

Remove transverse gauge sensitivity before reconstructing strain

A resistance grid responds primarily to extension along its long axis. Its transverse sensitivity may still matter when a small shear strain is inferred from differences of readings. Model a gauge at angle θ as reporting $g_\theta = \epsilon_\theta + \xi \epsilon_{\theta+90^\circ}$ after longitudinal calibration, with known dimensionless transverse response ξ. Assume uniform infinitesimal in-plane strain over a rectangular $0^\circ, 45^\circ, 90^\circ$ rosette, identical temperature compensation and the same ξ for each grid. Readings and strains below are dimensionless.

At the two axial gauges the transverse direction of one is the longitudinal direction of the other. Sum and subtract their equations to separate mean extension from normal-strain difference:

$$g_0 + g_{90} = (1 + \xi)(\epsilon_x + \epsilon_y),$$

$$g_0 - g_{90} = (1 - \xi)(\epsilon_x - \epsilon_y).$$

At 45° and 135°, small-strain transformation gives normal strain $(\epsilon_x + \epsilon_y)/2 \pm \gamma_{xy}/2$, where γ_{xy} is engineering shear strain. Inserting both projections into the calibrated reading gives

$$2g_{45} = (1 + \xi)(\epsilon_x + \epsilon_y) + (1 - \xi)\gamma_{xy}.$$

Remove the trace using the first equation and recover the corrected shear and principal strains:

$$\gamma_{xy} = \frac{2g_{45} - g_0 - g_{90}}{1 - \xi},$$

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \frac{1}{2}\sqrt{(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2}.$$

If $\xi = 0$ this reduces to the ideal rosette. As ξ approaches unity all readings respond to the trace and directional information becomes poorly conditioned. A common temperature offset cancels from this shear combination but changes the inferred mean strain if no thermal correction is made.

WORKED EXAMPLE

Recover a strain tensor from raw readings

For $\xi = 0.0200$ suppose $(g_0, g_{45}, g_{90}) = (298, 200, -94)$ microstrain. The true strain sum is $204/1.02 = 200$ microstrain, while its difference is $392/0.98 = 400$ microstrain. Thus $\epsilon_x = 300$ and $\epsilon_y = -100$ microstrain. The corrected engineering shear is $(400 - 298 + 94)/0.98 = 200$ microstrain. The principal values are 100 ± 223.6, or 323.6 and -123.6 microstrain.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Determine which calibration error matters for a pure shear test

Apply a reference pure engineering shear of 500 microstrain in the same axes. The corrected model predicts $g_0 = g_{90} = 0$ and $g_{45} = (1 - 0.0200)(500)/2 = 245$ microstrain. The ideal-gauge inversion would report 490 microstrain, low by 2.0%. A uniform offset of 12 microstrain on every channel leaves this shear unchanged, separating the transverse response from the common offset in a calibration test.

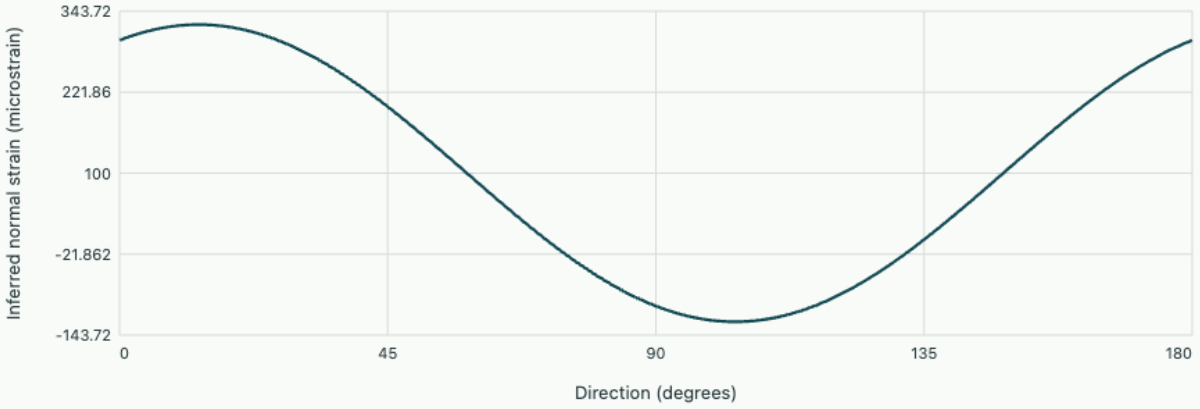
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Transverse sensitivity ξ	**0.02**
Raw0° reading (microstrain)	**298**
Raw45° reading (microstrain)	**200**
Raw90° reading (microstrain)	**-94**



Calibrated strain tensor

εx=300.000, εy=-100.000, γxy=200.000 microstrain. Engineering shear is twice the off-diagonal tensor strain.

What a bonded gauge actually measures

A resistance gauge aligned with a line on a free surface records extension along that line. A single gauge cannot distinguish uniaxial tension from combined stretching and shear arranged to give the same directional extension. Bond three gauges at different angles near enough together that the local strain is approximately uniform. A rectangular rosette uses directions of zero, 45 and 90 degrees, all referred to the same positive surface axis.

Under infinitesimal deformation, a unit direction vector $n = (\cos\theta, \sin\theta)$ experiences normal strain $n^T \boldsymbol{\varepsilon} n$. Engineering shear strain γ_{xy} is twice the off-diagonal strain tensor component. Expanding the

directional expression yields

$$\varepsilon_\theta = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy}\sin\theta\cos\theta.$$

The zero and 90° readings give $\varepsilon_x, \varepsilon_y$ directly. The middle reading gives

$$\gamma_{xy} = 2\varepsilon_{45} - \varepsilon_x - \varepsilon_y.$$

A gauge at minus 45° measures the opposite shear contribution. A sketch of the physical gauge labels avoids reversing the inferred shear sign during data reduction.

Introduce the local stress assumption

At an unloaded outer surface the normal and shear tractions on that surface vanish. With z normal to it, this gives local plane stress. For isotropic linear elasticity and mechanical strains obtained after temperature correction, invert the plane-stress compliance equations and use the shear modulus for the remaining component.

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y),$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x),$$

$$\tau_{xy} = \frac{E\gamma_{xy}}{2(1+\nu)}.$$

A strain rosette measures only the in-plane quantities. A nonzero through-thickness strain is consistent with plane stress because the in-plane stresses cause Poisson contraction. Beneath a contact patch the local traction condition changes, so the same stress recovery would need additional information.

Interpret the readings on a plate

Suppose a steel surface gives $400,\ 250,\ -100$ microstrain at zero, 45 and 90 degrees. The recovered shear strain is $500 - 400 + 100 = 200$ microstrain. With $E = 210\,\mathrm{GPa}$ and $\nu = 0.30$, the stresses are $85.38,\ 4.62,\ 16.15\,\mathrm{MPa}$ for x, y, xy, respectively. The shear is positive under the stated gauge orientation. Converting microstrain to 10^{-6} before multiplying by MPa is essential.

The principal stresses are the eigenvalues of this symmetric matrix. Its mean normal stress is 45.0 MPa, and its Mohr-circle radius is

$$\sqrt{[(85.38-4.62)/2]^2 + 16.15^2} = 43.50\,\mathrm{MPa}.$$

Thus the two in-plane principal stresses are approximately 88.5 and 1.5 MPa. The major direction is $\tfrac{1}{2}\operatorname{atan2}(2\tau, \sigma_x - \sigma_y) = 10.9^\circ$ from the zero gauge. A strain reading of –100 microstrain along y

therefore does not imply compressive normal stress along that axis: Poisson contraction from the dominant x load accounts for much of that reading.

Resolve a uniform thermal signal

Uniform heating of an unconstrained isotropic plate adds the same free thermal strain $\alpha\Delta T$ to each gauge reading, after allowing for the gauge instrumentation itself. It cancels from the inferred engineering shear because $2 - 1 - 1 = 0$, but it persists in both recovered normal components. For a 20 K temperature rise and linear expansion $12 \times 10^{-6}\,\mathrm{K}^{-1}$, subtract 240 microstrain from each reading before applying the mechanical plane-stress relation.

Reconstruct a surface strain from a delta rosette

The zero, 45 and 90 degree rosette is convenient but not the only useful placement. Introduce the strain-circle coordinates $m = (\varepsilon_x + \varepsilon_y)/2,\ d = (\varepsilon_x - \varepsilon_y)/2,\ g = \gamma_{xy}/2$. Then each directional reading is $e_\theta = m + d\cos 2\theta + g\sin 2\theta$. Every installed gauge supplies one linear equation in these three unknowns. Gauge directions that are nearly parallel give nearly dependent equations, so a small error in an individual reading can then cause a large error in a recovered component.

For a delta rosette take directions $0, 60, 120^\circ$. The doubled angles are equally spaced round the strain circle, so their cosine and sine values each sum to zero. Adding all three readings gives

$$m = (e_0 + e_{60} + e_{120})/3.$$

The first reading then gives

$$d = e_0 - m;$$

subtracting the last two cancels both m and their identical cosine terms and gives

$$g = (e_{60} - e_{120})/\sqrt{3}.$$

The in-plane principal strains and their physical orientation follow without a material assumption:

$$e_{1,2} = m \pm \sqrt{d^2 + g^2},$$
$$\theta_1 = \tfrac{1}{2}\,\mathrm{atan}\,2(g, d).$$

The half-angle is necessary because a straight gauge has the same reading after a physical rotation of 180°. For an isotropic elastic solid the principal stress directions coincide with these strain directions. Orthotropic laminates can give stress and strain axes that differ, so converting a strain circle into stresses would require the material compliance in the specimen’s axes.

OneSimulation · How does a small gauge error become a large transverse strain?

A bonded gauge observes the normal strain along its axis. Reconstruct the two normal components and the engineering shear from three readings. Give only the middle reading a positive bias of 5 με. The factor multiplying the shear in the observation matrix follows from rotating a symmetric tensor with $\epsilon_{xy} = \gamma_{xy}/2$.

$$\epsilon_\theta = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta \cos\theta.$$

OneSimulation solves the same three-unknown system for two different layouts. At 0°, 45° and 90°, the last gauge directly measures transverse strain. At 0°, 10° and 20°, all gauges see mostly the longitudinal direction. The reconstructed transverse strain now misses by −155.8 με. Fitting three readings exactly supplies no independent check on accuracy.

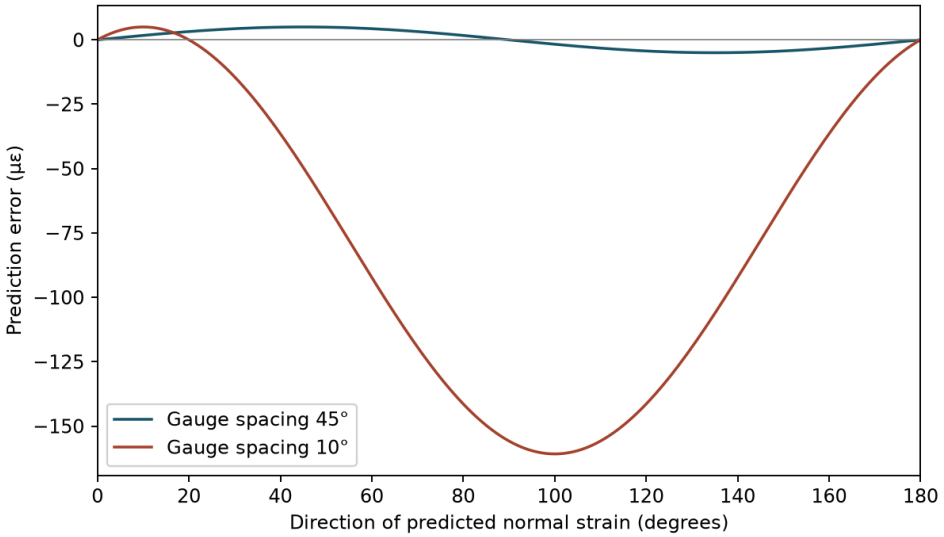
```
"""Compare strain-rosette geometries with the same middle-gauge error."""
import matplotlib.pyplot as plt
import numpy as np
from onesim.numerics.linear.matrices.dense_linear_algebra import solve_dense

true_strain = np.array([100., -60., 80.])  # ex, ey, engineering shear in με
bias = 5.0  # με added to the middle gauge
rotation = np.arange(0, 181)

def gauge_rows(degrees):
    angle = np.deg2rad(degrees)
    c, s = np.cos(angle), np.sin(angle)
    return np.column_stack((c**2, s**2, c*s))

fig, ax = plt.subplots(figsize=(7, 4), constrained_layout=True)
for spacing, colour in [(45, "#19596a"), (10, "#a5442e")]:
    layout = gauge_rows([0, spacing, 2*spacing])
    readings = layout @ true_strain
    assert np.allclose(solve_dense(layout, readings), true_strain)
    readings[1] += bias
    inferred = solve_dense(layout, readings)
    error = gauge_rows(rotation) @ (inferred - true_strain)
    ax.plot(rotation, error, color=colour, label=f"Gauge spacing {spacing}°")
    print(f"{spacing}°: error at 90° = {error[90]:.2f} με")

ax.axhline(0, color="0.5", linewidth=0.7)
ax.set(xlabel="Direction of predicted normal strain (degrees)",
       ylabel="Prediction error (με)", xlim=(0, 180))
ax.legend()
fig.savefig("engineering-mechanics.png", dpi=170)
```



Change the middle-gauge bias and rerun. Its error remains 5 με at that gauge; the much larger error occurs in directions far outside the narrow set of measured angles.

Download the Python example

Vary the parameters of this exact example in the chapter's interactive calculation.

Check an installed rosette using an independent gauge

On a loaded aluminium plate a delta rosette reads $e_0 = 350,\ e_{60} = 200,\ e_{120} = -100$ microstrain. The mean is 150 microstrain; $d = 200$ and $g = 300/\sqrt{3} = 173.2$ microstrain. Thus $\varepsilon_x = 350, \varepsilon_y = -50, \gamma_{xy} = 346.4$ microstrain. The strain-circle radius is $\sqrt{200^2 + 173.2^2} = 264.6$ microstrain. Principal strains are 414.6 and –114.6 microstrain, with largest extension along $\theta_1 = 20.45^\circ$ from the zero gauge.

Before converting these figures to stress, predict an independent reading at 90°. The directional equation gives $m - d = -50$ microstrain, which can be checked with a fourth gauge. If it instead reads +50 microstrain, possible explanations include angular mislabelling, an adhesive defect or a spatial strain gradient across the footprint. In fact the maximum local slope of the directional reading with respect to angle is $2\sqrt{d^2 + g^2}$ per radian. A two-degree placement error can therefore contribute at most about $529(2\pi/180) = 18.5$ microstrain for this field to first order. An observed 100 microstrain discrepancy would be too large to explain by that tolerance alone. This cross-check assesses the measurement geometry before uncertainty in elastic moduli enters the interpretation.

Close to an equibiaxial strain state both d and g are small, so angle estimates from $\operatorname{atan}2(g, d)$ can jump substantially between repeated readings. That sign of numerical sensitivity has a physical interpretation: when the strain circle collapses, every direction is principal. The nearly equal principal strain values may still be well determined even though a reported principal orientation is almost arbitrary. Quoting the circle radius together with the angle gives a reader enough information to decide whether the orientation is useful.

Interactive calculation · Change the inputs of this example

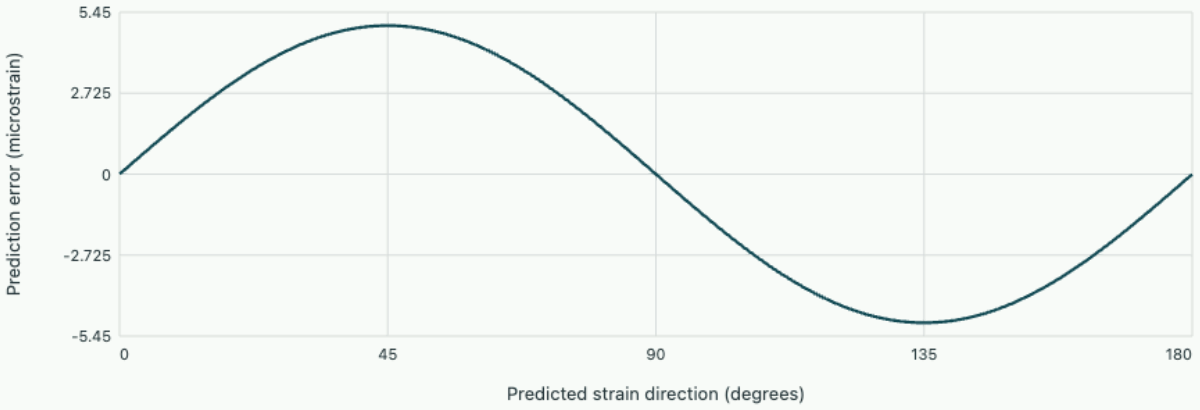
Change the parameters of this reference plot and compare its limiting cases. The fixed illustration remains beside the derivation; the adjustable calculation exposes the same dependence on geometry, material properties or loading.

Open this live calculation

Try the calculation

Gauge angular spacing (degrees) **45**

Middle-gauge bias (microstrain) **5**



Reconstructed tensor error

The0° gauge is exact; all error comes from the middle gauge. At90° prediction error=-0.000 microstrain. Crowding gauge angles amplifies the same measurement error.

Problem sheet 5 · Redundants and measured strains

Combine displacement compatibility with equilibrium and distinguish strain measurements from stress components.

A cantilever with a spring prop

A released cantilever tip displacement under the specified distributed load is 14 mm downward; the tip flexibility is 4 mm/kN. It rests on an initially unloaded spring of stiffness 0.50 kN/mm. Find the spring reaction and tip displacement. Repeat for a rigid support initially 5 mm below the undeformed tip.

▼ **Reveal the worked solution**

For the spring, $14 - 4R = R/0.50$, hence $R = 2.333\,\mathrm{kN}$ and tip movement $R/k = 4.667\,\mathrm{mm}$. With the rigid gapped support, the freely deflected beam would reach the support, so $14 - 4R = 5$ and $R = 2.25\,\mathrm{kN}$. In both cases the positive reaction verifies compressive contact.

A point load on a propped beam

A cantilever of length $L = 2.4\,\mathrm{m}$ carries a downward point load $P = 6\,\mathrm{kN}$ at midspan and has a prop at the far tip, fixed at its original height. By virtual work, find the released tip displacement in terms of EI, then determine prop reaction and the fixed-end moment magnitude.

▼ **Reveal the worked solution**

Use actual moment $P(a - x)$ for $x < a = L/2$ and zero elsewhere. With tip unit-load moment $L - x$, integrating gives

$$\delta = Pa^2(3L - a)/(6EI) = 5PL^3/(48EI).$$

The tip flexibility is $L^3/(3EI)$, giving $R = 5P/16 = 1.875\,\mathrm{kN}$. The external fixed-end moment magnitude is

$$Pa - RL = 7.2 - 4.5 = 2.70\,\mathrm{kN\,m}.$$

Identify a nearly pure shear state

An aluminium surface has readings $+50, +400, -50$ microstrain at zero, 45 and 90°. Use $E = 70\,\mathrm{GPa}$, $\nu = 0.33$ and plane stress. Find engineering shear strain, shear stress and the principal stresses. Why can a large 45° reading occur with small strains along both coordinate axes?

▼ **Reveal the worked solution**

The shear strain is $2(400) - 50 - (-50) = 800$ microstrain, giving $\tau = 70000(0.0008)/(2.66) = 21.05\,\mathrm{MPa}$. The normal stresses are $\pm 2.63\,\mathrm{MPa}$, and the mean normal stress is zero. The principal stresses are

$$\pm\sqrt{2.63^2 + 21.05^2} = \pm 21.22\,\mathrm{MPa}.$$

Shear changes the diagonal lengths of a material square even when its side lengths change little.

Find a mechanism before adding a diagonal spring

A joint is first held only by horizontal bars with combined axial stiffness 12 MN/m. Can it resist a 1.5 kN load directed upwards in a linear truss model? Add a bar running up and to the right at 45° from its fixed anchor to the joint. With axial stiffness 6 MN/m, find both displacements and the diagonal force.

▼ **Show the worked solution**

With only horizontal bars the free stiffness matrix is $\mathrm{diag}(12, 0)$ MN/m, so no finite solution exists for a nonzero vertical load. The added diagonal contributes 3 MN/m to all four entries, giving rows $(15, 3)$ and $(3, 3)$. In kN and mm the load vector is $(0, 1.5)$. The first row gives

$$u_y = -5u_x.$$

The second then gives

$$u_x = -0.125,\ u_y = 0.625\,\mathrm{mm}.$$

Diagonal extension is $(0.625 - 0.125)/\sqrt{2} = 0.3536$ mm and tension is 2.12 kN. The diagonal exerts 1.50 kN downwards and 1.50 kN leftwards on the joint. The horizontal bars exert 1.50 kN rightwards, so the joint balances the applied upward load.

Revisit chapter 9 · Assemble a displacement model of a redundant braced joint.

Explore the governing equations after solving the sheet: Compatibility and redundant supports; From a strain rosette to principal stresses. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Chapter 11 · Yield fronts and plastic bending capacity

Track the spread of yielding through a rectangular section and distinguish the first-yield moment from the fully plastic moment.

FROM FIRST PRINCIPLES

Separate first yield from the plastic moment of a section

Take a homogeneous symmetric beam section in pure bending, with plane sections remaining plane and elastic-perfectly plastic material of yield stress σ_y. Let c be the largest distance from the neutral axis. For small curvature κ longitudinal strain varies as κy, so the elastic stress distribution and bending moment are

$$\sigma_x = E\kappa y,$$
$$M = \int_A \sigma_x y \, dA = EI\kappa.$$

First yield occurs when the outermost stress magnitude reaches σ_y. Continuing to bend can still raise the moment as deeper fibres reach yield. In the ideal fully plastic limit, the tensile and compressive blocks have stresses $\pm\sigma_y$. Symmetry makes their axial forces cancel. The two distinct moment scales are therefore

$$M_y = \sigma_y I/c,$$
$$M_p = \sigma_y \int_A |y| \, dA.$$

For a rectangle of breadth b and total depth $2c$, the moments become $2b\sigma_y c^2/3$ and $b\sigma_y c^2$. Their ratio $3/2$ measures the stress redistribution available in the section. A complete frame mechanism generally needs several suitable plastic hinges, so section yield alone does not determine the collapse load.

DERIVATION AND APPLICATIONS

Predict springback after partial plastic bending

A bent coupon can retain curvature after the applied moment is removed. Consider a homogeneous rectangular section of breadth b, half-depth c, Young's modulus E and yield stress σ_y. Assume an elastic-perfectly plastic material in monotone pure bending, plane sections and symmetric tension/compression response. Take curvature $\kappa > 0$ and magnitudes on the upper half. After yielding begins the elastic core extends from the centre to $y_e = \sigma_y/(E\kappa)$; beyond it stress has saturated.

Integrate stress times lever arm separately in the core and yielded zones, then double by symmetry:

$$M = 2b\left[\int_0^{y_e} E\kappa y^2 \, dy + \int_{y_e}^{c} \sigma_y y \, dy\right] = b\sigma_y(c^2 - y_e^2/3).$$

At first yield

$$\kappa_y = \sigma_y/(Ec);$$

the fully plastic limiting moment is

$$M_p = b\sigma_y c^2.$$

Eliminating y_e relates peak moment to peak curvature:

$$M/M_p = 1 - \tfrac{1}{3}(\kappa_y/\kappa)^2, \qquad \kappa \geq \kappa_y.$$

Upon unloading, initially all fibres respond elastically. The removed moment corresponds to a change $M/(EI)$ in curvature, with $I = 2bc^3/3$. Since $M_p/(EI) = 3\kappa_y/2$, the residual curvature is

$$\kappa_r = \kappa - \frac{M}{EI} = \kappa - \tfrac{3}{2}\kappa_y\left[1 - \tfrac{1}{3}(\kappa_y/\kappa)^2\right].$$

At first yield the predicted residual is zero. The derivation assumes the unloading increment does not drive any fibre into reverse yield; that can be checked by its surface stress change. Real strain hardening or different moduli during unloading change the inferred springback.

WORKED EXAMPLE

Calculate the unloaded radius of a bent strip

Take $E = 200$ GPa, $\sigma_y = 250$ MPa and thickness $2c = 5.00$ mm. First-yield curvature is 0.500 m⁻¹. Bend to $\kappa = 1.00$ m⁻¹ so only the middle half of the depth remains elastic. The peak moment is $(11/12)M_p$, unloading removes 0.6875 m⁻¹, and $\kappa_r = 0.3125$ m⁻¹. The strip springs back from a 1.00 m radius to a radius of 3.20 m.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Check for reverse yielding and residual surface stress

At that same peak, the tensile surface was at 250 MPa. Elastic unloading changes its stress by $Ec(0.6875) = 343.75$ MPa in the opposite direction, leaving -93.8 MPa. This compressive residual remains below the 250 MPa reverse yield magnitude, consistently with elastic unloading. The unloaded part carries no applied moment even though its surface stress is nonzero; residual stresses elsewhere across the section balance its moment.

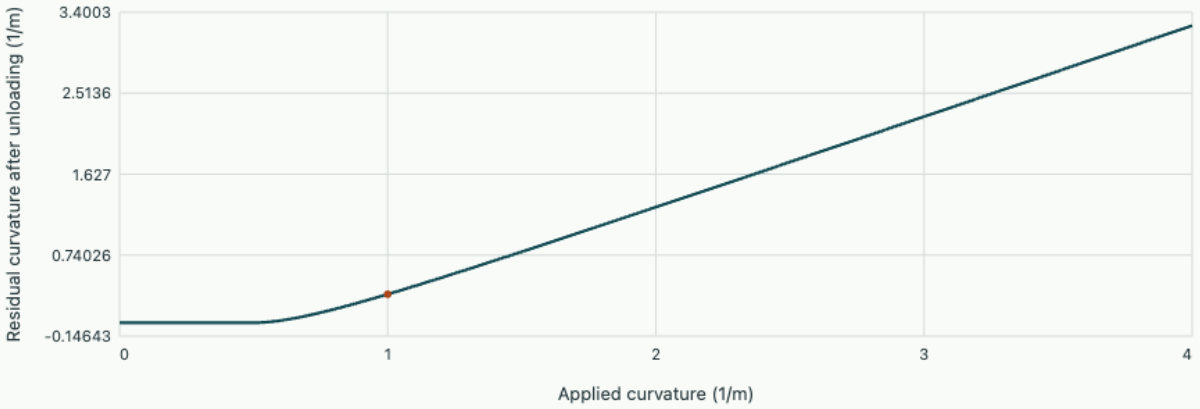
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Thickness (mm)	5
Yield stress (MPa)	250
Maximum curvature (1/m)	1



Elastic-perfectly-plastic strip Chosen bend

First-yield curvature 0.500 /m; retained curvature 0.313 /m. Elastic unloading is assumed; reverse yielding must be checked separately.

First yield occurs at a surface

A beam can retain moment capacity after its first surface fibres yield. Consider a straight rectangular bar of width b, total depth $2c$, equal tensile and compressive yield stresses σ_Y, and an elastic, perfectly plastic law. Assume pure bending about its centroid, small strains and a compact section that avoids local buckling. Plane sections remain plane, so longitudinal strain varies as κy. Symmetry keeps the neutral axis at mid-depth for this material and section.

While the whole section is elastic, $\sigma = E\kappa y$. As curvature increases, the first yield occurs when $E\kappa_Y c = \sigma_Y$. Integrating stress times moment arm gives the elastic first-yield moment $M_Y = EI\kappa_Y = \sigma_Y I/c$. A section with a narrow outer flange or unequal tensile and compressive material behaviour requires a fresh stress distribution and force equilibrium.

Integrate the elastic core and yielded layers

At a greater curvature let $a = \sigma_Y/(E\kappa)$ be the half-depth of the remaining elastic core. Stress in the core grows linearly as $\sigma_Y y/a$. Outside the core it remains at the yield magnitude. Each half of the section contributes the same bending moment. Splitting the integral at the moving yield front gives

$$M = 2b\left[\int_0^a \frac{\sigma_Y y^2}{a}\,dy + \int_a^c \sigma_Y y\,dy\right] = b\sigma_Y(c^2 - a^2/3).$$

At $a = c$ this reproduces $M_Y = (2/3)b\sigma_Y c^2$. As a approaches zero, M tends to $M_p = b\sigma_Y c^2$, which is 1.5 times first yield for this solid rectangle. The yield fronts advance towards the centre while progressively larger curvature increments produce smaller changes of moment. The bending stiffness at first loading is therefore history dependent after yielding begins.

Quantify the onset and extent of plastic deformation

A 20 mm wide bar is 40 mm deep, with $E = 200\,\mathrm{GPa}$ and $\sigma_Y = 250\,\mathrm{MPa}$. Its first-yield curvature is $250/(200000 \times 20) = 6.25 \times 10^{-5}\,\mathrm{mm^{-1}}$, or 0.0625 m⁻¹. First yield occurs at 1.333 kN m and the limiting plastic moment is 2.000 kN m. At twice the yield curvature the elastic core half-depth has fallen to 10 mm. Substitution gives

$$M = 20(250)[400 - 100/3] = 1.833\,\mathrm{kN\,m}.$$

Upon unloading, assume the stress changes elastically without reverse yield. The immediate moment–curvature slope is then EI, so the recovered curvature is $M/(EI)$, where $I = 20(40)^3/12 = 106667\,\mathrm{mm^4}$. It is $8.59 \times 10^{-5}\,\mathrm{mm^{-1}}$. The peak curvature was $1.25 \times 10^{-4}\,\mathrm{mm^{-1}}$, leaving about 0.0391 m⁻¹ residual curvature. A permanent bend can therefore coexist with zero applied bending moment. The residual internal stresses balance their own force and moment across the section.

Compare equal areas in different locations

Redistributing material towards the outer fibres changes the reserve between first yield and full plasticity. In the limiting idealisation of two very thin equal flanges of area A_f at $y = \pm c$, essentially all material reaches yield together. The first-yield and plastic moments both approach $2A_f\sigma_Y c$. A shape factor of 1.5 is specific to the solid rectangle and cannot be carried over to this ideal flange pair.

Build a collapse mechanism from plastic hinges

A section that reaches its plastic moment can undergo substantial further rotation in an ideal perfectly plastic model. A structure may still support additional external load because other sections can take up incremental bending. Collapse in a rigid–plastic beam idealisation occurs when enough hinges form to

allow a compatible mechanism. Use virtual work at collapse: the external forces do work through compatible virtual displacements, and each plastic hinge dissipates M_p times its relative rotation. Elastic strain energy changes are neglected in this limit calculation.

Take a uniform beam fixed at both ends under a concentrated downward force P at midspan. Assume equal plastic moments of magnitude M_p in sagging and hogging, compact sections and adequate rotation capacity. Consider hinges at the two clamps and at midspan, joined by two rigid half-spans. A small downward virtual centre movement δ rotates each half through $\theta = 2\delta/L$. The clamp hinges each rotate by θ, while the centre hinge rotates through 2θ because its adjacent halves slope in opposite directions. Work equivalence gives

$$P\delta = M_p(\theta + 2\theta + \theta) = 4M_p(2\delta/L),$$

$$P_c = 8M_p/L.$$

This kinematically admissible mechanism gives an upper bound on true plastic collapse load under these assumptions. Here a matching statically admissible diagram can be constructed: end moments are $-M_p$, the centre moment is $+M_p$, and constant shear on each half gives

$$P/2 = 4M_p/L.$$

The matching bounds identify the exact ideal collapse load for this geometry and loading.

Separate first yield, section capacity and structural collapse

A mild-steel rectangular strip 30 mm wide and 60 mm deep spans 1.20 m between fixed clamps. Take yield stress 240 MPa. The first-yield section moment is $M_Y = \sigma_Y b h^2/6 = 4.32\,\mathrm{kN\,m}$, and the fully plastic moment is

$$M_p = \sigma_Y b h^2/4 = 6.48\,\mathrm{kN\,m}.$$

For a central point force, elastic symmetry and compatibility give clamp and centre moment magnitudes $PL/8$. First yield thus occurs at $8(4.32)/1.20 = 28.8\,\mathrm{kN}$ and the ideal mechanism load is

$$8(6.48)/1.20 = 43.2\,\mathrm{kN}.$$

The factor of 1.5 here comes from the rectangular section’s reserve as yielding spreads. The three critical sections have equal absolute moment in this symmetric loading.

Release the right clamp rotation and the mechanism changes. For the simpler limiting case with both ends pinned, only one interior plastic hinge is needed and the moment maximum is $PL/4$. The same section would collapse at $4(6.48)/1.20 = 21.6\,\mathrm{kN}$, half the fixed-ended result. The comparison explains why a connection described as fixed must actually develop bending moment and plastic rotation before its contribution is credited. A brittle welded detail, lateral instability or a shear failure can prevent the assumed redistribution. The calculation supplies a limit for the stated deformation mode, not a prediction of every possible failure mechanism.

The location of the mechanism can be checked in an overload test by measuring rotations near both supports and on either side of midspan. A large rotation at a single connection before substantial rotation elsewhere indicates that the assumed distribution of hinge resistance is not being realised.

Study the full derivation

Springback and residual stress after plastic bending · Stress, deformation and structural measurement

After a rectangular strip has been bent beyond first yield, removing its end couple does not return it to its original curvature. Work out the loading and unloading paths fibre by fibre. Then explain why an unloaded strip can contain a substantial stress field with zero net force and zero net bending moment.

Chapter 12 · Crack driving force and fracture assessment

Use an energy balance and a stress-intensity model to distinguish crack-controlled fracture from yielding of the gross section.

FROM FIRST PRINCIPLES

Relate crack extension to the energy released by elastic compliance

Consider a slowly loaded linear elastic specimen of thickness B, applied force P, measured displacement δ and crack coordinate a. Write $C(a) = \delta/P$. The projected area convention is $dA = B\,da$. On a fixed crack the linear loading curve stores strain energy equal to the triangle below its force–displacement graph:

$$U = \tfrac{1}{2} P\delta = \tfrac{1}{2} P^2 C(a).$$

For a dead force held constant while the crack extends, the external force performs additional work $P\,d\delta$. Subtract the load potential $P\delta$ from the stored energy and differentiate with respect to area to obtain the driving force for extension:

$$\Pi = U - P\delta = -\tfrac{1}{2} P^2 C,$$

$$G = -\left.\frac{d\Pi}{dA}\right|_P = \frac{P^2}{2B}\frac{dC}{da}.$$

The units are J/m². An increase of compliance can free energy to create new surfaces and to dissipate energy near the crack. A material resistance measured under matching conditions is needed to predict onset. In linear elastic mode I, the local intensity relation $G = K_I^2/E'$ connects this global energy measure with the near-tip field.

Background · Stress intensity

$$K_I = Y\sigma\sqrt{\pi a}$$

Opening-mode intensity has units of stress times the square root of length. Geometry and the definition of crack length are part of the calibration.

DERIVATION AND APPLICATIONS

How machine compliance changes the energy available to a crack

A tensile fixture can release stored elastic energy as a crack grows even when the grips do not move. To quantify this, treat a specimen of thickness B with crack coordinate a and measured opening compliance $C(a) = \delta_s/P$. Use a single projected crack-area increment $B\,da$ consistent with the stated convention. Let an elastic testing machine in series have constant compliance C_m, and assume quasistatic linear elastic loading with negligible dissipative processes outside the crack tip. At fixed total grip opening Δ the force must change as the specimen compliance changes:

$$\Delta = P(C + C_m),$$

$$U = \tfrac{1}{2}P^2(C + C_m) = \frac{\Delta^2}{2(C + C_m)}.$$

No further external work enters at fixed Δ. Differentiate the stored energy with respect to projected area. Since C_m is independent of the crack coordinate, only the specimen derivative appears:

$$G = -\frac{1}{B}\frac{dU}{da}\bigg|_{\Delta} = \frac{\Delta^2}{2B(C + C_m)^2}\frac{dC}{da} = \frac{P^2}{2B}C'(a).$$

At a given instantaneous force this has the familiar force-based value, but its variation during crack extension depends on compliance control. Take the idealized measured relation $C = ca^3$ over a limited range. Logarithmic differentiation at fixed imposed opening gives

$$\frac{1}{G}\frac{dG}{da} = \frac{2}{a} - \frac{6C}{a(C + C_m)}.$$

For constant fracture resistance, a decreasing driving force after slight growth favours arrest. In this compliance model it occurs when $C_m < 2C$. This local comparison assumes a small crack extension within the calibrated range and a material whose fracture resistance is not changing appreciably.

WORKED EXAMPLE

Estimate driving force from a coupon measurement

At $a = 40.0$ mm take $C = 2.00$ micrometres/N, $C_m = 1.00$ micrometre/N, $B = 10.0$ mm and $\Delta = 0.600$ mm. The force is 200 N, and the local cubic slope is $C' = 3C/a = 1.50 \times 10^{-4}$ N⁻¹. Substitution yields $G = 300$ J/m². The calculated logarithmic slope is -50.0 m⁻¹, predicting about a 5% decrease for an initial additional millimetre of extension.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Compare a compliant test frame at the same force

Increase machine compliance to 6.00 micrometres/N and impose $\Delta = 1.60$ mm so the starting force stays 200 N. The initial G remains 300 J/m², but the slope is now $+12.5$ m⁻¹. A millimetre of additional crack length initially raises the driving force by about 1.25%. The specimen itself has not changed; the frame now delivers a different load history as the crack extends.

Interactive calculation · Change the inputs of this example

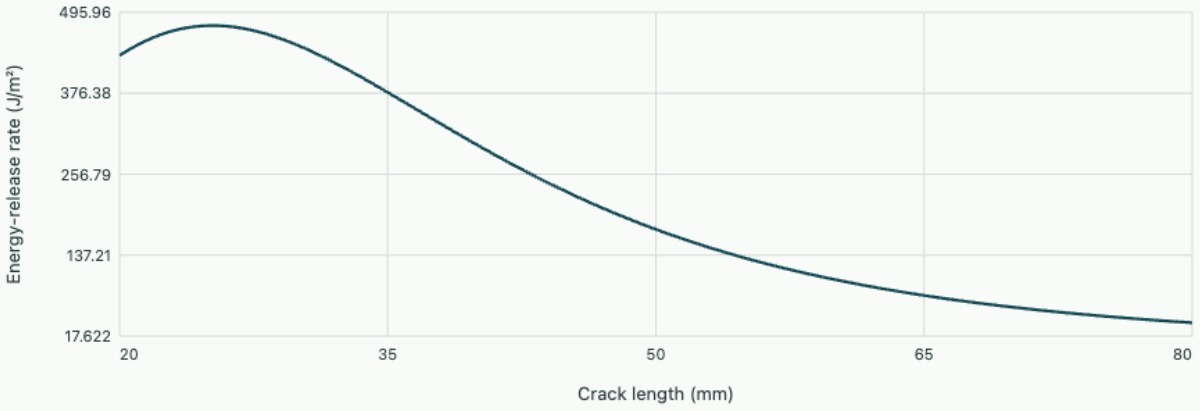
Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Machine compliance (µm/N) **1**

Crosshead opening (mm) **0.6**



Fixed crosshead displacement

At40 mm crack length, G=300.000 J/m². Specimen compliance is2 µm/N at40 mm and varies cubically; width10 mm.

A sharp crack changes the question

An uncracked tensile coupon can often be assessed by comparing nominal stress with a yield stress. A sharp pre-existing crack concentrates the stress into a small region near its tip. Within linear elastic fracture mechanics the leading opening-mode field has magnitude proportional to $K_I/\sqrt{2\pi r}$ at

distance r ahead of the tip. The unbounded mathematical stress signals the limits of the elastic tip description. A material toughness measured under comparable conditions instead sets a threshold for crack extension.

A common representation is $K_I = Y\sigma\sqrt{\pi a}$, where a is the specified crack dimension, σ is a stated remote stress, and Y describes the geometry and loading. For a central crack in a sufficiently wide plate a is the half-length and Y approaches one. For a short edge crack in a wide plate a is its depth and a common approximation is $Y \approx 1.12$. Finite width, bending or a different crack shape changes the factor.

Relate compliance to an energy release rate

In a linear test specimen let load P, opening displacement u, and compliance $C(a) = u/P$ refer to the same load point. At fixed load the stored elastic energy is $P^2C/2$. The external load does work $P\,du = P^2\,dC$ as the crack extends. The released potential energy is the difference, $P^2dC/2$. For a through crack with one advancing front and specimen thickness B, its projected area increases by $B\,da$, giving

$$G = \frac{P^2}{2B}\frac{dC}{da} = \frac{K_I^2}{E'},$$

$$E' = E \ (\text{plane stress}),$$

$$E' = E/(1-\nu^2) \ (\text{plane strain}).$$

The compliance argument explains how a distant load–displacement measurement can identify energy available at a crack. Define the crack area consistently: a symmetric specimen in which two tips advance together changes projected area twice as quickly for the same increment of half-length.

Infer a critical flaw size

Take a wide plate with an edge crack of depth 3.0 mm, tensile remote stress 160 MPa and illustrative applicable toughness $K_c = 35\,\mathrm{MPa}\sqrt{\mathrm{m}}$. The estimate $K_I = 1.12(160)\sqrt{\pi(0.003)} = 17.4\,\mathrm{MPa}\sqrt{\mathrm{m}}$ uses metres for crack depth. Holding geometry factor and stress fixed, the critical depth is $a_c = [35/(1.12\times160)]^2/\pi = 0.0121\,\mathrm{m}$, about 12.1 mm.

At twice the stress this depth falls by a factor of four, to approximately 3.0 mm. Before using that extrapolation, check remaining net section, applicability of the wide-plate geometry and whether plasticity near the crack is small compared with crack and ligament dimensions. Toughness depends on temperature, constraint and loading rate. An inspection interval would additionally require a crack-growth law, a load history and a detection limit. The calculated critical depth describes one load state and cannot by itself predict the number of cycles remaining.

Interpret a crack-length uncertainty

At fixed remote stress and geometry factor a small relative error in crack depth gives approximately half that relative error in stress intensity, because $dK/K = da/(2a)$. A 0.3 mm uncertainty on a 3 mm measurement therefore changes this estimated intensity by about five percent.

Integrate stable fatigue growth up to inspection

Fracture toughness characterises an instability at a given load. Damage tolerance also asks how long a detected crack takes to reach an inspection or retirement size under repeated loading. For illustration, suppose tests for the service material and stress ratio support a Paris relation $da/dN = C(\Delta K)^m$ in its intermediate stable growth range. Use crack depth a in metres and intensities in MPa$\sqrt{\text{m}}$. The coefficient C must correspond to those units, stress ratio and environment. This empirical law generally changes near threshold and near final instability.

For constant stress range $\Delta\sigma$, fixed geometry factor Y and exponent $m = 3$, substitute $\Delta K = Y\Delta\sigma\sqrt{\pi a}$. Crack advance per cycle then scales as $a^{3/2}$; the same stress history grows a larger crack faster. Separate variables between measured depth a_i and removal depth a_f. Integrating yields

$$N = \frac{2(a_i^{-1/2} - a_f^{-1/2})}{C(Y\Delta\sigma)^3\pi^{3/2}}.$$

This prediction decreases to zero as the two depths approach each other. It scales with the inverse cube of stress range at fixed bounds. If the geometry factor changes appreciably with crack depth, retain $Y(a)$ inside the integral. The maximum stress in the cycle is needed separately to compare $K_{\max}$ with applicable toughness, because it is the range that drives the adopted fatigue law.

Compare two inspection thresholds on the same plate

Consider a wide plate with $Y = 1.12$, $\Delta\sigma = 80\,\text{MPa}$, $\sigma_{\max} = 140\,\text{MPa}$, $m = 3$, and illustrative $C = 2.0 \times 10^{-12}\,(\text{m/cycle})/(\text{MPa}\sqrt{\text{m}})^3$. Suppose the measured growth-law range includes depths between 2 and 8 mm for this loading. At 2 mm,

$$\Delta K = 1.12(80)\sqrt{\pi(0.002)} = 7.10\,\text{MPa}\sqrt{\text{m}};$$

at 8 mm it doubles to 14.2. The growth rate at the upper limit is consequently eight times the lower-limit rate.

The integral denominator is $2.0 \times 10^{-12}(89.6)^3\pi^{3/2} = 8.01 \times 10^{-6}$ in the stated units, and the numerator is

$$2(22.36 - 11.18) = 22.36\,\text{m}^{-1/2}.$$

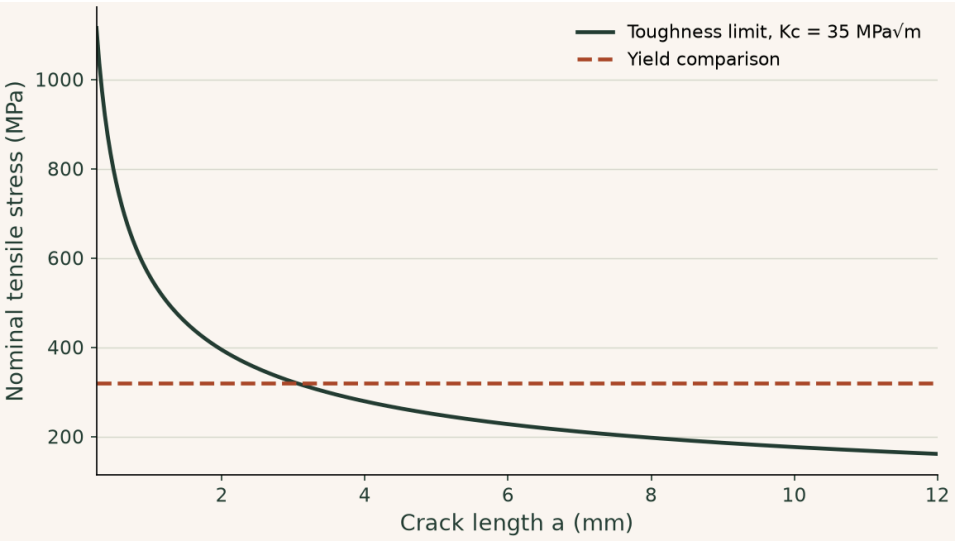
The estimated interval is 2.79×10^{6} cycles. If the smallest reliably detected crack were 4 mm, the available interval to 8 mm would fall to $2(15.81 - 11.18)/(8.01 \times 10^{-6}) = 1.16 \times 10^{6}$ cycles. Improving early detection has a measurable value in this model. At retirement depth $K_{\max} = 24.9\,\mathrm{MPa}\sqrt{\mathrm{m}}$; compare that independently with the applicable toughness. Uncertainty in detection, loading, environment and growth coefficients would enter an actual inspection decision rather than treating this calculated mean growth interval as a guaranteed service interval.

A useful sensitivity check is to increase the entire stress range by ten percent while keeping the inspection depths and coefficient fixed. The Paris estimate falls by $1/1.1^{3} = 0.751$, reducing the 2.79 million-cycle interval to about 2.10 million cycles. If the maximum stress also rises by ten percent, retirement intensity increases by the same ten percent to 27.4 MPa$\sqrt{\mathrm{m}}$. Those two changes affect different checks and cannot be combined into a single linear safety multiplier. A record of both stress range and stress maximum is therefore more informative than a peak-only reading for planning inspections.

Both quantities can be recovered from the two measured stress extremes if the counting convention and gauge reference are retained.

Which flaw sizes make fracture the controlling limit?

Consider a wide plate with a small edge crack, geometry factor $Y = 1.12$ and conditional toughness $K_c = 35\,\mathrm{MPa}\sqrt{\mathrm{m}}$. Rearrange $K = Y\sigma\sqrt{\pi a}$ and compare with a 320 MPa yield scale. Convert the crack length to metres.



The two estimates intersect near a = 3.04 mm. As cracks grow, the estimated fracture stress falls as the inverse square root. Applying this simple comparison to a real component requires geometry-specific factors, valid toughness data and a check on plastic-zone size.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

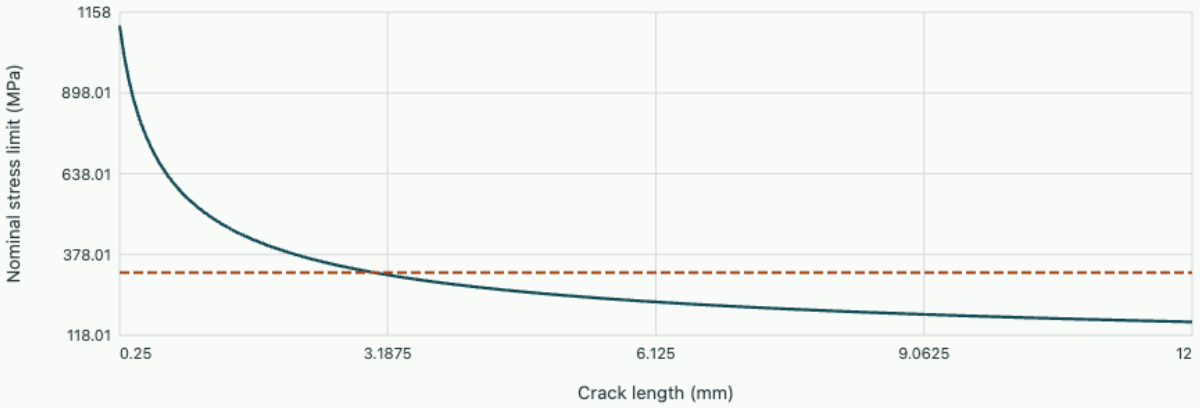
Vary toughness and yield scale independently. Locate the crack length where the controlling estimate changes.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Conditional toughness (MPa$\sqrt{m}$)	**35**
Geometry factor Y	**1.12**
Yield scale (MPa)	**320**



Conditional fracture Yield comparison

The two estimates cross at 3.036 mm. This comparison does not replace geometry and plastic-zone validity checks.

Problem sheet 6 · Plastic deformation and crack driving force

Check the limits of an elastic calculation using yielding, residual deformation and fracture energy.

How much of the cross-section has yielded?

A rectangular strip 12 mm wide and 30 mm deep has elastic–perfectly plastic yield stress 200 MPa. Determine the first-yield and plastic moments. At a curvature three times the yield curvature, find the depth of the elastic core and resulting moment.

▼ **Reveal the worked solution**

With $c = 15\,\mathrm{mm}$,

$$M_p = b\sigma_Y c^2 = 540\,\mathrm{N\,m},$$

$$M_Y = 2M_p/3 = 360\,\mathrm{N\,m}.$$

The elastic half-depth is $a = c/3 = 5\,\mathrm{mm}$, giving a 10 mm deep core. The moment is

$$12(200)(15^2 - 5^2/3) = 520\,\mathrm{N\,m}.$$

Yield has spread far inward for a comparatively small increase over the first-yield moment.

Changing the inspection threshold

A wide plate with a central through crack has applicable toughness 45 MPa$\sqrt{\mathrm{m}}$ at the service temperature. With remote tension 180 MPa and $Y = 1$, calculate critical half-length and full length. If inspection finds a full crack length of 16 mm, what remote stress gives the critical intensity?

▼ **Reveal the worked solution**

The critical half-length is $a_c = (45/180)^2/\pi = 0.0199\,\mathrm{m}$, or 19.9 mm; the full length is 39.8 mm. The measured half-length is 8 mm, so

$$\sigma_c = 45/\sqrt{\pi(0.008)} = 284\,\mathrm{MPa}.$$

Both predictions assume small-scale yielding and adequate plate width. Using full length inside the square root would underpredict the critical stress.

Infer release rate from two readings

A specimen with one advancing crack front has thickness 10 mm. Its measured compliance increases from 0.0250 to 0.0270 mm/kN as crack length rises by 1.0 mm. Under a fixed load of 4 kN estimate release rate. For plane stress with $E = 70\,\mathrm{GPa}$, find the corresponding intensity.

▼ **Reveal the worked solution**

The slope is

$$0.0020\,\mathrm{kN}^{-1} = 2.0 \times 10^{-6}\,\mathrm{N}^{-1}.$$

Thus

$$G = 4000^2 (2.0 \times 10^{-6})/[2(0.010)] = 1600\,\mathrm{J/m^2}.$$

Then

$$K = \sqrt{EG} = 10.58\,\mathrm{MPa}\sqrt{\mathrm{m}}.$$

This uses a local finite-difference slope, so significant curvature of the compliance data would call for measurements at closer crack lengths.

Separate load capacity from serviceability

A simply supported steel beam of span 2.00 m carries a 2.00 kN centre load. Its solid rectangular section is 40.0 mm wide by 80.0 mm deep. Take E = 200 GPa, allowable bending stress 120 MPa and a service deflection limit L/400. Find the admissible centre load. Would rotating the section by 90° change which limit governs?

▼ **Worked solution**

Symmetry gives reactions P/2 and the peak bending moment PL/4. With the 80 mm side vertical the second moment of area is $bh^3/12 = 1.707 \times 10^{-6}$ m⁴ and the elastic section modulus is $I/(h/2) = 4.267 \times 10^{-5}$ m³. Under the specified load the centre moment is 1000 N m and the extreme-fibre stress is 23.4 MPa.

Integrating the elastic curvature with zero displacement at the supports and zero slope at midspan gives $\delta = PL^3/(48EI)$. At 2.00 kN this is 0.977 mm. The stress limit gives 10.24 kN and the deflection limit of 5.00 mm also gives 10.24 kN. Their coincidence is particular to these dimensions and limits.

On rotating the beam, the new second moment is one quarter of the original and the section modulus one half. The stress limit now allows 5.12 kN, while deflection allows only 2.56 kN. The material, mass

and support reactions stayed the same. The calculation assumes elastic bending and effective lateral restraint; connection and local bearing capacities require separate data.

Explore the governing equations after solving the sheet: Yield fronts and plastic bending capacity; Crack driving force and fracture assessment. Compare the assumptions and units before transferring your chosen parameters; the controls may cover a different numerical range from the written exercise.

Chapter 13 · From cut forces to local equilibrium

A support reaction tells you the total force entering a part. To predict what an inclined internal surface carries, keep track of both the orientation of the cut and the direction of the force across it.

A force that depends on the surface you choose

Imagine a small triangular wedge cut out of a thin, slowly loaded plate. Its sloping face has unit outward normal $\mathbf{n} = (n_x, n_y)$ and area A. The two faces normal to the coordinate axes have areas An_x and An_y when both components are positive. Name the internal force per unit area on a face its traction. Opposite faces of the same cut carry equal and opposite tractions. Gravity and acceleration act on volume, which shrinks faster than area as the wedge shrinks. Dividing its force balance by A therefore gives a relation that no longer contains its size.

$$\mathbf{t}(\mathbf{n}) = n_x \mathbf{t}(\mathbf{e}_x) + n_y \mathbf{t}(\mathbf{e}_y) = \boldsymbol{\sigma}\mathbf{n}.$$

The columns of the stress matrix are the two coordinate-face tractions. Their linear combination supplies the force on any inclined face. A small square additionally needs zero net moment. To leading order, the shear forces on opposite faces form two couples of opposite sign and equal lever arm. Their balance requires $\sigma_{xy} = \sigma_{yx}$ for an ordinary material without internal body couples.

Background · Traction

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma}\mathbf{n}$$

Specify the plane normal before projecting the force per area into normal and shear parts. A stress component is not a net force until integrated over an area.

Why a gradient of stress carries a body force

Now use a tiny rectangular element of side lengths dx, dy and unit thickness. The right-face contribution to horizontal force is $\sigma_{xx}(x + dx, y)dy$; the left face contributes its negative evaluated at x. Their difference is $\partial_x \sigma_{xx}\, dx\, dy$ to leading order. Resolve the upper and lower shear tractions in the same way. Add a prescribed horizontal force per volume b_x and divide by volume. Repeating in the vertical direction produces two independent equations.

$$\partial_x \sigma_{xx} + \partial_y \sigma_{xy} + b_x = 0,$$
$$\partial_x \sigma_{xy} + \partial_y \sigma_{yy} + b_y = 0.$$

These equations say nothing yet about how much the material stretches. They can test an assumed stress field before the elastic modulus has even been measured. A candidate must also reproduce the applied boundary tractions. Satisfying the interior equations alone leaves many possible stresses.

Worked example · the grip on a diagonal bonding plane

At a measurement point let $\sigma_{xx} = 60, \sigma_{yy} = 20, \sigma_{xy} = 10$ MPa. The outward normal is $(0.6, 0.8)$. Matrix multiplication gives $\mathbf{t} = (44, 22)$ MPa. Projection along the normal gives $0.6(44) + 0.8(22) = 44.0$ MPa tension. Projection on the unit tangent $(-0.8, 0.6)$ gives -22.0 MPa. Reversing the tangent changes the sign of reported shear. Reversing the surface normal reverses the traction vector itself. For a uniform 100 mm² patch the force has components 4.40 and 2.20 kN. Using only normal stress would miss the tangential load that the adhesive must transmit.

Worked example · a specimen with a known stress gradient

Stand a bar upright, support it at its bottom and leave the top free. Use upward y, length L, constant density ρ, and tension positive. The vertical equation reads $d\sigma_{yy}/dy - \rho g = 0$. Integrating from the free top gives

$$\sigma_{yy} = \rho g (y - L).$$

For a 3.0 m aluminium bar of density 2700 kg/m³ the base stress is −79.5 kPa. Its upward support traction is positive because the bottom face has downward outward normal. A uniform cross-sectional area of 200 mm² gives a base reaction of 15.9 N, equal to the weight. The local and global checks involve different quantities but must agree.

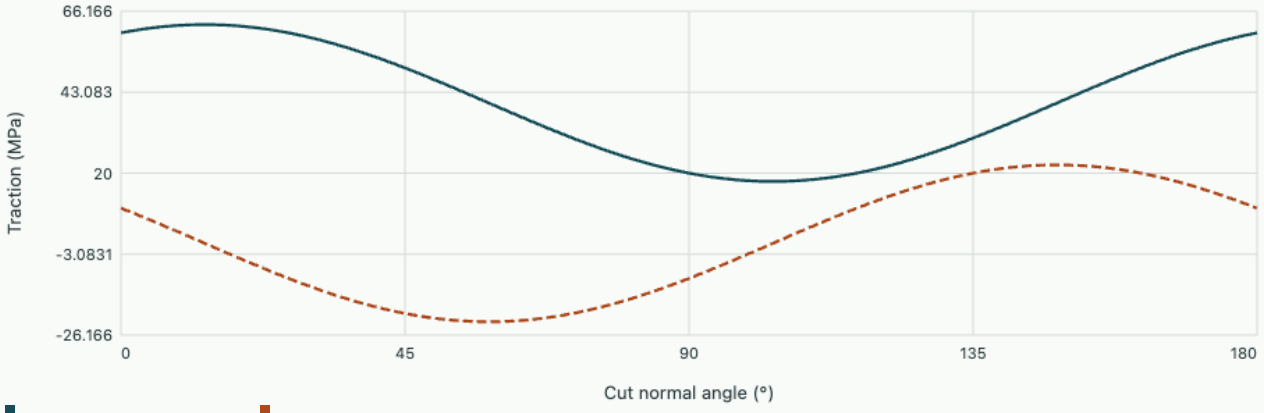
Find the angles where the shear traction passes through zero. Compare each with an extremum of the normal traction. Change the sign of the imposed xy shear, sketch the new angles before moving the slider, and check whether the two extreme normal stresses change.

Open this experiment online

Try the calculation

Normal stress xx (MPa) 60

Shear stress xy (MPa) 10



On the cut with normal (0.6, 0.8), the normal traction is 44.00 MPa and the signed shear is -22.00 MPa.

Tutorial · test a candidate before using a material law

In a region with x, y in metres, propose $\sigma_{xx} = ax,\ \sigma_{xy} = -ay,\ \sigma_{yy} = 0$, where $a = 2.0$ MPa/m. Does the field need a body force? Find the traction at $x = 0.30,\ y = 0.20$ m on a face normal to positive x.

▼ **Check the balance and the boundary reading**

The horizontal divergence is $a - a = 0$. The vertical divergence is zero because the shear has no x variation. No body force is needed. The face sees the first column of the matrix, $(0.60, -0.40)$ MPa. A traction-free exterior at that location would reject this field even though it satisfies both interior equations.

Further reading · A. F. Bower, Applied Mechanics of Solids, §2.3, internal forces and stress

Chapter 14 · How a lateral restraint changes a measured modulus

A short compression test can report a larger stiffness than a tensile coupon cut from the same material. Derive the difference from the strains that the fixture permits.

Separate changes of volume and changes of shape

For small strain, uniaxial tension σ gives longitudinal strain σ/E and transverse strain $-\nu\sigma/E$. These are experimental definitions of Young's modulus and Poisson's ratio for an isotropic material. In a linear solid the three normal coordinate loading cases can be superposed. Under tensile stresses positive, this gives

$$E\epsilon_x = \sigma_x - \nu(\sigma_y + \sigma_z),$$
$$E\epsilon_y = \sigma_y - \nu(\sigma_x + \sigma_z),$$
$$E\epsilon_z = \sigma_z - \nu(\sigma_x + \sigma_y).$$

Add the three equations. The sum of the small normal strains is the relative volume change because the product $(1+\epsilon_x)(1+\epsilon_y)(1+\epsilon_z)$ retains only first-order terms. Under compressive pressure p all three normal stresses are $-p$. Hence

$$\frac{\Delta V}{V} = -\frac{3(1-2\nu)}{E}p = -\frac{p}{K},$$
$$K = \frac{E}{3(1-2\nu)}.$$

Large bulk stiffness can coexist with moderate shear stiffness. Near $\nu = 0.5$, changing volume becomes difficult, while changing shape can remain easy. This distinction matters when interpreting a pad compressed between smooth plates and the same pad enclosed within a rigid sleeve.

Derive the stiffness in a close-fitting sleeve

Apply axial stress σ_z to a specimen with both lateral strains constrained to zero. Assume uniform deformation away from the ends and identical conditions in x and y. Symmetry gives

$$\sigma_x = \sigma_y = s.$$

The first strain law becomes $0 = (1-\nu)s - \nu\sigma_z$, so the sleeve must supply a lateral stress even though no lateral load was specified at the start.

$$s = \frac{\nu}{1-\nu}\sigma_z,$$
$$\frac{\sigma_z}{\epsilon_z} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} = M.$$

The constraint modulus M increases rapidly near incompressibility. It is the ratio of a particular stress to a particular strain in this test. The material constants in the original three-dimensional law continue to apply in both fixtures. A supposedly free compression test with strong friction on the platens may fall between ideal free and fully confined behaviour.

Worked example · recover the sleeve reaction

A linearly elastic sample has $E = 30$ MPa and $\nu = 0.40$. Its thickness is 20 mm and the axial stress is −0.30 MPa. With free sidewalls the axial strain is −0.0100 and the contraction is 0.200 mm. Inside the ideal rigid sleeve, $M = 64.29$ MPa, giving contraction $20(0.30/64.29) = 0.0933$ mm. Each lateral stress is −0.200 MPa. Substitution into the original x-strain equation gives zero, a direct test of the force that was inferred from compatibility.

Worked example · infer a bulk property independently

Compress the same material hydrostatically by 0.15 MPa. The measured constants predict $K = 50.0$ MPa and a volume loss of 0.300%. A 40 cm³ specimen would lose 0.120 cm³. Using its sleeve modulus in this calculation would underestimate the volume change. In hydrostatic compression all three strains are equal. In the sleeve only one strain can be nonzero. A constitutive test must specify what is imposed on the other faces and what the measurement records.

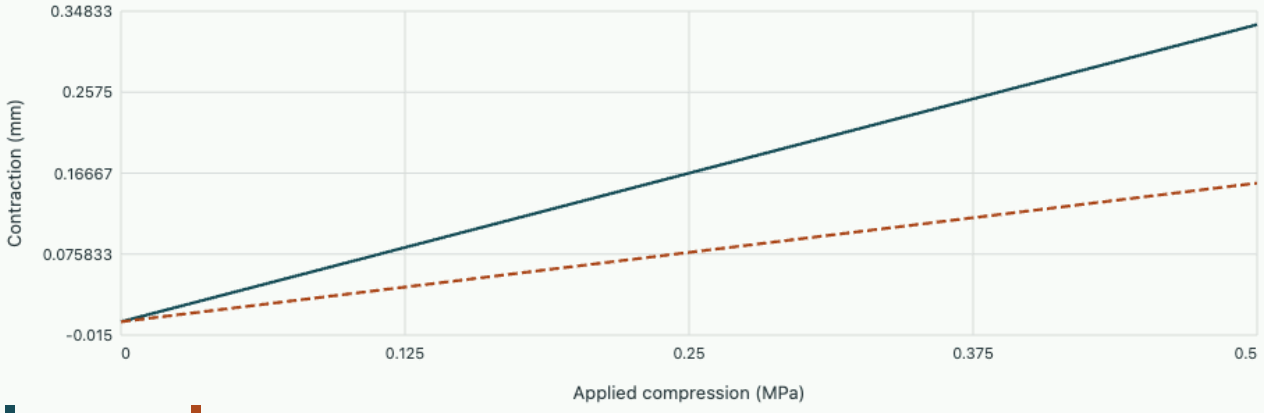
Start with zero Poisson ratio. Explain why the two compression curves coincide. Raise the ratio to 0.45 and read the force required for equal movement in the two tests. Which part of the constitutive law causes the slopes to separate?

Open this experiment online

Try the calculation

Young's modulus (MPa) **30**

Poisson ratio **0.4**



At 0.30 MPa the 20 mm specimen contracts by 0.200 mm freely and 0.093 mm in the sleeve. The sleeve modulus is 64.29 MPa.

Tutorial · identify a Poisson effect from two tests

An axial test gives $E = 24$ MPa. A hydrostatic test gives $K = 40$ MPa. Infer the Poisson ratio and predict the axial stress required for 0.5% compression in an ideal sleeve. Which additional measurement would reveal slipping or a clearance in the sleeve?

▼ **Reconstruct the constraint**

Solving $E = 3K(1 - 2\nu)$ gives $\nu = 0.40$. The constraint modulus is

$$24(0.60)/(1.40 \times 0.20) = 51.43\,\mathrm{MPa}.$$

For $\epsilon_z = -0.005$ the stress is –0.257 MPa. The ideal lateral strain is zero everywhere in the uniform test section. A transverse displacement measurement along the midsection tests this condition directly. Near the ends the same reading can include fixture effects.

Further reading · A. F. Bower, Applied Mechanics of Solids, §3.2, linear elastic material behaviour

Chapter 15 · Radial displacement and hoop stress in a thick ring

As a soft sleeve presses outward on a ring, the inside and outside radii move by different amounts. A cut through the thickness reveals why replacing the entire wall by one hoop stress loses information.

Let one radius displace by a small amount

Take a long annular section or a thin annular disk loaded uniformly around its circumference. Here use the disk idealisation with free faces and plane stress. Let a, b be its inner and outer radii. Axial displacement effects near contact edges are excluded. If the material point at r moves outward by $u(r)$, a radial line length dr changes by du, while a circumference $2\pi r$ changes by $2\pi u$. Dividing each change by the original length gives

$$\epsilon_r = u'(r),$$

$$\epsilon_\theta = u/r,$$

$$\sigma_r = \frac{E}{1-\nu^2}\left(u' + \nu\frac{u}{r}\right),$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu u'\right).$$

On a small annular wedge the two radial faces have different areas. The two hoop tractions also have a net radial component, since their directions differ by the angle of the wedge. Resolving radial forces and dividing by its volume leaves $d\sigma_r/dr + (\sigma_r - \sigma_\theta)/r = 0$. Inserting the material equations eliminates both stresses.

$$u'' + \frac{u'}{r} - \frac{u}{r^2} = 0,$$

$$u = C_1 r + C_2/r.$$

To see where the two terms come from, try $u = r^n$. All terms scale as r^{n-2}; their coefficients add to $n^2 - 1$. Both $n = 1$ and $n = -1$ are admissible because the domain excludes $r = 0$. For a solid disk the singular term would have to vanish at its centre.

Use pressure as a radial traction

The radial and hoop stresses take the forms $A - B/r^2$ and $A + B/r^2$. Let compressive inner contact pressure be p, and the outside be free. The normal stress conditions are $\sigma_r(a) = -p$, $\sigma_r(b) = 0$. These two equations fix both constants.

$$A = \frac{p a^2}{b^2 - a^2},$$

$$B = \frac{p a^2 b^2}{b^2 - a^2},$$

$$u(r) = \frac{(1-\nu) A r + (1+\nu) B / r}{E}.$$

Stress depends on geometry and the applied tractions. Displacement additionally depends on stiffness. The outer surface can therefore carry tensile hoop stress while having zero radial traction. A strain gauge aligned around that surface responds to the hoop direction.

Worked example · compare inside and outside readings

Take $a = 20,\ b = 40$ mm, $p = 3.0$ MPa, $E = 70{,}000$ MPa and $\nu = 0.30$. The constants are $A = 1.00$ MPa and $B = 1600$ MPa mm². Inner hoop stress is

$$1 + 1600/400 = 5.00\,\mathrm{MPa}.$$

Outer hoop stress is 2.00 MPa. The outer radial displacement is

$$40(2/70000) = 0.001143\,\mathrm{mm}.$$

At the inner wall $\epsilon_\theta = (5 - 0.30(-3))/70000$; multiplying by 20 mm gives 0.001686 mm. A uniform dilation fitted to the outer reading would predict the wrong opening at the inner face.

Worked example · why a thin-wall balance still gets a resultant right

Cut the same ring along a diameter. The mean hoop stress over the wall must carry the projected inner contact force. Integrating the distribution gives $\int_a^b (A + B/r^2)dr = A(b-a) + B(1/a - 1/b) = 60$ N/mm of axial depth. Dividing by wall thickness gives 3.00 MPa, precisely $pa/(b-a)$. That average underpredicts the inner peak by 40%. Global balance is satisfied in both descriptions. A local comparison requires the radial distribution.

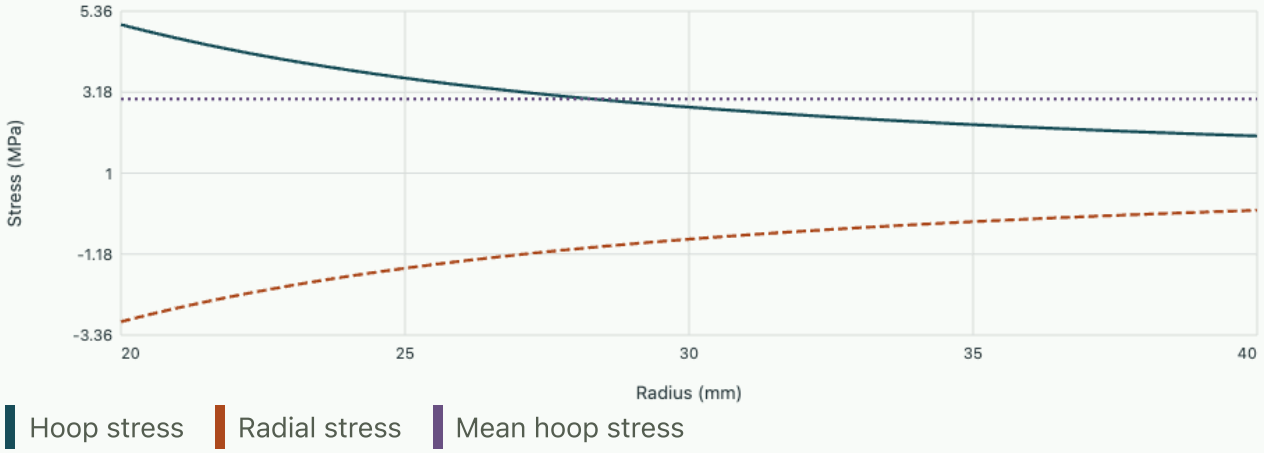
Keep the inside pressure fixed and increase the outer radius. Track both the inner hoop stress and the wall average. Before inspecting the thin-wall end of the slider, predict which pair of stress estimates will draw closest in relative terms.

Open this experiment online

Try the calculation

Outer radius (mm) 40

Inner pressure (MPa) 3



The inside hoop stress is 5.00 MPa. The wall average is 3.00 MPa, while the radial stress vanishes at the outer boundary.

Tutorial · recover contact pressure from an outer strain

For the same ring, an outer circumferential gauge records 50 microstrain. Infer p and the inner hoop stress. What geometric assumption lets one gauge determine this field?

▼ **Invert the surface relation**

At the free surface $\sigma_\theta(b) = E\epsilon_\theta = 3.50$ MPa. As $\sigma_r(b) = 0$, the hoop value is

$$2A = 2pa^2/(b^2 - a^2) = 2p/3.$$

Thus $p = 5.25$ MPa and $\sigma_\theta(a) = 8.75$ MPa. Circumferentially uniform contact and known radii reduce the unknown traction to one number. A point load or an out-of-round sleeve would require more observations.

Further reading · A. F. Bower, Applied Mechanics of Solids, §4.1, axisymmetric elasticity

Study the full derivation

Contact pressure between elastic cylinders · Stress, deformation and structural measurement

Two polished cylindrical rollers touch along a narrow strip. Dividing load by a nominal full face area gives little indication of the actual pressure. Use the local shape of the gap and elastic half-space compliance

to obtain the pressure distribution, and decide whether a measured contact width can determine material stiffness.

Chapter 16 · Creep, relaxation and what a holding test measures

Two polymer specimens can show the same initial force and quite different readings ten minutes later. Change the quantity held fixed and the same material reveals a different time scale.

Connect a spring to a dashpot and keep track of the variables

An ideal spring obeys $\sigma = E\epsilon_s$. A linear dashpot obeys $\sigma = \eta\dot{\epsilon}_d$, where the viscosity η has units Pa s. These elements describe recoverable storage and dissipative flow. Start with a Maxwell element by putting them in series. Both carry the same stress and their strains add, so differentiating total strain gives

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}.$$

A fixed strain sets the left side to zero. Stored spring deformation slowly transfers to the dashpot while total extension stays fixed, producing a decaying stress. Under a constant nonzero stress the dashpot instead accumulates strain indefinitely. This second prediction makes the fluid-like long-time behaviour of the Maxwell element easy to test.

Add an equilibrium load path

Place a spring of modulus E_0 parallel to the Maxwell element and call the Maxwell stress s. Both branches now have the same total strain. The measured stress is $E_0\epsilon + s$. Replace s by $\sigma - E_0\epsilon$ in the series equation and gather the derivatives:

$$\dot{\sigma} + \frac{E_1}{\eta}\sigma = (E_0 + E_1)\dot{\epsilon} + \frac{E_0 E_1}{\eta}\epsilon.$$

In a sudden small strain step of magnitude ϵ_*, a finite dashpot cannot change strain instantaneously, so both springs respond at first. Holding the extension then allows the Maxwell stress to relax to zero. Conversely, a sudden stress step loads both springs at first but ends with the equilibrium spring carrying the complete applied stress. Solving the constant-coefficient equations with these initial and final conditions gives

$$\sigma(t) = \epsilon_*[E_0 + E_1 e^{-t/\tau_r}],$$

$$\tau_r = \eta/E_1,$$

$$\epsilon(t) = \frac{\sigma_*}{E_0} + \left[\frac{\sigma_*}{E_0 + E_1} - \frac{\sigma_*}{E_0}\right] e^{-t/\tau_c},$$

$$\tau_c = \frac{\eta(E_0 + E_1)}{E_0 E_1}.$$

The two exponential times differ because under load control the equilibrium spring gradually takes a different share of the force. At $t = 0$ the effective modulus is $E_0 + E_1$ in either test.

Worked example · identify parameters from one record

Impose 1% strain. Stress starts at 0.90 MPa and eventually settles at 0.30 MPa. The final value gives $E_0 = 30$ MPa and the drop gives $E_1 = 60$ MPa. Suppose the excess above the plateau falls to $1/e$ of its initial value after 40 s. The viscosity is then $60(40) = 2400$ MPa s. After 80 s the stress prediction is

$$0.30 + 0.60 e^{-2} = 0.3812 \,\mathrm{MPa}.$$

Subtracting the plateau before fitting a logarithm is essential: the logarithm of the total stress has a changing slope.

Worked example · make a prediction the fit did not use

In an independent test impose a stress of 0.45 MPa. Strain jumps to $0.45/90 = 0.005$ and approaches $0.45/30 = 0.015$. The predicted creep time is

$$40(90/30) = 120 \,\mathrm{s}.$$

After one minute the prediction is

$$0.015 - 0.010 e^{-0.5} = 0.00893.$$

A model that used the 40 s relaxation time for creep would yield 0.01277, readily distinguishable if the load has settled. Both experiments should use the same temperature because molecular mobility, and therefore the viscosity, can be highly temperature dependent.

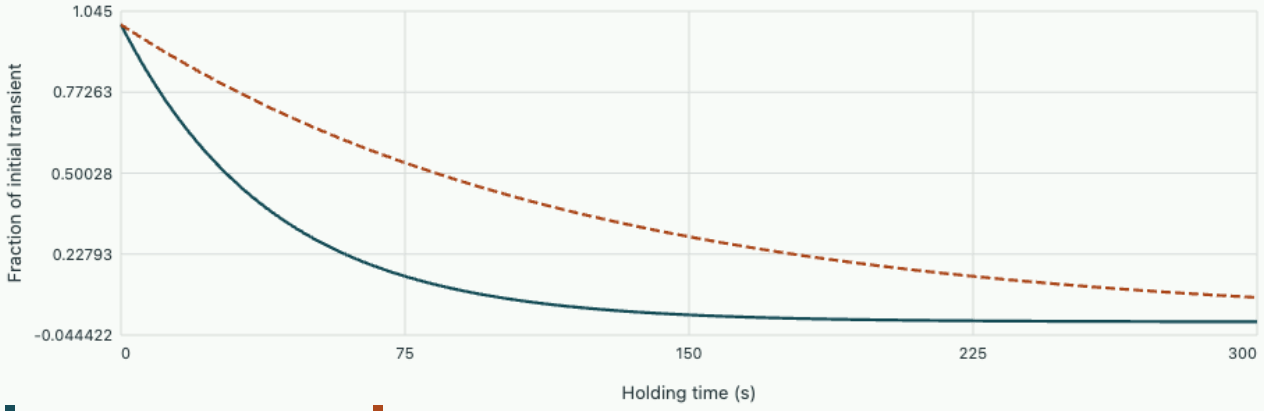
Double viscosity while leaving both spring moduli fixed. Predict which curve reaches its original one-minute value after two minutes. Next change the parallel spring alone and use the normalized curves to distinguish the two characteristic times.

Open this experiment online

Try the calculation

Viscosity (MPa s) **2400**

Parallel spring modulus (MPa) **30**



Relaxation excess stress Creep strain still to develop

The relaxation time is 40.0 s and the creep time is 120.0 s. At 1% fixed strain the stress after 80 s is 0.381 MPa.

Tutorial · interrupt a relaxation experiment

Release the prescribed 1% strain instantaneously back to zero after 40 s. Calculate the stress immediately before and after release. What should happen over a further 40 s while zero extension is held?

▼ **Track the spring jumps**

Before release the stress is

$$0.30 + 0.60/e = 0.5207\,\mathrm{MPa}.$$

At the abrupt negative strain step the dashpot strain cannot jump. Both springs together change stress by $-0.01(90) = -0.90$ MPa, leaving –0.3793 MPa. At zero total strain the equilibrium spring carries zero and the remaining compressive Maxwell stress relaxes with the original 40 s time. It reaches –0.1395 MPa one relaxation time later.

Further reading · A. F. Bower, Applied Mechanics of Solids, §3.6, small-strain viscoelasticity

Chapter 17 · How a continuum balance becomes a finite-element stiffness matrix

A single bar element can predict an end displacement exactly and still miss the stress field inside. The difference follows directly from the balance it is asked to satisfy.

Turn a differential balance into work against a test displacement

Let a straight small-strain bar of length L have area A, Young's modulus E, displacement $u(x)$ and distributed axial load $q(x)$ in N/m. Tensile axial resultant is $N = EAu'$. Force balance on a slice gives $N' + q = 0$. Clamp the left end and prescribe a right-end force P. Multiply the balance by any smooth virtual displacement v satisfying $v(0) = 0$, integrate along the bar, and integrate the derivative of N by parts.

$$\int_0^L EA\,u'v'\,dx = P\,v(L) + \int_0^L qv\,dx.$$

The boundary term at the clamp disappears because that displacement is prescribed. The unknown reaction can later be recovered by equilibrium. This formulation requires only first derivatives of trial displacements. Piecewise linear displacements are therefore admissible even though their slopes may jump at internal nodes.

Compute the stiffness rather than memorising it

On an element from $x = 0$ to h, interpolate the two nodal displacements linearly:

$$u = (1 - x/h)u_1 + (x/h)u_2.$$

Let the test displacement have the same two basis functions. A virtual change of the first or second node gives the corresponding row of the balance. Both derivatives are constant with opposite signs.

$$B = \begin{bmatrix} -1/h & 1/h \end{bmatrix}, \qquad K_e = \int_0^h B^T EAB\,dx = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

For a constant distributed load, each basis function integrates to $h/2$, so each node receives equivalent load $qh/2$. The two rows of stiffness sum to zero. A uniform translation stores no strain energy. At a shared node the neighbouring virtual-work contributions add because both elements refer to the same unknown displacement.

Worked example · compare a nodal value with the stress in a cell

Take a 1 m bar with $EA = 2.0$ MN, uniform $q = 1200$ N/m and free right end. Exact balance integrated from $N(L) = 0$ gives

$$N = q(L - x).$$

Integrating strain from the clamp produces $u = q(Lx - x^2/2)/(EA)$. The free-end displacement is 0.300 mm. With a single element the free row of the discrete system has stiffness EA/L and load $qL/2$, giving the same displacement. Yet its piecewise constant axial force is

$$EAu(L)/L = 600\,\mathrm{N}.$$

The exact force ranges from 1200 N at the clamp to zero at the free end. The numerical force represents the mean over the cell.

Worked example · see what doubling resolution changes

Divide the bar into two equal elements. The middle node receives 600 N from both halves together, while the free node receives 300 N. The right element carries 300 N. Equilibrium of the middle node then puts 900 N in the left element. Their extensions are $900(0.5)/(2 \times 10^6) = 0.225$ mm and 0.075 mm. Nodal displacements agree with the exact quadratic at both nodes. The largest one-sided stress-resultant error within either element is now 300 N instead of 600 N. Comparing only tip displacements would hide this improvement.

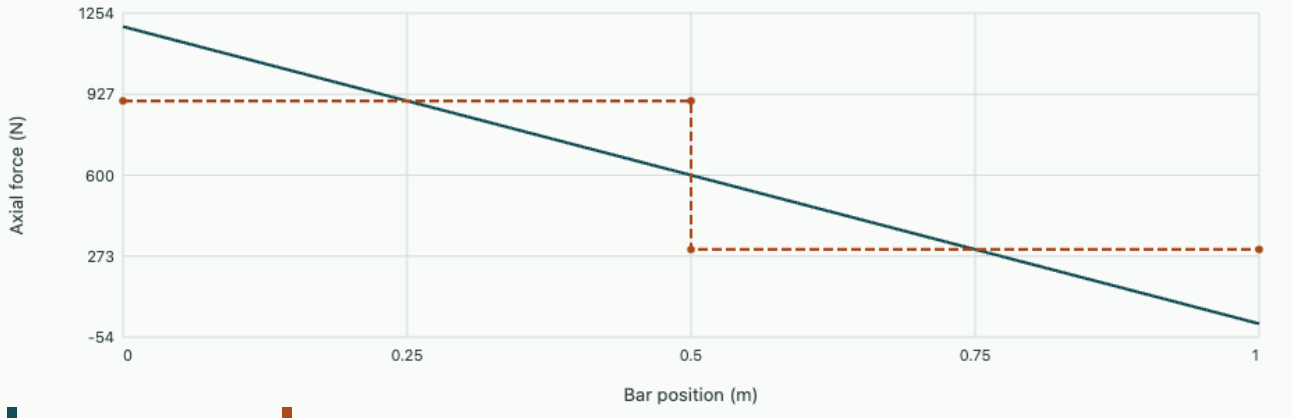
Double the number of cells until the force error is below 100 N for the baseline load. Inspect the values on both sides of an element boundary and explain why agreement in tip displacement alone could have ended this refinement study too early.

Open this experiment online

Try the calculation

Number of elements **2**

Distributed load (N/m) **1200**



The tip displacement is 0.300 mm on every mesh here. Maximum axial-force error is 300.0 N with 2 elements.

A related OneSimulation calculation

Compare the bar's nodal displacement with a OneSimulation truss assembled from axial members. Run the area-doubling case and check the predicted inverse-area change in displacement.

Run the truss stiffness comparison in OneSimulation

Tutorial · find the support reaction from the discarded row

For the two-element calculation, recover the left reaction using the assembled loads and displacement. Then derive an expression for the maximum axial-force error with n equal elements. Why can two different mesh studies give two different conclusions?

▼ **Evaluate the clamp row and the local profile**

The left internal nodal force is -900 N. The left external distributed-load share is $+300$ N. Equilibrium requires an additional reaction -1200 N. On an element of width L/n the exact linearly varying force changes by qL/n; its constant midpoint approximation misses an endpoint by $qL/(2n)$. Tip displacement and maximum local force answer different engineering questions. Mesh convergence must therefore name its measured quantity.

Further reading · A. F. Bower, Applied Mechanics of Solids, §8.1, finite-element equations for elastic solids

Chapter 18 · Finding a displacement when no load acts in its direction

A displacement gauge measures the lateral travel of a truss joint under a vertical load. A force balance determines the bar forces, but cannot directly give the sideways motion. Introduce a dummy load, differentiate the stored energy and recover the gauge reading with its sign.

Link an external force coordinate to stored strain energy

For a linear elastic pin-jointed truss, neglect support movement and temperature changes and assume small displacements. Bar i has constant axial force N_i, length L_i, modulus E_i and area A_i. Integrating its force against its extension gives

$$U_i = N_i^2 L_i/(2E_i A_i).$$

Superposition makes each force linear in the external loads. A small increment of load changes the equilibrium configuration. For a linear elastic conservative system the complementary energy equals U and its derivative with respect to an independent applied force gives displacement along that force.

$$U = \sum_i \frac{N_i^2 L_i}{2E_i A_i},$$

$$\delta_Q = \frac{\partial U}{\partial Q} = \sum_i \frac{N_i n_i L_i}{E_i A_i},$$

$$n_i = \frac{\partial N_i}{\partial Q}.$$

The derivative n_i is the member force produced per unit trial load in the requested direction, with all other independent loads held fixed. Solve this trial equilibrium even if $Q = 0$ physically. Setting $Q = 0$ before differentiating erases its entire contribution. As a dimensional check, n_i is dimensionless when Q and N_i both use newtons, and the sum has dimensions of length.

Compute the two-bar flexibility matrix

A joint hangs from two overhead pins by straight bars slanting up-left and up-right, each at 45° to horizontal. Positive joint coordinates are rightward and downward. Let V be the applied downward force and H the rightward force; bars carry tension positive. Equilibrium gives

$$N_L = (V + H)/\sqrt{2},\ N_R = (V - H)/\sqrt{2}.$$

Define bar axial compliances $c_i = L_i/(E_i A_i)$. Insert the forces into the energy and differentiate with respect to H and V independently:

$$\begin{bmatrix} \delta_H \\ \delta_V \end{bmatrix} = \frac{1}{2} \begin{bmatrix} c_L + c_R & c_L - c_R \\ c_L - c_R & c_L + c_R \end{bmatrix} \begin{bmatrix} H \\ V \end{bmatrix}.$$

The cross coefficients are equal because the mixed derivatives of the stored energy commute. This is Maxwell–Betti reciprocity. A cross displacement can be negative even though both direct compliances and stored energy are positive. The eigenvalues of this particular flexibility matrix are c_L, c_R, corresponding to the two independent bar-extension directions.

Predict a sideways gauge reading and reverse the test

Take $c_L = 0.0020,\ c_R = 0.0040$ mm/N. At $V = 100$ N and $H = 0$, both members carry 70.71 N. The vertical displacement is 0.300 mm, and the horizontal displacement is −0.100 mm. The more compliant right bar stretches farther; the joint moves left and down to accommodate both bar extensions. Half the vertical load times its displacement gives 15 N mm of strain energy. Summing the two bar energies independently gives the same result.

Apply instead $H = 100$ N and $V = 0$, assuming the slender right bar remains stable under compression. The vertical reading is −0.100 mm and horizontal motion is +0.300 mm. Thus the lateral response to the vertical test predicts the vertical response to a lateral test of equal magnitude. Equalizing bar areas would remove the cross response if the lengths and moduli matched. In practice a pin with clearance or a cable that goes slack can break the assumed linear bilateral response.

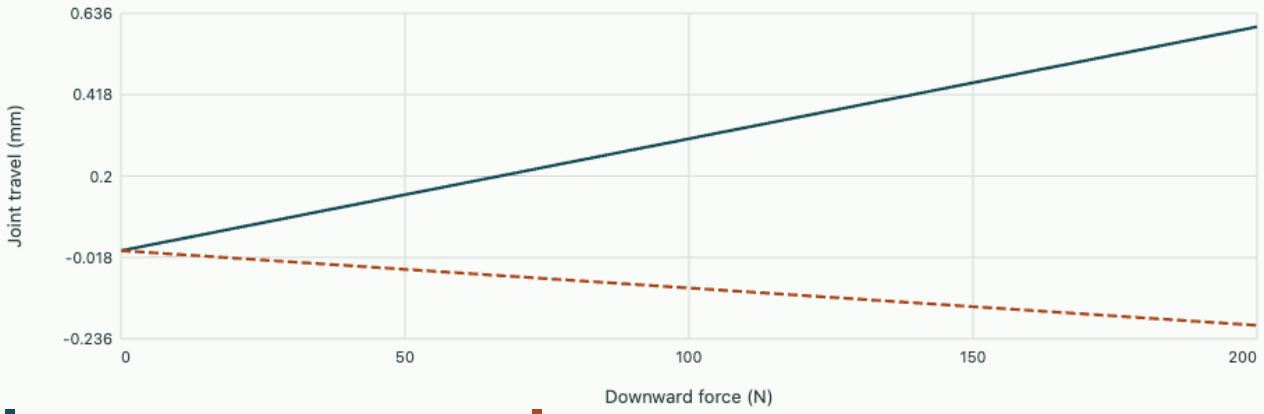
Exchange the left and right compliances. Before looking at the new graph, mark which readout should reverse sign and which should remain equal. Then set the two values alike and identify the symmetry that removes cross-coupling.

Open this experiment online

Try the calculation

Left bar compliance (µm/N) 2

Right bar compliance (µm/N) 4



At 100 N down the joint travels 0.300 mm down and -0.100 mm right. By reciprocity the latter also gives downward travel under 100 N right.

A related OneSimulation calculation

Use the course truss to test the same flexibility scaling on a larger assembly. Double every member area at fixed loads and geometry, then compare the joint displacement with the separately calculated chord force.

Run the OneSimulation truss experiment

Tutorial · identify the two member compliances from one load case

A 200 N vertical test gives +0.50 mm down and +0.10 mm right. Estimate each compliance and predict horizontal travel under simultaneous 60 N rightward and 100 N downward loading. How can one additional gauge reading test reciprocity?

▼ **Show the identification and loading prediction**

Direct compliance is $0.50/200 = 0.0025$ mm/N and cross compliance is $0.10/200 = 0.0005$ mm/N. Adding and subtracting gives $c_L = 0.0030,\ c_R = 0.0020$ mm/N. The predicted horizontal motion is

$$0.0025(60) + 0.0005(100) = 0.200\,\mathrm{mm}.$$

A 200 N rightward test should give +0.10 mm down. A disagreement after changing load direction suggests checking seating and the linear-elastic domain before assigning a single compliance to each bar.

Further reading · David Roylance, MIT 3.11 · truss displacements, strain energy and Castigliano's theorem

Chapter 19 · Why a bonded pair of strips bends under uniform heating

Two perfectly bonded strips would expand by different amounts if separated. After bonding they must share a continuous longitudinal strain field. Solve simultaneously for its mean and its curvature and recover stresses that can be present even when the entire strip is free.

Separate total strain into free and constrained parts

Consider a narrow long strip of width b and total thickness $2t$, with layers of equal thickness on either side of $z = 0$. The upper layer has expansion coefficient α_+, the lower α_-, and both have the same E here. Start stress-free, then impose a uniform temperature increase ΔT. With plane sections remaining plane, define signed curvature by writing longitudinal strain $\epsilon = \epsilon_0 + \kappa z$. Stress is proportional to the difference from the free thermal strain:

$$\sigma_\pm(z) = E[\epsilon_0 + \kappa z - \alpha_\pm \Delta T].$$

Perfect adhesion enforces continuity of longitudinal displacement at the interface. It permits longitudinal stress to jump there because the stress-free thermal strain changes discontinuously. Traction continuity across the horizontal interface concerns its normal and shear traction components. This beam model describes the middle portion away from free ends where interfacial shear transfers load between layers.

Use both a zero axial force and a zero moment

For an unconstrained strip no external force or moment acts on its section. The thickness integral of stress must vanish, and its first moment must also vanish. Because z is measured from the symmetric section midpoint, the odd integrals of $\epsilon_0 z$ and κz cancel. Separate the thermal integrals over the two layers:

$$2t\epsilon_0 = t(\alpha_+ + \alpha_-)\Delta T,$$

$$\frac{2t^3}{3}\kappa = \frac{t^2}{2}(\alpha_+ - \alpha_-)\Delta T.$$

Hence $\epsilon_0 = (\alpha_+ + \alpha_-)\Delta T/2$ and $\kappa = 3(\alpha_+ - \alpha_-)\Delta T/(4t)$. The higher-expansion side takes the greater total strain at large positive z. For unequal moduli, the two equilibrium equations still work but must retain E inside the integrals; symmetry no longer removes every strain–curvature coupling term.

A tiny curvature can coexist with measurable stress

Let $t = 0.50$ mm, $E = 100$ GPa, $\alpha_+ = 20$ and $\alpha_- = 10$ micrometres per metre per kelvin, with $\Delta T = 40$ K. The mean strain is 600 microstrain, curvature is 0.600 m⁻¹, and the radius of curvature is 1.67 m. Across a free 100 mm length with one end setting the reference tangent, the small-slope tip offset magnitude is

$$\kappa L^2/2 = 3.0\,\mathrm{mm}.$$

A free single layer with either coefficient would expand without this curvature.

At the interface the upper layer has stress $100\,\mathrm{GPa}(600-800)10^{-6}=-20$ MPa and the lower +20 MPa. At the outer surfaces the bending term contributes ±300 microstrain; the upper stress becomes +10 MPa and the lower −10 MPa. If a fixture holds the same strip flat while allowing average extension, the mean strain is still 600 microstrain but the layer stresses are uniform ±20 MPa. The fixture must now supply a restraining moment. Compare the stress fields at fixed temperature before interpreting a strain-gauge reading as stress.

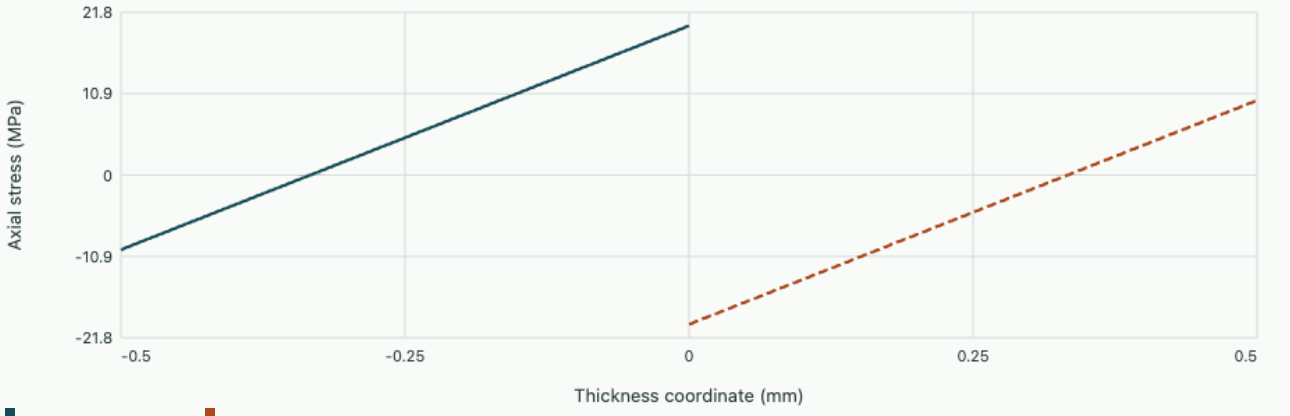
First set both expansion coefficients to 10 and heat the strip. Next restore the mismatch and cool by the same temperature magnitude. Locate stress reversals at the interface and within each layer. Which of these reversals would a gauge on an external face detect?

Open this experiment online

Try the calculation

Temperature change (K) **40**

Upper expansion coefficient (µm/m/K) **20**



Free curvature is 0.600 m⁻¹. With a 100 mm reference tangent, the small-slope tip offset is 3.00 mm. The two interface stresses are 20.0 and -20.0 MPa.

Tutorial · infer thermal mismatch from shape

For $t=0.40$ mm a free 80 mm strip bends 1.20 mm from its initial tangent after a 30 K rise. Estimate the expansion-coefficient difference and the upper-interface stress magnitude for $E=100$ GPa. State what independent measurement identifies the average expansion coefficient.

▼ **Recover mismatch and mean expansion separately**

The curvature magnitude is $2(0.00120)/(0.080)^2=0.375$ m⁻¹. Inversion gives $|\Delta\alpha|=4t\kappa/(3\Delta T)=6.67\times10^{-6}$ K⁻¹. The interface stress magnitude is

$$E|\Delta\alpha|\Delta T/2 = 10.0\,\mathrm{MPa}.$$

Curvature alone determines only the difference. Measuring strain on both external surfaces and averaging removes the bending term and gives the common mid-plane strain from which the mean expansion coefficient follows.

Further reading · David Roylance, MIT 3.11 · laminate force and moment resultants and thermal strains

Chapter 20 · How a beam spreads a point load through an elastic support

A long rail rests on a strip of elastic material. An instrument foot applies a vertical load near the middle. Which part of the strip carries that load? If the rail were completely rigid the answer would depend on its full length. A flexible rail selects a local length through the competition between bending and support compression. Derive that length, calculate a displacement that can be measured by a dial gauge, and recover the answer independently by minimising the energy of a discrete beam.

Replace many small supports by a force per length

Let x measure distance along the beam and $w(x)$ displacement positive downward from the unloaded configuration. A span dx moves downward by w and experiences an upward foundation reaction of magnitude $kw\,dx$. The coefficient k has units N/m². If a manufacturer gives a compression modulus per area and per deflection in N/m³, multiply it by the effective contact width before using it in this one-dimensional description. An error here changes the apparent spreading length as well as the deflection. Doubling an actual pad width also changes how the contact pressure is distributed across the rail, which the line model cannot resolve.

A Winkler foundation is a set of independent local springs in a continuum limit. Adjacent springs communicate through the bending rail, while the substrate itself transmits no horizontal shear between their tops. This is appropriate for a row of distinct pads represented on length scales large compared with their spacing. A continuous elastic half-space can spread load inside the substrate as well. Treating both systems as the same foundation merely because they give the same displacement under one foot can produce different predictions for two feet placed some distance apart.

For a slender, uniform Euler–Bernoulli rail take the change of curvature to be w''. The bending energy per unit length is $EI(w'')^2/2$. The elastic support stores $kw^2/2$ per unit length. With a point load P at the origin, the total potential energy is

$$\Pi[w] = \frac{1}{2}\int_{-\infty}^{\infty} \left[EI(w'')^2 + kw^2\right] dx - Pw(0).$$

Vary the shape by adding a small test displacement η that vanishes sufficiently far away. The variation of the bending term is $\int EIw''\eta''dx$. Integration by parts twice moves the derivatives onto EIw''. Equating the remaining interior coefficient of η to the applied work gives

$$EIw'''' + kw = P\delta(x).$$

The right-hand side is a load per length; its integral over any small interval containing zero is the finite force P. The free equation applies everywhere away from the foot. A narrow real foot agrees with the point-load idealisation only while its footprint is short relative to the calculated spreading length.

Build the decaying deflection and determine its amplitude

The spatial response satisfies $EIr^4 + k = 0$ for a trial field $w = e^{rx}$. Define $\beta = (k/4EI)^{1/4}$. On the positive half-line the roots with negative real part are $-\beta \pm i\beta$. Decay and oscillation therefore have the same spatial scale. Symmetry about the load requires zero slope at the origin. Applying this condition to the combination of sine and cosine gives $w = Ae^{-\beta x}(\cos \beta x + \sin \beta x)$ for $x > 0$.

To find A, integrate the beam equation across the loading point. The support force over this short interval vanishes as the interval shrinks, leaving a jump in EIw''' equal to P. Reflection symmetry makes the third derivative odd. Its right-hand value is $4A\beta^3$, so $8EI\beta^3 A = P$. Substituting the definition of β produces the same amplitude written as $P\beta/(2k)$. Reflecting the one-sided shape gives the complete result:

$$w(x) = \frac{P\beta}{2k} e^{-\beta|x|}\big(\cos \beta|x| + \sin \beta|x|\big),$$

$$\ell = \beta^{-1}.$$

A units check provides a quick guard against interchanging the line and surface moduli. The ratio EI/k has units m⁴, so ℓ is a length. At fixed k a stronger rail spreads the reaction more widely. At fixed EI a harder support makes the footprint narrower yet reduces the central displacement, since $w(0)$ varies as $k^{-3/4}$. These exponents also tell us which property can be inferred most reliably from which observable. A deflection reading alone mixes the effects of the foundation and the bending stiffness.

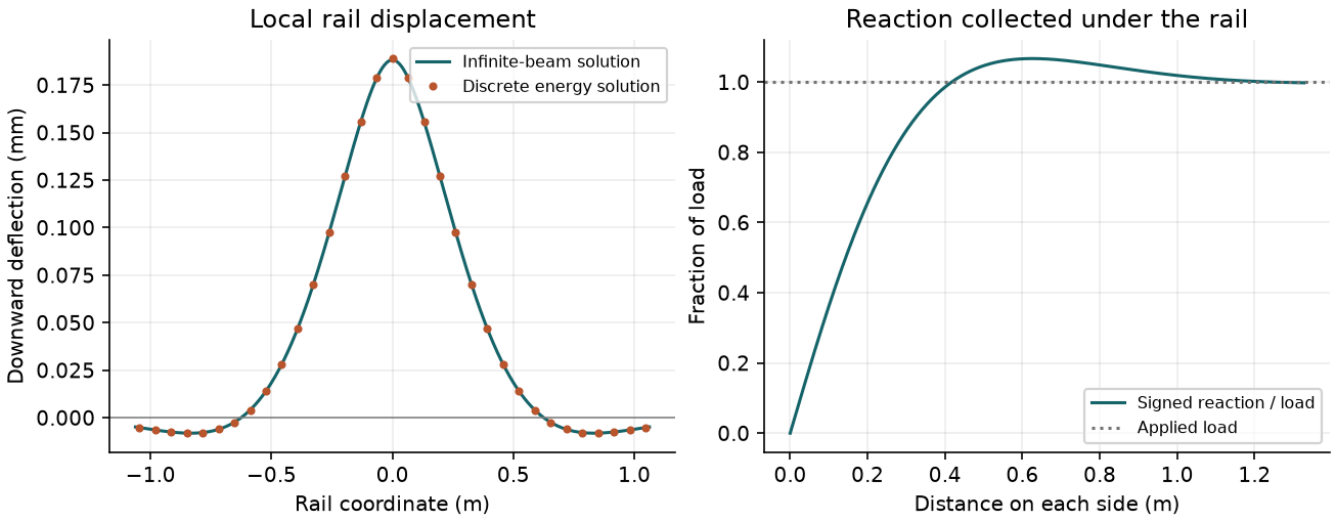
Open editable notebook Download .ipynb

Integrate the footprint that a row of load cells would measure

Imagine force sensors beneath a symmetric interval $[-d, d]$. The reading is a signed integral of kw. Transform it to $s = \beta x$, integrate the product of the exponential with the sine and cosine, and use the symmetry. The load fraction becomes

$$\frac{R(d)}{P} = 1 - e^{-\beta d}\cos(\beta d).$$

Its limiting value is one, as required by vertical equilibrium. At one spreading length the interval already carries about 80.1% of the applied force. At $d = \pi\ell/2$ it carries exactly the applied load. This does not mean every reaction beyond that position vanishes. Additional positive and negative reactions there integrate to zero. The cumulative curve reaches a maximum where w first crosses zero, at $\beta d = 3\pi/4$. These are distinct spatial locations and represent different observable events.



A displacement probe sees the beam shape on the left. Load cells summed outward from the centre see the signed reaction on the right. The small overshoot reveals the tensile part of the bilateral support model.

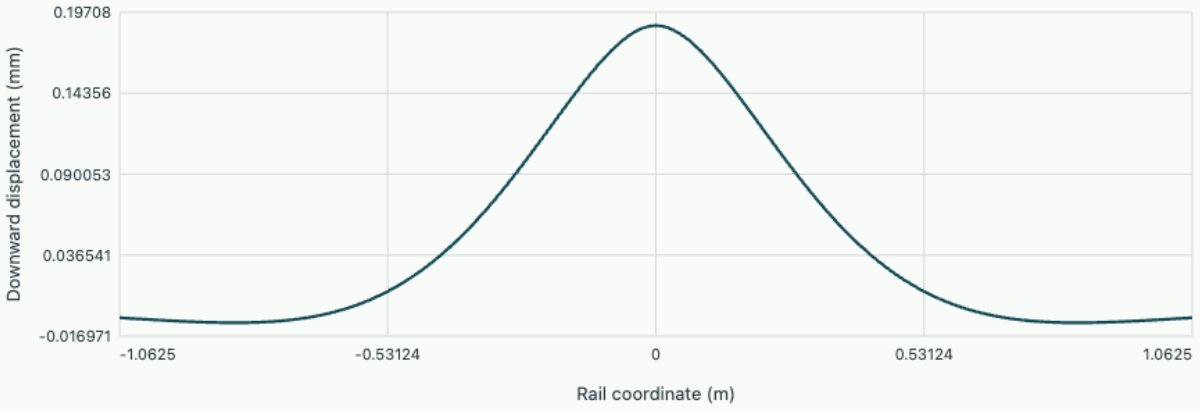
Interactive calculation · Foundation beam

Change the physical and numerical inputs of the notebook model. Compare the updated curves and balance checks with the recorded notebook figures. Each plot uses the same equations and assumptions documented in that notebook; its readout names any analytical or reduced-model reference.

Open this live calculation

Try the calculation

Foundation stiffness (MN/m²)	1.5
Point load (N)	150
Beam depth (mm)	20



Infinite-beam displacement

Decay length 0.266 m; central deflection 0.1882 mm. Within one decay length the signed foundation reaction is 0.801 of the applied force.

Download this example's editable notebook

For an aluminium rail 40 mm wide and 20 mm deep, choose $E = 70$ GPa and $k = 1.5$ MN/m². A central 150 N load gives $I = 2.667 \times 10^{-8}$ m⁴, $EI = 1867$ N m², $\ell = 0.2656$ m and $w(0) = 0.1882$ mm. Sensors extending 0.266 m on each side then measure approximately 120 N. The centre support intensity is $kw(0) = 282$ N/m. This is a line load; a rough mean pressure across the 40 mm width is 7.06 kPa. Resolving a nonuniform transverse pressure requires an additional contact model.

One can estimate a local coefficient without knowing the applied force precisely. Divide a neighbouring displacement by the central displacement. The amplitude drops out, leaving an equation for β from the measured ratio. At $x = \ell$ the ideal ratio is approximately 0.508. Once this length has been identified, combine it with the accurately calibrated centre displacement and the applied load to estimate $k = P/(2\ell w(0))$. Repeating the inference on either side of the foot tests the assumed symmetry.

Reconstruct the field by minimising a grid energy

Put equally spaced nodes on a long finite interval and impose zero displacement at the end supports. Replace each curvature by the second difference of the displacement divided by h^2. If D_2 contains these differences on the interior nodes, minimising the resulting quadratic energy yields

$$(EID_2^TD_2 + kI_d)\mathbf{w} = \mathbf{f}.$$

Here I_d is the identity matrix for the degrees of freedom. The central component of $\mathbf{f}$ is P/h because the discrete energy carries the integration weight h. Setting the central component to P would make the total force disappear under refinement.

The notebook puts the end supports eight decay lengths from the centre. With 65 interior unknowns the centre differs from the infinite-beam answer by about 1.44%. With 129 it differs by 0.38%. The finer result tests both a numerical approximation and a finite-domain approximation. To separate them, first hold the length fixed while refining, then extend the interval while holding h nearly fixed. The optional OneSimulation linear solver takes the energy matrix directly. Changing the algebraic solver and changing the physical boundary condition test different questions.

Predict the interference between two instrument feet

A long optical table carries equal 150 N loads at two feet spaced one decay length apart. Mark the rail under one of them and install a displacement gauge there. The first load acting alone produced 0.1882 mm of deflection. The second acts at a distance of one decay length, so its contribution at the gauge is the central value multiplied by $e^{-1}(\cos 1 + \sin 1) = 0.5083$. Superposing the two gives 0.2839 mm. The maximum need not occur exactly under either foot. At the midpoint each foot lies half a decay length away, so the midpoint displacement is

$$2(0.1882)e^{-1/2}(\cos 0.5 + \sin 0.5) = 0.3096\,\mathrm{mm}.$$

Two locally smaller contributions produce a larger combined movement between the load points.

To reproduce the calculation on the grid, place two force entries in the load vector. If a foot lies between nodes, distribute the force linearly between its two neighbours. For a foot a fraction ξ of the way across an element of length h, give the left and right entries $(1-\xi)P/h$ and $\xi P/h$. Their weighted sum recovers the force, and their first moment recovers its position. Simply moving each foot to a nearby node changes the experimental geometry each time the mesh changes. For closely spaced loads that can masquerade as a change in stiffness. To test superposition in a real assembly, record each foot alone, sum the readings at the same gauge locations and compare with applying both weights together.

Exercises on inference, contact and material choice

1. Double the section depth at fixed width. Predict the new decay length and central deflection before running the notebook.

▼ **Work through the solution**

The second moment increases by eight. The length increases by $8^{1/4} = 1.682$, to 0.447 m. Since the same total force now spreads farther, $w(0)$ decreases by the same factor, to 0.112 mm. A simple unsupported beam with fixed length would exhibit a different stiffness exponent. Its support positions supply an independent length, while the foundation here selects the active span.

---

2. A free-standing rail on an unbonded pad appears to have a very small upward deflection far from the loaded foot. Can you use the negative foundation pressure predicted there?

▼ **Work through the solution**

For a support that can only push, a negative total pressure is impossible. With no preload the bilateral solution first requests tension at a distance $3\pi\ell/4$, about 0.626 m. A unilateral calculation must determine the contact set and impose zero pressure where separation occurs. With a substantial compressive preload, a negative *incremental* reaction may remain compatible with contact. Check the total pressure before judging that case.

---

3. Only the central displacement and its half-width are available. How could the observations distinguish a soft rail from a soft foundation?

▼ **Work through the solution**

Fit the normalised spatial profile to obtain ℓ. A measured half-width corresponds to the positive root of $e^{-s}(\cos s + \sin s) = 1/2$ on the first lobe, with s close to one. The observed amplitude gives

$$k = P/(2\ell w(0)).$$

Finally obtain $EI = k\ell^4/4$. A single displacement supplies one equation for two unknowns. A spatial measurement supplies the missing independent relation.

Read the foundation boundary-value problem

The NPTEL mechanics chapter on beams on elastic foundations develops the differential equation and matching conditions for other loading configurations. Compare a finite beam end with the decay condition imposed in this calculation.

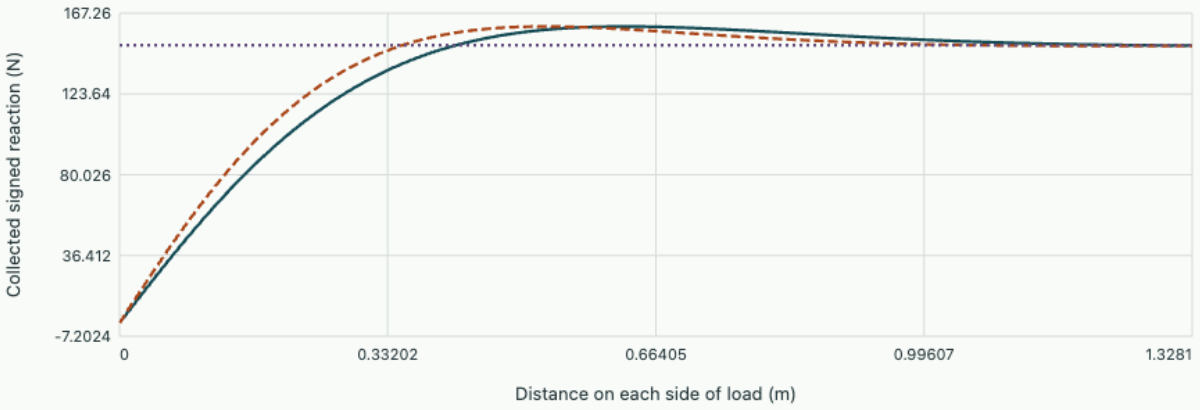
Interactive calculation · where the foundation carries the load

Change stiffness and rail depth to alter the decay length, then integrate the signed foundation reaction. Compare the cumulative force with the applied point load.

Open this live calculation

Try the calculation

Foundation stiffness (MN/m²)	1.5
Point load (N)	150
Beam depth (mm)	20



Selected foundation | Twice foundation stiffness | Applied load

Within one decay length: 120.19 N. The signed cumulative reaction can exceed the applied load because the bilateral linear foundation also pulls downward where the beam lifts; a contact-only bed requires another model.

Download the editable notebook and compare both observables

Problem sheet 7 · Calibrating a compression fixture through three independent readings

A teaching rig uses a hydraulic piston to compress a circular polymer pad. Its readouts are a hoop strain gauge on an annular calibration disk and a displacement encoder on the crosshead. Predict what each instrument reads and diagnose a deliberately inconsistent run.

1 · Infer the applied force from a radial strain

Fluid at gauge pressure p acts independently on the inner circumference of a thin calibration disk and on a frictionless piston of area 100 mm². The disk has inner radius 20 mm, outer radius 40 mm, free flat faces and modulus 70 GPa. An outer circumferential gauge reads +9.52 microstrain. Find the pressure, piston force and inner hoop stress. Explain why the piston area and the disk area enter different steps.

▼ **Recover pressure and close the disk force balance**

The outer radial traction is zero. In plane stress, the strain there gives

$$\sigma_\theta(b) = E\epsilon_\theta = 0.6664\,\mathrm{MPa}.$$

In the axisymmetric disk $\sigma_\theta(b) = 2pa^2/(b^2 - a^2) = 2p/3$, so $p \simeq 1.00$ MPa. Piston thrust is $pA_p = 100$ N. Inner hoop stress is

$$p(a^2 + b^2)/(b^2 - a^2) = 1.67\,\mathrm{MPa}.$$

The disk senses pressure through its curved surface. Its wall geometry enters the gauge calibration, while the piston converts that common fluid pressure into an axial force.

Set the inner pressure to 1 MPa and check both boundary stresses

2 · Separate pad compression from fixture compliance

The pad has area 400 mm², initial thickness 20 mm, $E = 30$ MPa and $\nu = 0.40$. Its sides are constrained by a close-fitting rigid cylindrical sleeve. A steel tie bar of length 300 mm and axial stiffness product $EA = 2.00$ MN is in the load path. The encoder adds pad compression and bar elongation in magnitude. Predict crosshead displacement, then repeat for free lateral expansion.

▼ **Resolve both compliances at the same measured force**

The compressive stress magnitude is

$$100/400 = 0.250\,\mathrm{MPa}.$$

In the sleeve the effective modulus is $M = E(1 - \nu)/[(1 + \nu)(1 - 2\nu)] = 64.29$ MPa, so compression is 0.0778 mm. Bar elongation is

$$FL/(EA) = 0.0150\,\mathrm{mm}.$$

The predicted crosshead motion is 0.0928 mm. Free sidewalls give 0.1667 mm pad compression and 0.1817 mm total. Each component carries 100 N. Adding its displacement, rather than averaging its modulus with the other component, implements the series load path.

Compare sleeved and free compliance at the selected material constants

3 · Interpret an anomalous displacement

With the same pressure reading the encoder now moves 0.125 mm. Subtract the calibrated bar extension and infer a pad modulus under the observed constraint. Could a pressure offset alone explain the readings if the strain gauge and the bar calibration are both trusted? Describe a second measurement that discriminates between partial lateral restraint and extra contact compliance.

▼ Diagnose what the two signals identify

The inferred pad motion is $0.125 - 0.015 = 0.110$ mm, giving apparent modulus $0.250(20)/0.110 = 45.5$ MPa. This value lies between the free and ideal-sleeve results. A pressure change large enough to explain the excess motion under the ideal model would also change the hoop strain. Incomplete lateral restraint could increase true pad strain. A settling contact could instead add displacement to the encoder without changing the pad strain. A transverse mid-height gauge tests the sleeve assumption; a direct axial pad extensometer separates pad deformation from contact settlement. A model fit to the encoder alone cannot identify both effects.

---

4 · Change only the load level

Double the pressure within the same linear elastic idealization. Predict the outer strain, encoder displacement for the sound sleeved assembly, and the ratio of bar displacement to total motion. State an observation that would contradict the model.

▼ Predict the second operating point

The outer strain doubles to about 19.0 microstrain, thrust reaches 200 N and total motion doubles to 0.1856 mm. The bar accounts for $0.0150/0.0928 = 16.2\%$ at both loads. A changing fraction from direct extensometer measurements would point to a load-dependent stiffness, slip or contact seating. The proportional predictions hold while geometry, constraint and material response remain linear.

Problem sheet 8 · What stress does a curved flexure actually carry?

A thin rectangular strip is formed into a circular arc whose radial depth is no longer small compared with its radius. It is loaded by equal and opposite in-plane end couples in a laboratory bending fixture. Derive the neutral radius, determine which face sees the larger stress and design a strain-gauge check that distinguishes initial curvature from accidental axial loading.

Use the unequal lengths of the original fibres

Consider a homogeneous circular segment of inner radius $r_i = 30$ mm, outer radius $r_o = 50$ mm and out-of-plane width $b = 10$ mm. Far from its ends assume cross-sections remain plane, radial stress is small and material response is linearly elastic. Why may a pure couple give a nonlinear stress distribution through the depth? Derive its form and impose zero axial force.

▼ **Derive the shape using axial equilibrium**

Two initially radial sections an angle $d\theta$ apart contain fibres with original lengths $r\,d\theta$. If the section rotation changes, geometry gives a change in arc length that is linear in radius within the plane-section model. Dividing by the original arc length leaves circumferential strain of the form $A + B/r$. For constant Young modulus stress has the same form. The inner fibres are initially shorter, so equal increments of end displacement need not give equal strains.

Write the stress as $\sigma = C(r_n/r - 1)$ with positive inner stress for the chosen moment direction. Its zero is the neutral radius. Zero resultant force requires $r_n \int_A dA/r = A_s$ where $A_s = b(r_o - r_i)$. For a rectangular section this gives

$$r_n = \frac{r_o - r_i}{\ln(r_o/r_i)} = 39.152\ \text{mm}.$$

The centroid is at 40 mm; the neutral fibre lies 0.848 mm inward of it. Placing the neutral axis at the centroid would violate axial force balance in this constitutive geometry. Reversing the applied couple reverses stress signs without changing the calculated neutral radius.

Enforce the bending couple and compare face stresses

For a 12 N m couple determine the stress at both radial faces and compare with the straight-beam formula. Choose moment sign explicitly and check that the integrated stress field can deliver the applied couple.

▼ **Use a moment integral about the centroid**

Take $M = \int_A (\bar{r} - r)\sigma\,dA$ so positive M gives inner tension. Because the axial resultant vanishes, this equals minus the integral of $r\sigma$. Substitution gives

$$M = CA_s(\bar{r} - r_n).$$

Put $e = \bar{r} - r_n$, obtain $C = M/(A_s e)$, and hence predict inner tension 21.59 MPa and outer compression 15.36 MPa in magnitude. A straight beam would have a centroidal second moment $bd^3/12$, with $d = r_o - r_i$, and equal face magnitudes $6M/(bd^2) = 18.00$ MPa. That estimate under-predicts the larger inner stress by 16.6 percent relative to the curved-beam result.

For steel of modulus 210 GPa the calculated circumferential gauge strains are approximately plus 103 and minus 73 microstrain. A gauge averaging over a long length near a fixture contact may not see the pure-couple region. Place it well clear of the ends and align it with the local tangent. The material yield check must use actual material properties and any local stress concentration; the smooth flexure model gives nominal elastic stresses.

Use the radius sweep and a pair of strain gauges

Predict how both face stresses change as the inner radius grows with the same depth, width and moment. Compare a gauge pair that records plus 110 and minus 66 microstrain with the predicted pair. Which simple unwanted load might explain the common shift?

▼ **Separate a strain difference from a strain offset**

For $d/r_i \ll 1$, expand the logarithm to find

$$e \simeq d^2/(12\bar{r}).$$

The stress law approaches the linear straight-beam distribution. The two formulas on the graph therefore approach one another as the radius increases without changing the physical applied moment. The pair's measured strain difference is 176 microstrain, essentially the predicted difference. Both measured readings exceed their predictions by about seven microstrain, consistent with a small common tensile strain or a common temperature offset. In a simple superposition a common mechanical stress of approximately 1.5 MPa corresponds to about 300 N on this section.

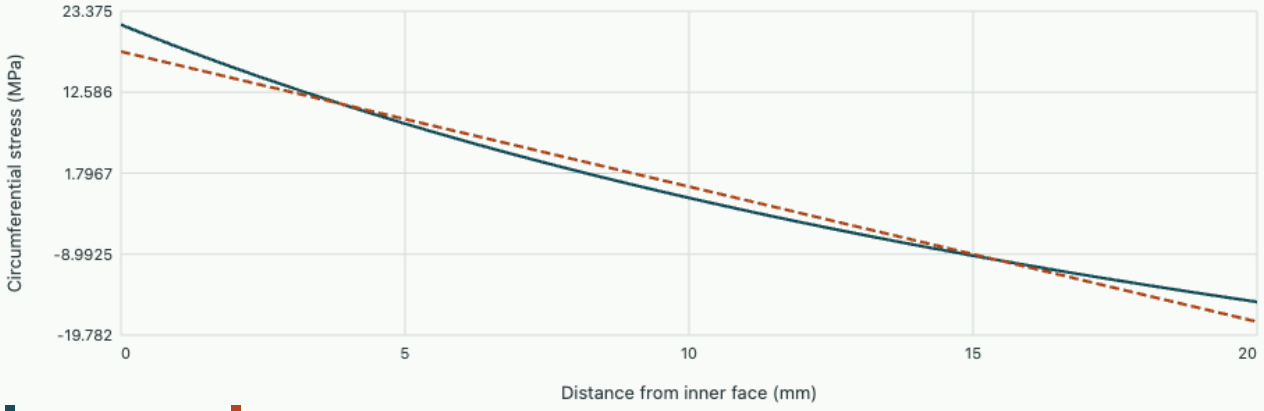
Open this experiment online

Try the calculation

Inner radius (mm) 30

Section depth (mm) 20

Applied moment (N m) 12



The neutral radius is 39.15 mm and peak inner stress is 21.59 MPa, compared with 18.00 MPa for a straight beam.

Decide which independent measurement would settle the discrepancy

A technician suggests increasing the moment until the two gauge traces agree with their predicted absolute values. Would this establish the model? Explain how to distinguish a parasitic axial force, thermal strain and fixture end effects with separate measurements.

▼ **Compare reversal and geometry controls**

Repeat at positive and negative moments and fit signed slopes and intercepts for each gauge. Ideal elastic bending slopes have opposite signs and unequal magnitudes fixed by radius. A common offset that persists at zero moment points toward preload, temperature or electronic zero; a common shift proportional to moment can indicate a force introduced by the loading fixture. Add an independent axial load cell or a temperature-compensated unstrained reference gauge as appropriate. Moving duplicate gauges around the arc tests whether the discrepancy localises near the end contacts.

The radial-stress approximation is strongest in a region of modest curvature far from applied forces and interfaces. At very large depth-to-radius ratio or close to a concentrated contact, compute a two-dimensional elasticity solution with compatible loading and compare circumference-averaged gauge predictions. Refinement of a one-dimensional curved-beam mesh alone cannot recover a missing contact stress or out-of-plane warping.

NPTEL Advanced Solid Mechanics · bending of curved beams

Notebook practice · How a beam spreads a point load through an elastic support

Stress, deformation and structural measurement

Start from balance, compatibility and material response. Work through structural predictions with alternative load or support conditions and assess the measurements needed to test them.

1. ## Torsion in a thin-walled two-cell section

   Two closed cells share one slender web. The applied couple gives only their combined torque, yet the common web can carry much less shear than either exterior wall. Derive the missing compatibility equation, calculate a twist and test which measurements distinguish the two circulations.

2. ## Contact pressure between elastic cylinders

   Two polished cylindrical rollers touch along a narrow strip. Dividing load by a nominal full face area gives little indication of the actual pressure. Use the local shape of the gap and elastic half-space compliance to obtain the pressure distribution, and decide whether a measured contact width can determine material stiffness.

3. ## Springback and residual stress after plastic bending

   After a rectangular strip has been bent beyond first yield, removing its end couple does not return it to its original curvature. Work out the loading and unloading paths fibre by fibre. Then explain why an unloaded strip can contain a substantial stress field with zero net force and zero net bending moment.

4. ## Why an open channel twists under a transverse load

   A channel can twist even when a force passes through its centroid. Trace the longitudinal shear flow around its flanges, find the load line that produces bending without torsion, and design a loading experiment that separates the stiffness in bending from eccentricity relative to this shear centre.

5. ## Where does a circular opening concentrate stress?

   Make a hole in a flat sheet pulled along one direction. Derive the conditions on the empty edge, compute the circumferential stress and check how quickly the disturbance decays as a strain gauge moves into the sheet.

6. ## Which members carry a load through a truss?

   A weighing frame can have perfect external force balance and still contain a failed load path. Follow the reactions into individual bars, cut across the middle of the frame, and carry the same reasoning into three dimensions.

7. ## Where does the neutral axis of an angle section lie?

A downward tip load can bend an angle section sideways. Construct its full area-moment matrix and use measured strains to distinguish coupling of bending axes from an unintended applied torque.

8. When does Euler’s column load cease to govern?

A stocky strut may yield while its elastic eigenvalue still looks comfortable. Separate a bifurcation of a straight member, stress amplification under eccentric loading and loss of tangent stiffness in a yielded section.

Stress, deformation and structural measurement · 1. Torsion in a thin-walled two-cell section

Two closed cells share one slender web. The applied couple gives only their combined torque, yet the common web can carry much less shear than either exterior wall. Derive the missing compatibility equation, calculate a twist and test which measurements distinguish the two circulations.

Obtain one torque equation from traction

Consider an initially straight prismatic member of length much greater than its section dimensions. The walls are thin relative to cell dimensions, shear modulus is uniform and Saint-Venant torsion is established away from ends. Define shear flow $q = \tau t$, the tangential force transmitted per unit member length along the wall midline. In a closed single cell without a longitudinal body load, equilibrium makes its circulatory shear flow constant. Each tangential force around the contour contributes a torque, and the line integral of the moment arm gives twice the enclosed midline area. Consequently $T = 2Aq$.

For two cells choose positive circulation with the same sense in both. Superposition gives

$$T = 2A_1 q_1 + 2A_2 q_2.$$

On the shared wall the two contour directions oppose each other, so its actual flow is $q_1 - q_2$ when viewed along cell one's circulation. Adding the two flows there would violate the chosen orientation. This torque equation alone cannot determine two unknown circulations. Treating each as proportional to area would be a guess about stiffness, not a derivation.

Enforce equal twist on both closed contours

The tangential shear strain integrated around a closed contour is related to axial twist rate θ'. Under the thin-wall Saint-Venant assumptions, warping displacement is single-valued after a closed circuit, leaving $2A\theta' = \oint q/(Gt)\,ds$. Let C_1, C_2 denote integrals of ds/t along the two exterior contours excluding their common wall and C_w its shared-wall integral. Compatibility then requires

$$2A_1 G\theta' = C_1 q_1 + C_w(q_1 - q_2),$$
$$2A_2 G\theta' = C_2 q_2 + C_w(q_2 - q_1).$$

The common twist is a geometrical condition because the cells belong to one connected section. With different materials replace each $ds/(Gt)$ term individually; pulling out one common G would then be invalid. For equal exterior geometry the equations give equal flows and zero net web shear, while the web can still stabilise shape and carry other load cases.

Take heights 40 mm and widths 60 and 40 mm, with 1 mm walls throughout, $G = 27$ GPa and $T = 10$ N m. The areas are 2400 and 1600 square millimetres. Exterior and common geometric coefficients are

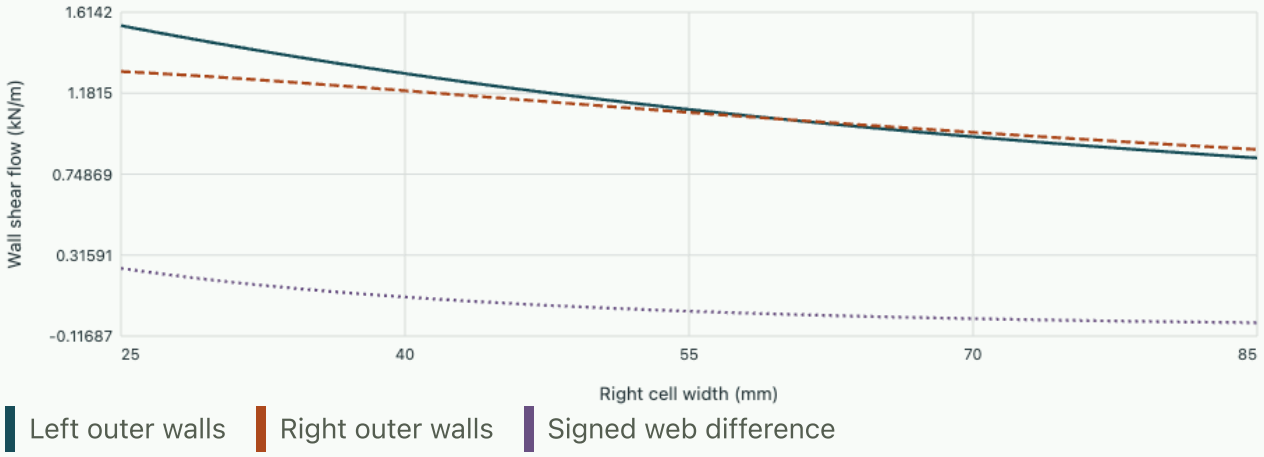
$$C_1 = 160, C_2 = 120, C_w = 40.$$

Solving gives flows 1.287 and 1.195 kN/m, a shared difference 0.092 kN/m, and twist rate 0.001617 rad/m. The exterior wall stresses equal their flow divided by thickness and are therefore 1.287 and 1.195 MPa. Over a one-metre span the ideal rotation is only 0.0926 degrees.

Open the live experiment

Try the calculation

Left cell width (mm)	60
Web thickness (mm)	1
Torque (N m)	10



At 40 mm right width the two circulations are 1.287 and 1.195 kN/m. The shared web carries their signed difference, 0.092 kN/m.

Background · Why does a shared wall carry the difference of the cell flows?

Trace a positive circulation counterclockwise around each closed cell. Along the common wall the two traces point in opposite directions. Give the common wall the positive direction of cell 1. Its physical shear flow is then $q_1 - q_2$; it is this flow divided by thickness that produces the local shear stress. For a common wall of length l and constant thickness t the stored shear energy per unit beam length is

$$U'_{\text{common}} = \frac{l}{2Gt}(q_1 - q_2)^2.$$

For identical cells under symmetric torsion, equal circulations leave the common wall unstressed in this ideal model even though the outer walls carry torque. If the common wall is very thin, its large compliance increasingly penalises unequal circulations. This energy check fixes the relative sign without depending on how the cross-section was drawn.

Which measurement fixes which resistance?

Sweep the right width through the fixed left width. The shared flow should pass smoothly through zero while the outer walls continue transmitting the global torque. Increase the web thickness. Its C_w decreases and it can transfer a given difference at smaller shear strain. The exterior walls and compatibility still control the complete response; the common wall need not carry a large flow simply because it is thick. Measure external twist and at least one wall strain if the individual circulations are the quantities of interest.

A diagonal strain-gauge pair on a thin wall gives an approximate shear strain when the gauges sample in-plane strain at plus and minus forty-five degrees. Their difference rejects common normal strain to first order. At a predicted 0.092 MPa common-wall shear, the strain is only about 3.4 microstrain for the stated modulus. A one-axis gauge zero error can therefore overwhelm the common-wall signal even while exterior torque and twist appear accurate. Confirm the sign by reversing the applied couple.

Check the wall flows by minimising strain energy

A second derivation makes an effective independent numerical test. Under a torque constraint choose the two cell flows so that the sum of their area moments gives the required torque. Their elastic energy per unit length is the sum of three terms, one from each group's exclusive walls and one from the shared wall:

$$U' = \frac{1}{2G}\left(C_1 q_1^2 + C_2 q_2^2 + C_w (q_1 - q_2)^2\right).$$

Perturb the first circulation while adjusting the second to keep torque constant. At the physical solution this variation cannot lower the energy. The resulting stationarity condition reproduces equal twist, including the minus sign on the common wall. A program that accidentally squares the sum on that wall would predict positive strain energy there even for two identical cells at equal flow. This unit test takes no measured twist as input and pinpoints a common implementation error. A second comparison differentiates the minimum energy with respect to applied torque and obtains twist per unit length. Because all wall stresses are linear in torque in this model, doubling torque doubles twist and quadruples stored energy. In a bench specimen, rapid departures from this scaling can reveal slip at a seam or local buckling before individual strain gauges identify which wall has changed.

Tutorial · Test a symmetry prediction numerically

Make both cells 50 mm wide with the same height and identical exterior walls. Find each circulating flow at 10 N m. Does changing only shared-web thickness affect ideal torsional twist in this exactly symmetric case?

▼ Use symmetry before solving two equations

Each area is 0.0020 square metres. Equal flows give

$$q = T/[2(A_1 + A_2)] = 1250\,\mathrm{N/m}.$$

The net web flow vanishes identically and hence its shear-compliance contribution drops out of both contour equations. With unchanged exterior geometry the ideal twist is unchanged by web thickness in this case. A measured change could reflect local distortion, end constraints, material changes during fabrication or other loads outside the thin-wall torsion model.

Tutorial · Infer torque from imperfect shear gauges

In the original unequal section calibrated exterior gauges indicate 1.30 and 1.18 MPa. Recover torque and predict shared-wall shear. Why is the torque result less sensitive to the gauge fluctuations than the difference?

▼ **Propagate the measured flows**

At one millimetre thickness the flows are 1300 and 1180 N/m. Torque is $2(0.0024 \cdot 1300 + 0.0016 \cdot 1180) = 10.016$ N m. Predicted shared flow is 120 N/m, or 0.120 MPa. Independent small errors affect a sum of two large positive contributions only modestly in relative terms, while the shared difference is a much smaller residual. Check twist compatibility as a separate residual instead of concluding from the accurate torque alone that both gauge readings are correct.

Tutorial · Identify a missing effect at a clamped end

A very short specimen twists less than the prediction and shows high longitudinal stresses near the clamp. Which model assumptions are questionable, and how could two specimen lengths test them without changing cross-section?

▼ **Compare distributed and end contributions**

Restrained warping, local load introduction and section distortion can create additional stresses and stiffness near the clamps. Fabricate geometrically similar fixtures at two or more lengths and compare differential rotation between interior stations per unit gauge length. An approximately constant interior slope matching the derived twist with endpoint departures supports a local end effect. If the slope itself varies throughout, check actual wall thickness, joint continuity, shear modulus and distortion. Increasing the number of contour integration points in an otherwise prismatic free-warping model does not recover a missing boundary mechanism.

MIT Ship Structural Analysis and Design · Saint-Venant torsion of multicell sections

Stress, deformation and structural measurement · 2. Contact pressure between elastic cylinders

Two polished cylindrical rollers touch along a narrow strip. Dividing load by a nominal full face area gives little indication of the actual pressure. Use the local shape of the gap and elastic half-space compliance to obtain the pressure distribution, and decide whether a measured contact width can determine material stiffness.

Replace the undeformed outlines by their local curvatures

Take two long parallel cylinders pressed together by total force F uniformly per engaged length L. Ignore tangential traction, adhesion, end effects and plastic deformation. In a cross-section perpendicular to the axes the local undeformed gap grows as $x^2/(2R^*)$, with $1/R^* = 1/R_1 + 1/R_2$ for two externally convex cylinders. A flat counterpart corresponds to infinite radius. For an internal conforming surface the signed curvature changes, but nearly conformal contact can violate the small-patch assumptions used here.

Normal deformations of both elastic bodies add. For isotropic half-spaces in line contact their combined modulus obeys $1/E^* = (1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2$. Replacing this by one body's modulus overestimates stiffness whenever the partner also deforms significantly. For two identical steels with $E = 210$ GPa and $\nu = 0.30$, $E^* = 115.4$ GPa. An elastically rigid counterpart would give roughly twice that value for the same first body.

Why the square-root distribution fits the quadratic gap

In two-dimensional half-space elasticity a line element of pressure produces a logarithmic normal surface displacement up to an arbitrary reference. Differentiating the displacement removes that reference and gives a singular kernel proportional to the inverse horizontal separation. In contact the two deformed faces conform, so the derivative of their combined displacement has magnitude x/R^*. Using a consistent compressive sign convention, the elastic equation for $|x| < a$ is

$$\frac{x}{R^*} = \frac{2}{\pi E^*}\,\mathrm{PV}\int_{-a}^{a} \frac{p(s)}{x-s}\,ds.$$

The trial shape $p(s) = p_0\sqrt{1 - s^2/a^2}$ gives a linear principal-value integral $\pi p_0 x/a$, exactly matching the gap slope. Normal force per unit length $W = F/L$ is the integral of this pressure, the area of a semiellipse: $W = \pi a p_0/2$. Combining the two conditions determines both quantities,

$$a = \sqrt{\frac{4WR^*}{\pi E^*}},$$

$$p_0 = \frac{2W}{\pi a}.$$

The full width is $2a$. Central pressure exceeds the average over the strip by $4/\pi$. The normal approach of ideal infinite cylinders has a logarithmic outer-length dependence, so absolute displacement from a frame is less cleanly predicted by this local half-space construction than pressure or width. Include

whole-body and fixture compliance before using crosshead displacement to infer a local contact modulus.

Use a width series and check its scaling

For $R^* = 10$ mm, $E^* = 115$ GPa and $W = 20{,}000$ N/m the predicted half-width is 47.1 micrometres and peak pressure about 271 MPa. Doubling force at constant length increases both width and pressure by a factor of $\sqrt{2}$. A nominal area built from the full millimetre-scale roller width would understate this pressure dramatically. Conversely, comparing peak hydrostatic or normal pressure directly with uniaxial yield stress is not a rigorous yield condition: evaluate the multiaxial subsurface stress and actual material strength for that separate question.

A pressure-sensitive film or optical contact image has finite resolution and can broaden the visible strip. Fit the square of measured half-width against W. A nonzero intercept, load-dependent slope or hysteresis on unloading can reveal optical threshold bias, initial crowning or irreversible deformation. Surface roughness can produce sparse microcontacts inside a broader nominal strip. Compare several axial positions to detect end loading or misalignment before interpreting the width as uniform line contact.

Compare another geometry-dependent elastic stress distribution in the thick-cylinder experiment

Compare a resolved pressure trace with total applied force

The pressure texture of a two-dimensional Hertz contact gives more information than one width measurement. Predict the integrated line load by integrating the measured pressure over transverse position and compare it with a load cell divided by contact length. The model semiellipse has mean pressure across its width equal to one quarter of pi times its peak. A sensor with limited spatial resolution tends to round the central peak and move the apparent edges. If its response kernel has unit area and the full footprint is recorded, however, convolution preserves the integrated force. Disagreement in area suggests gain error, incomplete coverage or a departure from uniform loading along the roller length.

Try the selected loading and then double it while keeping radius and modulus fixed. The recorded footprint broadens, so rescale the transverse axis by each run's own calculated half-width. Also divide pressure by its own peak. The two normalised shapes should collapse. That collapse tests shape and linear elasticity, while the scale factors separately test the force and length calibration. Repeating with the load applied closer to an end changes the axial uniformity assumption; a contact trace at the middle may then give a different inferred load from a trace close to the end.

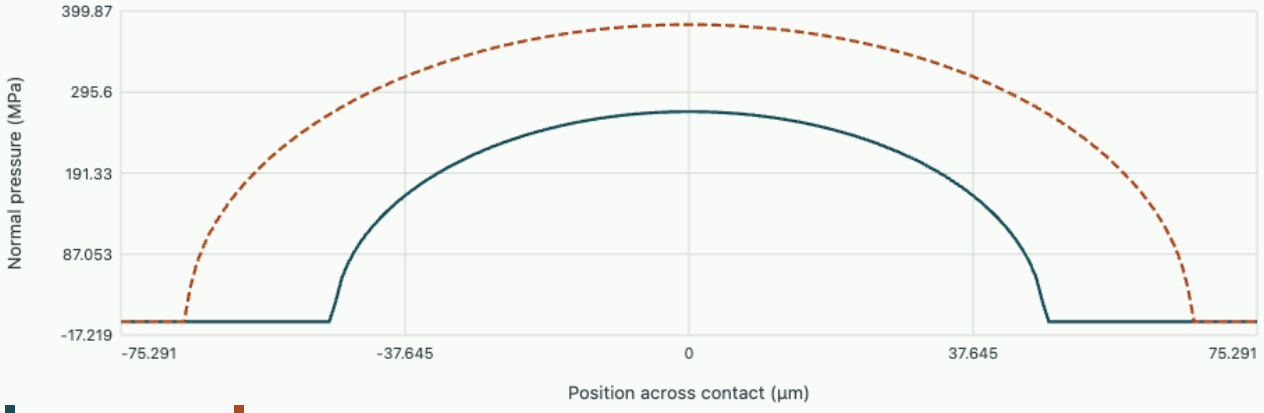
Open the live experiment

Try the calculation

Line load (kN/m) **20**

Reduced radius (mm) **10**

Reduced modulus (GPa) **115**



At the selected load the half-width is 47.1 µm and the peak pressure is 270.6 MPa.

Tutorial · Infer a modulus and estimate its systematic sensitivity

A roll pair with $R^* = 12.0$ mm is loaded at 15.0 kN/m. A camera gives a full width of 96.0 micrometres. Infer E^*. If optical blur makes the true full width five percent smaller than the reported value, by what factor does the inferred modulus change?

▼ **Convert full width before inverting**

Using $a = 48.0$ micrometres gives

$$E^* = 4WR^*/(\pi a^2) = 99.5\,\mathrm{GPa}.$$

Replacing a by $0.95a$ raises the inferred modulus by $1/0.95^2 = 1.108$, about eleven percent. Treating the quoted full width as a would instead underestimate modulus fourfold. Independent optical calibration and known line load both matter to a meaningful inverse measurement.

Tutorial · Change the engaged roller length

A fixed total force is spread uniformly over twice the original length. Predict changes in central pressure, half-width and total nominal contact area. Which change would reveal a failure of uniform load sharing?

▼ **Track load per length and the physical area**

The line load halves, so width and pressure each fall to $1/\sqrt{2}$ of their initial values. Total nominal area $2aL$ rises by $\sqrt{2}$, preserving total force after integrating pressure. If measured width at the centre remains nearly fixed while a large near-end maximum develops, the assumption that F/L is spatially uniform is suspect. Roller bending, tapered clearance or imposed misalignment can redistribute line load even at unchanged total force.

Tutorial · Can width identify both material moduli?

A laboratory wants to estimate E_1 and E_2 using the same pair at several loads. Is this possible from ideal widths alone if both Poisson ratios are known? Design a measurement that supplies additional information.

▼ **Identify the one combination present in the forward equation**

Every load yields the same effective reciprocal modulus combination. A load sweep tests the square-root law and improves precision but does not separate the two material moduli. Pair one unknown with a reference body of independently measured modulus and known curvature, then solve for the first unknown. A second arrangement constrains the other. Use multiple loads to verify elasticity and repeatability in each arrangement; a permanent flat left after unloading invalidates the reversible Hertzian inversion.

MIT Mechanics and Materials II · elastic deformation and Hertzian contact measurements

Stress, deformation and structural measurement · 3. Springback and residual stress after plastic bending

After a rectangular strip has been bent beyond first yield, removing its end couple does not return it to its original curvature. Work out the loading and unloading paths fibre by fibre. Then explain why an unloaded strip can contain a substantial stress field with zero net force and zero net bending moment.

Locate the elastic core and compute the applied moment

Consider a straight rectangular strip of width b, half-depth c and Young modulus E, in pure bending with plane sections remaining plane. Assume symmetric elastic-perfectly plastic response with yield magnitude σ_y, small strain, no hardening and no unloading-induced reverse yielding for the excursion being considered. Symmetry puts the neutral axis at mid-depth. Under increasing curvature κ the axial strain is κy; stress increases linearly until it reaches the signed yield plateau. First yield occurs at $\kappa_y = \sigma_y/(Ec)$.

At greater curvature define $y_e = \sigma_y/(E\kappa)$, the half-thickness of the remaining elastic core. Integrate stress times moment arm over the upper half and double. The core contributes $2bE\kappa y_e^3/3$ and the yielded layer contributes $b\sigma_y(c^2 - y_e^2)$. Substituting the definition of y_e gives

$$M = b\sigma_y(c^2 - y_e^2/3),$$

$$M_y = \frac{2}{3} b\sigma_y c^2,$$

$$M_p = b\sigma_y c^2.$$

The limiting fully plastic moment is one and a half times first-yield moment. It is approached as the elastic core shrinks; ideal plane-section theory does not justify arbitrary curvature once strains and geometry cease to be small. A load-controlled perfectly plastic model can also become indeterminate at a fully developed plastic hinge. A known curvature history is a cleaner basis for calculating springback in this ideal test.

Subtract an elastic stress field to satisfy moment release

During a purely elastic unloading of moment magnitude M, every fibre responds with slope E. The curvature decreases by $M/(EI)$, where $I = 2bc^3/3$. The remaining curvature and signed residual stress profile are

$$\kappa_r = \kappa - \frac{M}{EI},$$

$$\sigma_r(y) = \sigma_{\mathrm{load}}(y) - \frac{M}{I} y.$$

The antisymmetric residual stresses have zero net force. Their bending resultant is zero because the second term removes exactly the original moment. A free strip can therefore remain curved even though its internal stress has neither an axial nor a bending resultant. Its plastic strain varies through the

thickness and makes the unstrained reference of each fibre incompatible with one common straight shape.

Take $E = 200$ GPa, $\sigma_y = 250$ MPa, $b = 20$ mm and $c = 2.0$ mm. The yield curvature is 0.625 per metre, first moment 13.33 N m and plastic limit 20.0 N m. At curvature 1.25 per metre, $y_e = 1.0$ mm and applied moment 18.33 N m. Bending stiffness is 21.33 N square metres and elastic curvature recovery 0.859 per metre. The unloaded curvature is about 0.391 per metre. The surface formerly in tension ends up in compression of about 93.75 MPa, well below reverse yield under this idealised law.

Compare elastic-core growth and plastic moment in the course's bending chapter

Measure curvature and residual stress with different observables

A photograph of the unloaded centreline can determine residual curvature but cannot recover the full through-thickness stress profile. A single strain gauge bonded before loading measures total strain change at its location; separating plastic strain from elastic residual strain requires a constitutive model or an independent release measurement. If a thin surface layer is removed, the remaining section can change curvature as its previously self-equilibrated stress resultants are unbalanced. That change contains information about the removed residual stress but needs a recalculated section geometry.

Measure moment and curvature through loading and initial unloading as separate data sets. The small unloading slope gives an independent estimate of EI, while the first slope departure during loading constrains yield. Friction at fixture rollers, nonuniform moment and axial tension during forming can produce an asymmetric neutral axis that the simple pure-couple solution cannot fit. Reversing the strip or comparing strains on both faces is an inexpensive diagnostic.

Look inside the strip after the applied moment vanishes

In the plot compare the stress at the surface with that on the boundary of the surviving elastic core. The applied stress there reaches yield but the subsequent elastic unloading stress is proportional to distance from the middle. The surface therefore experiences a greater stress change during unloading than the point at the edge of the core. This permits a tensile residual just inside a compressive outer skin on one side. Reflection about the middle reverses all the signs. A surface strain reading alone cannot reveal that self-equilibrated pattern.

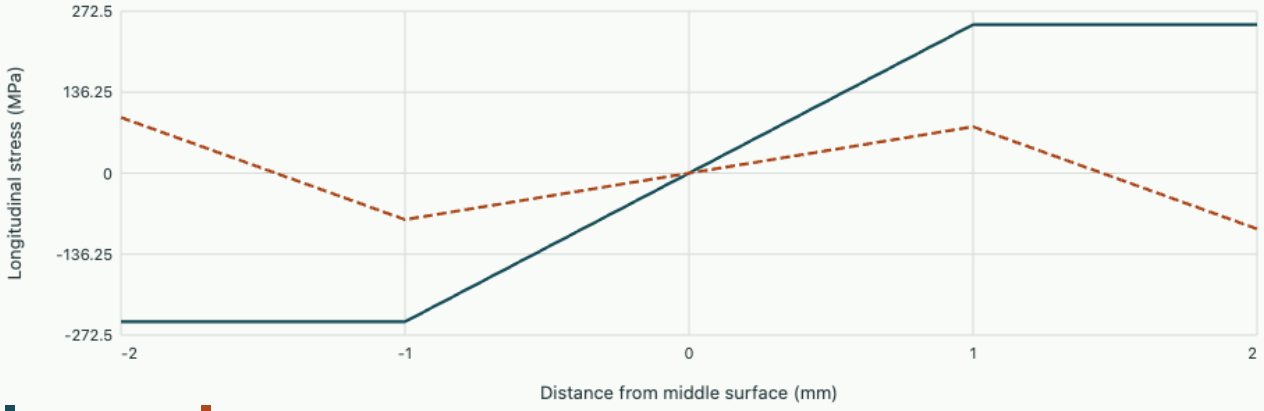
Open the live experiment

Try the calculation

Applied curvature / first-yield curvature **2**

Yield stress (MPa) **250**

Half-thickness (mm) **2**



Residual curvature is 0.391 per metre. The previously tensile surface has residual stress -93.75 MPa.

Check the area under the residual stress over the entire depth. Antisymmetry makes the axial resultant vanish. The moment also vanishes, which follows because subtracting the elastic unloading distribution subtracts exactly the applied moment. The two cancellation tests are stronger than checking the stress at one point. They can be applied to a numerical fibre calculation even when hardening changes the shape of each branch. If an experiment instead clamps the strip straight after bending, the imposed curvature differs from free springback and a reaction moment appears. Compare the same unloading boundary condition before fitting a hardening coefficient.

Tutorial · Predict a residual profile at the elastic-core edge

For the numerical strip above calculate the residual stress at $y = +1.0$ mm after unloading, and compare its sign with the outer surface. Where in the originally tensile outer layer is residual stress zero?

▼ **Use the loading plateau and the unloading slope**

The point at the core boundary reached plus 250 MPa. Its elastic unloading change is $E(M/EI)y = 171.875$ MPa, so residual is plus 78.1 MPa. At the outer surface the unloading change is twice that and residual negative. Within the former tensile plateau the zero occurs at $y = \sigma_y/[E(M/EI)] = 1.455$ mm. The sign change is necessary for the residual field to carry no net moment, although the exact value follows from its detailed profile.

Tutorial · Recover a yield stress from first moment and depth

A second strip of width 15 mm and total thickness 3.0 mm first deviates reproducibly from linear bending near 6.00 N m. Infer yield stress assuming the same model, and estimate the consequence of a two-percent thickness calibration error.

▼ **Use the section modulus and its sensitivity**

With $c = 1.5$ mm, $\sigma_y = 3M_y/(2bc^2) = 267$ MPa. Yield stress scales inversely with squared thickness, so a two-percent overestimate of thickness biases stress downward by nearly four percent. A small proportionality deviation can also reflect fixture or material nonlinearity before conventional yield. A tensile specimen or repeated unload–reload cycles provide independent evidence of irreversibility.

Tutorial · Test whether elastic unloading remains admissible

How would you detect reverse yielding or a Bauschinger effect when applying this springback calculation to a previously prestrained alloy? What can and cannot be repaired by substituting a fitted effective Young modulus?

▼ **Check the stress path before reusing the formula**

Compute the signed unloaded stresses throughout the section and compare the full unloading path against measured reverse yield behaviour. A reverse-yield threshold lower than the virgin value can be reached before the nominal moment is fully removed. Observe changes of unloading slope and hysteresis in cyclic bending or in a tension–compression test on similarly prestrained material. A fitted scalar modulus might reproduce one final curvature yet fail residual stresses and subsequent reverse loading. A history-dependent constitutive law with an appropriate hardening rule is then required for transferable predictions.

MIT Techniques for Structural Analysis and Design · material response and plastic deformation

Stress, deformation and structural measurement · 4. Why an open channel twists under a transverse load

A channel can twist even when a force passes through its centroid. Trace the longitudinal shear flow around its flanges, find the load line that produces bending without torsion, and design a loading experiment that separates the stiffness in bending from eccentricity relative to this shear centre.

Find the internal horizontal force in each flange

Use a thin channel made of a vertical web of midline height h and two horizontal flanges of midline width b projecting to the same side. Take uniform thickness $t \ll h, b$, ignore corner radii and flange through-thickness bending, and consider a vertical shear resultant V. Symmetry places the shear centre on the horizontal axis through mid-height but does not place it on the web or centroid. Set the web midline at $x = 0$ and both flanges at positive x.

For bending about the horizontal centroidal axis, the thin-wall second moment is

$$I_x \simeq th^3/12 + 2bt(h/2)^2.$$

The usual beam shear-flow relation $q = VQ/I_x$ follows from balancing the longitudinal stress change of a cut-off strip over an incremental beam length, using $dM/dz = V$ with a consistent beam-axis orientation. At a free flange edge the cut-off area vanishes and the flow is zero. A distance s inwards from that free edge, $Q \simeq tsh/2$. The flow increases linearly toward the web.

Integrating the flow over one flange gives a horizontal resultant magnitude $F_f = Vthb^2/(4I_x)$. Top and bottom flange forces are opposing, so they contribute zero net horizontal force but a couple $F_f h$ about the web line. The web carries the net vertical shear and its contribution has zero moment about its own midline in this thin-wall representation. Balance the internal couple with the moment of the external vertical force about the web. The shear centre is on the side opposite the flanges, at

$$x_{\mathrm{sc}} = -e,$$

$$e = \frac{th^2b^2}{4I_x} = \frac{3b^2}{h + 6b}.$$

The sign describes the stated channel opening direction. Reflecting the channel reverses the sign in a fixed apparatus. The result is independent of uniform thickness in this thin-wall limit because both shear flow first moment and bending second moment scale with thickness. It does not imply that the torsional stiffness or stress is independent of thickness.

Measure a load line outside the material

Take $h = 60$ mm, $b = 30$ mm and $t = 2$ mm. The shear centre lies 11.25 mm behind the web. The thin-wall centroid is at $x_c = b^2/(h + 2b) = 7.50$ mm on the opening side. A vertical load of 100 N applied through the centroid therefore introduces a torsional moment magnitude $100(0.01875) = 1.875$ N m about the shear centre. A cantilever channel clamped against warping can show both Saint-Venant twist and additional restrained-warping stresses, especially near the clamp.

Mount an adjustable loading arm across the tip and apply several known vertical forces at signed positions x. In a sufficiently linear regime the measured twist under fixed load is proportional to $x - x_{\mathrm{sc}}$. Fit its zero crossing; this eliminates the need to know absolute torsional rigidity in the first estimate. A force applied through an external bracket can reach a line outside the actual material. Ensure the bracket supplies a force and no unintended direct couple, and take readings away from local load introduction.

Compare the open-channel result with twist compatibility in a closed multicell member

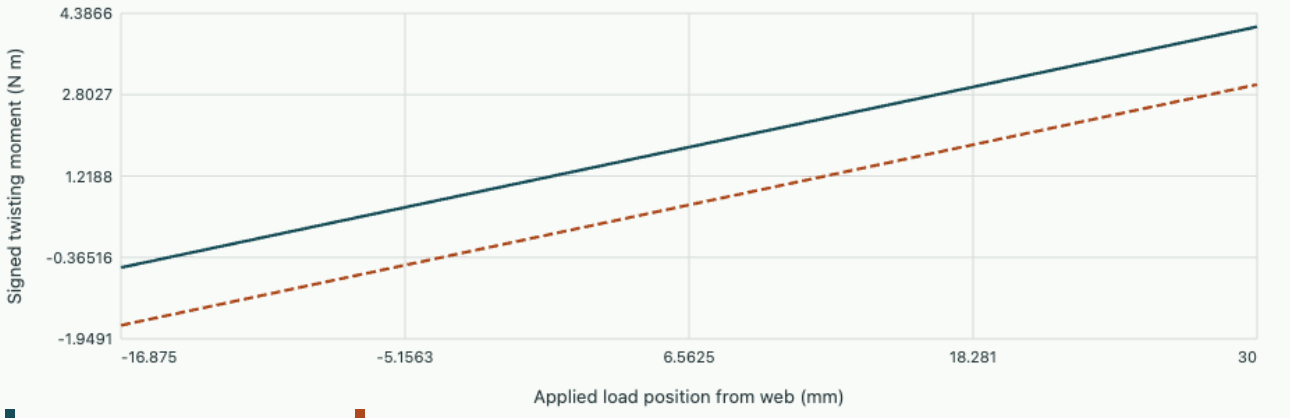
Find the same shear centre with a signed load-position sweep

Keep the vertical force fixed and move its line of application laterally while recording twist at a section a known distance from a clamp. In a member with constant torsional compliance in the test range, the slope of twist versus lateral position contains both the force and compliance. Its zero crossing locates the shear centre without requiring a separate torsional stiffness measurement. The second curve in the model shows the moment computed about the web. Its zero crossing is set by the coordinate origin. Only the other crossing predicts a vanishing twist of the section in the simple beam idealisation.

Open the live experiment

Try the calculation

Midline web height (mm)	60
Flange width (mm)	30
Vertical load (N)	100



The zero-twist line lies 11.25 mm behind the web. A centroidal load produces 1.875 N m of torque.

Repeat the sweep with the force reversed and the same attachment bracket. A constant apparatus torque shifts the two inferred zero positions in opposite directions because the applied torque per unit position changes sign. Their mean can remove a load-independent offset. A bracket whose compliance

changes asymmetrically under reversal requires a separate correction. Measure twist with two gauges spanning a known width so that common vertical translation cancels. If the two gauges are far apart along the beam instead of across the same section, differing bending slopes can contaminate the apparent twist and shift the crossing.

Tutorial · Recover the shear-centre location from two angular readings

At fixed load an adjustable fixture gives twist minus 0.10 degrees when the load lies at $x = -15.0$ mm and plus 0.30 degrees at $x = 0$. Infer the shear-centre coordinate and predict twist when the force passes through the centroid of the example.

▼ **Use signed interpolation and check the geometric result**

The slope is $0.40/15 = 0.0267$ degrees per millimetre. The zero crossing is $-15.0 + 0.10/0.0267 = -11.25$ mm, matching the predicted shear centre. At $x = 7.50$ mm the lever arm is 18.75 mm and the predicted twist 0.500 degrees. This inference assumes no independent angular offset. Reversing load lifts out an offset unchanged by force sign, while moving the bracket with no force checks alignment.

Tutorial · Predict the effect of making the flanges shorter

At unchanged web height and thickness, halve flange width to 15 mm. Predict both centroid and shear-centre location. Is their separation simply half its old value?

▼ **Recompute both first moments**

The centroid is $225/(60 + 30) = 2.50$ mm, while $e = 3(225)/(60 + 90) = 4.50$ mm. Their distance becomes 7.00 mm instead of 18.75 mm. Both formulas contain a squared flange width and a changing perimeter or second moment, so halving width does not simply halve the eccentricity. In the formal limit of vanishing flange width channel torsional resistance also becomes very small, requiring careful control of imperfection and fixture compliance.

Tutorial · Distinguish eccentric bending from torsional warping

A short channel shows nonzero mean twist even at the predicted load line and opposite longitudinal strains on the free flange tips near its clamp. Could changing shear modulus alone move the predicted shear centre? Suggest two specific measurement changes.

▼ **Test the assumptions that control the zero crossing**

For this homogeneous thin elastic channel the centre follows force and moment equilibrium of bending shear and ideal geometry; changing one common shear modulus principally changes torsional compliance rather than moving that ideal geometric point. Compare two specimen lengths and measure twist slopes at several interior stations to identify restrained end warping. Reverse channel orientation while keeping the apparatus fixed to separate a channel-related eccentricity from a fixed bracket couple. If the discrepancy follows the specimen, measure actual corner radii, wall thickness asymmetry and local distortion and use a suitable open-section beam or shell model before attributing every measured longitudinal strain to pure flexure.

MIT Structural Analysis and Control · torsion–flexure and the shear centre

Stress, deformation and structural measurement · 5. Where does a circular opening concentrate stress?

Make a hole in a flat sheet pulled along one direction. Derive the conditions on the empty edge, compute the circumferential stress and check how quickly the disturbance decays as a strain gauge moves into the sheet.

Keep the free edge and the distant traction in one solution

Take an isotropic linear-elastic sheet, long and wide relative to a circular through-hole of radius a. At a large distance apply uniaxial tension S along x. The empty edge can carry neither normal nor tangential traction, so $\sigma_{rr} = \tau_{r\theta} = 0$ at $r = a$. The circumferential stress $\sigma_{\theta\theta}$ is parallel to this edge and need not vanish. Writing uniaxial tension in polar coordinates introduces only a constant angular sector and a sector in $\cos(2\theta)$ and $\sin(2\theta)$.

In plane elasticity an Airy stress function satisfies equilibrium automatically and must be biharmonic for a homogeneous isotropic solid without body forces. The circular-hole solution can be built by adding decaying terms in those same two angular sectors to the distant uniform stress. One convenient potential is

$$\Phi = \frac{S}{4}\left[r^2 - 2a^2\log(r/a) - (r^2 - 2a^2 + a^4/r^2)\cos(2\theta)\right].$$

Differentiate twice with respect to radius to obtain the hoop stress. Writing $u = (a/r)^2$ gives

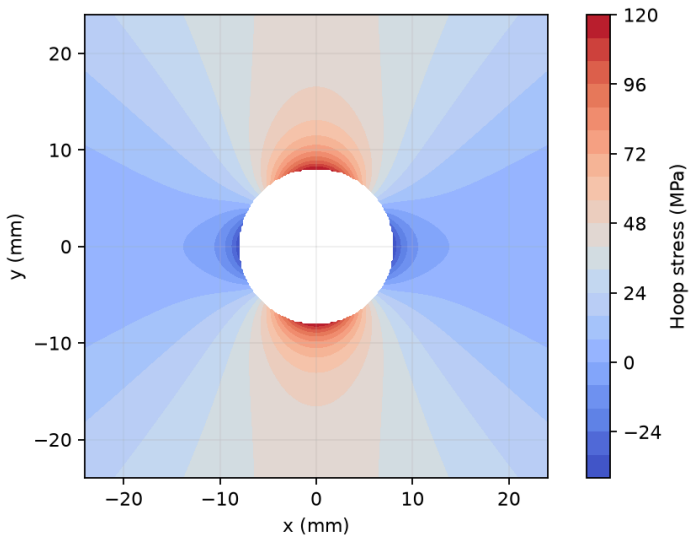
$$\sigma_{\theta\theta} = \frac{S}{2}[1 + u - (1 + 3u^2)\cos(2\theta)].$$

The radial and shear stress expressions in the calculation cell follow from $\sigma_{rr} = \Phi_r/r + \Phi_{\theta\theta}/r^2$ and $\tau_{r\theta} = -\partial_r(\Phi_\theta/r)$. Check both edge tractions explicitly; imposing zero hoop stress instead would erase the stress concentration we aim to measure.

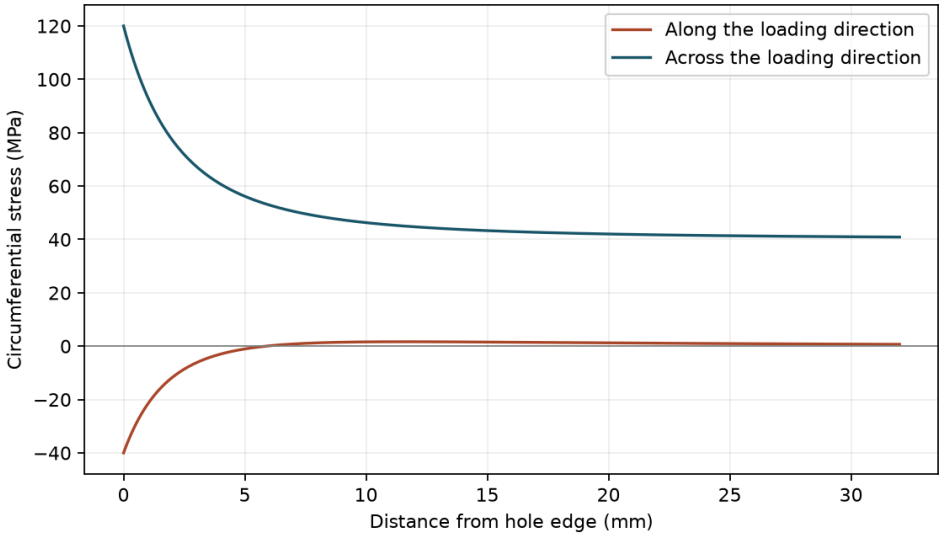
Locate a gauge and interpret its reading

On the edge $u = 1$, hence $\sigma_{\theta\theta} = S[1 - 2\cos(2\theta)]$. With $S = 40$ MPa, gauges at the top and bottom of the opening align with tensile hoop stress of 120 MPa. At the point facing along the remote pulling direction the hoop stress is compressive at −40 MPa. The nominal far-field stress refers to the gross, undisturbed section. A factor based on a finite specimen's reduced net section would use a different reference.

Along the transverse radius, moving from $r = a$ to $r = 2a$ reduces the ratio to $1 + u/2 + 3u^2/2 = 1.21875$. A gauge whose active length samples a substantial fraction of the hole radius therefore averages a gradient. Failure to read a threefold concentration at a distant or finite-area gauge does not by itself invalidate the elastic solution.



The white interior represents removed material. The colour field is one stress component expressed relative to the local circumferential direction.



The horizontal coordinate measures actual distance outward from the wall of the hole. Moving the gauge outward changes the inferred peak.

Exercise · rotate the sensor

At what angle between zero and 90 degrees does the edge hoop stress change sign?

▼ **Work through the solution**

Set $1 - 2\cos(2\theta) = 0$. Thus $2\theta = 60^\circ$ and $\theta = 30^\circ$. This is a zero of the tangential normal stress on a boundary already free of radial and shear traction.

Exercise · change the radius

Double the hole radius at the same remote stress in an effectively infinite sheet. What changes in the peak and the stress gradient in MPa per millimetre?

▼ **Work through the solution**

The peak remains $3S$ because the stress field depends only on r/a and angle. The physical length scale doubles, hence an otherwise corresponding spatial gradient halves. Finite edge distances, plastic yielding and rounded specimen boundaries can introduce new scales and change these conclusions.

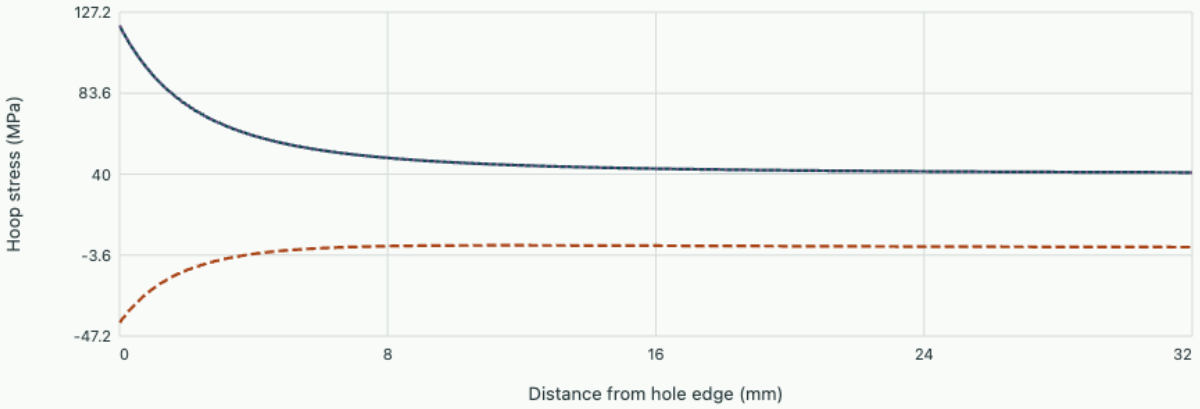
Interactive calculation · Hole stress

Change the physical and numerical inputs of the notebook model. Compare the updated curves and balance checks with the recorded notebook figures. Each plot uses the same equations and assumptions documented in that notebook; its readout names any analytical or reduced-model reference.

Open this live calculation

Try the calculation

Hole radius (mm)	8
Remote tension (MPa)	40
Radial-ray angle (degrees)	90



Selected edge hoop stress 120.000 MPa; peak around the hole is 120.000 MPa. Infinite isotropic plate, circular traction-free hole, remote uniaxial stress.

Download this example's editable notebook

Open editable notebook Download .ipynb

Stress, deformation and structural measurement · 6. Which members carry a load through a truss?

A weighing frame can have perfect external force balance and still contain a failed load path. Follow the reactions into individual bars, cut across the middle of the frame, and carry the same reasoning into three dimensions.

Earn the two-force assumption

Ideal pins transmit forces through their centres. Apply loads at joints, neglect member self-weight or lump it at joints, and treat each initially straight bar as hinged at both ends. Moment balance on that bar requires its two end forces to be collinear. A single scalar N now describes each member. Positive N pulls away from the joint. For a bar joining positions x_i and x_j, use $e_{ij} = (x_j - x_i)/|x_j - x_i|$. Reversing the joint reverses its column direction.

$$\sum_j N_{ij} e_{ij} + R_i + P_i = 0.$$

At an unloaded joint with two noncollinear bars their coefficients must both vanish. With three bars, two collinear and no applied load, project normal to their shared line to eliminate both and identify the third as a zero-force member. A brace classified this way may still resist the next load case or restrain buckling.

Read a two-panel bridge from both ends

Place bottom joints A, B, C at (0,0), (2,0), (4,0) m, and D, E at (1,1), (3,1). Connect AB, BC, AD, DB, BE, EC and DE. Pin A, put C on a vertical-reaction roller. Apply 6 kN down at D and 2 kN down at E. Taking moments about A gives $4C_y = 6(1) + 2(3)$, hence $C_y = 3$ and $A_y = 5$ kN. At A, the vertical projection gives $N_{AD} = -5\sqrt{2}$ kN, and horizontal equilibrium gives $N_{AB} = 5$ kN. At C the corresponding forces are $N_{EC} = -3\sqrt{2}$ and $N_{BC} = 3$ kN.

Cut AB, DB and DE immediately left of B and take moments on the left fragment about B. AB and DB intersect there. The reaction at A contributes –10 kN m, the downward load at D contributes +6 kN m, and the horizontal force in DE contributes $-N_{DE}(1\,\mathrm{m})$. Thus $N_{DE} = -4$ kN. Solving at D gives $N_{DB} = -\sqrt{2}$ kN. At E the corresponding diagonal BE is in tension at $+\sqrt{2}$ kN. The opposite signs are physically measurable under this asymmetric loading.

When a section cut can determine three forces

Three equilibrium equations suffice only if the three cut-member force lines supply independent force and moment conditions. If all three lines meet at one point, their moments about that point vanish together. Choose another cut or a second free body.

Lift the joint into three dimensions

Place three feet at equal azimuthal intervals on a circle of radius 1 m. Pin the apex 1 m above its centre. Measure the horizontal 0° direction toward foot 1. The directions from apex to feet are $(\cos\theta, \sin\theta, -1)/\sqrt{2}$. A downward load W and horizontal load H give $N_2 = N_3$ by symmetry. Taking horizontal and vertical projections yields

$$N_1 = -\frac{\sqrt{2}}{3}(W + 2H),$$

$$N_2 = N_3 = \frac{\sqrt{2}}{3}(H - W).$$

At $W = 6$ and $H = 3$ kN, all three bars are in compression. At $H = 6$ kN two members unload simultaneously. Pins capable of taking tension and unanchored compression-only feet then predict different admissible states. The notebook resolves all three joint equations and plots that boundary.

Interactive experiment · Rotate a loaded tripod and change its geometry

Move the height, foot radius and horizontal load independently, then turn the load through a full circle. The browser solves the three joint-equilibrium equations for every setting. The plot sweeps horizontal load while the spatial view shows the selected operating point. Each bar points from the apex to its own foot, so a positive signed force pulls the apex toward that foot.

$$L = \sqrt{r^2 + h^2},$$

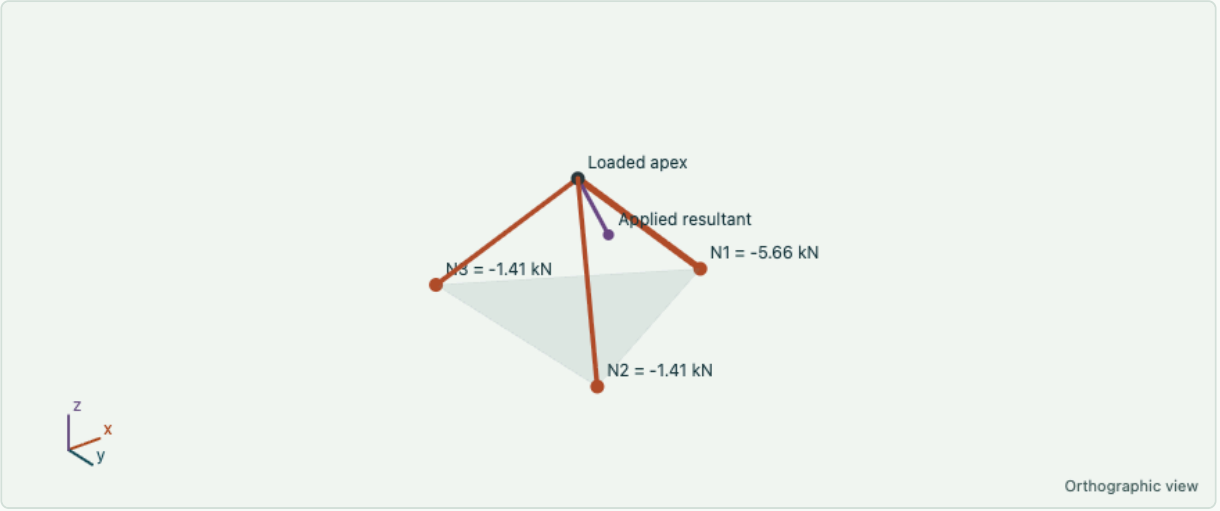
$$N_i = -\frac{WL}{3h} - \frac{2HL}{3r}\cos(\theta_i - \alpha).$$

First keep *h = r = 1 m*, *W = 6 kN* and azimuth zero to recover the notebook. Then lower the apex: the vertical load requires increasingly large compression. Rotate the horizontal load by 60° and identify the first member that would require an anchored foot. Rotate the 3D view to inspect the distinct force directions; bar colour records tension or compression, not a stress contour. This ideal pin-jointed model omits deformation and buckling.

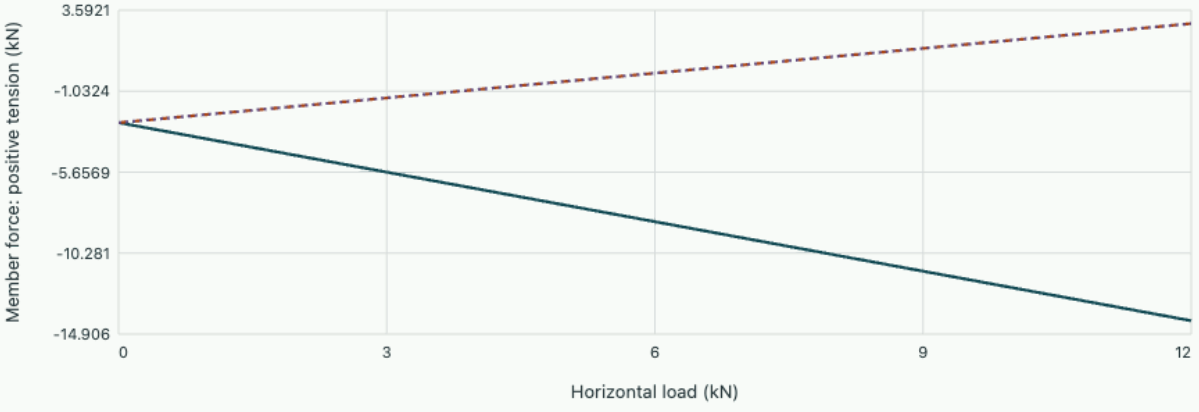
Open this experiment online

Try the calculation

Apex height (m)	1
Foot radius (m)	1
Downward load (kN)	6
Horizontal load (kN)	3
Horizontal load azimuth (degrees)	0



Three-dimensional pin-jointed tripod at the selected load. Orange members carry compression; teal members carry tension; grey means unloaded. The purple segment shows the applied force direction with a symbolic length. Members retain their undeformed geometry. Axes: x = x (m); y = y (m); z = z (m).



At the selected load: N1 = -5.657 kN; N2 = -1.414 kN; N3 = -1.414 kN. Joint force residual 1.3e-15 kN. All members are in compression or unloaded.

Continue in the OneSimulation notebook · Download its Python model

OneSimulation experiment · Two legs unload as the horizontal apex force reaches the vertical load

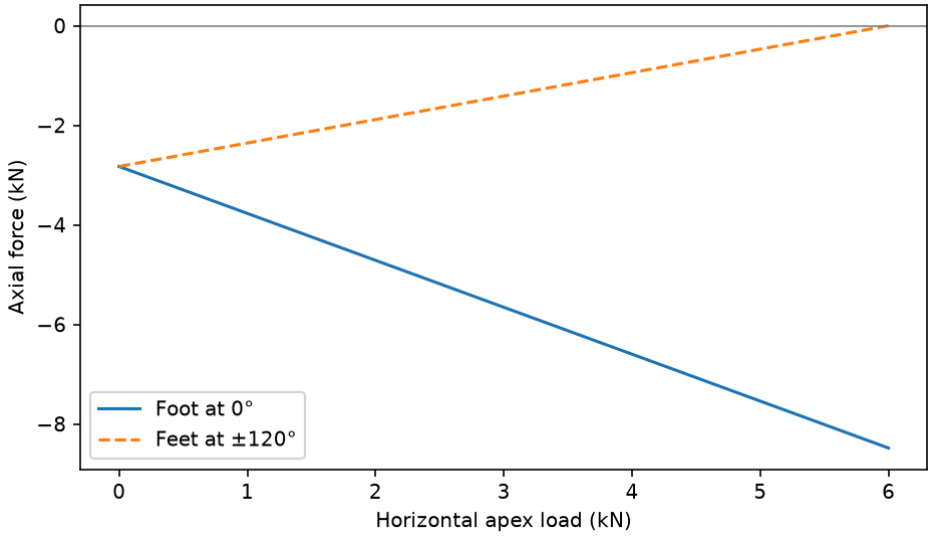
Run the notebook with OneSimulation 0.2.0, NumPy, SciPy and Matplotlib. Change the input data and compare your run with the recorded result.

```
"""Resolve measured loads on a three-dimensional tripod into signed member forces."""
import numpy as np
import matplotlib.pyplot as plt
from onesim.numerics.nonlinear import LeastSquares

angles = np.deg2rad([0, 120, 240])
# Columns point from the loaded apex toward a foot; positive force is tension.
directions = np.array([np.cos(angles), np.sin(angles), -np.ones(3)]) / np.sqrt(2)
horizontal = np.linspace(0, 6, 61)  # kN
forces = []
for load in horizontal:
    external = np.array([load, 0.0, -6.0])
    fit = LeastSquares().solve(lambda force: directions @ force + external, [-2.0]*3)
    assert np.linalg.norm(directions @ fit.x + external) < 1e-6
    forces.append(fit.x)
forces = np.array(forces)
print("H=3 kN: signed member forces [kN]", np.round(forces[30], 3))
print("H=6 kN: signed member forces [kN]", np.round(forces[-1], 3))
```

Reference run

```
H=3 kN: signed member forces [kN] [-5.657 -1.414 -1.414]
H=6 kN: signed member forces [kN] [-8.485  0.     0.   ]
```



Two legs unload as the horizontal apex force reaches the vertical load

Download editable notebook · Download Python script

Exercises on force paths

Move both bridge loads to E while retaining their combined magnitude. Which support reaction and which top-chord force change?

▼ **Worked solution**

The right reaction is $8(3)/4 = 6$ kN and the left is 2 kN. Taking moments about B on the same left fragment gives $-2(2) - N_{DE} = 0$, so DE still carries −4 kN. An unchanged chord force does not imply unchanged diagonals.

Remove BE from the original frame. Does external equilibrium establish rigidity?

▼ **Worked solution**

The pin and roller provide three independent reaction components. Five planar joints admit ten equilibrium rows against six members and three reactions. There cannot be full rank. The original load may no longer be supported without motion. For geometries satisfying the count, independence must still be checked.

What happens when the apex height tends to zero at fixed radius and vertical load?

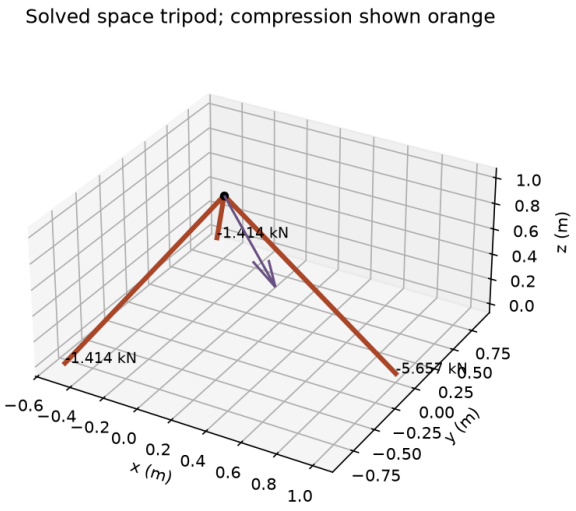
▼ **Worked solution**

Every vertical component of a member direction tends to zero. Maintaining finite upward resultant requires unbounded compression before the idealised matrix loses rank at zero height. Strength, deformation and buckling constrain the real frame much earlier.

Further reading · Hibbeler, Engineering Mechanics: Statics in SI Units, 15th ed., Chapter 6, especially sections 6.1–6.5.

Spatial notebook output · inspect the same solved model in3D

The downloadable notebook now also draws the solved member forces or discrete beam eigenvectors in three dimensions. Its spatial figure below is produced by the actual OneSimulation run.



Recorded spatial output; the beam view enlarges transverse dimensions to make its mode shape legible.

Download the executed notebook with3D output

Stress, deformation and structural measurement · 7. Where does the neutral axis of an angle section lie?

A downward tip load can bend an angle section sideways. Construct its full area-moment matrix and use measured strains to distinguish coupling of bending axes from an unintended applied torque.

Balance two bending moments with one stress plane

Consider a straight, homogeneous prismatic beam in small-deflection Euler–Bernoulli bending. Coordinates x, y lie in the section, with their origin at the area centroid. Plane sections remain plane and zero axial resultant eliminates the constant part of the stress. Write $\sigma = ax + by$. Fix signs by defining $M_x = \int y\sigma\, dA$ and $M_y = -\int x\sigma\, dA$. These signs follow from the cross product of position with axial traction. Then

$$\begin{bmatrix} I_{yy} & I_{xy} \\ I_{xy} & I_{xx} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -M_y \\ M_x \end{bmatrix}, \quad I_{xy} = \int_A xy\, dA.$$

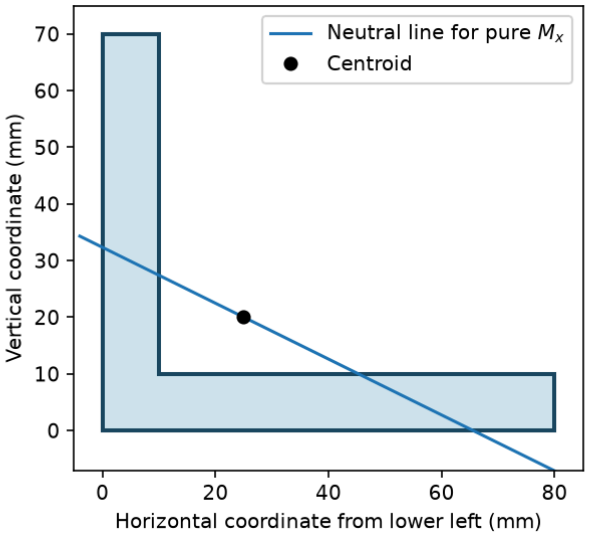
Rotating coordinates to an orthogonal eigenbasis diagonalises this matrix. Its positive determinant is the Cauchy–Schwarz inequality applied to the coordinates. Under pure M_x, $a/b = -I_{xy}/I_{yy}$, so the neutral line $y = -(a/b)x$ generally slopes. A moment aligned with a principal axis recovers ordinary uniaxial bending.

Build the section before inverting the moments

Use an 80 by 10 mm horizontal rectangle and a 10 by 60 mm upright above its left edge. The centroid measured from the lower left corner is (25,20) mm. Translating each rectangle to the centroid gives $I_{xx} = 606667$, $I_{yy} = 851667$, and $I_{xy} = -420000$ in mm⁴. For instance, the horizontal part contributes $800(40-25)(5-20) = -180000$ mm⁴ to the product. Adding independent values of $bh^3/12$ would omit these offsets.

Take $M_x = 1$ kN m and $M_y = 0$. With moments in N mm, the result is $a = 1.2343$ and $b = 2.5029$ MPa/mm. The neutral line has slope −0.4932. At the horizontal flange corner (80,10) the predicted longitudinal tension is 42.86 MPa. The two principal area moments are approximately 291667 and 1166667 mm⁴. These are useful geometry checks because a rigid rotation cannot change them.

> **If the tip load misses the shear centre**
>
> An eccentric transverse force can supply both a bending moment and a torque. The formula above predicts longitudinal bending stress in a section away from load introduction. A separate twist measurement or a loading couple with no net transverse force helps isolate the source of a sideways signal.



The L section and its neutral line under positive bending about the horizontal axis. The centroid is outside the material.

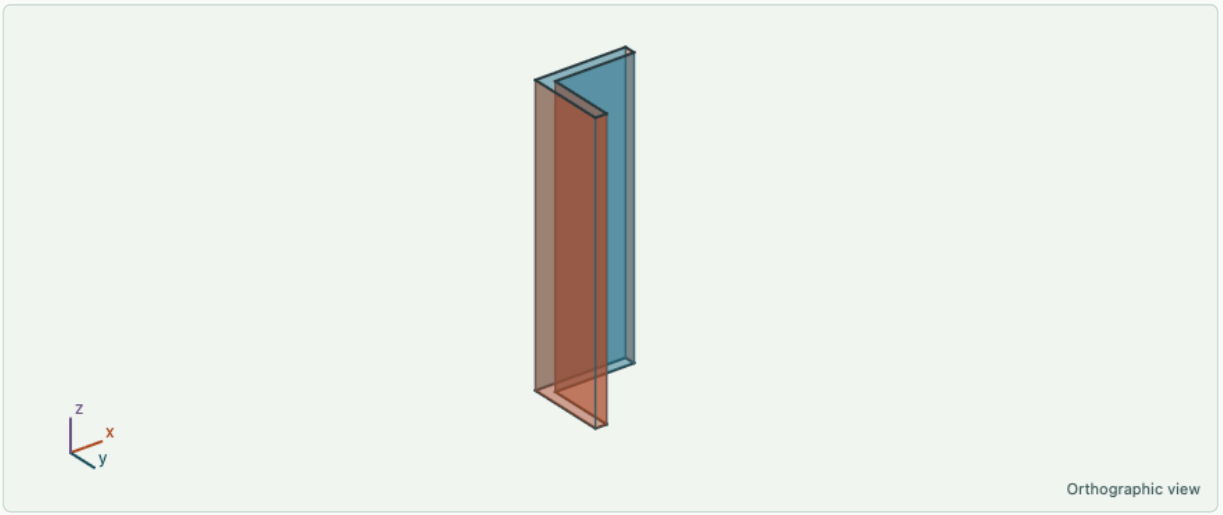
Interactive calculation · Turn the bending couple on an asymmetric section

Change Mx and My independently. The stress plane, perimeter stresses and spatially coloured beam faces are recomputed from the coupled area-moment matrix. A pure moment about a geometric axis need not bend about that axis.

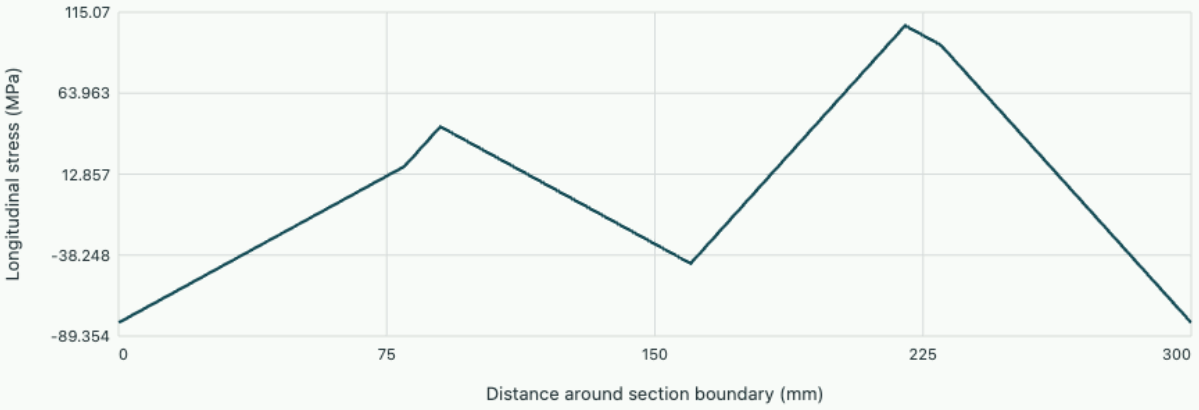
Open this live calculation

Try the calculation

Bending moment Mx (kN m)	1
Bending moment My (kN m)	0



Three-dimensional 250 mm segment of the L section under a pure end-moment couple. Orange surface patches have tensile longitudinal stress and blue patches compressive stress at their midpoints; colours are qualitative. Geometry is undeformed. The full area-moment coupling is solved, as in the linked OneSimulation notebook. Axes: x = section x (mm); y = section y (mm); z = beam axis (mm).



Signed boundary stress

Stress coefficients a=1.2343 and b=2.5029 MPa/mm. At (80,10) mm, stress 42.857 MPa. Bending moments are constant along this prismatic segment.

Exercises on signed strain readings

A strain gauge is glued at (0,0) mm. What signal should it give for steel with $E = 200$ GPa?

▼ **Worked solution**

Its stress is $1.2343(-25) + 2.5029(-20) = -80.9$ MPa. Its axial strain is −405 microstrain. A wrong choice of corner origin would falsely predict zero.

Can two axial gauges recover both bending moments?

▼ **Worked solution**

Write their two rows as (x_i, y_i) relative to the centroid and solve for a, b from $E\epsilon_i$. Recovery requires $x_1 y_2 - x_2 y_1 \neq 0$. Two gauges on the same centroidal line give proportional information even if they lie on opposite sides of the section.

Further reading · Hibbeler, Mechanics of Materials, 11th ed., Section 6.5 on unsymmetric bending.

OneSimulation notebook · solved spatial stresses, code and output

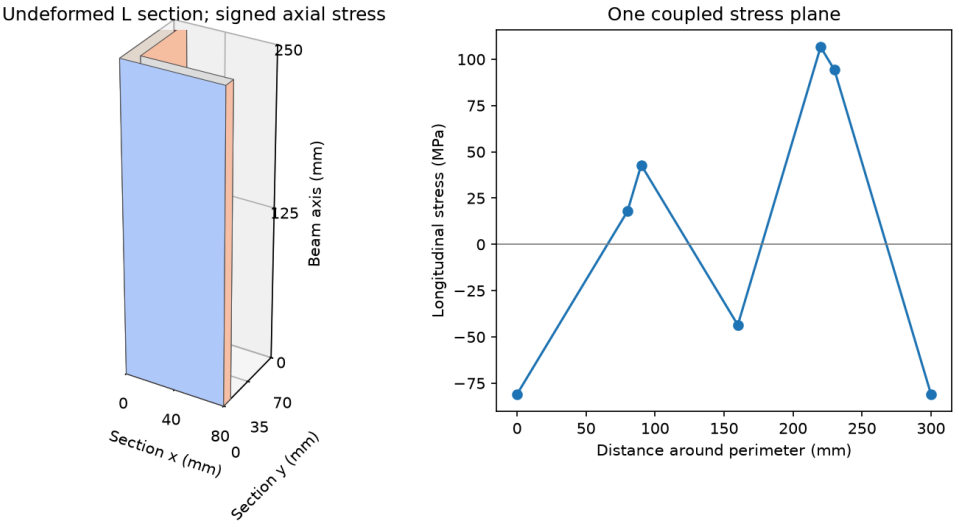
The code solves the illustrated stress model with OneSimulation and checks its equilibrium or eigensystem. The figure is the output of that code.

```
"""Resolve a biaxial moment with OneSimulation and inspect a 3D prismatic L section."""
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d.art3d import Poly3DCollection
from onesim.numerics.linear.matrices.dense_linear_algebra import solve_dense, symmetric_eigensystem

# mm, N, MPa. Two nonoverlapping rectangles; moments follow the chapter signs.
rectangles = [(800., 40., 5., 80., 10.), (600., 5., 40., 10., 60.)]
area = sum(row[0] for row in rectangles)
centroid = np.array([sum(A*x for A,x,y,w,h in rectangles),
                     sum(A*y for A,x,y,w,h in rectangles)])/area
Ixx = sum(A*h*h/12+A*(y-centroid[1])**2 for A,x,y,w,h in rectangles)
Iyy = sum(A*w*w/12+A*(x-centroid[0])**2 for A,x,y,w,h in rectangles)
Ixy = sum(A*(x-centroid[0])*(y-centroid[1]) for A,x,y,w,h in rectangles)
matrix = np.array([[Iyy,Ixy],[Ixy,Ixx]])
Mx, My = 1e6, 0.0  # N mm; change these two inputs as in the live view
coefficients = solve_dense(matrix,np.array([-My,Mx]))
values, axes = symmetric_eigensystem(matrix)
assert np.allclose(matrix@coefficients,[-My,Mx])
assert np.allclose(axes@np.diag(values)@axes.T,matrix)
outline = np.array([[0,0],[80,0],[80,10],[10,10],[10,70],[0,70],[0,0]],dtype=float)
stress = (outline-centroid)@coefficients
print('Centroid [mm]:',centroid)
print('Stress-plane coefficients [MPa/mm]:',np.round(coefficients,6))
print('Principal area moments [mm^4]:',np.round(values,3))
print('Stress at (80,10) mm [MPa]:',round(stress[2],6))

fig = plt.figure(figsize=(9,4.2),layout='constrained')
ax = fig.add_subplot(121,projection='3d')
faces=[];colors=[]
scale=max(abs(stress))
for i in range(len(outline)-1):
    p,q=outline[i:i+2]
    faces.append([[*p,0],[*q,0],[*q,250],[*p,250]])
```

```
    colors.append(plt.cm.coolwarm(.5+.5*(stress[i]+stress[i+1])/2/scale))
ax.add_collection3d(Poly3DCollection(faces,facecolors=colors,edgecolors='0.3',linewidths=.5))
ax.set(xlim=(0,80),ylim=(0,70),zlim=(0,250),xlabel='Section x (mm)',ylabel='Section y
(mm)',zlabel='Beam axis (mm)',title='Undeformed L section; signed axial stress')
ax.set(xticks=[0,40,80], yticks=[0,35,70], zticks=[0,125,250])
ax.tick_params(axis='x', pad=1)
ax.tick_params(axis='y', pad=8)
ax.tick_params(axis='z', pad=1)
ax.zaxis.labelpad = 16
ax.yaxis.labelpad = 18
ax.set_box_aspect((80,70,250))
ax2=fig.add_subplot(122)
arc=np.r_[0,np.cumsum(np.linalg.norm(np.diff(outline,axis=0),axis=1))]
ax2.plot(arc,stress,'o-')
ax2.axhline(0,color='0.5',lw=.8)
ax2.set(xlabel='Distance around perimeter (mm)',ylabel='Longitudinal stress (MPa)',title='One
coupled stress plane')
fig.savefig('complete-engineering-unsymmetric3d.png',dpi=170)
plt.show()
```



Recorded output from the executed OneSimulation notebook; change its inputs to reproduce the browser study.

Download the executed editable notebook · Download the Python model

Stress, deformation and structural measurement · 8. When does Euler’s column load cease to govern?

A stocky strut may yield while its elastic eigenvalue still looks comfortable. Separate a bifurcation of a straight member, stress amplification under eccentric loading and loss of tangent stiffness in a yielded section.

Compare slenderness against the material stress scale

For a pin-ended straight column with effective length L_e, radius of gyration $r = \sqrt{I/A}$ and uniform modulus E, the homogeneous transverse balance admits a nonzero mode when $P_E = \pi^2 EI/L_e^2$. The corresponding average stress falls as the inverse square of $\lambda = L_e/r$. If this value exceeds the proportional limit, part of the constitutive relation assumed by the eigenproblem is already invalid.

One engineering interpolation for intermediate steel columns is a Johnson parabola starting at yield stress S_y. Write $\sigma_J = S_y - c\lambda^2$ and require value and first derivative to match Euler’s curve at the join. Eliminating c and the joining slenderness gives

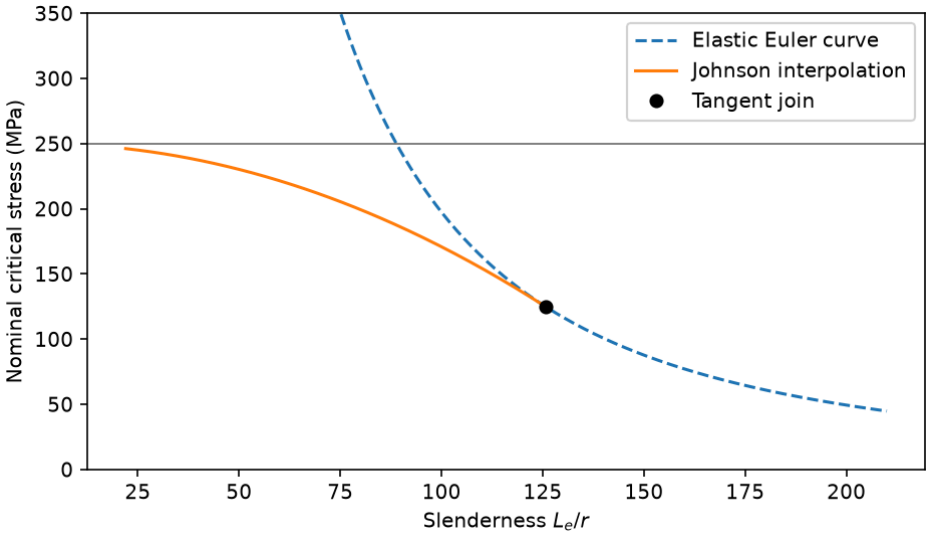
$$c = \frac{S_y^2}{4\pi^2 E},$$

$$\lambda_c = \sqrt{\frac{2\pi^2 E}{S_y}}.$$

For $E = 200000$ MPa and $S_y = 250$ MPa the join is 125.7. At $\lambda = 70$ the interpolation predicts 211.2 MPa whereas elastic Euler predicts 402.8 MPa. The parabola is a material/design model. Measured residual stress, end restraint and a prescribed column curve can change the permissible design load.

Why a tangent modulus appears

Linearise a measured compressive material curve about the current stress. An incremental strain changes stress by $E_t d\epsilon$. Under uniform continued loading, replacing E by $E_t(\sigma)$ in the incremental bending resistance leads to an implicit tangent-modulus estimate. Unloading fibres may respond with a different modulus. Check which constitutive path the test actually follows.



Elastic buckling overpredicts the stress of the intermediate column. The two model curves meet with a common tangent at slenderness 125.7.

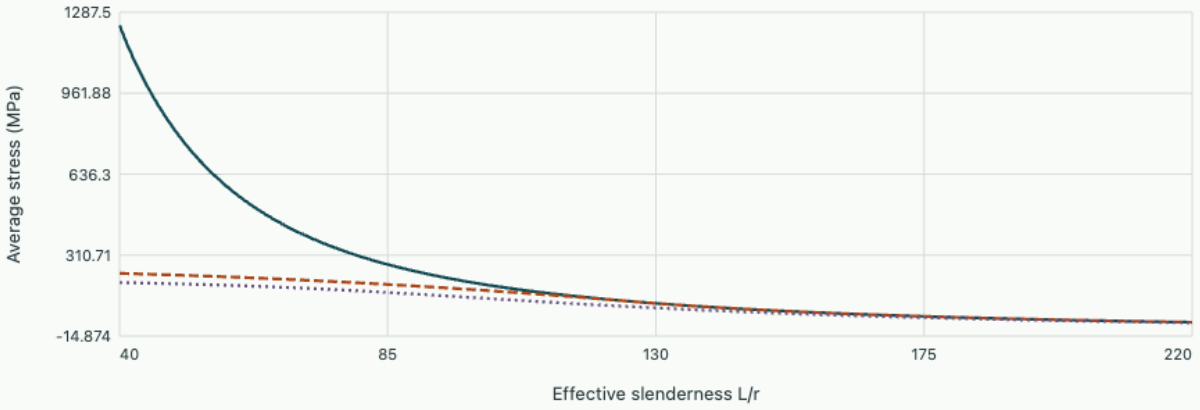
Interactive calculation · Compare yielding, eccentricity and an elastic eigenvalue

Change elastic stiffness and yield stress before adding eccentricity. The secant branch solves the first-fibre-yield equation below its elastic pole; it is not a post-buckling strength model.

Open this live calculation

Try the calculation

Elastic modulus (GPa)	200
Yield stress (MPa)	250
Eccentricity parameter ec/r²	0.2



Euler elastic eigenstress | Johnson–Euler interpolation | First yield with eccentricity

Smooth Johnson/Euler join at slenderness 125.664. At L/r=100 eccentric first yield occurs at 138.945 MPa. These are conditional model estimates, not an industrial column rating.

Derive the secant amplification under eccentric load

Apply equal end eccentricities e on the same side of an ideal pin-ended member of length L. A section at lateral deflection v sees moment magnitude $P(e+v)$. Taking positive v toward the load line gives $EIv'' = -P(e+v)$. Set $k^2 = P/EI$, use symmetry $v'(0) = 0$ at midspan, and enforce $v(L/2) = 0$ at a pin. Thus $e + v(x) = e\cos(kx)/\cos(kL/2)$. At the outer fibre of distance c_f,

$$\sigma_{\max} = p\left[1 + \frac{ec_f}{r^2}\sec\left(\frac{\pi}{2}\sqrt{p/p_E}\right)\right],$$

$$p = P/A,$$

$$p_E = P_E/A.$$

For $\lambda = 100$ and $ec_f/r^2 = 0.20$, $p_E = 197.39$ MPa. Solving the monotone pre-buckling branch for first yield at 250 MPa gives $p = 138.94$ MPa. The formula locates first fibre yield while the member remains nominally elastic. Post-yield redistribution needs an updated material and section model.

Combine an initial bow, locked-in stress and incremental stiffness

Make the constitutive assumption explicit with four equal-area longitudinal fibres at $y = \pm a$ and $\pm a/3$. Use compression positive. At zero external load, both outer fibres have residual compression +60 MPa and both inner fibres have residual tension −60 MPa. The initial resultant force and bending moment vanish, but subsequent yielding starts at different applied loads. Take $E = 200000$ MPa, compressive knee $S_y = 250$ MPa and continued-loading tangent $E_t = 4000$ MPa. Plane sections remain plane. The numerical model below linearises lateral response around the uniformly compressed reference path.

For axial strain increment $d\epsilon_0$ and curvature increment $d\kappa$, every fibre sees $d\sigma_i = E_i(d\epsilon_0 + y_i d\kappa)$. Summing force and moment gives the tangent matrix with $A_t = \sum E_i A_i$, $B_t = \sum E_i A_i y_i$ and $D_t = \sum E_i A_i y_i^2$. Holding incremental axial force fixed and eliminating $d\epsilon_0$ yields effective bending rigidity $D_b = D_t - B_t^2/A_t$. Here equal opposite fibres have the same tangent on the straight reference path, so $B_t = 0$. The outer and inner fibre pairs contribute 90% and 10% of geometric I, respectively.

The outer pair first reaches 250 MPa when $E\epsilon_0 = 190$ MPa. The inner fibres then carry 130 MPa and the mean applied stress is 190 MPa. Before this point $D_b/I = 200000$ MPa. Afterwards $D_b/I = 0.9(4000) + 0.1(200000) = 23600$ MPa until the inner pair reaches its knee at mean stress 251.2 MPa. With $a = 12$ mm, $r = 8.944$ mm and $L/r = 32$, the elastic and second-stage tangent eigenstresses $p_t = \pi^2 D_b/(AL^2)$ are 1927.66 and 227.46 MPa. The latter lies inside its assumed reference-load interval.

Now prescribe a small initial half-sine bow of amplitude $w_0 = 0.050$ mm, without load eccentricity. Approximate the additional lateral displacement by the same half-sine. At midspan the moment is Pw where w denotes the total amplitude, including the initial bow. Differentiating equilibrium gives $D_b(\pi/L)^2 dw = w\,dP + P\,dw$. Thus the amplification law for a branch of constant tangent is

$$(p_t - p)dw = w\,dp,$$

$$w(p) = w(p_a)\frac{p_t - p_a}{p_t - p}.$$

Carry the actual displacement across each change in stiffness. At $p = 190$ MPa the first branch gives $w = 0.05547$ mm. On the second branch $w(210) = 0.05547(227.46 - 190)/(227.46 - 210) = 0.1190$ mm. Replacing E by E_t at zero load would lose the prior history and use the wrong fraction of yielded material.

> **Where the tangent estimate stops being reliable**
>
> An initial bow makes yielding start at slightly different loads on individual fibres and vary along the span. This small-imperfection model evaluates tangents on the symmetric reference path and loses accuracy near its asymptote. For a finite imperfection, update fibres and unloading moduli station by station and follow the equilibrium path. The calculated 227.46 MPa is a tangent-reference threshold, not a measured collapse strength.

Keeping all reference data, estimate deflection at 220 MPa and identify which measurement best tests this model.

▼ **Worked solution**

The second branch gives $w = 0.05547(37.46)/(7.46) = 0.278$ mm. Opposed axial strain gauges at midspan, a lateral displacement reading and the initial unloaded shape separate compression, curvature and imperfection. Divergence from the predicted growth can reveal nonuniform yielding before the tangent asymptote.

Exercises on competing limits

At the join λ_c, what fraction of yield stress do both laws give?

▼ **Worked solution**

Substitute the expression for λ_c into either branch. Both give $S_y/2 = 125$ MPa and have the same negative slope, a direct check on the interpolation.

A 400 mm² pin-ended specimen has the geometry of the eccentric example. What maximum force first produces a nominally yielded fibre? What if the eccentricity doubles?

▼ **Worked solution**

The first force is $138.94 \times 400 = 55.58$ kN. In the secant bracket the coefficient increases from 0.2 to 0.4. Holding p fixed raises peak stress, so the new admissible root is lower. Re-solve before using it as a load rating.

Further reading · Hibbeler, Mechanics of Materials, 11th ed., Chapter 13, especially sections 13.4–13.7.

Elastic members assembled into mechanisms

Choose a load path, propagate its compliance and determine where contact or a bonded connection changes the boundary conditions.

1. Shaft reactions, twist and rotating bending
2. Preload, joint stiffness and separation
3. Tolerances, interference and torque transmission
4. Locating a part with six contacts and predicting datum error
5. Sizing the leaves of a compliant translation guide
6. Why increasing the length of a bonded overlap gives diminishing returns
7. When a suspended capacitor loses its stable equilibrium
8. Virtual work and the forces hidden by a constraint
9. Moving constraints, boundary work and a changing energy

Shaft reactions, twist and rotating bending

Machine elements and contact mechanics

Trace power and belt forces into shaft reactions, then check the shaft section for yielding, twist and cyclic bending.

FROM FIRST PRINCIPLES

Derive bending and torsional stress at a round shaft surface

A rotating shaft transmits torque through circumferential shear and can carry transverse loads through bending. Consider a slender, solid, circular shaft of diameter d under small elastic deformation. In pure bending, plane sections stay plane so longitudinal strain varies linearly with distance y from the neutral axis. In circular torsion, shear strain varies linearly with radius r. Choosing stress gradients to reproduce the applied moment and torque gives

$$M = \int_A y\sigma \, dA,$$

$$\sigma = My/I,$$

$$T = \int_A r\tau \, dA,$$

$$\tau = Tr/J.$$

Here $I = \int_A y^2 dA = \pi d^4/64$ and $J = \int_A r^2 dA = \pi d^4/32$. Taking the surface distance as $d/2$ gives the nominal maximum components.

$$|\sigma_b|_{\max} = \frac{32|M|}{\pi d^3},$$

$$|\tau_t|_{\max} = \frac{16|T|}{\pi d^3}.$$

Torque can be inferred from mechanical power using $T = P/\omega$, while bending moment requires transverse force and support equilibrium. A stationary transverse load produces alternating bending at a material point as the shaft rotates. The formulas assume a smooth section; shoulders and keyways alter local stress.

Background · Torsional stiffness

$$k_\theta = GJ/L$$

For a uniform elastic circular shaft use its polar second moment. A noncircular section generally needs a distinct torsion constant and warping analysis.

DERIVATION AND APPLICATIONS

Distribute a fixed-end shaft torque by twist compatibility

A shaft supported against rotation at both ends has an extra reaction torque that cannot be found from overall moment equilibrium alone. Consider two coaxial circular segments meeting at a collar. Apply a static torque T at the collar. The left and right outer ends are held fixed. Assume linear elastic shear, small twist, fully bonded interfaces and torque transfer without slip. Segment j has length L_j, shear modulus G_j and polar second moment J_j. Common collar angle θ twists each segment through the same magnitude.

$$\theta = \frac{T_L L_L}{G_L J_L} = \frac{T_R L_R}{G_R J_R},$$
$$T_L + T_R = T.$$

The symbols T_L, T_R are positive magnitudes of the moments resisting the applied torque. Define torsional stiffness $k_j = G_j J_j / L_j$ and substitute into equilibrium. The two segments behave as parallel torsional springs when observed from the collar.

$$\theta = \frac{T}{k_L + k_R},$$
$$T_L = \frac{k_L}{k_L + k_R} T,$$
$$T_R = \frac{k_R}{k_L + k_R} T.$$

For a solid round shaft $J_j = \pi d_j^4 / 32$. Local surface shear is proportional to torque times outer radius divided by J_j. The increase of torque carried by a stiffer segment therefore does not translate directly into the same factor of stress.

$$\tau_{j,\max} = \frac{T_j d_j}{2 J_j} = \frac{16 T_j}{\pi d_j^3}.$$

If one support becomes torsionally soft, its torque approaches zero and the stiffer side carries nearly all of T. The fixed-end assumption is especially important in using rotation measurements to identify a support reaction.

WORKED EXAMPLE

Split torque across two lengths of shaft

Two segments have the same 20.0 mm diameter and shear modulus 80.0 GPa, with lengths 0.200 and 0.400 m. The shorter segment is twice as stiff. A 90.0 N m collar torque splits into 60.0 and 30.0 N m.

The shorter segment reaches $16(60)/[\pi(0.0200)^3] = 38.2$ MPa surface shear. Its twist is $60(0.200)/(80.0 \times 10^9 J) = 0.00955$ rad, shared by the collar.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Select a diameter for equal torque share

Another shaft uses identical materials, with the right segment twice as long as the left. To carry equal torques require

$$d_R^4/L_R = d_L^4/L_L.$$

If $d_L = 25.0$ mm, $d_R = 25.0(2)^{1/4} = 29.7$ mm. Each carries half the applied torque, but the right surface stress is only $(25.0/29.7)^3 = 0.595$ of the left because its larger diameter provides greater torsional section modulus.

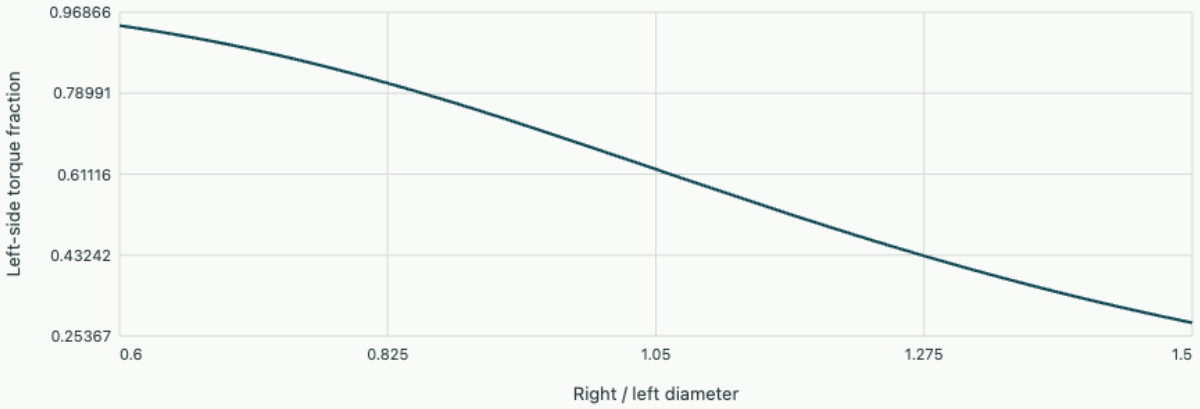
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Left length (m)	0.2
Right length (m)	0.4
Applied collar torque (N m)	90



Compatible torsion

At equal20 mm diameters, torque shares 60.000 and 30.000 N m. Equal sharing requires diameter ratio 1.189.

Rotation determines the torque, contacts determine the bending

A shaft transmits a given power through torque at a specified speed. Mechanical work in a small rotation is $dW = T\,d\theta$, so power is $T\omega$, with angular speed in radians per second. Belt tensions act at opposite sides of a pulley. Their difference gives the net driving torque. For two nearly parallel belt spans pulling in the same transverse direction their sum loads the shaft sideways. A belt with greater pretension can increase bearing loads although the transmitted power remains constant.

$$\omega = 2\pi n/60,$$

$$T = P/\omega = (F_1 - F_2)r,$$

$$F_\perp \simeq F_1 + F_2.$$

Choose bearing centres as ideal simple supports and identify where each torque enters or leaves the shaft. A section between an input coupling and an output pulley carries the transmitted torque. Beyond the output pulley it may carry transverse shear while carrying virtually zero torque. Mark the bending and torque diagrams separately so the combination at a critical cross-section represents loads that actually coexist there.

Determine the stress components from section geometry

For a solid circular shaft the bending second moment is $I = \pi d^4/64$, and its polar moment is $J = 2I$. The elastic bending stress at the outer fibre is $M(d/2)/I$; the torsional shear stress is $T(d/2)/J$. Under proportional elastic loading, the plane-stress von Mises expression then gives a static yield check for a ductile material.

$$\sigma_b = 32M/(\pi d^3),$$

$$\tau_t = 16T/(\pi d^3),$$

$$\sigma_v = \sqrt{\sigma_b^2 + 3\tau_t^2}.$$

The cubic diameter dependence makes a shoulder important even before its local stress concentration is considered. These equations describe a smooth prismatic section and nominal stresses. At a fillet, the maximum stresses may be estimated from geometric concentration factors appropriate to bending and torsion. For cyclic design a fatigue concentration factor may differ from the elastic geometric factor because sensitivity to the notch depends on material and radius.

Size the loading at the pulley seat

A 120 mm radius pulley sits at the middle of a 0.40 m span and draws 6.0 kW at 600 rpm. The two belt forces are parallel and their sum is 1800 N. The driving coupling lies immediately outside the left bearing. The shaft is 32 mm in diameter. From $\omega = 62.83\,\mathrm{rad/s}$, the transmitted torque is 95.5 N m and the tension difference is

$$95.5/0.120 = 795.8\,\mathrm{N}.$$

Combining difference and sum gives 1298 and 502 N.

Each bearing carries 900 N. Taking moments about a cut just to the left of the pulley gives

$$M = 900(0.20) = 180\,\mathrm{N\,m}.$$

There the shaft still transmits full torque. Substitution gives 56.0 MPa bending and 14.8 MPa torsional shear. Thus the nominal von Mises stress is 61.6 MPa, giving a nominal yield safety factor of 4.06 for steel with 250 MPa yield strength. This calculation has not yet accounted for a keyway or the pulley seat fillet.

Follow one material point through a revolution

A fixed vertical belt pull produces a steady bending moment in the laboratory. Mark a point on the rotating shaft surface. It passes through the tensile and compressive bending fibres once each per turn, so that its bending stress is fully reversed with amplitude 56.0 MPa. Steady driving torque supplies mean shear. A fatigue test at zero mean axial load describes only one part of this combined loading history. A shaft that spends a long time at constant speed can accumulate many stress cycles while the torque meter shows a level trace.

The twist angle is a further service requirement. Integrate $d\theta/dx = T/(GJ)$ along the torque-carrying length. With $G = 80\,\mathrm{GPa}$, $J = 1.029 \times 10^{-7}\,\mathrm{m}^4$, and a 0.20 m torque-carrying span, the relative twist is 0.00232 rad, or 0.133°. Bending stress varies as the inverse cube of diameter and twist at fixed torque as the inverse fourth power. A diameter selected from a strength limit can still give an unacceptable positioning error at the driven component.

Use the static deflection to estimate a shaft critical speed

The shaft diameter required to transmit torque need not satisfy the dynamic constraints of a rotating machine. For a first screening, model a relatively light simply supported shaft carrying a heavy disc at midspan. Treat the disc as a concentrated translating mass and neglect shaft mass, gyroscopic coupling, bearing flexibility and damping at first. Under a static transverse force F, bending theory gives midpoint deflection $\delta = FL^3/(48EI)$, hence lateral stiffness $k_s = 48EI/L^3$. With gravity loading only from the disc, $\delta_g = mg/k_s$. The ideal bending natural frequency then follows without explicitly measuring disc mass:

$$\omega_n = \sqrt{k_s/m} = \sqrt{g/\delta_g}.$$

A small rotating mass eccentricity e produces lateral forcing $me\Omega^2$ synchronous with speed. The undamped critical speed occurs when Ω equals ω_n. Gravity sag itself is a static offset in vertical operation of this horizontal shaft; its magnitude is useful because it reveals the ratio m/k_s. For a shaft with distributed mass or several mounted elements, one measured sag at a disc need not represent the relevant modal inertia. A compliant test stand also contributes to the measured deflection and can lower the assembled critical speed.

Adding simple viscous damping gives synchronous orbit-amplitude ratio $X/e = r^2/\sqrt{(1-r^2)^2 + (2\zeta r)^2}$, $r = \Omega/\omega_n$, for the ideal isotropic single-disc model. It approaches unity far above critical speed but can be much larger near it.

Check a shaft after increasing disc mass

A machine shaft carrying a balanced test disc deflects 0.40 mm at its centre when the disc’s weight is applied. The fixture was tared for its own sag. The estimated natural frequency is $\sqrt{9.81/0.00040} = 156.6\,\mathrm{rad/s}$, corresponding to 1496 rpm. A production target of 1200 rpm gives

$r = 0.802$. With small residual eccentricity $e = 0.050\,\mathrm{mm}$ and damping ratio 0.04, the predicted orbit amplitude is

$$0.050(0.802)^2 / \sqrt{(1 - 0.802^2)^2 + (0.08 \times 0.802)^2} = 0.089\,\mathrm{mm}.$$

A new disc doubles the dominant mass at the same shaft stiffness. Sag doubles to 0.80 mm and the estimated critical speed falls to 1057 rpm. The machine must now pass through that speed during startup and operates above it, despite the torque capability of the shaft being essentially unchanged. To restore its original critical speed under the doubled mass while retaining the span and material, stiffness must double. For a smooth solid circular shaft $I \propto d^4$, so

$$d_{\mathrm{new}} / d_{\mathrm{old}} = 2^{1/4} = 1.189.$$

A 30 mm shaft would need about 35.7 mm under this model. Restoring the natural frequency alone does not fix a bearing or coupling alignment issue, and a run-up through resonance requires transient analysis if the dwell time is significant relative to the decay time.

The static stiffness estimate assumes the midpoint force and the displacement gauge describe the same work-conjugate location and direction. An overhung disc or a gauge near a support can give a very different apparent stiffness. Recording the shaft layout alongside the sag value keeps the screening calculation reproducible when the assembly changes.

Why a single universal joint produces a twice-per-revolution speed fluctuation

A Hooke joint lets two intersecting shafts transmit rotation at a fixed misalignment angle β. Its cross carries two perpendicular trunnions. Choose the input axis as z, tilt the output axis about x, and measure input rotation θ and output rotation ϕ from aligned angular references. A unit vector along the first trunnion is $(\cos\theta, \sin\theta, 0)$. The second rotates in the tilted output plane and can be written $(-\sin\phi, \cos\beta\cos\phi, \sin\beta\cos\phi)$. Their dot product vanishes, giving $\tan\phi = \cos\beta\tan\theta$. Follow a continuous angular branch over a full turn.

Differentiate this constraint before averaging. Using the tangent relation to remove ϕ from the derivative gives

$$\frac{\omega_{\mathrm{out}}}{\omega_{\mathrm{in}}} = \frac{d\phi}{d\theta} = \frac{\cos\beta}{1 - \sin^2\beta\sin^2\theta}.$$

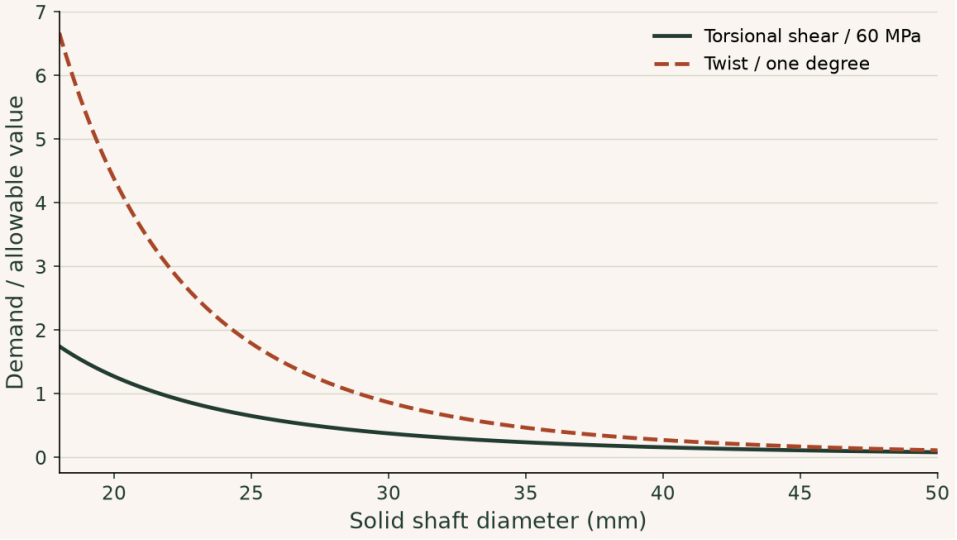
The extremes are $\cos\beta$ and $1/\cos\beta$. Both shafts complete one revolution in the same time, yet the instantaneous ratio changes twice per revolution. For small β in radians it is $1 - (\beta^2/2)\cos 2\theta + O(\beta^4)$. This identifies a second-order dependence on small misalignment and a forcing frequency twice the shaft frequency.

A test rig drives a single joint at a constant 600 rpm with $\beta = 20^\circ$. The driven shaft ranges from $600\cos 20^\circ = 563.8\,\mathrm{rpm}$ to $600/\cos 20^\circ = 638.5\,\mathrm{rpm}$. Its speed ripple repeats at

$2(600/60) = 20.0\,\mathrm{Hz}$. A large downstream inertia will therefore demand an alternating accelerating torque even if the external load is steady. A double-joint arrangement can cancel the kinematic ripple at the final shaft: with parallel outer shafts, equal joint angles and correctly phased forks, the second joint implements the inverse angular mapping of the first. The intermediate shaft still fluctuates, so bearing and torsional loads within the assembly merit separate treatment.

Will shaft strength or angular stiffness set the diameter?

A solid shaft transmits 120 N·m over 0.80 m with shear modulus 80 GPa. Limit nominal shear to 60 MPa and end rotation to one degree. Compare $\tau = TR/J$ and $\phi = TL/(GJ)$ only after dividing each quantity by its own allowable.



Strength requires about 21.7 mm, while the twist limit requires about 28.9 mm. The stiffness requirement controls this ideal span. The different diameter powers, three and four, also make the two margins respond differently to a small machining change.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

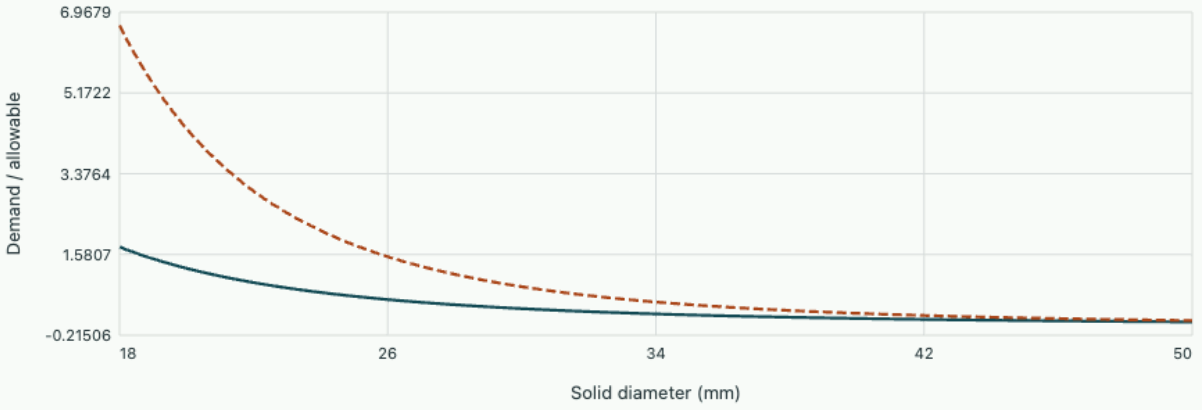
Raise torque and length separately. Compare how much diameter the strength and angular-stiffness conditions require.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Torque (N m)	120
Length (m)	0.8
Allowable twist (degrees)	1



Strength requires 21.677 mm; twist requires 28.929 mm.

Preload, joint stiffness and separation

Machine elements and contact mechanics

Compute how a tensile service load changes bolt force and clamp force, and determine when a closed-joint calculation ceases to apply.

FROM FIRST PRINCIPLES

Derive bolt and clamp increments from compatible deformation

A tightened bolt stretches while the clamped members compress. Before external loading their force magnitudes equal the preload F_i. Apply a separating load P and assume both components respond elastically and the interfaces stay closed. A small opening displacement δ stretches the bolt additionally and relieves the members by the same displacement. With stiffnesses k_b, k_m the increments are

$$\begin{aligned} \Delta F_b &= k_b \delta, \\ -\Delta F_m &= k_m \delta, \\ P &= \Delta F_b - \Delta F_m = (k_b + k_m)\delta. \end{aligned}$$

Here member force F_m is positive in compression. Eliminate the displacement to find both service forces and the fraction C of external load reaching the bolt.

$$\begin{aligned} C &= \frac{k_b}{k_b + k_m}, \\ F_b &= F_i + CP, \\ F_m &= F_i - (1 - C)P. \end{aligned}$$

Setting $F_m = 0$ predicts incipient separation at $P = F_i/(1 - C)$. Once the interface can no longer supply compressive reaction, the closed-joint stiffness partition ceases to describe the load path. The derivation explains why service bolt tension grows by only a fraction of applied load before separation.

Background · Joint stiffness fraction

$$C = k_b/(k_b + k_m)$$

Before separation, bolt and clamped-member stiffnesses decide what fraction of a separating external force increases the bolt load.

DERIVATION AND APPLICATIONS

Predict preload loss caused by settlement of a bolted joint

A tightened joint may lose clamp load before it sees any external service force. Model the bolt by axial stiffness k_b and the compressed members by stiffness k_m. Assume elastic incremental behaviour, closed contact, fixed temperature and a nut that does not rotate after tightening. Initial preload F_i stretches the bolt and compresses the members. A small permanent settlement s at asperities or interfaces consumes part of the initial imposed interference.

$$\Delta_{\text{installed}} = F_i \left(\frac{1}{k_b} + \frac{1}{k_m} \right),$$
$$\Delta_{\text{remaining}} = \Delta_{\text{installed}} - s.$$

The same preload passes through the bolt and compressed stack before an external tensile load is applied. Recovering their combined elastic displacement therefore reduces the preload according to their series stiffness. The result applies as long as positive clamp load remains.

$$F_r = F_i - k_{\text{eq}} s,$$
$$k_{\text{eq}} = \left(\frac{1}{k_b} + \frac{1}{k_m} \right)^{-1}.$$

Now apply an external separating force P. In a closed joint the bolt takes fraction $C = k_b/(k_b + k_m)$ of that force and the clamp decreases by $(1 - C)P$. Setting the residual clamp to zero gives the revised separation threshold.

$$F_{\text{clamp}} = F_r - (1 - C)P,$$
$$P_{\text{sep}} = \frac{F_i - k_{\text{eq}} s}{1 - C}.$$

Settlement is a displacement, so a more compliant bolt can reduce the preload lost per micrometre. Other design constraints, including stiffness, load introduction and fatigue, remain relevant when altering bolt geometry.

WORKED EXAMPLE

Audit a joint after a measured bedding-in displacement

Let $k_b = 200$ kN/mm, $k_m = 800$ kN/mm and $F_i = 24.0$ kN. The series stiffness is 160 kN/mm. A settlement of 15.0 µm reduces preload by $160(0.0150) = 2.40$ kN, leaving 21.6 kN. Since $C = 0.200$, the new separation load is 27.0 kN. Before settlement it would have been 30.0 kN.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Find an allowable settlement from a service condition

A separate joint has $k_b = 120$ kN/mm, $k_m = 360$ kN/mm and initial preload 18.0 kN. At maximum separating force 12.0 kN it must retain 5.00 kN clamp. The required residual preload is

$$5.00 + 0.750(12.0) = 14.0\,\mathrm{kN}.$$

The allowable preload loss is 4.00 kN and $k_{\mathrm{eq}} = 90.0$ kN/mm, yielding $s_{\max} = 0.0444$ mm. The stated requirement is more restrictive than merely keeping contact closed.

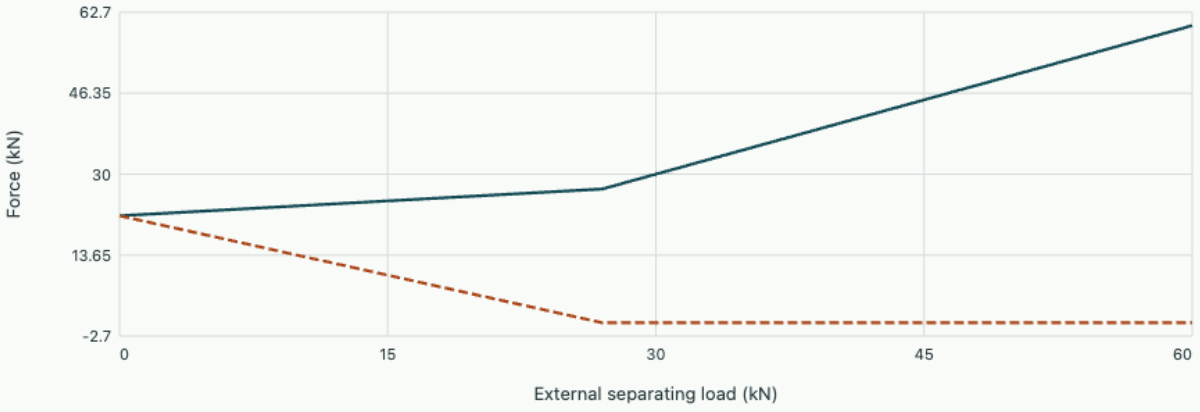
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Installed preload (kN)	24
Bolt stiffness fraction	0.2
Settlement (µm)	15



Retained preload 21.600 kN; separation at 27.000 kN. Equivalent settlement stiffness160 kN/mm is held fixed while the load fraction is explored.

Establish the state before the external load arrives

A tensioned bolt in an assembled flange has already elongated. The material clamped between head and nut has already shortened. In the ideal concentric joint these initial forces have equal magnitude F_i, as a free-body diagram of either flange shows. Tightening stores elastic energy in both parts. Calling a joint unloaded usually means that the external service load is zero. The thread root can still carry a significant tensile stress.

Represent the effective bolt grip by a tensile spring k_b and the compressed members by a spring k_m. For a uniform segment the axial stiffness is EA/L; threaded and plain segments in series have added compliances. The member stiffness depends on the path through which compressive force spreads from the head and washer. A thin flexible flange may also bend. For this derivation consider one bolt, an axial load introduced at the outside faces and no eccentricity or prying. Let clamp force denote a positive magnitude in compression.

Background · How does a small loss of grip thickness reduce preload?

Surface roughness can flatten under load after tightening. Suppose this reduces the unloaded thickness of the clamped stack by s while the nut stays on the same thread position. The recovered bolt elongation and the elastic recovery of the clamped members must together accommodate that lost thickness.

$$s = \Delta F \left(\frac{1}{k_b} + \frac{1}{k_m} \right),$$

$$\Delta F = \frac{k_b k_m}{k_b + k_m}\, s.$$

With bolt stiffness 200 kN/mm, member stiffness 800 kN/mm and settlement 0.010 mm, preload falls by 1.6 kN. This estimate assumes the stack is still in compression and both springs retain their tangent stiffness. A longer, more compliant bolt loses less preload for the same settlement because a given length change produces a smaller force change.

Apply compatibility to the load increment

An external separating force P allows the bolt to extend a little while the clamped members recover some of their initial compression. As long as the faces remain in contact, the extension and recovery have the same magnitude x. The bolt-force increase is $k_b x$ and the clamp-force decrease is $k_m x$. Force equilibrium across the joint gives

$$P = F_b - F_m = (k_b + k_m)x.$$

Eliminate x and introduce the load fraction C.

$$C = \frac{k_b}{k_b + k_m},$$
$$F_b = F_i + CP,$$
$$F_m = F_i - (1 - C)P.$$

The slopes of bolt force and clamp-force magnitude have opposite signs. The joint opens in this ideal model when F_m reaches zero, at $P_{\text{sep}} = F_i/(1 - C)$. Continuing the same line beyond that point would predict tension between unbonded faces. Once open, the bolt must carry the full external tensile force and the load slope changes to unity. Both expressions give the same bolt force at the instant of separation.

Check a service cycle and a separate overload

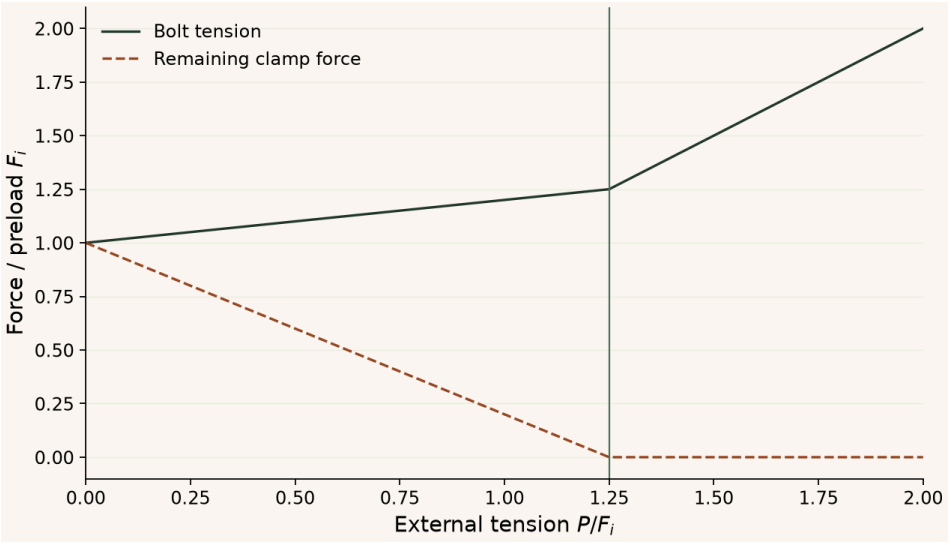
Take $k_b = 180\,\text{kN/mm}$, $k_m = 720\,\text{kN/mm}$, $F_i = 25\,\text{kN}$ and bolt tensile stress area $A_t = 58.0\,\text{mm}^2$. A pressure pulsation creates a separating force cycling between 4 and 16 kN. Here $C = 0.20$, hence bolt force ranges from 25.8 to 28.2 kN. Clamp force falls from 21.8 to 12.2 kN and remains positive. The bolt’s alternating tensile stress is the force range divided by twice its tensile stress area.

$$\sigma_a = (28.2 - 25.8)10^3/(2 \times 58.0) = 20.7\,\text{MPa}.$$

At the maximum ordinary load the calculated tensile stress is 486 MPa. If the specified proof stress is 600 MPa, the axial proof-load benchmark is $600(58.0) = 34.8\,\text{kN}$. The estimated separation threshold is

$$25/0.8 = 31.25\,\text{kN}.$$

A singular tensile overload of 34 kN opens the interface and brings the bolt to 34 kN, just below the supplied axial proof benchmark. It can nevertheless break the seal at the flange face. Different requirements can control the same event.



For a load-sharing factor of 0.20 the joint opens at 1.25 times its initial preload. The slope of bolt force against external force changes at separation.

Use attainable preload, including its scatter

Increasing preload by 3 kN raises both initial forces and delays separation by $3/(1-C) = 3.75\,\mathrm{kN}$. It also raises the peak bolt stress by $3000/58 = 51.7\,\mathrm{MPa}$ throughout the closed regime. Reducing bolt stiffness relative to the members can lower the alternating force. For example a longer effective grip can increase bolt compliance. Recheck the residual clamp and separation conditions whenever changing stiffness.

If tightening produces a range of preloads, assess separation with the low end and proof stress with the high end. An assembly tolerance of ±3 kN around the nominal value gives a minimum separation threshold of $22/0.8 = 27.5\,\mathrm{kN}$. The largest ordinary bolt force would be $28 + 0.2(16) = 31.2\,\mathrm{kN}$. The preload scatter enters these checks even though the external service force is perfectly known. In a real flange, surface settling, differential thermal expansion and off-axis loading can change the available clamp force after assembly.

Resolve tightening torque into thread lift and friction

Joint preload depends on the tightening method as well as bolt and flange stiffness. For an elementary torque estimate, unwrap one turn of the thread at pitch diameter d_2. Its lead l over the circumferential distance πd_2 defines helix angle $\lambda = \arctan[l/(\pi d_2)]$. For a square thread lifting axial load F_i, resolving an applied tangential force and Coulomb friction on the incline gives

$$F_{\tan} = F_i \tan(\lambda + \phi),$$

where $\tan\phi = \mu_t$. For a symmetric V-thread of half-angle α, a common small-helix approximation increases the effective friction to $\mu_t/\cos\alpha$ because the flanks carry a larger normal force.

Under the head or nut, sliding friction at effective radius r_b adds torque $\mu_b F_i r_b$. A simple total tightening estimate is

$$T \simeq F_i[(d_2/2)\tan(\lambda + \arctan(\mu_t/\cos\alpha)) + \mu_b r_b].$$

The first term includes ideal screw lift and thread friction; the second is bearing-face friction. In the limit of zero friction the thread term reduces to $F_i l/(2\pi)$, as verified by equating work over one revolution to axial load times lead. This lower bound also shows why most measured tightening torque can disappear into friction rather than useful bolt elongation. Effective bearing radius depends on contact geometry and pressure distribution, and the coefficients may differ between production lots.

Determine preload sensitivity to lubrication at fixed torque

Use an illustrative single-start thread with $d_2 = 9.0$ mm, $l = 1.50$ mm, flank half-angle 30° and effective bearing radius 7.0 mm. The helix angle is

$$\arctan[1.5/(\pi 9)] = 3.04^\circ.$$

With $\mu_t = \mu_b = 0.15$, the effective thread friction angle is 9.83°. The thread torque arm is $4.5\tan(12.87^\circ) = 1.03\,\mathrm{mm}$, and bearing friction contributes another 1.05 mm. A tightening torque of 40 N m therefore suggests $F_i = 40000/(1.03 + 1.05) = 19.2\,\mathrm{kN}$. The frictionless lift portion would be only $19.2(1.50)/(2\pi) = 4.59\,\mathrm{N\,m}$.

If lubrication reduces both coefficients to 0.08 at the same applied torque, the effective thread friction angle becomes 5.28°. The torque arms are now $4.5\tan(8.32^\circ) = 0.658$ mm and 0.560 mm. Implied preload rises to

$$40000/1.218 = 32.8\,\mathrm{kN}.$$

A tensile area of 58 mm² would then carry about 566 MPa before the service force arrived, compared with 331 MPa originally. An old torque setting may therefore raise preload substantially after a change of coating or lubricant. A direct elongation measurement can help distinguish this friction change from loss of clamp due to settling, although the effective bolt grip and modulus must also be known.

The axial stretch provides an independent expected signal. Suppose the effective uniform tensile length is 50 mm with $E = 210\,\mathrm{GPa}$ and area 58 mm². The two preload estimates imply elastic elongations $F_i L/(EA) = 0.079$ and 0.135 mm. An instrument resolving about 0.01 mm could distinguish the two states in this example if its gauge length includes the correctly modelled bolt segments. Thread and shank regions of different areas contribute as elements in series. A clamp-force reading taken after surfaces have settled can also differ from a tightening-time elongation record without a change in friction coefficient.

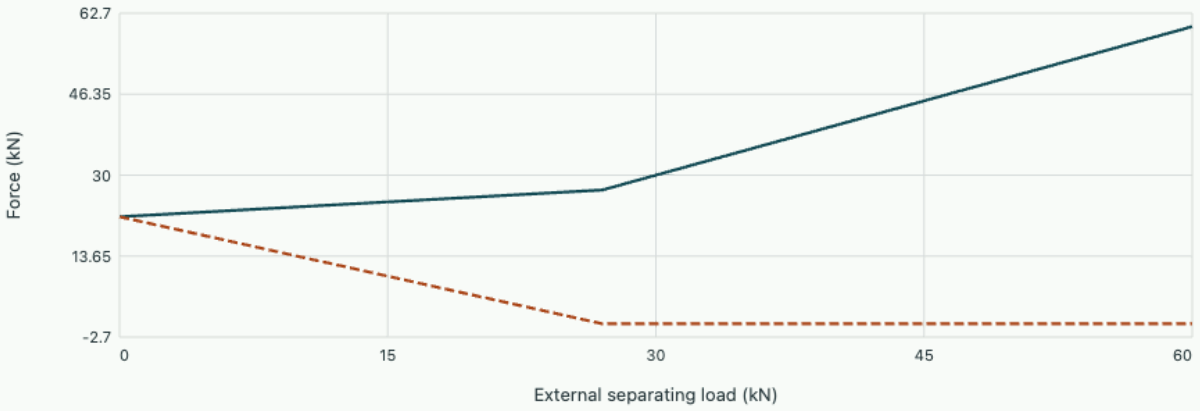
Interactive calculation · Change the inputs of this example

Change the parameters of this reference plot and compare its limiting cases. The fixed illustration remains beside the derivation; the adjustable calculation exposes the same dependence on geometry, material properties or loading.

Open this live calculation

Try the calculation

Installed preload (kN)	24
Bolt stiffness fraction	0.2
Settlement (µm)	15



Retained preload 21.600 kN; separation at 27.000 kN. Equivalent settlement stiffness160 kN/mm is held fixed while the load fraction is explored.

Tolerances, interference and torque transmission

Machine elements and contact mechanics

Turn shaft and hub limits into a range of contact pressures and check whether a cylindrical interference fit can transmit torque.

FROM FIRST PRINCIPLES

Relate diametral interference to frictional holding torque

An unassembled shaft can have a larger free diameter than its mating hub bore. Define positive diametral interference $I = d_s - d_h$. Under elastic assembly the shaft surface moves inwards and the bore outwards until their radii agree. Assume axisymmetry, uniform pressure p, nominal diameter d, contact length L, and combined radial compliance c_r giving total radial displacement per unit pressure. Compatibility gives

$$I/2 = |u_s| + u_h = c_r p,$$

$$p = \frac{I}{2c_r}.$$

Under ideal Coulomb friction, the local shear traction at impending uniform circumferential slip is μp. Every area element of the cylindrical contact has moment arm $d/2$. Integrating the limiting traction yields

$$T_{\lim} = \int_A \frac{d}{2}\mu p \, dA = \frac{\pi}{2}\mu p L d^2.$$

Uniform axial sliding instead has limiting force $F_{\lim} = \mu p \pi d L$. These rely on the same available friction traction, so simultaneous axial and torsional loading must share that capacity. The factor of two in compatibility matters because manufacturing limits usually specify diameters, while surface displacements are radial.

DERIVATION AND APPLICATIONS

Predict how differential thermal expansion changes an interference fit

A shaft and hub assembled with a press fit can lose contact pressure when their temperatures change. Let d be their common nominal diameter at reference temperature and $I_0 > 0$ the initial diametral interference, defined as free shaft diameter minus free hub-bore diameter. Assume small strains, uniform temperature in each part, free axial expansion and elastic response while contact is closed. Let α_s, α_h be the linear expansion coefficients and $\Delta T_s, \Delta T_h$ the temperature changes. To first order in strain and interference-to-diameter ratio,

$$I(T) = I_0 + d(\alpha_s \Delta T_s - \alpha_h \Delta T_h).$$

If c_r is the combined radial compliance per unit pressure, the radial mismatch is $I/2$. Positive pressure requires positive interference. When the calculated interference becomes negative the unloaded circular parts have clearance; the elastic closure formula then ceases to apply.

$$p(T) = \frac{\max[0, I(T)]}{2c_r}.$$

For uniform axial contact length L and constant friction coefficient μ, the maximum tangential traction is μp. Integrating that traction over the cylindrical surface, with moment arm $d/2$, translates a temperature difference into a torque capacity.

$$T_{\lim} = \frac{\pi}{2}\mu p L d^2,$$

$$\frac{T_{\lim}(T)}{T_{\lim}(0)} = \frac{\max[0, I(T)]}{I_0}.$$

This ratio assumes the elastic moduli, friction coefficient and dimensions appearing outside the interference stay effectively constant. Thermal gradients that distort the circular geometry can invalidate the uniform-pressure assumption before mean interference reaches zero.

WORKED EXAMPLE

Estimate capacity loss after uniform warming

A 50.0 mm steel shaft in an aluminium hub has $I_0 = 40.0$ µm. Their expansion coefficients are 12.0 and 23.0 µm/(m K). A common 40.0 K temperature rise reduces interference by $0.0500(11 \times 10^{-6})(40.0) = 22.0$ µm. Remaining interference is 18.0 µm. A measured reference capacity of 120 N m scales to $120(18/40) = 54.0$ N m in this model.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Choose a temporary assembly temperature

A separate 80.0 mm bore with expansion coefficient 18.0×10^{-6} K⁻¹ has 50.0 µm reference interference. With the shaft kept at reference temperature, a desired assembly clearance of 20.0 µm requires bore growth of 70.0 µm. The uniform hub temperature rise is

$$70.0 \times 10^{-6} / [0.0800(18.0 \times 10^{-6})] = 48.6\,\mathrm{K}.$$

The nominal interference returns after both parts settle back to the reference temperature, provided the material response and surfaces remain elastic.

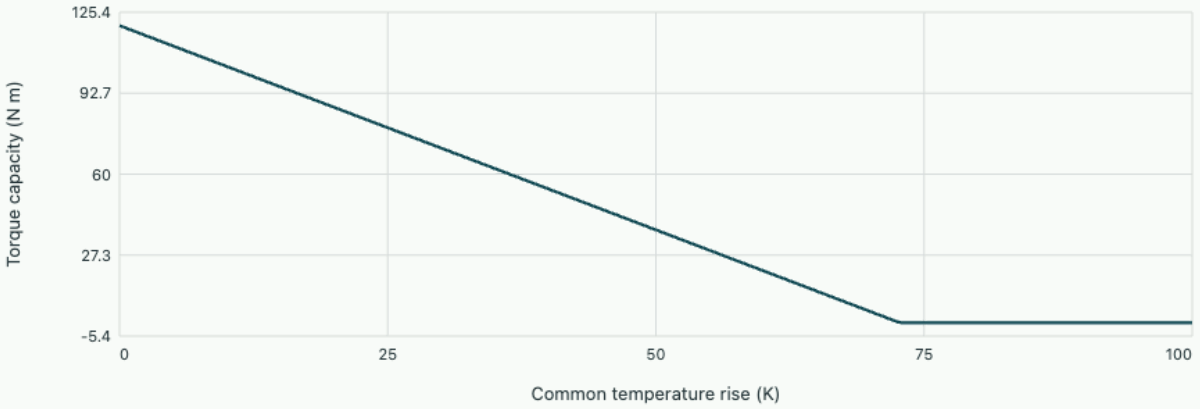
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

Initial interference (µm)	40
Hub expansion coefficient (µm/m K)	23
Reference torque capacity (N m)	120



Friction capacity under common warming

Contact is lost at common warming 72.727 K in this50 mm steel-shaft/hub model. Tensile contact pressure is replaced by zero; no artificial negative torque capacity is allowed.

What thermal clearance survives the tolerance stack?

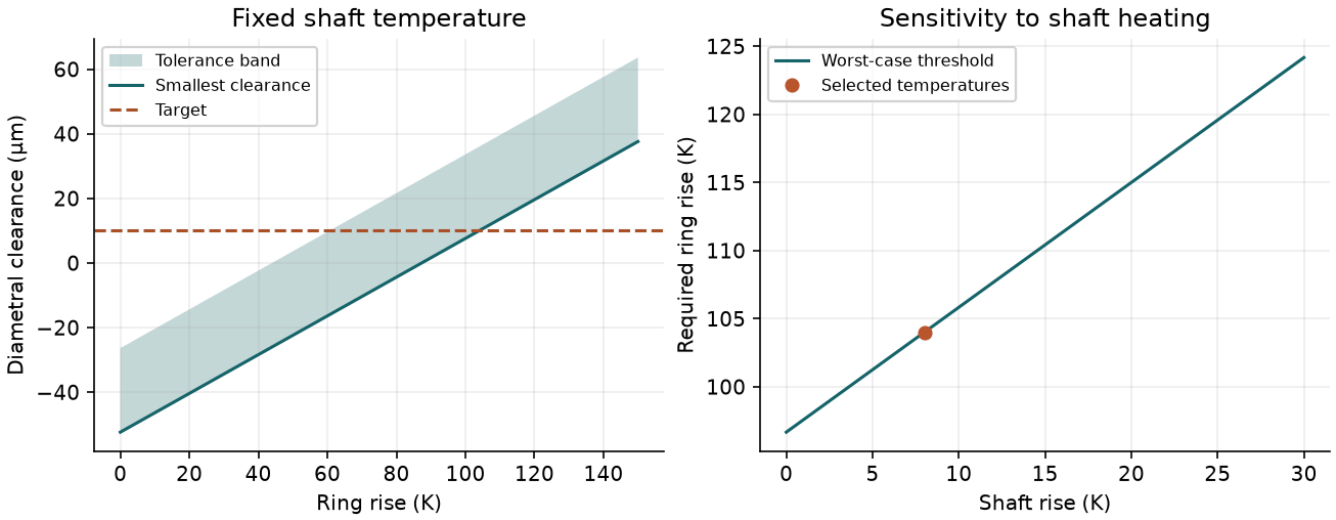
At room temperature a ring bore is slightly smaller than its mating shaft. An employee measures a temperature rise of 100 K at the ring and sees that the nominal pair should slide together. A second assembly made to the same drawings nevertheless has very little clearance. Resolve this apparent disagreement by carrying the diameter tolerance and the warming of the shaft through the thermal expansion calculation.

Write the initial interference as $i = d_{\mathrm{shaft},0} - d_{\mathrm{bore},0}$, positive for a press fit. In free thermal expansion a length changes by $\alpha d \Delta T$. Thus the available diametral clearance before engagement is the hot bore diameter less the current shaft diameter. Replacing their near-equal nominal diameters in the small thermal corrections by the common scale d gives the first-order relation

$$C = -i + d(\alpha_r \Delta T_r - \alpha_s \Delta T_s).$$

All terms are diametral. Using a desired radial clearance in this formula would understate the diametral target by a factor of two. The common-diameter approximation drops terms of order $i\alpha\Delta T$; products at that scale can matter for very tight metrology. Temperature rises here are relative to the same inspection reference.

Take $d = 50$ mm, interference between 22 and 48 µm, a ring expansion coefficient of 12×10^{-6} per kelvin and a shaft coefficient of 11×10^{-6} per kelvin. If the ring warms by 100 K its bore gains 60.0 µm. A shaft warming by only 8 K has already gained 4.4 µm. The remaining clearance consequently spans 7.6 to 33.6 µm. The largest assembly clears easily by eye, while the worst-case pair falls short of a specified 10 µm diametral clearance even though both nominal operating temperatures were observed.



A single temperature does not identify the available clearance. The width of the tolerance band survives a uniform shift of the entire band by thermal expansion.

Set the smallest clearance equal to the design target and solve for the ring rise. The example gives

$$(10 + 48 + 4.4)/0.60 = 104\,\mathrm{K}.$$

This calculation identifies the ideal free-diameter threshold. It does not set a permissible material or process temperature; gradients, heat loss before alignment, surface finish and mounting constraints require their own measurements. The notebook uses two independent temperature inputs so that the effect of the shaft can be examined directly.

For a population of assemblies it is tempting to quote the centre of the clearance range. With the nominal midpoint interference of 35 µm, the example predicts 20.6 µm of clearance. That value says nothing about the probability of meeting the target unless there is a measured distribution of part diameters. The two tolerance limits are geometric bounds. If each shaft is measured before matching it to a ring, substitute the measured interference into the same equation and retain only the uncertainty of that measurement. An assembly process using unmatched stock has to accommodate the limiting combination of the two drawings.

If both parts have the same thermal expansion coefficient and experience the same 100 K rise, what does this first-order calculation predict?

▼ **Work through the solution**

The two expansions cancel at the common nominal-diameter scale, leaving essentially the original negative clearance. Using exact distinct initial diameters gives a slightly *larger* interference, scaled by their common expansion factor. In the default example an error of just 10 K in the presumed shaft temperature shifts the clearance by 5.5 µm. Compare that sensitivity with the required assembly margin before reducing the model to a single ring thermometer.

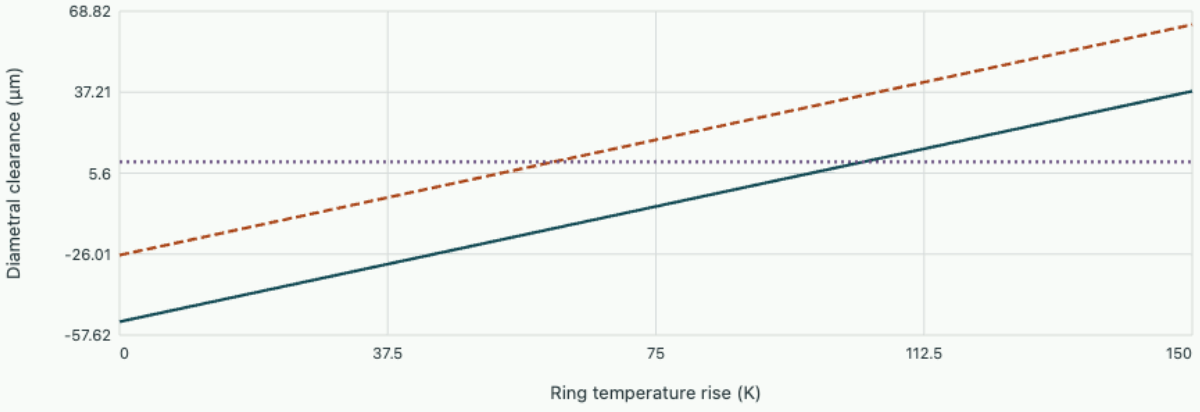
Interactive calculation · Fit clearance

Vary the notebook inputs and compare both plotted observables. The same equations generate the browser curves and the notebook reference; fixed material, geometry and model assumptions are stated in the readout.

Open this live calculation

Try the calculation

Largest interference (µm)	48
Shaft rise (K)	8
Required clearance (µm)	10



Worst-case ring rise 104.0 K. Diameter50 mm, ring expansion12×10⁻⁶/K, shaft11×10⁻⁶/K; lower interference22 µm.

Download the editable notebook for this example

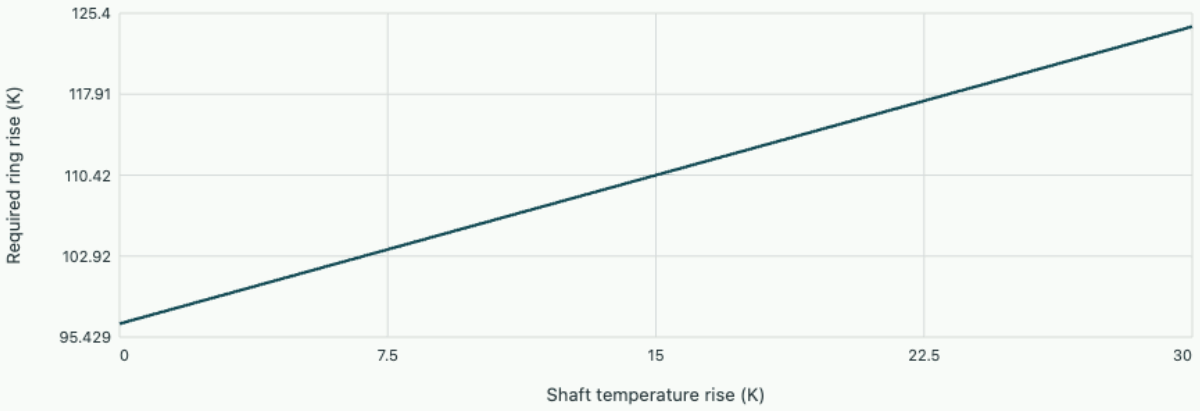
Interactive calculation · Fit threshold

Vary the notebook inputs and compare both plotted observables. The same equations generate the browser curves and the notebook reference; fixed material, geometry and model assumptions are stated in the readout.

Open this live calculation

Try the calculation

Largest interference (µm)	48
Shaft rise (K)	8
Required clearance (µm)	10



Worst-case assembly threshold

Worst-case ring rise 104.0 K. Diameter50 mm, ring expansion12×10⁻⁶/K, shaft11×10⁻⁶/K; lower interference22 µm.

Download the editable notebook for this example

Open editable notebook Download .ipynb

Convert manufacturing limits into assembly conditions

A shaft and a hub can share a nominal diameter while their manufactured diameters differ by several tens of micrometres. Name the diameters explicitly before subtracting them. For the present joint use positive diametral interference $I = d_s - d_h$, where d_s is the free shaft diameter and d_h the free bore. Assemble the smallest shaft with the largest hole to obtain the minimum interference. The opposite pair gives the maximum. Worst-case limits do not require a probability distribution for the production process.

Consider a 40 mm connection with shaft limits 40.030 to 40.045 mm and bore limits 40.000 to 40.015 mm. The minimum difference is 0.015 mm and the maximum 0.045 mm. Every permitted pair interferes. The radial mismatch is half the diametral interference because the radius changes on both sides of a diameter. Confusing these conventions introduces a factor of two in the inferred pressure and torque.

Relate geometric mismatch to elastic deformation

When the parts return to their common assembly temperature, contact pressure compresses the shaft and expands the bore. Assume cylindrical surfaces, a short open-ended elastic hub with negligible axial stress, a solid elastic shaft and approximately uniform pressure away from the ends. Define the radial compliance c_r as the sum of the inward shaft and outward hub displacements per unit interface pressure. Compatibility across the common radius then reads

$$\delta_r = I/2 = |u_s| + u_h = p\,c_r,$$

$$p = \frac{I}{2c_r}.$$

The combined compliance can be found from an elastic cylinder calculation or a verified test of geometrically identical parts. Providing it separately permits a torque calculation without assuming an infinitely stiff hub. A thinner hub generally expands further at the same pressure. Its larger compliance reduces the pressure developed by a given interference. Surface roughness that flattens during assembly can also reduce the effective interference relative to a measurement over the original asperities.

Integrate friction over the contact surface

A narrow surface patch of area dA can support a tangential friction force up to $\mu p\,dA$. All such forces act at radius $r = d/2$ and oppose impending rotation. Under uniform contact pressure and fully mobilised Coulomb friction, integrating around the circumference and along the overlap length L gives the limiting torque. For a purely axial pull the same friction forces act along the axis and their direct sum gives the limiting pull-out force.

$$T_{\lim} = \int r\mu p\,dA = \frac{\pi}{2}\mu p L d^2,$$

$$F_{\lim} = \mu p \pi d L.$$

Take $L = 35\,\mathrm{mm}$, $\mu = 0.12$, and $c_r = 0.50\,\mu\mathrm{m/MPa}$ in the 40 mm joint. The radial mismatches range from 7.5 to 22.5 micrometres, so calculated pressures range from 15 to 45 MPa. At the lower limit the tangential force capacity is $0.12(15)\pi(40)(35) = 7917\,\mathrm{N}$, and the torque capacity is $7917(0.020) = 158\,\mathrm{N\,m}$. At the upper limit pressure and capacity are three times larger. Use the lowest pressure to test transmission and the highest to evaluate hub stress.

Provide thermal clearance and account for combined service loads

Suppose only the hub is heated, so its bore grows approximately by $\alpha d \Delta T$. To slip the worst-case pair together with a further 0.020 mm of diametral assembly clearance, the bore must grow by $0.045 + 0.020 = 0.065\,\mathrm{mm}$. With $\alpha = 12 \times 10^{-6}\,\mathrm{K^{-1}}$ a uniform temperature rise of $0.065/[40(12 \times 10^{-6})] = 135\,\mathrm{K}$ is required in this ideal expansion calculation. Heat loss during transfer changes the actual temperature at insertion.

A shaft may also carry a sustained axial pull while delivering torque. Under the uniform-traction model the axial and circumferential components of the surface friction force share one bound. At the lower interference with a 4.0 kN pull the remaining circumferential capacity is $\sqrt{7917^2 - 4000^2} = 6832\,\mathrm{N}$, or about 137 N m. Checking the separate pure-load capacities would miss this interaction. If repeated microslip, an uncertain friction coefficient or stiffness near the hub edges is important, measurements of the actual assembly can determine which assumptions need closer investigation.

Derive interference pressure from thick-cylinder compliance

The radial compliance used above can be related to actual dimensions and materials. Consider a long enough cylindrical fit for a uniform central contact region, with free axial expansion so a plane-stress approximation is used for each cross-section. Let the common nominal radius be a, outer hub radius b, and positive pressure p. The solid shaft experiences uniform in-plane compression $\sigma_r = \sigma_\theta = -p$. Consequently its surface circumferential strain is $-p(1-\nu_s)/E_s$, giving radial shrinkage magnitude $pa(1-\nu_s)/E_s$.

For an elastic hub with free outside, radial equilibrium $d\sigma_r/dr + (\sigma_r - \sigma_\theta)/r = 0$ and axisymmetric compatibility give Lamé fields $\sigma_r = A - B/r^2,\ \sigma_\theta = A + B/r^2$. Setting $\sigma_r(a) = -p,\ \sigma_r(b) = 0$ gives bore hoop stress $\sigma_\theta(a) = p(b^2 + a^2)/(b^2 - a^2)$. Use $\epsilon_\theta(a) = [\sigma_\theta(a) - \nu_h \sigma_r(a)]/E_h = u_h(a)/a$. Adding hub expansion and shaft contraction leads to

$$I/2 = pa\left[\frac{1-\nu_s}{E_s} + \frac{(b^2+a^2)/(b^2-a^2) + \nu_h}{E_h}\right],$$

where I is the free diametral interference. The factor of two and the sign of radial pressure both affect the result. Significant axial restraint, plastic deformation, a hollow shaft or edge-pressure peaks require a different compliance model.

Compare thick and thin hubs at the same interference

A steel shaft and hub both have $E = 210\,\mathrm{GPa}$, $\nu = 0.30$. Their contact radius is 20 mm, hub outer radius 40 mm and measured diametral interference 30 micrometres. In mm–MPa units the combined radial compliance is

$$20[0.70 + (40^2 + 20^2)/(40^2 - 20^2) + 0.30]/210000 = 0.000254\,\mathrm{mm/MPa}.$$

Pressure is

$$0.015/0.000254 = 59.1\,\mathrm{MPa}.$$

The tensile hoop stress at the hub bore is

$$59.1(5/3) = 98.5\,\mathrm{MPa}.$$

For engagement length 25 mm and $\mu = 0.12$, torque capacity is $2\pi\mu p a^2 L = 446\,\mathrm{N\,m}$ approximately.

Now reduce the outer radius to 26 mm with the same parts at the fit surface. The hub geometry factor becomes $(26^2 + 20^2)/(26^2 - 20^2) = 3.90$, giving combined radial compliance $20(1 + 3.90)/210000 = 0.000467\,\mathrm{mm/MPa}$. Pressure falls to 32.2 MPa and torque capacity falls to about 243 N m. Yet bore hoop stress rises to $32.2(3.90) = 126\,\mathrm{MPa}$ because the thinner hub amplifies hoop demand more strongly. Reducing contact pressure alone is not evidence that the hub becomes safer. The same dimensional change can reduce torque capacity and increase peak elastic stress, so both checks should use the geometry of the actual mating parts.

For ductile elastic parts, the hub bore should also be assessed in its multiaxial state. In the thicker example its principal stresses are approximately $+98.5, 0, -59.1$ MPa under plane stress. The von Mises value is $\sqrt{98.5^2 + 59.1^2 + 98.5(59.1)} = 138\,\mathrm{MPa}$, above the tensile hoop stress by itself. For the thinner hub, using $+126, 0, -32.2$ MPa gives approximately 145 MPa. A comparison with an allowable uniaxial stress must therefore state the criterion applied. Once significant yielding occurs the linear compliance used to determine pressure is itself affected, and enlarging the nominal interference will no longer guarantee a proportional increase in transmitted torque. A pull-out or torque-to-slip test on representative surfaces checks the independent assumption about friction.

For an interference test, measuring the outside hub diameter change offers an additional observable. At the outer surface radial stress vanishes, so its hoop strain depends directly on the local tensile hoop stress.

In fact the hub’s outside hoop stress from the same Lamé field is $2pa^2/(b^2 - a^2)$. For the 40 mm outer radius it is $2(59.1)(400)/(1600 - 400) = 39.4\,\mathrm{MPa}$, so the predicted 80 mm outside diameter increases by $80(39.4)/210000 = 0.0150\,\mathrm{mm}$. For the thinner hub it is $2(32.2)(400)/(676 - 400) = 93.3\,\mathrm{MPa}$, giving an outside diameter increase $52(93.3)/210000 = 0.0231\,\mathrm{mm}$. The thinner outer wall moves more under a smaller interface pressure. Observing this change together with transmitted torque can separate an incorrect elastic pressure estimate from an incorrect friction coefficient.

For example, if the thick hub expands by the predicted 15 micrometres but slips at 330 N m under otherwise matching conditions, the implied mean friction coefficient is

$$0.12(330/446) = 0.089.$$

If the measured expansion were instead much smaller, an undersized effective interference, partial contact or a nonuniform fit would deserve investigation before calibrating friction. The micrometer reading is itself temperature sensitive: a uniform temperature difference of 10 K changes an 80 mm steel

diameter by about 9.6 micrometres at $12 \times 10^{-6}\,\mathrm{K}^{-1}$. Stabilise the temperatures before comparing the calculated elastic expansion with metrology.

Size the working flanks and shear section of a parallel key

A shaft can drive a removable hub through the flanks of a parallel key. The load path runs from the shaft keyway into one side of the key, across its shear section, and out of the opposite side into the hub. Let d be the shaft diameter. Represent the distributed tangential contact by the resultant $F = 2T/d$ at the nominal shaft radius. This model assumes a single fitted key carries the specified torque without credit for friction from a separate clamp.

There are two different projected areas. If b is the key width and L_e the effective engaged length, the shear plane has area bL_e. If h_e is the actual load-bearing radial height on the hub side, the projected bearing area on that flank is $h_e L_e$. The full key height includes material inside the shaft. Chamfers, end relief and clearance can further reduce the effective dimensions. Under a uniform-load sizing approximation,

$$\tau_{\mathrm{key}} = \frac{2T}{dbL_e},$$

$$p_{\mathrm{hub}} = \frac{2T}{dh_eL_e},$$

$$L_{e,\min} = \max\left(\frac{2T}{db\tau_{\mathrm{allow}}},\, \frac{2T}{dh_e p_{\mathrm{allow}}}\right).$$

For a conveyor pulley, $T = 180\,\mathrm{N\,m}$, $d = 40.0\,\mathrm{mm}$, $b = 12.0\,\mathrm{mm}$, $h_e = 3.30\,\mathrm{mm}$ and $L_e = 36.0\,\mathrm{mm}$. Work in N and mm, giving $F = 2(180000)/40 = 9000\,\mathrm{N}$. Mean shear is $9000/(12 \times 36) = 20.8\,\mathrm{MPa}$, whereas hub bearing pressure is

$$9000/(3.30 \times 36) = 75.8\,\mathrm{MPa}.$$

For design allowables of 45 and 90 MPa respectively, required engaged lengths are 16.7 and 30.3 mm. The proposed engagement passes these two static checks, with hub bearing pressure setting the longer minimum. A softer hub can therefore govern even when a much stronger key material is chosen. This calculation does not establish the shaft's fatigue strength at its keyway root. That requires the local shaft stress, particularly for rotating bending.

Locating a part with six contacts and predicting datum error

Machine elements and contact mechanics

A dimensionally accurate part can still be assembled in the wrong place. Use small rigid-body motions to work out what six contacts actually locate, then predict how dirt or a tilted guide appears in a measurement.

One frictionless contact supplies one scalar condition

Take the infinitesimal rigid motion of a body to consist of a translation $\mathbf{t}$ and a rotation vector $\boldsymbol{\varphi}$ in radians. A point initially at $\mathbf{r}_i$ moves by $\mathbf{t} + \boldsymbol{\varphi} \times \mathbf{r}_i$. A rigid locating pad at that point controls the component along its inward normal $\mathbf{n}_i$. If its normal height error is e_i, the displacement condition is

$$\mathbf{n}_i \cdot \mathbf{t} + (\mathbf{r}_i \times \mathbf{n}_i) \cdot \boldsymbol{\varphi} = e_i.$$

Stack the contact rows into a six-column matrix. Six independent rows locate a rigid part in three dimensions for small changes while the contacts remain seated. Three contacts in a plane support translation normal to the plane and two tilts. Two contacts on a second face locate one further translation and the remaining rotation. One on a third face sets the last translation. A preload must keep the unilateral contacts engaged. Six pads clustered along a single line would leave rotations undetermined despite the pad count.

Compute a tilted datum from three measured pad heights

Place the first three pads at coordinates $(0,0)$, $(a,0)$, $(0,b)$ on the nominal support plane. Normal displacement of a point (x,y) is $w = t_z + \varphi_x y - \varphi_y x$. Evaluate at the pads and subtract the origin row. The inverse relation gives

$$t_z = e_1,$$
$$\varphi_x = (e_3 - e_1)/b,$$
$$\varphi_y = -(e_2 - e_1)/a.$$

The predicted height of any feature is therefore an affine interpolation of the three heights. Outside their triangle it is an extrapolation and can magnify a small discrepancy. An additional hard pad away from the plane defined by the first three requires elastic deformation, a gap or loss of another contact. Its load sharing needs a stiffness model.

A particle under a pad and an encoder offset

Use $a = 120$ mm and $b = 80$ mm. A 12 µm particle raises only pad 2. Then $\varphi_y = -100$ µrad and $\varphi_x = t_z = 0$. A feature 180 mm from the origin in the x direction rises by 18 µm. Doubling the spacing a halves this sensitivity while leaving the particle height unchanged. The feature error varies with position and cannot be removed at every point by a single vertical zero adjustment.

Consider a horizontal translation stage whose encoder sits a vertical distance $d = 75$ mm from a tool point. A pitch change $\varphi_y = 80$ µrad produces a longitudinal difference $d\varphi_y = 6.0$ µm between the two lines of action. This is the first-order Abbe error derived from the same rigid-body rotation law. Encoder scale calibration alone cannot distinguish this error from translation at a single feature. Compare the extrapolated feature trace with the heights inside the support footprint.

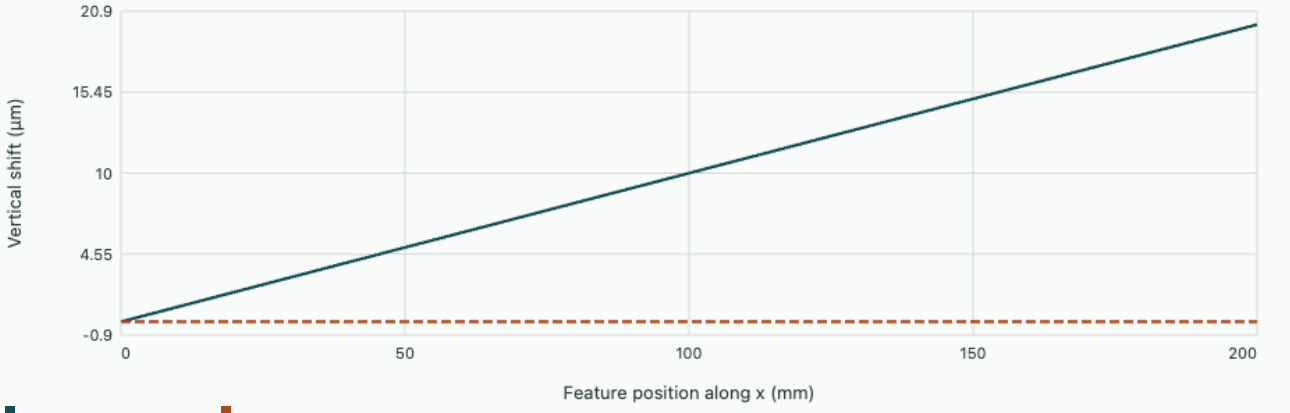
Fix one pad-height error and move pad 2 farther from the origin. Predict the shift halfway between the original pad positions versus the shift halfway between the new positions. Label the point whose physical location changes between those comparisons.

Open this experiment online

Try the calculation

Height error at pad 2 (µm) **12**

Pad spacing (mm) **120**



Pitch is -100.0 µrad. At 180 mm the feature shifts by 18.00 µm; at the second support it shifts by 12.0 µm.

Tutorial · reconstruct an entire datum plane

Pad heights at the three locations are 3, −3 and 11 µm. Find the two tilts and the vertical shift of a feature at (60,40) mm. What will a second probe at (120,80) mm read relative to nominal?

▼ **Show the plane reconstruction**

The differences are −6 µm over 120 mm and +8 µm over 80 mm, giving $\varphi_y = +50$ µrad and $\varphi_x = +100$ µrad. At the first feature the height is $3 - 3 + 4 = 4$ µm. At the second it is $3 - 6 + 8 = 5$ µm. The first point lies on the line joining pads 2 and 3, so its height is also their arithmetic mean. That geometric check detects a reversed rotation sign.

Further reading · MIT 2.72 · lecture 6 on constraint-based design

Sizing the leaves of a compliant translation guide

Machine elements and contact mechanics

A pair of thin leaves can guide a carriage with very little sliding friction. Their stiffness, stress and parasitic motion stem from the same bent shape. Derive those links before choosing a thickness or stroke.

Enforce the root and platform boundary conditions

Consider two identical rectangular leaves connecting a fixed base to a rigid guided platform. Each leaf has length L, width b, thickness t and bending stiffness $EI = Ebt^3/12$. Let the platform translate sideways by a small amount δ without rotating. Ignore shear deformation, axial tension and clamp compliance in the linear bending solution. Along each unloaded span $EIw'''' = 0$, so the transverse shape is a cubic polynomial. At x=0 displacement and slope vanish. At x=L impose $w = \delta$ and $w' = 0$. With $s = x/L$ the resulting shape and curvature are

$$w = \delta(3s^2 - 2s^3),$$

$$w'' = \frac{6\delta}{L^2}(1 - 2s).$$

The curvature changes sign at midspan. Each leaf has opposite end moments and maximal bending strain at both ends. A root modeled as a free pin would remove an essential slope condition and give the wrong stiffness.

Find guide force, surface stress and secondary motion

The end shear magnitude is $EI|w'''| = 12EI|\delta|/L^3$ per leaf. Because both leaves impose the same displacement their forces add. At the outer fibre the strain magnitude is $(t/2)|w''|$. Hooke's law gives

$$F = \frac{24EI}{L^3}\delta,$$

$$|\sigma|_{\max} = \frac{3Et|\delta|}{L^2}.$$

For inextensible leaves free to shorten their projected span, the arclength exceeds the projection by approximately $\frac{1}{2}\int_0^L (w')^2 dx$. Substituting the cubic shape gives $3\delta^2/(5L)$. This parasitic travel is second order and has the same direction for positive or negative displacement. Constraining that axial motion instead introduces an axial force and nonlinearity. At large stroke use an appropriate geometric model for both stress and stiffness.

A stiffness target can conflict with a stress target

Choose $L = 40$ mm, $b = 10$ mm, $t = 0.30$ mm and $E = 200$ GPa. For each leaf $I = 2.25 \times 10^{-14}$ m⁴. The two-leaf stiffness is 1688 N/m. At 1 mm stroke the required force is 1.69 N, peak stress is 112.5 MPa

and the free projected shortening is 15 µm. This small axial travel can matter when the stage is used for optical alignment.

Doubling the leaf thickness to 0.60 mm multiplies stiffness by eight, so 13.5 N is needed at the same stroke. Peak stress doubles to 225 MPa because surface fibres are farther from the neutral axis. In contrast, doubling width doubles stiffness without changing surface stress at a specified displacement. At a specified force the comparison changes because the stiffer guide travels less. Always identify which quantity is prescribed before drawing a design conclusion.

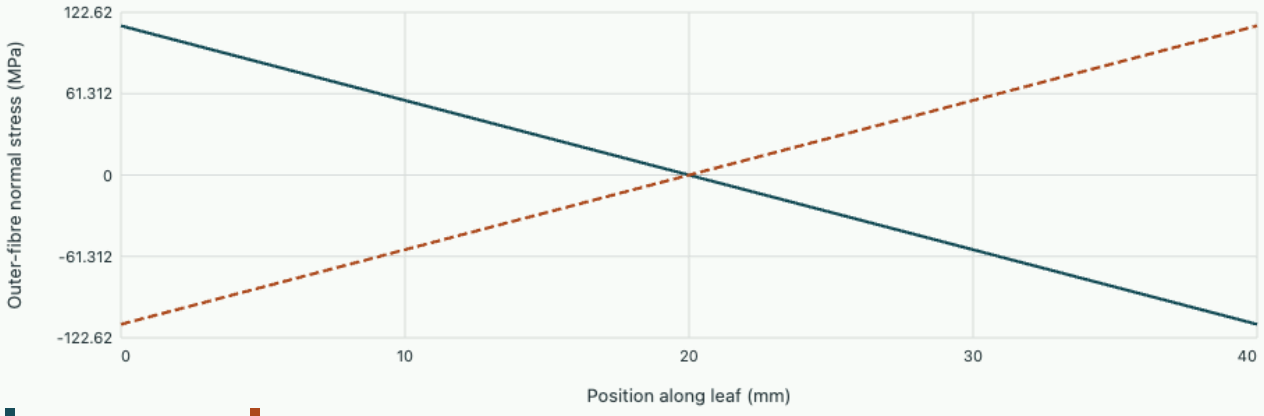
Hold the requested stroke fixed and double the leaf thickness. Check the ratios of force, peak stress and shortening against their different powers of thickness. Then reduce stroke until the original peak stress is recovered.

Open this experiment online

Try the calculation

Leaf thickness (mm) **0.3**

Stage stroke (mm) **1**



The guide force is 1.69 N, peak nominal stress is 112.5 MPa and projected shortening is 15.0 µm.

Tutorial · select stroke and compare an end moment measurement

Return to the original guide and restrict nominal peak bending stress to 90 MPa. Find the corresponding stroke, force and end-moment magnitude in one leaf. Estimate the projected shortening at that stroke.

▼ **Show the displacement-limited design**

Solving the stress expression gives $\delta = 0.80$ mm. Force is $1687.5(0.0008) = 1.35$ N. The end moment magnitude is $EI(6\delta/L^2) = 0.0135$ N m. The leaf end shear is 0.675 N; multiplying it by half the span gives the same end-moment magnitude. The predicted projected shortening is

$3(0.0008)^2/[5(0.040)] = 9.6$ µm. Local clamp stress concentrations require the actual corner geometry.

Further reading · MIT 2.72 · lecture 8 on flexure bearings

Why increasing the length of a bonded overlap gives diminishing returns

Machine elements and contact mechanics

Two slender straps transfer a tensile load through adhesive shear. Average stress decreases in inverse proportion to overlap length, but edge stress can approach a finite plateau. Derive the difference by coupling the axial extension of each strap to its neighbour.

Convert adhesive distortion into a distributed force

Consider identical adherends of thickness t, width b and Young's modulus E. A thin adhesive of thickness t_a and shear modulus G connects them over a length L. A tensile force P enters through strap 1 at the left and leaves through strap 2 on the right. Let n_1, n_2 be axial force per unit width, with $n_1 + n_2 = N = P/b$. Exclude bending and peel to isolate the symmetric extensional load-transfer mechanism. This idealization is more credible for an aligned or symmetrically balanced joint than for an eccentric single-lap joint.

Choose coordinates so the transmitted shear $\tau = -n_1' = n_2'$ is positive. Axial strains are $u_i' = n_i/(Et)$. Taking adhesive shear in the load-transfer direction as $\tau = (G/t_a)(u_2 - u_1)$ gives

$$\tau' = G(n_2 - n_1)/(t_a E t).$$

One more derivative eliminates the member forces:

$$\tau'' - \lambda^2 \tau = 0,$$

$$\lambda = \sqrt{\frac{2G}{t_a E t}},$$

$$\int_{-L/2}^{L/2} \tau \, dx = N.$$

The inverse transfer length $1/\lambda$ is a mechanical length scale. A stiff adhesive transfers load over a shorter distance. A stiffer or thicker adherend distributes the required relative extension over a larger region. Reversing coordinate labels reverses force derivatives consistently without changing the predicted stress magnitude.

Apply symmetry and recover the edge-to-mean ratio

Identical adherends loaded at opposite ends give a shear profile symmetric about the middle. Set $\tau = C\cosh(\lambda x)$ and choose C so its integral equals N. At the centre and edge this yields

$$\tau(x) = \frac{N\lambda \cosh(\lambda x)}{2\sinh(\lambda L/2)},$$

$$\frac{\tau_{\text{edge}}}{N/L} = \frac{\lambda L}{2}\coth\frac{\lambda L}{2}.$$

For a very short overlap the ratio approaches one because $\sinh y \sim y$. For a long overlap the edge stress approaches $N\lambda/2$ even while N/L continues to decrease. The central portion then carries little shear. The model computes a smooth nominal adhesive stress; it does not resolve singular edge detail, peel or a local failure criterion.

Compare an average-based estimate with the load-transfer profile

Take $P = 1000$ N, $b = 20$ mm, $E = 70000$ MPa, $t = 2$ mm, $t_a = 0.20$ mm and $G = 300$ MPa. The force per width is 50 N/mm and $\lambda = 0.1464$ mm⁻¹, giving a transfer length of 6.83 mm. With $L = 20$ mm the mean stress is 2.50 MPa, but the edge sees 4.07 MPa and the centre only 1.79 MPa.

At $L = 40$ mm mean stress halves to 1.25 MPa. The edge stress falls only to 3.68 MPa and the centre drops to 0.393 MPa. The long-overlap limiting edge stress is 3.66 MPa. Thus using mean stress alone would suggest substantial unused room at the ends. Doubling the width at fixed P, in contrast, halves N and every stress in this one-dimensional model. Manufacturing constraints and adhesion quality can still change which failure mechanism occurs first.

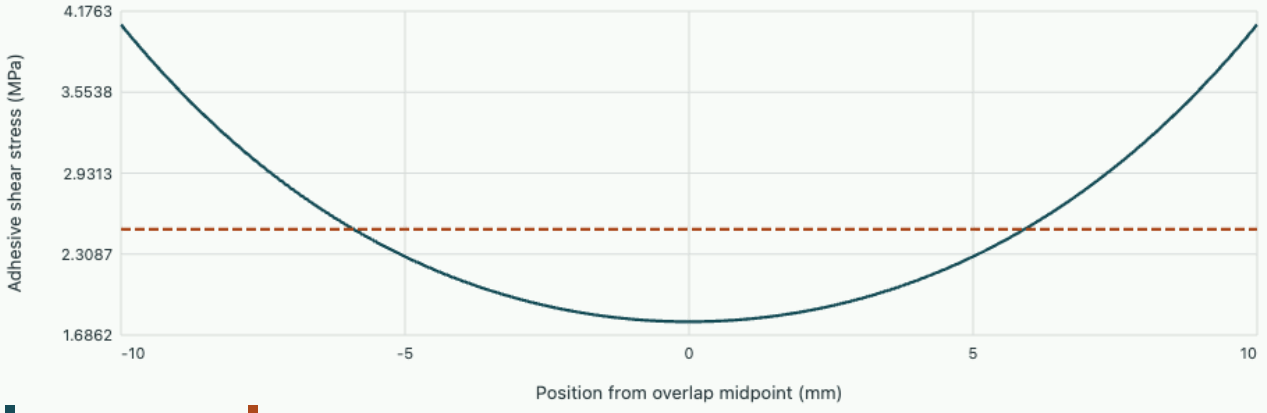
Double overlap length while keeping adhesive thickness fixed. Compare the percentage fall in centre and edge stress. Then increase adhesive thickness and predict where the newly transferred load is distributed along the overlap.

Open this experiment online

Try the calculation

Overlap length (mm) 20

Adhesive thickness (mm) 0.2



Shear-lag field Uniform average

Transfer length is 6.83 mm. The edge stress is 4.074 MPa compared with 1.789 MPa at the centre and 2.500 MPa averaged over the overlap.

Tutorial · tune load-transfer length using adhesive thickness

At the original E, t, G and force, a preliminary model calls for a long-overlap edge stress no greater than 3.0 MPa. Find the minimum adhesive thickness implied by that asymptote. Can a finite overlap at that exact thickness satisfy the same bound?

▼ **Invert the long-overlap limit**

The limit requires $\lambda \leq 2(3.0)/50 = 0.120$ mm⁻¹. Thus

$$t_a \geq 2G/(Et\lambda^2) = 0.298\,\mathrm{mm}.$$

For a finite length, $\coth(\lambda L/2) > 1$, so the edge stress at exactly that thickness stays slightly above 3.0 MPa. One must allow additional thickness, adjust the width, or evaluate a different load path. An adhesive change also alters deformation and peel, so this result is one design condition rather than a full joint qualification.

Further reading · Eindhoven University of Technology · adhesive single-lap joints and analytical shear-lag models

When a suspended capacitor loses its stable equilibrium

Electromechanics, optics and measurement

An electrode suspended on a spring can remain motionless at a bias that makes the calculated electrostatic force grow as it moves. Static force balance alone misses the voltage at which that equilibrium disappears. Deriving the small-signal frequency gives an independent measurement of the same instability.

Start with the electrical and mechanical control variables

A movable conducting plate of area A faces a fixed electrode. Its unstressed gap is g. Define positive displacement x toward the fixed plate, with spring force $-kx$, mass m and viscous damping coefficient b. For $0 \le x < g$ assume ideal parallel plates, negligible tilt and fringing, constant ϵ, and a linear elastic spring. The capacitance is $C(x) = \epsilon A/(g - x)$. The model ends when the plates approach contact or when the suspending structure departs from its specified elastic law. It cannot by itself calculate adhesion, contact resistance or post-contact motion.

Before differentiating electrical energy specify whether the source holds voltage or charge. At fixed charge Q, electrical energy is $Q^2/[2C(x)]$, and its negative displacement derivative gives the force toward increasing x. At fixed voltage V, a source supplies $V dQ$ as the plates move. The appropriate potential for mechanical equilibrium is the spring energy plus the field energy minus QV, which here is $\mathcal{U}_V = kx^2/2 - C(x)V^2/2$. Differentiating gives the outward spring reaction and the attractive electrical force $F_e = V^2 C'(x)/2 = \epsilon A V^2/[2(g - x)^2]$. An independent derivation multiplies the conductor pressure $\epsilon E^2/2$ by area with $E = V/(g - x)$. Both give a positive force. Replacing the source potential with the stored energy at fixed voltage would reverse its sign.

Locate all equilibria before choosing a branch

Define $s = x/g$ and $\Lambda = \epsilon A V^2/(2kg^3)$. The static force equation reduces to

$$\Lambda = s(1 - s)^2.$$

At a specified voltage one can find s by scanning the curve $s \mapsto s(1 - s)^2$ over the physically admissible interval $0 \le s < 1$. Its derivative is $(1 - s)(1 - 3s)$, giving an interior maximum $4/27$ at $s = 1/3$. For $0 < \Lambda < 4/27$ there are two physical roots. The smaller is a local minimum of $\mathcal{U}_V$ while the larger is a local maximum. Their merger is a fold of the static equilibrium branch:

$$x_{\mathrm{pi}} = g/3,$$

$$V_{\mathrm{pi}} = \sqrt{\frac{8kg^3}{27\epsilon A}}.$$

The often-quoted one-third displacement is the result for a linear spring, ideal voltage drive and this capacitance geometry. It is not a universal travel limit determined by the gap alone. Additional mechanical restoring forces or a different electrical boundary condition change the fold. Neither positive net electrical force nor the mere existence of a real solution is evidence of stable operation. A root finder

started near the almost closed gap can converge to the unstable large-displacement root with an extremely small static residual.

The two branches can be distinguished experimentally without deliberately reaching contact. At static equilibrium $kx = F_e$. Linearise the total force about x_0 while maintaining voltage at its prescribed value. The incremental restoring stiffness and corresponding weak-damping natural frequency are

$$k_{\text{eff}} = k - \frac{\epsilon A V^2}{(g - x_0)^3} = k\,\frac{1 - 3s_0}{1 - s_0},$$

$$f_{\text{small}} = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{m}}.$$

A small push toward the fixed electrode is restored only when $k_{\text{eff}} > 0$. The familiar electrostatic softening is the reason the smaller equilibrium is stable. The small-signal frequency tends to zero as the fold is approached. With appreciable damping the observed amplitude resonance and ringdown oscillation have slightly different frequencies; fit the full mechanical response rather than identifying an arbitrary peak with the undamped f_{small}.

Background · Can a sudden step pull in a statically stable actuator?

An ideal voltage step applied to a plate initially at rest can push it past the unstable equilibrium. During the motion at fixed voltage the effective potential per kg^2 is $u(s) = s^2/2 - b/[2(1 - s)]$, where $s = x/g$ and $b = \epsilon A V^2/(kg^3)$. Neglect damping and begin at $s = 0$. At the dynamical threshold the initial energy equals the saddle energy and the slope vanishes at that saddle.

$$u(s) = u(0) \implies b = s(1 - s),$$

$$u'(s) = 0 \implies b = 2s(1 - s)^2.$$

The two equations meet at $s = 1/2$, $b = 1/4$. The quasistatic fold gives $b = 8/27$. Consequently the undamped step threshold is only $\sqrt{27/32} \simeq 0.919$ times the slow-ramp pull-in voltage. A device operated between the two voltages can survive a ramp but collapse after an abrupt step. Dissipation and the actual rise time change this prediction.

Reconstruct the prepared state from a frequency measurement

Take a 100 µm square electrode in vacuum, $g = 2.00$ µm, $k = 2.00$ N/m and $m = 5.00 \times 10^{-10}$ kg. The predicted pull-in voltage is 7.32 V, the zero-bias capacitance 44.3 fF and the zero-bias natural frequency 10.07 kHz. On the low branch $s_0 = 0.20$ gives $x_0 = 0.400$ µm and voltage 6.80 V. Then $k_{\text{eff}} = 1.00$ N/m and the small-signal frequency is 7.12 kHz. A measured frequency ratio f/f_0 can be inverted via $s = (1 - r)/(3 - r)$, where $r = (f/f_0)^2$. The result is independent of absolute optical displacement calibration if the mass and underlying spring stiffness have not changed during the bias sweep.

At the same voltage the other physical root is $s = 0.48769$. A numerical force balance at this root would give the same supply reading but a negative tangent stiffness. The ideal voltage-controlled potential in units kg^2 is $u(s) = s^2/2 - \Lambda/(1 - s)$. The barrier height $kg^2[u(0.48769) - u(0.20)]$ for this example is 7.26×10^{-14} J. A sufficiently energetic disturbance can carry the plate out of the stable basin even below static pull-in. This is different from loss of the static equilibrium itself. A quasistatic model predicts the existence and local stiffness of the equilibrium; a transient simulation must additionally specify initial displacement, velocity, drive ramp and damping.

Change the preparation to fixed charge

Disconnect a perfect isolated capacitor after placing a known charge Q on it, neglecting leakage and parasitic capacitance. Then $F_Q = Q^2/(2\epsilon A)$ is independent of the remaining gap, because $D = Q/A$ is fixed. Static equilibrium is $x_0 = Q^2/(2\epsilon A k)$, with incremental stiffness exactly k. At the same initial voltage and gap the instantaneous force agrees with the voltage-controlled value, since $Q = C(x_0)V$. The derivatives governing subsequent motion differ. At $x_0 = 0.400$ µm the required charge in the numerical example is 0.376 pC. After ideal electrical isolation the small-signal frequency returns to the underlying elastic value even though the loaded static position stays the same.

Ideal charge isolation is delicate in a small device. A fixed parasitic capacitance C_p from the isolated node to a reference means the conserved charge belongs to $C(x) + C_p$. Motion changes the fraction of total charge residing on the actuator. The electrical energy becomes $Q_{\mathrm{total}}^2/[2(C + C_p)]$, so the force is $Q_{\mathrm{total}}^2 C'/[2(C + C_p)^2]$, generally displacement dependent. If $C_p \gg C$, the node voltage changes little with a small displacement and behavior is closer to voltage control. Interpreting an observed softening after disconnecting the supply as evidence against electromagnetic force balance would overlook this boundary change.

A different practical preparation puts an intentional series capacitor C_s between a fixed voltage supply and the actuator. Its series combination is $C_{\mathrm{ser}} = \epsilon A/[g - x + \epsilon A/C_s]$. For static mechanical work at fixed supply voltage use $F = V_s^2 C'_{\mathrm{ser}}/2$. Defining an effective electrical spacing $g_e = g + \epsilon A/C_s$, the static fold occurs at $x = g_e/3$ if this lies within the open physical gap. If $\epsilon A/C_s \geq 2g$, contact would occur before any such static fold in the ideal model. This calculation shows a direct connection between passive circuit design and allowed mechanical travel, while making clear that the maximum actuator voltage can differ from the voltage printed on the supply.

Why a sudden voltage change can reach contact below static pull-in

For a rapid electrical step the plate has a position and velocity inherited from the previous preparation. Suppose it starts at rest at its zero-bias position, the supply steps to a constant V much faster than a mechanical period, and mechanical damping is initially negligible. Immediately after the step, energy in the mechanical potential $u(s) = s^2/2 - \Lambda/(1 - s)$ is $u(0) = -\Lambda$ in units of kg^2. Motion under this constant voltage can reach the force barrier if the initial value exceeds the potential at the unstable

stationary point. This question can be solved by comparing two energies without integrating an entire time history.

Let s_b be the unstable root of $\Lambda = s_b(1 - s_b)^2$. At threshold the initial state is on the separatrix, the trajectory separating oscillations in the well from excursions to the closing gap. Set $u(0) = u(s_b)$, subtract the two expressions and cancel s_b from the nontrivial solution. The two conditions become

$$\Lambda = \frac{s_b(1 - s_b)}{2},$$

$$\Lambda = s_b(1 - s_b)^2.$$

Thus $s_b = 1/2$, $\Lambda = 1/8$, and the ideal undamped step threshold is

$$V_{\mathrm{step}}/V_{\mathrm{pi}} = \sqrt{27/32} = 0.919.$$

In our example this is 6.72 V. A slow increase to 6.80 V leads to the observed stable 0.40 µm deflection, whereas a sudden ideal voltage step to the same value from rest at the origin contains enough energy to escape the well. Starting on the biased equilibrium before applying a small perturbation is a different experiment. The direction of the threshold shift makes physical sense: damping removes mechanical energy during the first excursion, and a slow ramp allows the equilibrium to move with the plate. The undamped value specifies a particular ideal initial-value experiment, rather than a voltage rating for an unspecified drive waveform.

One can check a simulated step by monitoring $m\dot{x}^2/2 + \mathcal{U}_V$ after the supply settles. With viscous damping its derivative is $-b\dot{x}^2$, so any sustained increase during a constant-voltage segment points to a numerical or modelling error. Before the step ends there is an explicit drive term $-CV\dot{V}$ in this energy balance. The mechanical potential already includes the source work. Adding $V\dot{Q}$ to the same balance a second time gives an artificial energy imbalance. A transient solver should stop at a physically specified minimum gap instead of marching across the singularity; a statement about contact time depends on what counts as contact.

A finite source resistance changes what a ringdown measures

A dc supply with series resistance R can maintain the static voltage accurately while failing to hold the actuator voltage fixed over a mechanical cycle. Let its steady node voltage be V, with equilibrium capacitance $C = C(x_0)$. A small sinusoidal displacement δx causes current $i\omega(C\,\delta v + VC'\delta x)$ into the actuator. The small-signal supply voltage is zero, so $\delta v = -Ri$. Eliminating current gives

$$\delta v = -\frac{VC'}{C}\frac{i\omega\tau}{1 + i\omega\tau}\delta x,$$

$$\tau = RC.$$

Linearise $F_e = V^2C'/2$ allowing both voltage and position to vary. The direct derivative is $K_{e0} = V^2C''/2$, while the change from node voltage is $VC'\,\delta v$. For an ideal parallel plate $C''/2 = (C')^2/C$, so the two terms combine to the simple complex relation $\delta F_e = K_{e0}\delta x/(1 + i\omega\tau)$.

At low frequency the electrical force follows displacement with its fixed-voltage slope. At high frequency the slope vanishes, as it must for an isolated ideal plate carrying fixed charge over one oscillation. At intermediate frequency a motion-induced current dissipates heat in the resistor.

Bring δF_e to the left side of the mechanical equation. Its real part subtracts $K_{e0}/[1+(\omega\tau)^2]$ from the mechanical stiffness; its imaginary part adds $K_{e0}\tau/[1+(\omega\tau)^2]$ to the viscous damping coefficient. This damping is positive for a passive resistor. At $s = 0.20$ the static electrical stiffness is 1.00 N/m, the operating capacitance is 55.3 fF and $R = 400$ MΩ gives

$$\tau = 22.1\ \mu\text{s}.$$

A 7.2 kHz interrogation has $\omega\tau \simeq 1$. Only about half the static electrical slope survives and the added damping is roughly $11\ \mu\text{N s/m}$. Treat these as frequency-dependent coefficients in the driven susceptibility. Substituting the low-frequency stiffness into a ringdown fit at this frequency would give the wrong displacement when the fitted frequency is inverted. Fitting the complex response at several frequencies avoids that ambiguity.

An experiment can separate the circuit effect from mechanical nonlinearity. Keep the same static bias and displacement, insert two known series resistances, then measure both in-phase and quadrature response to a small sinusoidal force. The displaced equilibrium should remain essentially unchanged at dc if leakage stays small. A stiffness dispersion and dissipative peak that shift in proportion to $1/R$ support an electrical cause. Then repeat at a smaller oscillation amplitude. Amplitude-dependent frequency shift that survives the change in resistance can arise from nonlinear spring or force curvature, which the first-order susceptibility leaves out. A simple change to the external circuit has turned the slope of an energy curve into a prediction for a measured phase lag.

Compare force intersections for voltage and charge control

The initial gap is 2.00 µm, the area is a 100 µm square and the mass is 0.50 µg. Use the voltage ratio to move the force curve through the fold. At 93% of pull-in the low-displacement fixed-voltage intersection has a positive restoring slope. The high-displacement intersection has a negative slope. The charge-controlled line is prepared at the selected voltage at the original open gap. It gives the same initial force but a different subsequent curve. The readout reports both open-gap roots even if the upper root lies beyond the plotted interval. At the baseline stiffness, static pull-in is 7.32 V.

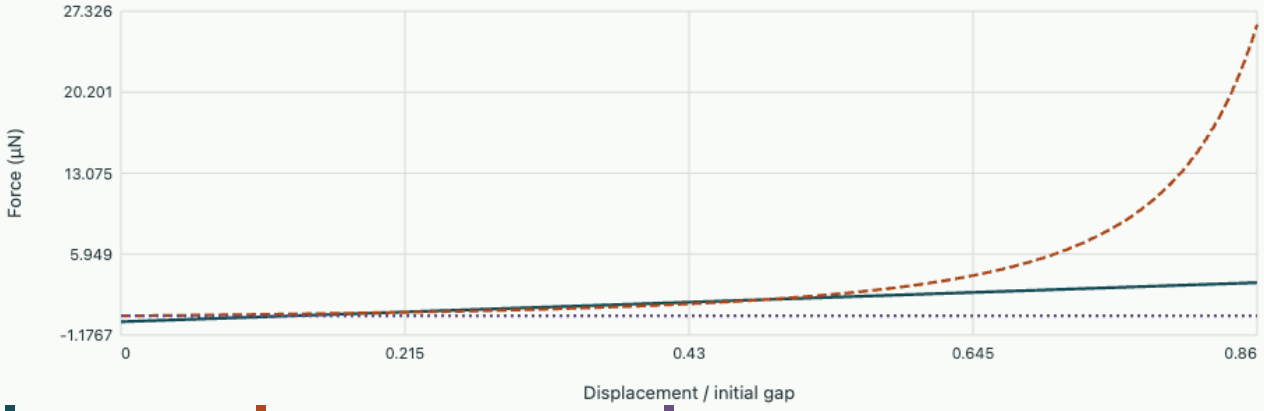
Change the spring stiffness while holding the voltage ratio fixed. The dimensionless locations of the roots stay in place, but their dimensional force and small-signal frequency change. This checks the scaling that eliminated the physical dimensions in the fold equation. Then set the ratio between the dynamic step threshold and unity. Use the force intersections to locate the potential maximum and sketch the area between attraction and spring reaction from the initial position. Its signed integral supplies the work which distinguishes a bounded excursion from crossing the maximum. The displayed curves predict initial-value behaviour only when combined with a choice of initial kinetic energy and damping.

Open this experiment online

Try the calculation

Bias as fraction of static pull-in **0.93**

Spring stiffness (N/m) **2**



Bias 6.805 V; static pull-in 7.317 V. Stable x/g = 0.200, unstable x/g = 0.487, small-signal frequency 7.11 kHz.

Tutorial · distinguish stability from force balance

Infer a gap fraction and a bias from a resonance

An otherwise identical actuator shows $f/f_0 = 0.80$ at a stable fixed voltage. Find the deflection as a fraction of gap, the voltage relative to pull-in, and the independent capacitance change. What could spoil this comparison?

▼ Work through the solution

Here $r = 0.64$ and $s = (1 - 0.64)/(3 - 0.64) = 0.1525$. The bias ratio is $\sqrt{(27/4)s(1-s)^2} = 0.860$ approximately. The parallel-plate capacitance should rise by $1/(1-s) - 1 = 18.0\%$. Measure the change relative to the active capacitance after removing any constant wiring capacitance. Disagreement can come from changing mode shape, a nonlinear spring, a tilted plate, a measurement frequency above the electrical equilibration bandwidth, or an incorrect extraction of mechanical stiffness from a strongly damped resonance. With a measured gap and mode shape these explanations yield different quantitative corrections.

Choose a series capacitor to move the static fold

The original 44.3 fF zero-bias actuator should remain on its stable slope as far as $x = 0.60g$, ignoring contact and other limits. Find the series capacitance whose fold falls exactly at this displacement. How

much supply voltage appears across the actuator at that point?

▼ **Work through the solution**

Set $g_e/3 = 0.60g$, giving $\epsilon A/C_s = 0.80g$ and $C_s = C(0)/0.80 = 55.4$ fF. At $0.60g$ the actuator capacitance is $C(0)/0.40 = 110.7$ fF. Series charge is identical in both capacitors, so

$$V_{\text{act}}/V_{\text{supply}} = C_s/(C_s + C_{\text{act}}) = 1/3.$$

Choosing exactly this value places operation at vanishing ideal stiffness, so a strictly positive measured restoring slope at that displacement requires a smaller series capacitance or a different margin. The result is sensitive to additional capacitance connected between the series node and ground.

Further reading

- Derive capacitor pressure from field stress and work

Virtual work and the forces hidden by a constraint

Analytical mechanics from constraints to invariants

A bead follows a wire, a pendulum remains the same length, and a rolling wheel obeys a relation between translation and rotation. Their reduced equations describe the motion but can conceal the largest forces in the apparatus. Determine which forces disappear under a virtual displacement, recover their values, and decide when the assumed motion ceases to be physically admissible.

Hold the clock fixed when defining a virtual displacement

A system of particles has Cartesian positions $\mathbf{r}_i$ and applied forces $\mathbf{F}_i$. A holonomic constraint is an equation $g_\alpha(\mathbf{r}_1, \dots, \mathbf{r}_N, t) = 0$. At a fixed instant an admissible infinitesimal change of configuration obeys $\sum_i \nabla_i g_\alpha \cdot \delta\mathbf{r}_i = 0$. The physical velocities obey a different relation, $\sum_i \nabla_i g_\alpha \cdot \dot{\mathbf{r}}_i + \partial_t g_\alpha = 0$. A virtual displacement compares geometrically allowed configurations at the same time. It has no prescribed duration and makes no claim that the system's momentum actually points along it.

The distinction becomes measurable for a bead on a moving horizontal table. Every allowed virtual displacement has zero vertical component, so the table's upward normal force gives zero virtual work. An upward-moving table can nevertheless raise the bead and perform positive ordinary work. Ordinary power involves the actual velocity, which includes the table motion. A careless statement that constraint forces do no work confuses two different operations. For a fixed ideal frictionless surface the real velocity is tangent as well, and vanishing virtual work then does imply vanishing ordinary power.

Let $\mathbf{R}_i$ be the reaction forces and use $m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i + \mathbf{R}_i$. Multiply by an admissible $\delta\mathbf{r}_i$ and sum over particles. We call the constraints ideal when $\sum_i \mathbf{R}_i \cdot \delta\mathbf{r}_i = 0$ for every allowed virtual displacement. Eliminating those reactions gives d'Alembert's principle,

$$\sum_i (\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i) \cdot \delta\mathbf{r}_i = 0.$$

This is a dynamical equation expressed through a static-looking virtual-work test. It applies instantaneously along the motion. It does not assert that the total force vanishes or that the acceleration is an independent virtual displacement. One advantage is that the detailed construction of a rigid support need not be known to determine the admissible tangential acceleration; its force can be calculated afterwards when designing the support.

Pass from virtual work to equations for coordinates

Choose independent coordinates q_a with $\mathbf{r}_i = \mathbf{r}_i(q, t)$ and independent virtual increments δq_a. The velocity is $\sum_a \partial_{q_a}\mathbf{r}_i\,\dot{q}_a + \partial_t \mathbf{r}_i$. With kinetic energy $T = \sum_i m_i\dot{\mathbf{r}}_i^2/2$, differentiation at fixed q, t gives

$$\partial_{\dot{q}_a} T = \sum_i m_i \dot{\mathbf{r}}_i \cdot \partial_{q_a}\mathbf{r}_i.$$

Differentiate in time. The product rule produces an acceleration term and a second term involving $d(\partial_{q_a}\mathbf{r}_i)/dt$. The latter equals $\partial_{q_a}\dot{\mathbf{r}}_i$, with the generalised velocities held fixed, and supplies $\partial_{q_a} T$.

Hence

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_a} - \frac{\partial T}{\partial q_a} = \sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_a}.$$

The projected applied force is

$$Q_a = \sum_i \mathbf{F}_i \cdot \partial_{q_a} \mathbf{r}_i.$$

Independence of the coordinate variations in d'Alembert's principle gives one equation $d(\partial_{\dot{q}_a} T)/dt - \partial_{q_a} T = Q_a$ per coordinate. When $\mathbf{F}_i$ comes from a potential $V(q,t)$, the conservative contribution is $-\partial_{q_a} V$. Other contributions such as a measured resisting torque remain explicit. A coordinate expressed as an angle has a generalised force in joules per radian; a coordinate expressed as a length has one in newtons. Checking $Q_a\,dq_a$ as work avoids mistaking a generalised force for a Cartesian component.

For a simple pendulum of length l and mass m, choose θ measured from downward vertical. Then $T = ml^2\dot{\theta}^2/2$ and $V = mgl(1-\cos\theta)$. The equation $ml^2\ddot{\theta} = -mgl\sin\theta$ follows at once. The string tension vanished under the angular displacement. It has not been shown to vanish physically. Projecting Newton's equation onto the direction from the bead toward the pivot gives

$$T_{\text{string}} = mg\cos\theta + ml\dot{\theta}^2.$$

This second observable is sensitive to speed even when the reduced angular acceleration depends only on angle.

Recover multipliers before choosing coordinates

Cartesian coordinates keep reaction directions explicit. Put all coordinates into a vector x, let M be the mass matrix and let G contain the rows ∇g_α. The tangent virtual displacements are precisely those satisfying $G\delta x = 0$. A reaction doing zero virtual work must be a linear combination of the normal covectors, $R = G^T\lambda$, when the constraints are independent. Newton's equation becomes

$$M\ddot{x} = F + G^T\lambda.$$

Differentiate the constraints twice along the trajectory: $G\ddot{x} + \gamma = 0$, where γ includes the known velocity, curvature and explicit time terms. Substituting for $\ddot{x}$ determines reactions from a much smaller system,

$$(GM^{-1}G^T)\lambda = -\gamma - GM^{-1}F.$$

For any vector c, $c^T GM^{-1}G^T c = (G^T c)^T M^{-1}(G^T c)$ is positive if G has independent rows and particle masses are positive. Dependent constraint equations make the individual multipliers ambiguous. For instance writing both $g = 0$ and $2g = 0$ declares the same geometrical restriction twice. Only the

combined reaction is then determined. The magnitude of a raw multiplier also depends on the units and normalisation of g. Reporting the force $G^T\lambda$ makes comparisons between alternative constraint descriptions meaningful.

Use $g = (x^2 + y^2 - l^2)/2 = 0$ for a planar pendulum, with y upward, so the gradient is the radius vector $\mathbf{r}$. Newton's equation reads $m\ddot{\mathbf{r}} = -mg\hat{y} + \lambda\mathbf{r}$, where the symbol g in the gravitational term denotes the acceleration magnitude. The acceleration constraint $\mathbf{r}\cdot\ddot{\mathbf{r}} = -v^2$ gives

$$\lambda = (mgy - mv^2)/l^2.$$

The inward tension is $-\lambda l = mv^2/l - mgy/l$. Since $y = -l\cos\theta$, this is the radial-force result already obtained. The multiplier in this convention has units of force per length. Using the radial-distance constraint instead would make its magnitude a force.

Background · What if the support can push but cannot pull?

Replacing a surface by an equality constraint assumes the object stays in contact. A particle sliding along the inside of a smooth vertical circular track illustrates the additional physical test. At the top, gravity and the inward normal reaction together must supply the centripetal acceleration.

$$N = \frac{mv^2}{R} - mg, \qquad N \geq 0.$$

At a top speed below $\sqrt{gR}$ the equality-constrained solution requires negative N. An ordinary track cannot exert that pull. Its normal force then falls to zero and the particle must follow free flight until it meets the track again. A bead threaded on a wire could receive the oppositely directed force and would follow a different trajectory. Both models have the same geometrical path during contact.

Recognise an inequality at a contact

A bead threaded on a stiff loop can experience reaction in either direction, whereas a string can pull only. A surface without adhesion can push only. Derive the constrained motion, calculate its reaction, and compare the required sign with the physical mechanism. When a calculated tension turns negative, replace the constrained trajectory by free flight until a new contact event. Continuing to integrate the pendulum equation would retain a mathematically valid circle and a physically unavailable pushing string.

A mass starts at the bottom of a frictionless vertical loop with speed v_b. Energy conservation gives $v_{\text{top}}^2 = v_b^2 - 4gl$, since the top is $2l$ higher. A string just remaining taut at the top requires $T_{\text{top}} = mv_{\text{top}}^2/l - mg \geq 0$, giving $v_b^2 \geq 5gl$. The energy-only criterion for reaching the elevation is $v_b^2 \geq 4gl$; the interval between these two values illustrates why a trajectory test must include reactions. With $l = 0.80\,\text{m}$, the threshold bottom speed is $6.26\,\text{m/s}$. At this threshold the bottom tension is $6mg$, six times the weight, even though the tension is about to vanish at the top.

Recover the forces on a wedge that is free to slide

Place a wedge of mass M on a smooth horizontal floor and a block of mass m on its smooth upper face. Let the slope descend to the right at angle α. Choose X as the wedge's horizontal position and s as distance down the face. The block's coordinates apart from fixed offsets are $x = X + s\cos\alpha,\ y = -s\sin\alpha$. Its relative velocity points down the slope but its absolute acceleration need not. The kinetic energy contains a mixed term $m\dot{X}\dot{s}\cos\alpha$. Dropping that term is equivalent to holding the wedge stationary while allowing it to recoil, two incompatible mechanical models.

The reduced equations are $(M+m)\ddot{X} + m\ddot{s}\cos\alpha = 0$ and $\ddot{s} + \ddot{X}\cos\alpha = g\sin\alpha$. The first is conservation of total horizontal momentum, since the block–wedge normal force is internal and the floor is frictionless. Elimination gives

$$\ddot{s} = (M+m)g\sin\alpha/(M + m\sin^2\alpha),$$

$$\ddot{X} = -mg\sin\alpha\cos\alpha/(M + m\sin^2\alpha).$$

The acceleration down the moving face exceeds $g\sin\alpha$ because the wedge moves left as the block slides right. Nevertheless the absolute vertical acceleration of the block remains less than or equal to free fall because the upper face pushes upward.

To find the block's normal force, project its absolute acceleration onto the outward unit normal $(\sin\alpha, \cos\alpha)$. The part proportional to $\ddot{s}$ cancels, leaving $\ddot{X}\sin\alpha$. Newton's equation along that direction gives

$$N = m(g\cos\alpha + \ddot{X}\sin\alpha) = mMg\cos\alpha/(M + m\sin^2\alpha).$$

With $M = 3.0\,\mathrm{kg}$, $m = 1.0\,\mathrm{kg}$, $\alpha = 30^\circ$, one finds $\ddot{s} = 6.04\,\mathrm{m/s^2}$,

$$\ddot{X} = -1.31\,\mathrm{m/s^2},$$

$$N = 7.84\,\mathrm{N}.$$

The normal based on a stationary wedge would be $8.50\,\mathrm{N}$. A force sensor under the moving face could distinguish the two models even when a short record of relative slope displacement cannot resolve the difference.

Which reaction does no virtual work here? The normal force on the block has virtual work $N\hat{n}\cdot(\delta X\hat{x} + \delta s\hat{t}) = N\sin\alpha\,\delta X$. The opposite normal force on the wedge gives $-N\sin\alpha\,\delta X$, so the sum cancels. Considering only the block and asserting zero work from the normal for arbitrary δX would be incorrect. At the real velocities the same reaction transfers energy between the bodies even though its net internal power on their pair is zero. The floor's normal reaction balances the vertical momentum change of the whole system. During the slide it is $(M+m)g - m\ddot{s}\sin\alpha$, less than total weight while the block accelerates downwards.

Take the limits as an additional test. At fixed m, α, increasing M to infinity returns the familiar fixed-wedge results. At a nearly horizontal face, downslope acceleration and wedge recoil vanish, restoring the full block weight as normal force. At a vertical face the normal vanishes and the block falls. These limits hold because the contact has no friction and can be maintained with a nonnegative normal. A

moving wedge of independently prescribed acceleration would replace the horizontal momentum equation by a control input, changing both the relative acceleration and the support force.

Distinguish rolling constraints from purely geometric restrictions

A cylinder of radius R, mass m and axial inertia I rolls without slip down a fixed rough incline. Choose coordinate s down the incline and spin angle ϕ in the corresponding sense. The constraint $\dot{s} = R\dot{\phi}$ is integrable in this one-dimensional motion, so it can be inserted as $s - R\phi = \text{constant}$. The resulting kinetic energy is $(m + I/R^2)\dot{s}^2/2$. The reduced equation gives

$$\ddot{s} = g\sin\alpha/[1 + I/(mR^2)].$$

Reconstruct the static friction from translation, $f = gm\sin\alpha - m\ddot{s}$, directed up the slope, and compare it with the available maximum $\mu_s mg\cos\alpha$.

For a homogeneous solid cylinder $I = mR^2/2$, the speed-up is $2g\sin\alpha/3$ and friction magnitude $mg\sin\alpha/3$. A ring of the same mass and radius has $I = mR^2$, half the gravitational downslope acceleration and half the downslope weight component as friction. At 30° the minimum static friction coefficients are respectively $\tan 30^\circ/3 = 0.192$ and $\tan 30^\circ/2 = 0.289$. Two objects on the same surface can therefore diverge qualitatively because the ring slips while the cylinder still rolls. Treating an assumed no-slip relation as exact without testing that inequality can produce an unphysical result despite apparently consistent energy conservation.

The friction on a rigid body rolling without slip on a fixed surface has zero power because its instantaneous point of contact is stationary. The force still changes both linear and angular momentum. On a moving conveyor the contact velocity is no longer zero in the laboratory; friction can supply mechanical energy even while the relative contact velocity vanishes. For a wheel steering on a plane the restrictions $\dot{x} = R\dot{\phi}\cos\psi,\ \dot{y} = R\dot{\phi}\sin\psi$ depend on heading ψ. In general they cannot be integrated into two configuration equations relating x, y, ϕ, ψ globally. d'Alembert's fixed-time allowed variations still project out ideal rolling reactions, but replacing those velocity constraints by invented holonomic coordinates gives the wrong dynamics. The distinction can be seen geometrically by rolling and steering through a closed sequence of steering operations that leaves a net displacement.

Find loads on two supports of a suspended bar

A rigid horizontal bar of mass m and length $2a$ is supported at its ends by two vertical inextensible light cords. At one instant both cords are vertical and the bar is horizontal. An extra downward load of magnitude F is applied at horizontal coordinate s relative to the centre. Ask first for a static experiment. The two vertical reaction forces T_L, T_R obey force balance $T_L + T_R = mg + F$ and moment balance $a(T_R - T_L) = sF$. This gives

$$T_R = (mg + F + sF/a)/2,$$

$$T_L = (mg + F - sF/a)/2.$$

The lever arms encode geometry that the vertical balance alone cannot provide.

The same result can be obtained from virtual variations if the reactions are kept as multipliers until the end. For small vertical translation y of the centre and small tilt θ, the endpoint vertical displacements are $\delta y - a\delta\theta$ on the left and $\delta y + a\delta\theta$ on the right. The load point displacement is $\delta y + s\delta\theta$. Before eliminating the endpoint constraints, project all applied and reaction forces against arbitrary independent $\delta y, \delta\theta$; vanishing of both coefficients gives the preceding balances. After imposing both endpoint constraints directly, these two variations vanish and the reduced model for vertical motion contains no equation from which the support loads can be read. Removing a degree of freedom can remove precisely the observable of interest unless reactions are reconstructed.

For $m = 2.0\,\mathrm{kg}$, $a = 0.40\,\mathrm{m}$, $F = 12\,\mathrm{N}$ and $s = 0.25\,\mathrm{m}$, the total load is $31.62\,\mathrm{N}$ and the load difference $7.50\,\mathrm{N}$. Hence $T_R = 19.56\,\mathrm{N}$, $T_L = 12.06\,\mathrm{N}$. Exchanging the load from right to left reverses the difference while preserving the sum. This symmetry is a calibrating manoeuvre for two load cells with different offsets: subtract readings from mirrored configurations to isolate the lever-arm response, and compare their sum to the known total weight and applied force.

Now imagine three vertical supports at $-a, 0, +a$. Their three tensions enter only two independent balances. Perfectly rigid constraints determine no unique allocation to the centre support. The ambiguity is physical; the stiffnesses, preloads and small height differences of real supports decide the result. With three identical unstrained linear springs and a centred load F, each spring extends equally if the bar stays level, so each carries one third of the total weight. If the middle spring is twice as stiff and all have the same unloaded height, it carries one half and each outer spring one quarter. Both answers satisfy exactly the same force and moment balances. Reporting a unique value from a singular ideal multiplier system would manufacture an unmeasured constitutive assumption.

Test an acceleration constraint against drift of the position itself

Solving $G\ddot{x} + \gamma = 0$ guarantees the correct instantaneous normal acceleration at a state lying on the constraint manifold with tangent velocity. It does not magically repair earlier position or velocity errors in approximate time integration. For a particle constrained to a circle, monitor $g = (r^2 - l^2)/2$, $\dot{g} = \mathbf{r} \cdot \mathbf{v}$, and the computed force. A small tangential displacement can give tiny instantaneous force error while the radius gradually drifts. Integrating an acceleration-level constraint twice without a position or velocity correction can cause substantial mismatch over many revolutions.

A simple projection illustrates what each correction does. Normalise an approximate position vector to length l, then remove the radial component of velocity by $\mathbf{v} \leftarrow \mathbf{v} - \hat{r}(\hat{r} \cdot \mathbf{v})$. The corrected point and tangent satisfy the constraints, but the velocity correction removes kinetic energy $m(\hat{r} \cdot \mathbf{v})^2/2$, and the position change can alter potential energy. Projection is therefore a numerical operation whose physical invariant errors must be measured. A structure-preserving constrained integrator builds the position and momentum conditions into the step and often gives better long-time behaviour. Compare convergence of reaction, constraint residual and energy separately instead of treating a rounded radius as proof of a correct force.

A pendulum on a free cart tests the complete mass matrix

Mount a pendulum pivot on a cart of mass M moving horizontally on a smooth track. The bob has mass m, length l and downward angle θ. Choose the cart position X and use bob coordinates $X + l\sin\theta, -l\cos\theta$. The kinetic energy is $(M+m)\dot{X}^2/2 + ml\dot{X}\dot{\theta}\cos\theta + ml^2\dot{\theta}^2/2$. Two different off-diagonal mass terms would contradict its origin as a quadratic form; the same factor $ml\cos\theta$ multiplies the companion acceleration in each equation. This is a useful check on a manually assembled multibody model.

The equations are

$$(M+m)\ddot{X} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = 0,$$

$$l\ddot{\theta} + \ddot{X}\cos\theta + g\sin\theta = 0.$$

In the small-angle regime the horizontal centre of mass moves uniformly. Choosing zero total momentum gives

$$\dot{X} \simeq -ml\dot{\theta}/(M+m).$$

Substitute its derivative into the angular equation to obtain

$$\ddot{\theta} + [(M+m)g/(Ml)]\theta = 0.$$

Releasing the support raises the oscillation frequency because the effective angular inertia falls while the gravitational stiffness stays the same. The result is a statement about the relative pendulum angle, measured from an inertial vertical, with an unforced horizontally moving support.

For $M = 0.60\,\mathrm{kg}$, $m = 0.20\,\mathrm{kg}$ and $l = 0.50\,\mathrm{m}$, the small-angle period is

$$2\pi\sqrt{Ml/[(M+m)g]} = 1.229\,\mathrm{s}.$$

Fixing the cart with a clamp gives $1.419\,\mathrm{s}$ instead. At zero total horizontal momentum the cart displacement from its equilibrium is approximately $-0.125\,\theta$ metres, so a $0.10\,\mathrm{rad}$ angle corresponds to $12.5\,\mathrm{mm}$ of recoil. Simultaneous position measurements on cart and bob should reconstruct a horizontal centre of mass that is stationary to within tracking uncertainty. A missing cart inertia or a wrongly filled cross term can alter both period and recoil; checking just one number may not identify which error occurred.

The reaction on the moving bob can again be recovered. Its radial acceleration along the outward string direction is $\ddot{X}\sin\theta - l\dot{\theta}^2$. Thus the tension is $T = m[g\cos\theta - \ddot{X}\sin\theta + l\dot{\theta}^2]$. The acceleration of the support appears explicitly. Reusing the fixed-pivot formula at all angles would omit a term that vanishes accidentally at the bottom. To test the sign, observe a rightward tilted pendulum whose pivot accelerates right: the support's acceleration has an outward projection along the tilted string, reducing the inward tension needed for the same relative motion. Substituting a directly measured cart acceleration provides an independent load-cell prediction.

Follow a force eliminated by a constraint

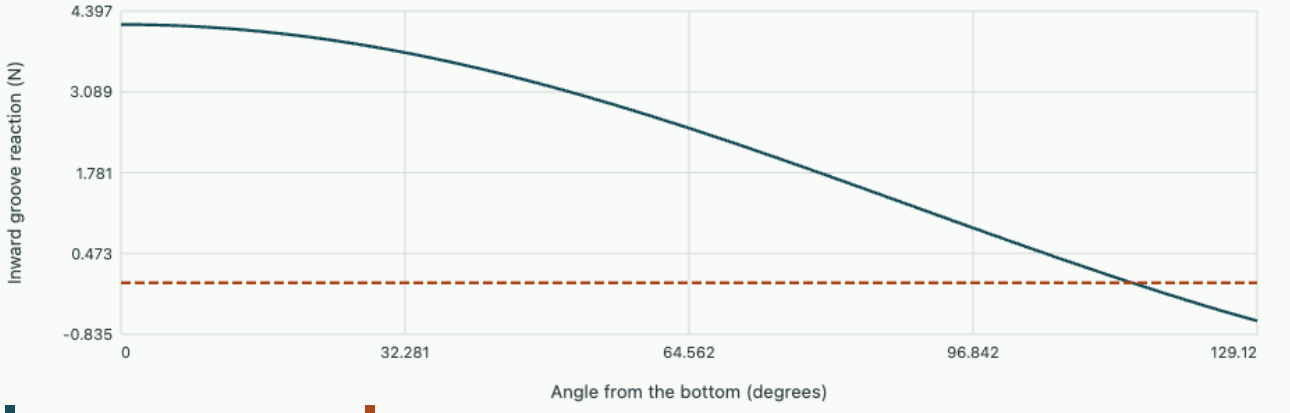
Predict when a two-sided circular groove must reverse its normal force. Check the prediction against the exact energy and radial balances.

Open this experiment online

Try the calculation

Bottom speed (m/s) **4**

Loop radius (m) **0.5**



Reachable angle 129.12°. Inward groove reaction there -0.619 N. One-sided loop threshold v_0 = 4.9523 m/s.

Decide which force a sensor actually reports

Reaction forces can be reported as tension along a member, normal traction at a contact, total support load on a floor, or a single Cartesian component read by a load cell. A rigid-body idealisation alone does not specify a sensor's orientation or the weight of its housing. For a fixed pendulum, for example, the string carries tension T, while a vertical force channel at its pivot reads the vertical string contribution $T\cos\theta$. A horizontal channel reads $T\sin\theta$. The force on the pivot is opposite the force exerted on the bob. Selecting the wrong sign or leaving out the fixture's own weight can create a discrepancy even when the dynamical calculation is correct.

Suppose an ideal load cell with its sensitive axis vertical is tared with the fixture installed but the $0.40\,\mathrm{kg}$ bob removed. When the bob is held at 60° by a separate purely tangential actuator and is otherwise stationary, the radial balance gives

$$T = mg\cos 60^\circ = 1.96\,\mathrm{N}.$$

The pivot's vertical reading attributable to the string is $0.981\,\mathrm{N}$. The tangential actuator supplies the remaining vertical support as well as a horizontal component. Once the actuator releases,

instantaneously the same tension and reading hold because radial speed is still zero. At the bottom of the subsequent swing the load jumps to the dynamical value obtained earlier, even though no upper fixture mass has changed.

A student hangs the bob at the same 60° angle using a separate horizontal cord instead of a tangential actuator. Should the pivot cell read the preceding $0.981\,\mathrm{N}$?

▼ **Work through the solution**

The horizontal cord contributes no vertical support. Static balance now requires $T\cos 60^\circ = mg$, giving $T = 7.85\,\mathrm{N}$ and vertical pivot contribution $3.92\,\mathrm{N}$. The horizontal holding force has a nonzero radial projection and changes the radial-force balance. After the horizontal cord is removed the tension immediately changes to $mg\cos 60^\circ$ because no momentum impulse is required for a finite jump in force. A schematic that shows only the bob angle is insufficient to infer the support load without saying how the angle is maintained.

In the free cart experiment bearing drag is weak but measurable. Can an observed frequency shift distinguish it from a wrongly assumed cart mass?

▼ **Work through the solution**

A viscous drag changes the horizontal momentum balance and allows the formerly stationary centre of mass to move as momentum leaks to the track. A mass error in a drag-free model changes the relative period and recoil ratio without supplying an actual external impulse. Track the complete centre of mass using independently weighed masses, compare its momentum change with the integrated bearing force, and measure the decay envelope. The frequency alone is a single scalar and can often be fitted by several combinations of missing mass, spring compliance and drag.

Locate contact loss on a smooth convex surface

A small block starts sliding from near rest just to one side of the top of a smooth fixed sphere of radius R. Measure angle θ from upward vertical. Energy gives $v^2 = 2gR(1-\cos\theta)$ during contact, while the inward radial equation reads $mg\cos\theta - N = mv^2/R$. Thus $N = mg(3\cos\theta - 2)$. Contact first vanishes at $\cos\theta = 2/3$, or 48.2°. For a sphere of radius $0.50\,\mathrm{m}$, this occurs after a vertical drop $R/3 = 0.167\,\mathrm{m}$ and at speed $1.81\,\mathrm{m/s}$. A block balanced exactly at the top at exact rest is an unstable equilibrium in the ideal model; a small initial displacement or speed initiates the measured trajectory.

Once the block leaves, the same spherical coordinates can still be used to describe position, but $r = R$ is no longer an equation of motion. Continue with the measured tangential velocity and free gravitational acceleration. Enforcing the sphere condition beyond release would make the computed normal negative and predict a surface that pulls on the block. The zero normal, the geometric release angle and the subsequent ballistic arc provide three independent observables for the same force reconstruction.

Tutorial on measuring reactions

A $0.40\,\mathrm{kg}$ pendulum of length $0.75\,\mathrm{m}$ is released from rest at 60° from downward vertical. Predict the load-cell reading at the pivot when it passes the bottom. What part does the reduced equation alone omit?

▼ Work through the solution

The height change is

$$l(1-\cos 60^\circ) = 0.375\,\mathrm{m}.$$

Energy conservation gives

$$v^2 = 2g(0.375) = 7.36\,\mathrm{m^2/s^2}.$$

At the bottom the tension is $mg + mv^2/l = 7.85\,\mathrm{N}$, twice the weight. The tangential acceleration there vanishes because gravity points along the string, but the radial acceleration remains nonzero. A sensor measuring the string force sees this centripetal contribution directly.

A bead lies on a smooth moving plane with upwards acceleration a_z. Which acceleration first makes a passive contact impossible if the bead initially shares the plane's motion?

▼ Work through the solution

Vertical Newton balance requires $N - mg = ma_z$, or $N = m(g + a_z)$. At $a_z = -g$ the reaction vanishes. A more negative plane acceleration would require the plane to pull the bead down, so contact is lost and the bead follows free fall until the separation closes. For upward acceleration the normal exceeds weight. In all these cases a virtual displacement tangent to the plane produces zero virtual normal work.

Two code implementations use $(x^2 + y^2 - l^2)/2$ and $x^2 + y^2 - l^2$ as constraints for the same state. Their multipliers differ by a factor two. How can the physical result be tested?

▼ Work through the solution

The second gradient is twice the first at the same point. Its multiplier must be half the first multiplier for $G^T\lambda$ to coincide. Compare the reaction vector in newtons, its dot product with an independently chosen tangent direction, and the normal acceleration required by the geometry. A comparison of raw multiplier values alone would flag a difference that is entirely due to normalisation.

Compare a constrained trajectory with the motion and force calculations above.

David Tong develops virtual work and constrained coordinates in his classical dynamics course.

Moving constraints, boundary work and a changing energy

Analytical mechanics from constraints to invariants

A pendulum with a reeled string can gain energy while its support force does no virtual work. A bead on a spinning guide can accelerate without gravity. Follow the work done by the apparatus, derive the relation between a reduced energy and laboratory energy, and identify which quantity a measurement tests.

Parameterise the motion of a guide as well as the bead

Let a prescribed moving guide define positions $\mathbf{r}(q,t)$. Its relative velocity along the guide is $\mathbf{r}_q\dot q$ and its imposed local velocity is $\mathbf{r}_t$, where t is differentiated at fixed coordinate. The kinetic energy $T=m|\mathbf{r}_q\dot q+\mathbf{r}_t|^2/2$ includes a quadratic term in $\dot q$, a mixed term linear in $\dot q$, and a term independent of $\dot q$. Dropping the last two merely because the guide motion is prescribed generally changes the bead equation. A term independent of $\dot q$ may still depend on its position q.

For $L=T-V$ define $p=\partial_{\dot q}L$ and $\mathcal{E}=p\dot q-L$. Differentiating along a solution of the Euler–Lagrange equation gives $d\mathcal{E}/dt=-\partial_t L$ in the absence of extra nonconservative generalised forces. This identity involves the partial time derivative at fixed $q,\dot q$. The laboratory energy is

$$E_{\mathrm{lab}}=m|\dot{\mathbf{r}}|^2/2+V.$$

Direct substitution gives $\mathcal{E}=E_{\mathrm{lab}}-m\dot{\mathbf{r}}\cdot\mathbf{r}_t$ for a single particle and ordinary position potential. Thus a time-independent reduced Lagrangian can conserve a quantity different from the laboratory mechanical energy. Which observer supplies a force, and which independent variables are held fixed, determine the energy being tracked.

Work supplied by a steadily rotating straight slot

A smooth horizontal guide turns about its inner end with prescribed angle $\varphi=\Omega t$. Let r be the bead's distance from the pivot. The velocity is $\dot r\hat r+r\Omega\hat\varphi$, and the reduced Lagrangian is $m(\dot r^2+\Omega^2r^2)/2$. Its Euler–Lagrange equation is

$$\ddot r=\Omega^2 r.$$

With $r(0)=r_0,\dot r(0)=0$ the solution is $r_0\cosh\Omega t$ until the bead leaves the slot. In the laboratory no force acts along the radial tangent of the slot, yet radial acceleration in polar coordinates is $\ddot r-r\Omega^2$, precisely zero. The tangential acceleration $2\Omega\dot r$ is provided by the slot wall.

The wall reaction $R_\varphi=2m\Omega\dot r$ exerts a torque rR_φ on the bead. The driving motor balances the opposite reaction on the slot and supplies power $\Omega rR_\varphi=2m\Omega^2r\dot r$. Differentiating $E_{\mathrm{lab}}=m(\dot r^2+\Omega^2r^2)/2$ and using the radial equation gives the same power. The reduced energy $m(\dot r^2-\Omega^2r^2)/2$ is constant. Its difference from E_{lab} equals $-\Omega L_z$, with $L_z=mr^2\Omega$. This is a consequence of uniform imposed rotation, not a negative physical rotational kinetic energy.

For $m = 0.10\,\mathrm{kg}$, $r_0 = 0.050\,\mathrm{m}$ and $\Omega = 4.0\,\mathrm{rad/s}$, consider $t = 0.25\,\mathrm{s}$. The radial position is $77.2\,\mathrm{mm}$, radial speed $0.235\,\mathrm{m/s}$, and wall reaction $0.188\,\mathrm{N}$. The motor supplies about $0.0580\,\mathrm{W}$ at that instant. Laboratory energy has risen from $0.00200\,\mathrm{J}$ to approximately $0.00752\,\mathrm{J}$. Measuring only the angular motor speed would miss the incoming mechanical work. The torque on the drive shaft or the changing moment of inertia supplies the other observable required for an energy budget.

A pendulum with prescribed length

Let a point pendulum have length $l(t) > 0$ and angle θ from downward vertical. Using $x = l\sin\theta$, $y = -l\cos\theta$ gives $T = m(\dot l^2 + l^2\dot\theta^2)/2$ and $V = -mgl\cos\theta$ for a fixed gravitational datum. The angular Euler–Lagrange equation becomes

$$\frac{d}{dt}(ml^2\dot\theta) = -mgl\sin\theta,$$

$$\ddot\theta + 2\frac{\dot l}{l}\dot\theta + \frac{g}{l}\sin\theta = 0.$$

The term proportional to $\dot l\dot\theta$ comes directly from differentiating angular momentum about the pivot. With increasing length the angular speed at fixed angular momentum decreases; reeling in has the opposite effect. It can look like damping or amplification in the angular equation even with no friction. A mechanical energy balance distinguishes the reversible exchange with a reel from energy lost as heat.

The radial Newton equation, taking outward as positive, is $m(\ddot l - l\dot\theta^2) = mg\cos\theta - T_{\mathrm{string}}$. Hence $T_{\mathrm{string}} = m(g\cos\theta + l\dot\theta^2 - \ddot l)$. The tension power on the bob is $-T_{\mathrm{string}}\dot l$. This has the correct sign when a heavy nearly vertical bob is lowered slowly: $\dot l > 0$ and the gravitational potential decreases as the string takes energy out. At the bottom of a swing the tension is larger, so a reel shortening the string there does more positive work per unit length than it would at a turning point. Timing a cyclic length change can pump an oscillation while the guide returns to precisely its initial length.

With $m = 0.50\,\mathrm{kg}$, $l = 1.0\,\mathrm{m}$, $\dot\theta = 2.0\,\mathrm{rad/s}$, and the bob at the bottom, a short interval of constant-rate shortening at $\dot l = -0.030\,\mathrm{m/s}$ has tension $6.91\,\mathrm{N}$ and delivers $0.207\,\mathrm{W}$. When the bob is momentarily at rest at 30°, the tension for the same length and zero radial acceleration is $4.25\,\mathrm{N}$. Reversing the reel rate there would remove only $0.127\,\mathrm{W}$. A complete finite cycle requires integrating the time-varying tensions and checking the acceleration during each reel reversal.

Translate a pivot and separate real forcing from a coordinate choice

For a pivot translating horizontally at prescribed $X(t)$ and constant string length l, write $x = X + l\sin\theta$, $y = -l\cos\theta$. The kinetic energy includes $ml\dot X\dot\theta\cos\theta$. In the angular Euler–Lagrange equation the terms proportional to $\dot X\dot\theta\sin\theta$ cancel, leaving $l\ddot\theta = -g\sin\theta - \ddot X\cos\theta$. A constant pivot velocity therefore does not force the relative angle. Replacing the pivot velocity by its magnitude as a

driving force would wrongly violate Galilean invariance. A horizontal pivot acceleration does force the angle; it appears as the opposite inertial acceleration in the translating frame.

For $|\theta| \ll 1$ and $X = X_0 \cos \omega t$, the equation becomes

$$\ddot{\theta} + \omega_0^2 \theta = (X_0 \omega^2 / l) \cos \omega t,$$

$$\omega_0^2 = g/l.$$

At drive frequencies well below ω_0,

$$\theta \simeq -\ddot{X}/g.$$

The bob nearly aligns with the effective gravity in the moving frame. Far above ω_0, the particular solution is approximately $-X/l$; then the bob's absolute horizontal coordinate $X + l\theta$ changes little even while the pivot moves. Near resonance the undamped steady particular amplitude is inappropriate, and the initial transient or a physical damping model matters.

Calculate work done by a moving wall in an elastic collision

A particle of mass m moves along a line and collides elastically with a wall whose prescribed laboratory velocity is U during the short contact. In the instantaneous wall frame the incoming relative velocity $v^- - U$ reverses, so

$$v^+ = 2U - v^-.$$

In the laboratory the impulse is $J = m(v^+ - v^-) = 2m(U - v^-)$ and the kinetic energy change is $\Delta K = m[(v^+)^2 - (v^-)^2]/2 = UJ$. The energy gained by the particle equals wall velocity times the impulse delivered by the wall. A perfectly elastic contact still changes laboratory kinetic energy when the wall moves, since elasticity was defined for the relative normal velocity.

Suppose a particle of mass $0.20\,\mathrm{kg}$ moves right at $3.0\,\mathrm{m/s}$ into a right wall moving left at $0.50\,\mathrm{m/s}$. Its final velocity is $-4.0\,\mathrm{m/s}$, and the impulse on the particle is $-1.40\,\mathrm{N\,s}$. The particle's kinetic energy rises from 0.90 to $1.60\,\mathrm{J}$, exactly $UJ = 0.70\,\mathrm{J}$. The actuator keeping the wall on schedule supplies the energy. When the same wall recedes to the right at $0.50\,\mathrm{m/s}$, the outgoing velocity is $-2.0\,\mathrm{m/s}$ and kinetic energy falls by $0.50\,\mathrm{J}$. The difference is not symmetric under reversal of wall speed at fixed incoming particle velocity, because that reversal also changes the relative approach speed.

Now trap a particle between a fixed wall and a length $l(t)$ varying slowly compared with the collision period. To leading order a right-wall collision changes the particle's speed by $-2\dot{l}$, and such collisions occur about once in $2l/v$. Hence $dv/dt \simeq -(v/l)\dot{l}$, so vl is approximately constant. The one-dimensional action around a full bounce cycle is

$$J_{\mathrm{act}} = (2\pi)^{-1}(2mvl) = mvl/\pi.$$

This elementary wall argument explains the adiabatic action without requiring the smooth-force approximation at the instantaneous collision. If the wall speed varies quickly within a round trip, the collision phases matter and the averaged invariant can fail.

Starting with $l_0 = 0.40\,\mathrm{m}$ and speed $4.0\,\mathrm{m/s}$, slow compression to $0.30\,\mathrm{m}$ predicts $v \simeq 5.33\,\mathrm{m/s}$. Its kinetic energy increases by a factor $16/9$, supplied through the moving boundary. A pressure sensor at the wall measures impulse per time, $2mv$ times the collision frequency $v/(2l)$, giving average force mv^2/l at leading order. Multiplying this force by the inward speed gives the power required by the predicted energy law. An instantaneous compression while the particle is elsewhere would instead change no kinetic energy until a later collision. The final length alone does not determine the energy unless the rate protocol is specified.

Follow a string through a moving boundary in configuration and time

For a vertical moving pivot with height $Y(t)$, coordinates $x = l\sin\theta$, $y = Y - l\cos\theta$ give kinetic energy $m[\dot{Y}^2 + 2l\dot{Y}\dot{\theta}\sin\theta + l^2\dot{\theta}^2]/2$. Derive the angular equation by differentiating $p_\theta = ml^2\dot{\theta} + ml\dot{Y}\sin\theta$. The terms containing $\dot{Y}\dot{\theta}\cos\theta$ cancel as in horizontal translation; the remaining equation is

$$l\ddot{\theta} = -(g + \ddot{Y})\sin\theta.$$

A vertical acceleration changes the restoring coefficient multiplicatively, unlike a horizontal displacement whose small-angle effect is an additive forcing term. The distinction is visible even when the two pivot motions have identical amplitudes and frequencies.

For $Y = A\cos\Omega t$, the effective gravity oscillates as $g - A\Omega^2\cos\Omega t$. The perfectly vertical solution remains an exact solution at any drive amplitude. A small deviation need not remain small, because its linear equation has a periodic coefficient. An observed growth of swing from a tiny initial disturbance near twice the natural frequency is a parametric effect. A camera viewing the pivot and bob together can separate relative angle from large imposed vertical displacement. Estimating angular amplitude from absolute bob height alone introduces a component at the drive frequency unrelated to the swing.

At modest drive $A = 1.0\,\mathrm{mm}$, $\Omega = 20\,\mathrm{rad/s}$, the pivot acceleration amplitude is $0.40\,\mathrm{m/s^2}$, about four percent of gravity. A pendulum of length $0.10\,\mathrm{m}$ has small-angle frequency $9.90\,\mathrm{rad/s}$, putting this drive near twice its natural frequency despite the small displacement. For an exactly stationary vertical bob relative to the pivot, the string tension is $m(g + \ddot{Y})$ as long as that value is positive. A force measurement thus contains a direct drive-frequency signal even before transverse oscillation develops. Combining an angle sensor with a load cell helps distinguish contact loss from growth of parametric oscillation.

A fast vertical drive can also modify the slow stability of the inverted position. A quantitative treatment requires separating a fast small angular ripple from the slow tilt and averaging its feedback, with the support acceleration and frequency satisfying the relevant ordering. The instantaneous sign of $g + \ddot{Y}$ alone is insufficient to establish the stability of the inverted solution. An upright motion that briefly

samples a favourable sign each cycle may still drift away over many cycles. Compare initial perturbations at several drive phases and fit the envelope over complete periods before drawing a stability conclusion.

Treat the drive itself as a degree of freedom

A prescribed rotating slot conceals the energy source behind a fixed pattern speed. Replace the motor by a free rotating turntable with moment of inertia I_0, and let its angle φ evolve dynamically. In the horizontal frictionless problem the total kinetic energy is $m\dot{r}^2/2 + (I_0 + mr^2)\dot{\varphi}^2/2$. The angle is cyclic, so the total angular momentum $P = (I_0 + mr^2)\dot{\varphi}$ is constant. As the bead moves out, the turntable slows. The radial equation remains $\ddot{r} = r\dot{\varphi}^2$, but its coefficient now decreases according to the total angular momentum instead of remaining externally fixed.

Eliminate $\dot{\varphi}$ from the total energy to find

$$E = m\dot{r}^2/2 + P^2/[2(I_0 + mr^2)].$$

The reduced radial potential decreases outwards and feeds radial kinetic energy, but the total energy available is finite. Differentiating the potential gives a radial acceleration $P^2 r/(I_0 + mr^2)^2$, consistent with the original radial equation. One should not simply insert $\dot{\varphi} = P/(I_0 + mr^2)$ into the original kinetic Lagrangian before varying r; the correct fixed-momentum Routh reduction subtracts $P\dot{\varphi}$, reversing the sign of the eliminated rotational term. The energy and Lagrangian methods agree once their controlled quantities are distinguished.

With $I_0 = 0.020\,\mathrm{kg\,m^2}$, $m = 0.10\,\mathrm{kg}$, $r_0 = 0.10\,\mathrm{m}$, initial angular speed $4.0\,\mathrm{rad/s}$ and zero radial speed, total momentum is $0.084\,\mathrm{kg\,m^2/s}$ and energy $0.168\,\mathrm{J}$. At $r = 0.30\,\mathrm{m}$ the rotation rate has fallen to $0.084/0.029 = 2.90\,\mathrm{rad/s}$. The remaining radial kinetic energy is $0.168 - 0.084^2/(2 \cdot 0.029) = 0.0463\,\mathrm{J}$, giving radial speed $0.962\,\mathrm{m/s}$. Under a prescribed constant angular speed of four, the radial first integral instead gives $\dot{r} = 4\sqrt{r^2 - r_0^2} = 1.13\,\mathrm{m/s}$ there. The two runs started with identical positions and velocities but have different power connections.

The drive can also be modelled between those limits. Let a servo apply torque $\tau_{\mathrm{drive}} = K_{\mathrm{s}}(\Omega_{\mathrm{command}} - \dot{\varphi})$. Then $d[(I_0 + mr^2)\dot{\varphi}]/dt = \tau_{\mathrm{drive}}$, and the rate of increase of total mechanical energy is $\tau_{\mathrm{drive}}\dot{\varphi}$. A high control gain with finite response time can approximate the prescribed-speed case over part of a trajectory, yet a rapid radial change can cause a measurable speed sag. Recording shaft torque and speed together tests the drive rather than attributing its lag to a fictitious radial friction on the bead.

Infer drive work from a closed loop of length and measured tension

The reel-pendulum experiment produces synchronised records of string force and length. Over a period in which the length returns to its original value, the work on the bob from the reel is $-\oint T\,dl$. A graph of tension against length can enclose a signed area although the string force at each instant is perpendicular to the bob's allowed virtual angular displacement. That loop area measures energy

transferred by the imposed control. If the bob returns to its full mechanical state and there are no other energy sources, any net positive reel work must go into losses. If only the string length returns, the bob's energy can increase cycle after cycle.

Suppose the reel shortens by $20\,\mathrm{mm}$ during a high-tension interval averaging $7.0\,\mathrm{N}$, then pays out the same length while the mean tension is $4.5\,\mathrm{N}$. The net mechanical work on the bob is $0.020(7.0-4.5)=0.050\,\mathrm{J}$ for that cycle. A torque sensor on a reel of changing winding radius must convert torque to tension with the actual instantaneous radius. A force signal delayed relative to the length encoder can create a spurious enclosed area, especially when tension changes rapidly near a turning point. Repeating the measurement slowly with a known hanging calibration weight provides a direct check of polarity and timing.

Derive rapid-pivot stabilisation rather than replacing gravity by its mean

The vertical-pivot equation can be written $\ddot{\theta}+[g/l-(A\Omega^2/l)\cos\Omega t]\sin\theta=0$. The time average of the oscillatory coefficient is zero. Nevertheless a high-frequency pivot can alter slow stability because the small fast ripple in θ correlates with the fast acceleration. Let $\theta=\Theta+\xi$, where Θ varies little in a drive period and ξ is small. Keeping the leading rapidly oscillating term gives $\ddot{\xi}\simeq(A\Omega^2/l)\sin\Theta\cos\Omega t$, hence $\xi\simeq-(A/l)\sin\Theta\cos\Omega t$. Its amplitude scales with displacement ratio A/l, so large acceleration and small angular ripple can coexist.

Expand $\sin(\Theta+\xi)\simeq\sin\Theta+\xi\cos\Theta$, insert the fast solution, and average. Terms linear in the cosine average to zero but the product of ripple and imposed acceleration gives a mean cosine square of one half. To leading order the slow equation is

$$\ddot{\Theta}+\frac{g}{l}\sin\Theta+\frac{A^2\Omega^2}{2l^2}\sin\Theta\cos\Theta=0.$$

It derives from an effective potential per angular inertia $U_{\mathrm{eff}}/(ml^2)=-(g/l)\cos\Theta+[A^2\Omega^2/(4l^2)]\sin^2\Theta$, up to a constant. Linearising near $\Theta=\pi+\eta$ gives

$$\ddot{\eta}+[A^2\Omega^2/(2l^2)-g/l]\eta=0.$$

The inverted state is stable within the averaged approximation if $A^2\Omega^2>2gl$. The restoring contribution depends on squared pivot speed amplitude. Replacing effective gravity by its simple time average would miss it entirely.

Choose a rigid pendulum arm with $l=0.20\,\mathrm{m}$, drive amplitude $A=0.0040\,\mathrm{m}$, and $\Omega=600\,\mathrm{s}^{-1}$. The small displacement ratio is 0.020, the pivot speed amplitude is $2.4\,\mathrm{m/s}$, and $A^2\Omega^2=5.76\,\mathrm{m}^2/\mathrm{s}^2$, above $2gl=3.92\,\mathrm{m}^2/\mathrm{s}^2$. The predicted slow angular frequency about the inverted state is $\sqrt{5.76/(2\cdot0.20^2)-9.81/0.20}=4.79\,\mathrm{s}^{-1}$, far below the drive frequency. A slow ripple of tilt $0.10\,\mathrm{rad}$ has a leading drive-frequency micromotion of roughly $0.0020\,\mathrm{rad}$. A camera sampling at an unfortunate phase can hide this micromotion or alias it into apparent slow drift.

The rigid arm in this example is essential. An ordinary string cannot push its bob when an averaged upright configuration requires alternating compression and tension throughout a fast cycle. Check the actual instantaneous radial reaction and the physical actuator's ability to deliver the large acceleration. Also check drive-period, arm elasticity and support resonance before identifying an observed stable wobble with the averaged rigid-arm prediction. The leading formula assumes small ripple and a clear separation between Ω and all slow angular frequencies; near threshold the slow restoring coefficient is a small difference and higher-order effects or friction become relatively important.

Separate support work from energy in a moving coordinate

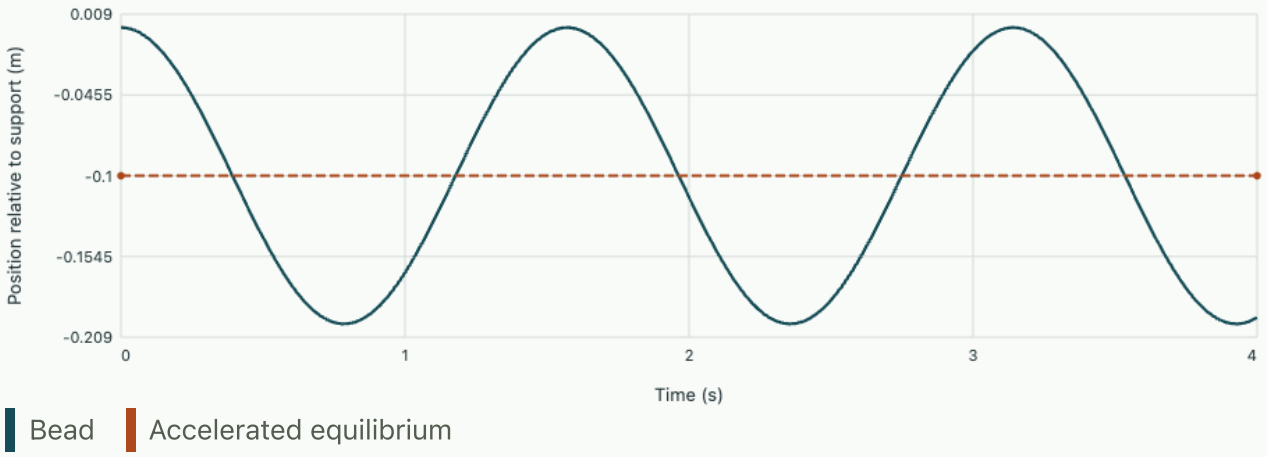
Prescribe an accelerating support. Compare the relative response with what a laboratory observer sees and identify how the drive supplies energy.

Open this experiment online

Try the calculation

Support acceleration (m/s²) **1.6**

Spring stiffness (N/m) **8**



The equilibrium is at -0.1 m. The first turning point is -0.2 m at 0.7854 s.

Distinguish two ways of moving an endpoint in Hamilton's principle

When a constraint itself is explicitly time dependent, the identity $\delta g = \nabla g \cdot \delta x = 0$ describes a virtual change at fixed time on the moving surface. A variational problem may instead allow the endpoint time to change. If a terminal point moves along a prescribed spacetime curve $q_b(t)$, the endpoint variation includes $dq_b = \dot{q}_b\, dt$; it is not an arbitrary tangent displacement at fixed laboratory clock. The boundary term in the on-shell action is $p\, dq - H\, dt$. Reading these two terms separately tells which physical terminal conditions a derivation has assumed.

For a free particle of mass m moving in one dimension from the origin at time zero to q at time T, the extremal action is $S = mq^2/(2T)$. Its derivatives are $S_q = mq/T = p$ and $S_T = -mq^2/(2T^2) = -H$. If the terminal marker moves as $q_b(T) = uT$, the action of the connecting extremal is $mu^2T/2$. Its derivative along that endpoint line is $pu - H = mu^2/2$, exactly the direct derivative. Holding the endpoint position fixed while differentiating instead gives $-mu^2/2$. Both derivative experiments are legitimate, but the different signs demonstrate why one must state what is fixed.

Can a numerical action fit identify momentum and energy from a single sequence of terminal markers all moving at speed u?

▼ **Work through the solution**

Along that one curve it returns the combination $pu - H$. Separating momentum and energy requires independent endpoint variations in position and time, or another dynamical relation. For the free-particle model the known quadratic Hamiltonian supplies such a relation. For an unknown effective system a single endpoint track can leave different combinations indistinguishable. A grid of nearby fixed-time endpoints estimates p, and a separate grid of terminal times at fixed place estimates $-H$. This measurement interpretation is the finite-dimensional counterpart of separating virtual reaction work from actual boundary power.

Infer whether a moving wall may be treated as prescribed

In an actual impact apparatus the wall has finite effective mass M. For a one-dimensional elastic impact with an initially moving wall, conservation of total momentum and reversal of relative velocity give

$$v^+ = [(m - M)v^- + 2MU]/(m + M),$$
$$U^+ = [2mv^- + (M - m)U]/(m + M).$$

Only the limit $M/m \to \infty$, or active feedback able to enforce the schedule on the collision time scale, gives the prescribed-wall expression used earlier. A motor that restores its target speed a second later may still behave as a finite inertia during the collision itself.

Take $m = 0.20\,\mathrm{kg}$, $M = 2.0\,\mathrm{kg}$, $v^- = 3.0\,\mathrm{m/s}$ and $U = -0.50\,\mathrm{m/s}$. The finite wall sends the particle back at approximately $-3.36\,\mathrm{m/s}$ and recoils to $+0.136\,\mathrm{m/s}$, compared with the prescribed-wall prediction of $-4.0\,\mathrm{m/s}$. Check that the initial and final total momenta both equal about $-0.40\,\mathrm{kg\,m/s}$. Measuring wall speed immediately before and after contact separates an imperfectly enforced boundary from an inelastic impact; both can reduce the magnitude of particle rebound relative to an ideal prescribed boundary.

Tutorial on a prescribed moving apparatus

The rotating slot is held at fixed Ω, but the bead is initially given inward speed $-\Omega r_0$. Predict its subsequent motion and whether the motor delivers or receives power.

▼ **Work through the solution**

The general solution is a combination of growing and decaying exponentials. These initial conditions select $r = r_0 e^{-\Omega t}$. The wall reaction $2m\Omega\dot{r}$ is negative, and its power $2m\Omega^2 r\dot{r}$ is also negative. The motor receives energy from the bead while maintaining its speed. The same geometric constraint can therefore inject or absorb energy depending on the actual trajectory. The origin is approached only asymptotically in the ideal solution.

At $l = 0.80\,\mathrm{m}$ and $\theta = 0$, a bob has zero angular speed and the prescribed reel tries to accelerate the length outward at $12\,\mathrm{m/s^2}$. Can a string enforce this motion?

▼ **Work through the solution**

The radial-force equation gives $T = m(g - 12)$, a negative tension. A string cannot push the bob outward. The reel pays out slack and the bob initially accelerates under gravity, so treating $l(t)$ as its exact radius becomes inadmissible. A rigid telescopic arm capable of compression would impose a different physical constraint and could enforce the same geometrical schedule.

A pivot is displaced sinusoidally by $2.0\,\mathrm{mm}$ at $1.0\,\mathrm{Hz}$ with $l = 0.25\,\mathrm{m}$. Is the low-frequency tilt formula suitable?

▼ **Work through the solution**

The natural angular frequency is $\sqrt{9.81/0.25} = 6.26\,\mathrm{rad/s}$, almost equal to the drive angular frequency $2\pi = 6.28\,\mathrm{rad/s}$. The low-frequency expansion requires a frequency much smaller than the natural one and fails here. Measure damping and initial conditions, or model a finite-time drive. A steady frictionless amplitude calculated from the small difference of squared frequencies is extremely sensitive to uncertainties in the length and drive frequency.

Compare a second rotating constraint and identify what the motor holds fixed.

David Tong's Lagrangian dynamics chapter develops explicitly time-dependent coordinate changes.

From stiffness to motion

Pass from a static deflection to a variational mode estimate and a transient stress wave. Check how choice of interpolation affects the answer.

1. Modal participation and absorber tuning
2. How a longitudinal pulse reflects from a change in rod section
3. Why the frequencies of a bending beam are not equally spaced
4. Estimating a cantilever frequency from energy
5. Weak forms and finite elements
6. Building a local residual estimator and refining where it matters
7. Why two finite-element wave solvers propagate the same mode at different speeds
8. Action density, stress waves and the boundary that reflects them

Modal participation and absorber tuning

Machine dynamics

Project external forcing onto vibration modes and determine when a vibration absorber suppresses displacement at a selected frequency.

FROM FIRST PRINCIPLES

Project a physical force onto a vibration mode

For small undamped motions a structure obeys $M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{f}(t)$ with symmetric mass and stiffness matrices. Its normal modes satisfy

$$K\boldsymbol{\phi}_j = \omega_j^2 M\boldsymbol{\phi}_j.$$

Expand displacement in these mode shapes, using time-dependent coefficients q_j. Premultiply the result by the transpose of one mode.

$$\mathbf{x} = \sum_j \boldsymbol{\phi}_j q_j,$$

$$\sum_j (\boldsymbol{\phi}_i^{\mathsf{T}} M\boldsymbol{\phi}_j\, \ddot{q}_j + \boldsymbol{\phi}_i^{\mathsf{T}} K\boldsymbol{\phi}_j\, q_j) = \boldsymbol{\phi}_i^{\mathsf{T}}\mathbf{f}.$$

For distinct frequencies, symmetry makes different modes orthogonal under both matrices. The corresponding off-diagonal products vanish, leaving a separate scalar oscillator for the selected mode.

$$m_i\ddot{q}_i + m_i\omega_i^2 q_i = Q_i(t),$$

$$m_i = \boldsymbol{\phi}_i^{\mathsf{T}} M\boldsymbol{\phi}_i,$$

$$Q_i = \boldsymbol{\phi}_i^{\mathsf{T}}\mathbf{f}.$$

The force projection has a direct interpretation as work per unit modal displacement. A symmetric load can have zero projection onto an antisymmetric mode despite a large total force. The displacement at a particular sensor is obtained only after adding its components from all excited modes.

Background · Modal participation

$$Q_j = \phi_j^{\mathsf{T}} f$$

A mode is excited according to its overlap with the force. Always state how its mode vector and modal mass were normalised.

DERIVATION AND APPLICATIONS

Determine how a point force appears in collocated modal measurements

The displacement measured at the force location can have a zero between two structural resonances. Consider an undamped linear structure with two relevant modes, distinct natural frequencies $\omega_1 < \omega_2$ and modal masses $m_j > 0$. Apply a harmonic point force of complex amplitude F at coordinate a and measure displacement at the same coordinate. For modal shape component $\phi_j(a)$, the generalised force is $\phi_j(a)F$, and reconstruction at the sensor supplies a second factor of that component.

$$q_j = \frac{\phi_j(a)F}{m_j(\omega_j^2 - \omega^2)},$$

$$H_{aa} = \frac{x_a}{F} = \frac{A_1}{\omega_1^2 - \omega^2} + \frac{A_2}{\omega_2^2 - \omega^2},$$

$$A_j = \frac{\phi_j(a)^2}{m_j}.$$

The residues A_j are positive when both modes are visible. Between the two poles the first denominator is negative and the second positive. Set the sum to zero and multiply through by both denominators to find the antiresonance.

$$A_1(\omega_2^2 - \omega_z^2) + A_2(\omega_1^2 - \omega_z^2) = 0,$$

$$\omega_z^2 = \frac{A_1\omega_2^2 + A_2\omega_1^2}{A_1 + A_2}.$$

A zero in the measured displacement can therefore result from cancellation of two finite modal displacements. Its position also reveals their relative participation without requiring the absolute force calibration.

$$\frac{A_1}{A_2} = \frac{\omega_z^2 - \omega_1^2}{\omega_2^2 - \omega_z^2}.$$

Separated force and response locations can give signed residues and change the interlacing. With damping a finite minimum commonly replaces the exact zero, so using its frequency as an exact antiresonance is an approximation.

WORKED EXAMPLE

Predict the dip of a two-mode stage

For $\omega_1 = 10.0$ rad/s, $\omega_2 = 20.0$ rad/s and $A_1 = 2A_2$, the squared zero frequency is $(2 \times 400 + 100)/3 = 300$ s⁻², hence $\omega_z = 17.3$ rad/s. At that frequency the first contribution is

$2A_2/(100-300) = -0.0100A_2$ and the second is

$$A_2/(400-300) = +0.0100A_2.$$

The sensor is stationary while both modes respond.

Change the inputs and inspect this calculation below

SECOND APPLICATION

Recover a relative participation from a measured minimum

A different collocated test gives resonances at 30.0 and 50.0 Hz and a negligible-damping minimum at 40.0 Hz. Ratios of squared angular frequencies can equally be expressed through squared frequencies in hertz. The residue ratio is

$$(40^2 - 30^2)/(50^2 - 40^2) = 7/9 = 0.778.$$

With equal modal masses the first shape magnitude at the force point is $\sqrt{7/9} = 0.882$ of the second.

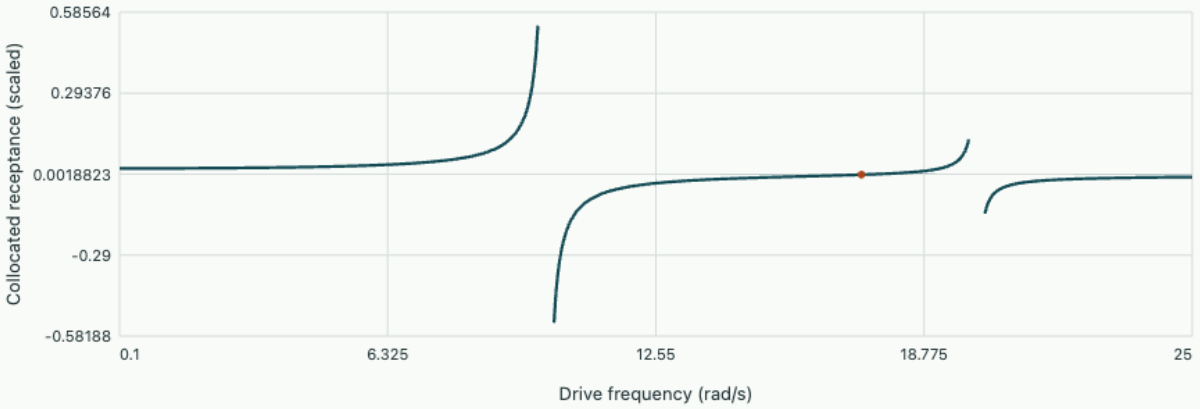
Interactive calculation · Change the inputs of this example

Use the adjustable quantities to test the two worked calculations above. The plot recomputes this derivation's balance or constitutive relation; its numerical readout reports the quantities needed for the forward calculation and the inverse or design question. Keep the fixed assumptions stated in the readout when comparing with the example values.

Open this live calculation

Try the calculation

First modal participation / second	2
First natural frequency (rad/s)	10
Second natural frequency (rad/s)	20



Two undamped modal terms Antiresonance

The cancellation lies at 17.321 rad/s between the two poles. Small neighbourhoods of the undamped poles are omitted from the curve.

Recover motion from mode amplitudes

Knowing a structure's natural frequencies does not yet tell you which frequencies a particular actuator will excite. For a linear undamped system write $M\ddot{x} + Kx = f(t)$ and let ϕ_j be a real mode shape solving $K\phi_j = \omega_j^2 M\phi_j$. Assume a complete set of independent modes and expand the displacement $x = \sum_j \phi_j q_j$. Multiplying the equation by ϕ_i^T uses mass and stiffness orthogonality to remove other modal coordinates.

$$m_i \ddot{q}_i + k_i q_i = \phi_i^T f(t),$$
$$m_i = \phi_i^T M \phi_i,$$
$$k_i = \omega_i^2 m_i.$$

Multiplying a mode shape by a constant changes the numerical modal mass and generalised force, while the reconstructed physical motion remains invariant. Thus a tabulated mode shape needs its normalisation if it is used to compute a physical response. A forcing pattern whose projection vanishes excites none of that mode from rest, even if the excitation frequency equals its natural frequency.

Select a mode by choosing the actuator pattern

Suppose two identical masses 2 kg each have mode shapes $(1, 1)$ and $(1, -1)$, with frequencies 10 and 20 rad/s. Both modal masses are 4 kg. A 12 N harmonic force on the first mass projects 12 N onto each mode in this normalisation. At a forcing frequency of 5 rad/s the undamped steady modal amplitudes are

$$12/[4(10^2 - 5^2)] = 0.0400\,\mathrm{m},$$
$$12/[4(20^2 - 5^2)] = 0.00800\,\mathrm{m}.$$

Recombination gives first and second physical amplitudes 0.0480 and 0.0320 m.

Now distribute the same total force as 6 N on each mass in phase. The first projection stays 12 N and the second becomes $6 - 6 = 0$. The masses now both move 0.0400 m. Relocating or phasing actuators can therefore change observed vibration without changing any mass, spring or natural frequency.

Cause the primary displacement to vanish

Attach an auxiliary mass m_a by spring k_a to a primary mass m_p grounded by spring k_p. A harmonic force of amplitude F acts on the primary mass. Seek displacement amplitudes X_p, X_a. The absorber equation is

$$-k_a X_p + (k_a - m_a \omega^2) X_a = 0.$$

At $\omega^2 = k_a/m_a$ this enforces $X_p = 0$. The primary equation then gives

$$X_a = -F/k_a.$$

The absorber’s spring force balances the excitation while the primary mass is stationary in this ideal steady response.

For $m_a = 0.50\,\mathrm{kg}$, a measured forcing frequency of 20 Hz requires

$$k_a = m_a(2\pi 20)^2 = 7896\,\mathrm{N/m}.$$

An 8 N forcing amplitude gives absorber travel amplitude 1.01 mm. The two coupled resonances bracket this tuned antiresonance; startup or a change of shaft speed can pass through them. Damping limits the resonant peaks but generally prevents the exact cancellation obtained from these undamped equations.

OneSimulation · How can two resonant modes cancel the primary displacement?

Apply a sinusoidal force to a 5 kg primary mass supported by a 500 N/m spring. An attached 0.5 kg absorber connects through a 50 N/m spring. At the undamped tuning frequency the absorber equation forces the primary displacement to zero. The full coupled system still has two distinct natural frequencies. With mass-normalised modes and the force and displacement at coordinate 1, the receptance is

$$\frac{X_1}{F_1} = \sum_j \frac{\phi_{1j}^2}{\omega_j^2 - \omega^2 + 2i\zeta_j\omega_j\omega}.$$

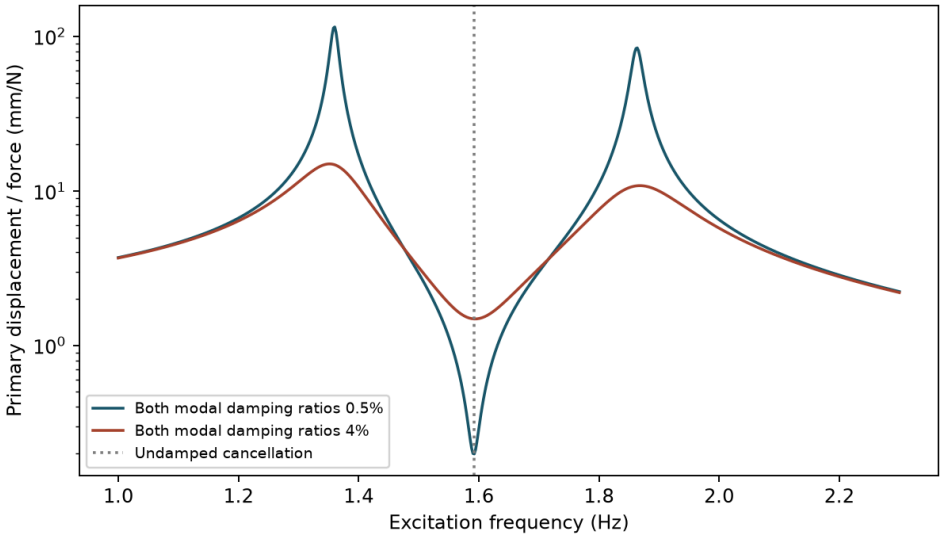
OneSimulation obtains the eigenpairs from the mass-scaled stiffness. The eigenfrequencies are 1.36 and 1.86 Hz. Between these frequencies the two undamped real contributions have opposite signs. They cancel at 1.59 Hz. The code checks this zero and independently verifies the static compliance. Increase the common modal damping ratio and inspect the notch: losses shift the modal phases, so the exact zero becomes a finite minimum.

```
"""A force on the primary mass excites two modes of a tuned absorber."""
import matplotlib.pyplot as plt
import numpy as np
from onesim.numerics.linear.matrices.dense_linear_algebra import symmetric_eigensystem

mass = np.array([5., 0.5])  # kg, primary and absorber
support, spring = 500., 50.  # N/m, support and connecting spring
stiffness = np.array([[support+spring, -spring], [-spring, spring]])
scale = np.diag(1/np.sqrt(mass))
omega2, shapes = symmetric_eigensystem(scale @ stiffness @ scale)
modes = scale @ shapes
residues = modes[0]**2      # applied force and measured motion both at mass 1
tuned = np.sqrt(spring/mass[1])
assert abs(np.sum(residues/(omega2 - tuned**2))) < 1e-12
assert np.isclose(np.sum(residues/omega2), 1/support)

frequency = np.linspace(1.0, 2.3, 600)  # Hz
omega = 2*np.pi*frequency[:, None]
fig, ax = plt.subplots(figsize=(7, 4), constrained_layout=True)
for damping, colour in [(0.005, "#19596a"), (0.04, "#a5442e")]:
    denominator = omega2 - omega**2 + 2j*damping*np.sqrt(omega2)*omega
    response = np.sum(residues / denominator, axis=1)
    ax.semilogy(frequency, 1000*abs(response), color=colour,
                label=f"Both modal damping ratios {100*damping:g}%")

ax.axvline(tuned/(2*np.pi), color="0.5", linestyle=":", label="Undamped cancellation")
ax.set(xlabel="Excitation frequency (Hz)", ylabel="Primary displacement / force (mm/N)")
ax.legend(fontsize=8)
fig.savefig("machine-dynamics.png", dpi=170)
```



A low reading at the primary mass can coexist with substantial absorber motion. The plotted response and static-limit check both refer to the force application point.

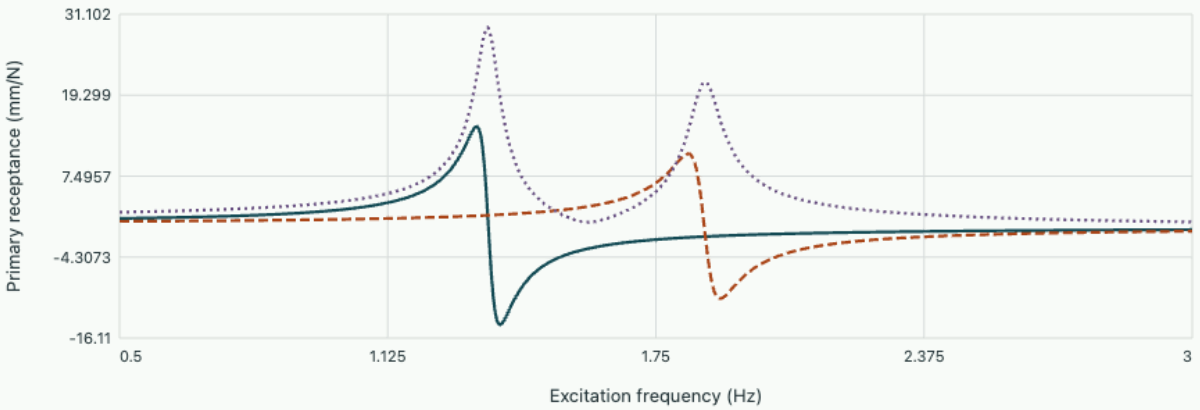
Download the Python example

Change the inputs of this exact example and compare the recalculated curves with the recorded OneSimulation output above. The browser evaluates the stated analytical or modal reference.

Open this live calculation

Try the calculation

Common modal damping ratio	0.02
Absorber mass (kg)	0.5
Absorber spring (N/m)	50



Mode1 real contribution Mode2 real contribution Magnitude of complex sum

Natural frequencies 1.360, 1.863 Hz. At undamped tuning 1.592 Hz, receptance is 0.77774 mm/N. Static compliance 2.000 mm/N. A common modal damping ratio modifies each modal phase; it is distinct from a physical dashpot attached only to the absorber.

Choose where to observe cancellation

At the tuned absorber frequency a sensor on the primary reports zero ideal steady displacement, while a sensor on the attached mass records finite motion. A force transducer between the bodies measures the full applied excitation in the opposing phase. Moving the sensor can therefore turn the same test from an apparent null into a readily measurable response, without altering the underlying eigenfrequencies.

Project a spatially distributed load onto a flexible member

A beam or string has motion distributed along its length. Its modal forcing is determined by a spatial integral, analogous to the vector dot product for discrete coordinates. For a taut uniform string of length L, linear density μ and constant tension H, small transverse displacement $w(x,t)$ obeys $\mu w_{tt} - H w_{xx} = p(x,t)$. The tension term follows from the difference in vertical tension components on a short segment. Assume both ends fixed in displacement and neglect bending stiffness, damping and the change of tension caused by stretching. The modes are $\phi_n = \sin(n\pi x/L)$, with frequencies $\omega_n = (n\pi/L)\sqrt{H/\mu}$.

Expand $w = \sum \phi_n q_n$, multiply by ϕ_j and integrate over the length. The mass integral is

$$m_j = \mu \int_0^L \phi_j^2 dx = \mu L/2.$$

Integration by parts of the tension term gives the modal stiffness $k_j = m_j \omega_j^2$, with the fixed-end boundary terms vanishing. The modal input becomes

$$Q_j = \int_0^L p \phi_j dx.$$

Under a uniform harmonic load amplitude p_0,

$$Q_j = \frac{p_0 L}{j\pi}[1 - (-1)^j].$$

Even modes have zero generalised force because their positive and negative portions cancel exactly. A point load at x_0 instead yields

$$Q_j = F \phi_j(x_0).$$

Moving an actuator or changing a load footprint can thus suppress a resonance by symmetry without changing the material, tension or natural frequency.

Estimate a cable’s displacement from its first few modes

A laboratory cable spans 2.0 m under 200 N tension and has linear density 0.50 kg/m. Its wave speed is $20\,\mathrm{m/s}$, so $f_1 = 20/(2L) = 5.0\,\mathrm{Hz}$ and $m_j = 0.50\,\mathrm{kg}$. Apply a distributed harmonic pressure-equivalent line load $p_0 = 0.40\,\mathrm{N/m}$ at 2.0 Hz. For mode one, $Q_1 = 2(0.40)(2)/\pi = 0.509\,\mathrm{N}$, and its dynamic stiffness is

$$0.50[(2\pi 5)^2 - (2\pi 2)^2] = 414.5\,\mathrm{N/m}.$$

Its modal amplitude is 1.23 mm.

The third mode has $Q_3 = 0.170\,\mathrm{N}$, $f_3 = 15$ Hz and amplitude $0.170/\{0.50[(2\pi 15)^2 - (2\pi 2)^2]\} = 0.0390\,\mathrm{mm}$. At the midpoint $\phi_1 = +1$ while $\phi_3 = -1$, so the two-term prediction is approximately 1.19 mm. The next odd mode contributes with the opposite sign again, but is far stiffer at this drive frequency. In the static limit an independent check follows from the parabolic string solution $w(L/2) = p_0 L^2/(8H) = 1.00\,\mathrm{mm}$. If a strong measured response appears at the symmetric load’s even-mode resonance, investigate load asymmetry, asymmetric support stiffness or a disturbed initial condition before assigning a nonzero ideal modal projection.

The spatial convergence pattern can be estimated before adding many terms. For odd mode number n well above the forcing frequency, the uniform-load projection decreases as $1/n$ while modal stiffness increases as n^2. Each displacement contribution therefore decreases roughly as $1/n^3$. A point actuator at midspan instead gives modal projections of essentially alternating constant magnitude for odd modes, producing only a $1/n^2$ displacement decay. Hence a truncation that accurately predicts the response to a distributed flow load can be less accurate for an impulsive local tap. Strain or slope predictions may require still more terms because a spatial derivative introduces an additional factor of mode number. Choose the retained modes for the actual load and observed quantity.

The uniform cable problem admits an independent harmonic calculation without modal truncation. Set $w(x,t) = W(x)\cos \Omega t$. The governing equation gives

$$W'' + \kappa^2 W = -p_0/H,$$

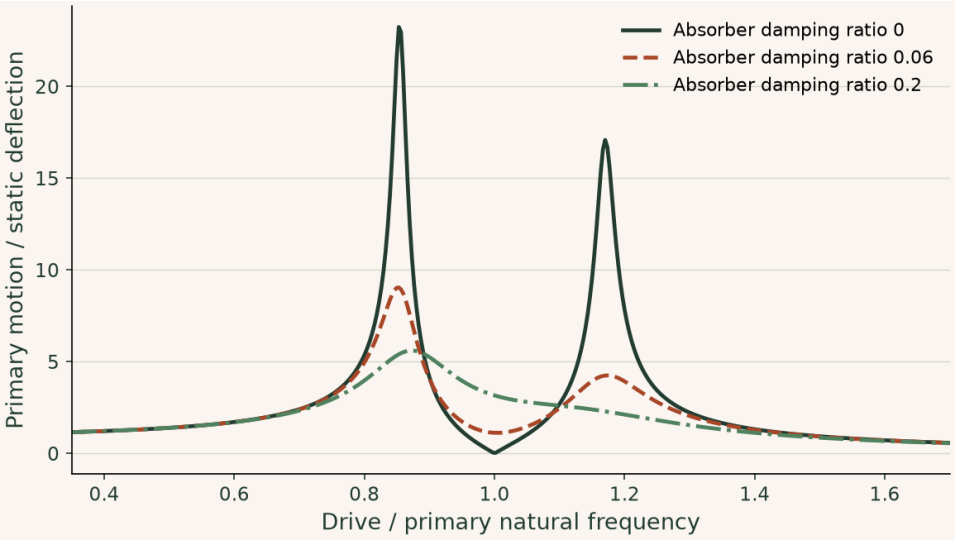
$$\kappa = \Omega\sqrt{\mu/H}.$$

Symmetry about midspan and fixed ends select $W(x) = [p_0/(\mu\Omega^2)]\{\cos[\kappa(x - L/2)]/\cos(\kappa L/2) - 1\}$, provided the denominator is nonzero. For the test above $\kappa L/2 = 0.6283$ and $W(L/2) = 0.40[1/\cos(0.6283) - 1]/[0.50(2\pi 2)^2] = 1.196\,\mathrm{mm}$. This direct boundary-value result confirms both the signs and the scale of the first two odd-modal contributions.

As the excitation approaches the first natural frequency, $\kappa L/2$ approaches $\pi/2$ and the undamped steady expression becomes singular. An actual time record may instead show progressively growing amplitude before damping or geometric effects intervene. At very low frequency expand $\sec q \simeq 1 + q^2/2$; substituting $q = \Omega L\sqrt{\mu/H}/2$ recovers the static midpoint displacement without subtracting two nearly equal numerical values. The modal and spatial calculations thus furnish independent checks in the static, off-resonant and resonant limits.

What happens to the notch of a tuned absorber?

Attach an absorber of mass 0.10 times the primary mass, tuned to the primary's undamped frequency. Include 2.5% primary damping. Solve the complex two-mass force balance for primary displacement under harmonic forcing; the absorber spring and dashpot act on relative motion.



Without absorber damping the primary displacement vanishes at the tuned frequency, with response peaks on both sides. Absorber damping fills the notch and changes those adjacent peaks. An installation with variable drive speed must check the entire band.

▶ **Read values from the plot**

Interactive calculation · change the inputs of this example

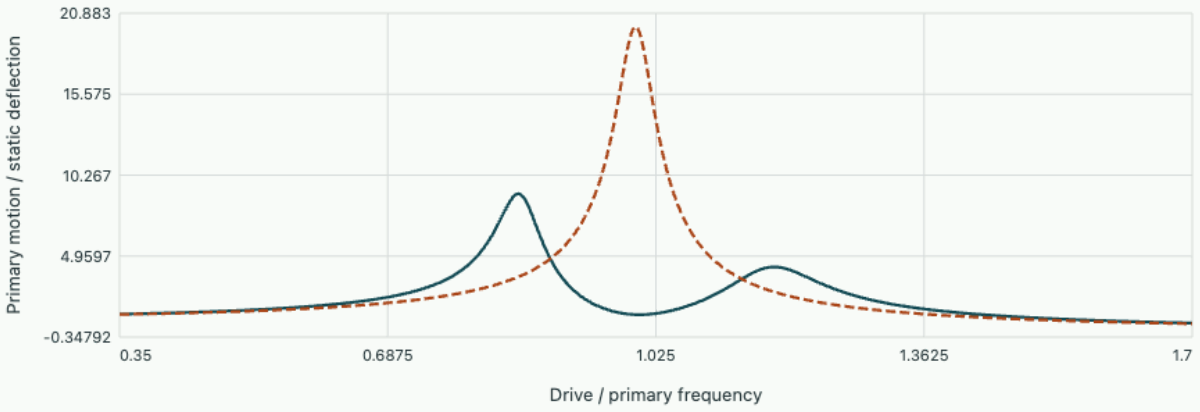
Detune the absorber, then add damping. Compare the cancellation notch with both adjacent peaks.

The curves below are recalculated from this section's equations in your browser. The fixed reference plot above remains available for comparison.

Open this live calculation

Try the calculation

Absorber / primary mass	**0.1**
Absorber damping ratio	**0.06**
Absorber / primary frequency	**1**



At the primary natural frequency, amplification is 1.125. A damped or mistuned absorber fills the exact undamped cancellation notch.

How a longitudinal pulse reflects from a change in rod section

Machine dynamics

A tap on one end of a rod reaches a receiver after a finite delay. Two rods joined end to end can even return a pulse before the receiver sees a far-end reflection. Derive propagation speed and distinguish the displacement amplitude from the energy carried by a pulse.

From the cut force to a travelling disturbance

Measure the small axial displacement $u(x,t)$ from a stress-free reference. In a slender homogeneous rod of area A, tensile force is

$$N = EAu_x.$$

A slice of length dx and mass $\rho A\,dx$ accelerates because its right and left forces differ by $N_x dx$. For constant properties, cancelling A gives

$$\rho A u_{tt} = (EAu_x)_x,$$

$$u_{tt} = c^2 u_{xx},$$

$$c = \sqrt{E/\rho}.$$

Either $u = f(x - ct)$ or $u = g(x + ct)$ satisfies this equation by differentiation. Curves of constant phase move right or left with speed c. Long-wavelength assumptions suppress lateral inertia and radial modes. Rod area cancels from wave speed but remains in the force carried at given particle velocity. For a right-moving pulse $v = u_t = -cu_x$, so tensile force $N = -Zv$, with mechanical impedance $Z = \rho Ac = EA/c$. The sign reverses for a left-moving wave.

Match velocity and force at a bonded junction

Let amplitudes I, R and T denote incident, reflected and transmitted particle velocity at the junction. A massless, perfectly bonded interface requires equal velocities and transmitted force. Apply the preceding sign convention separately to each travelling direction. The two equations determine two unknowns.

$$I + R = T,$$

$$-Z_1 I + Z_1 R = -Z_2 T,$$

$$\frac{R}{I} = \frac{Z_1 - Z_2}{Z_1 + Z_2},$$

$$\frac{T}{I} = \frac{2Z_1}{Z_1 + Z_2}.$$

The instantaneous power transported to the right is $-Nv = Zv^2$ for a single right-moving wave. For the same pulse shape the reflected energy fraction is $(R/I)^2$ and the transmitted fraction is $(Z_2/Z_1)(T/I)^2$. They sum to one. A transmitted velocity amplitude exceeding the incident amplitude can thus be compatible with conservation of energy.

A stepped rod and a pulse-echo observation

Join two aluminium rods of areas 80 and 160 mm², $E = 70$ GPa and $\rho = 2700$ kg/m³. Both have $c = 5092$ m/s, but the second impedance is twice the first. An incident velocity of 0.30 m/s produces a reflected −0.10 m/s and a transmitted 0.20 m/s. The reflected pulse carries one ninth of the energy. Reversing the experiment makes the reflection positive and the transmitted velocity 0.40 m/s for the same incident amplitude. The two energy fractions are unchanged on reversal.

A receiver at the tap detects a reflected pulse 0.40 ms after launch, before any far-end reflection contaminates it. The return path covers the distance twice, so the inferred junction is $c(0.00040)/2 = 1.02$ m away. Measure pulse velocity rather than strain when comparing the signs to the plot. Incident tensile force has sign opposite incident velocity with the convention used here.

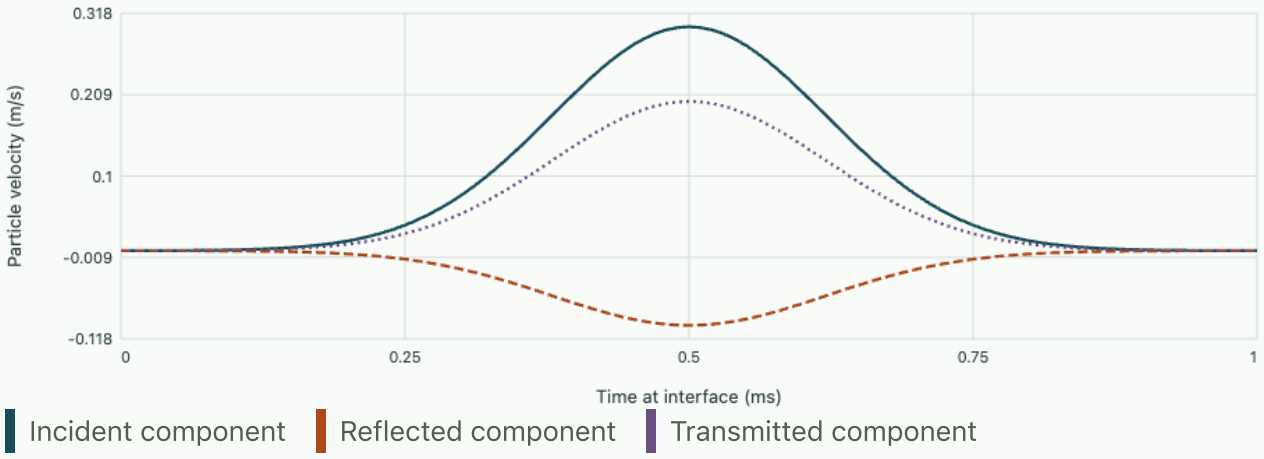
Keep the incoming pulse fixed and change the area ratio from 2 to 0.5. Determine which velocity component reverses sign and compare reflected energy fractions. Explain the difference by writing power balance for the two experiments.

Open this experiment online

Try the calculation

Second / first section area **2**

Incident peak velocity (m/s) **0.3**



Peak reflected velocity is -0.100 m/s and transmission is 0.200 m/s. The reflection carries 11.1% of the incident energy.

Tutorial · infer a second cross-section

On a uniform material an isolated junction returns 25% of the incident velocity amplitude with the same sign. Infer the area ratio and transmitted energy fraction. How could an energy-only measurement leave the geometry ambiguous?

▼ **Show the inverse relation**

Let $z = Z_2/Z_1 = A_2/A_1$. Then $0.25 = (1 - z)/(1 + z)$, so $z = 0.60$. The transmitted velocity is $1.25I$. Its energy fraction is

$$0.60(1.25)^2 = 0.9375.$$

Measuring only the reflected energy gives 0.0625 and loses the sign of the amplitude. A thickening by a factor $1/0.60$ would give the same energy reflection and the opposite sign in the velocity trace.

Further reading · Bower, Applied Mechanics of Solids · solutions to problems involving vibrations and wave propagation

Why the frequencies of a bending beam are not equally spaced

Machine dynamics

A distributed beam can bend into infinitely many shapes. Derive the relation between wavelength and oscillation frequency, then compare the measured response of a beam with the mode family predicted by its supports.

Combine transverse force, moment and curvature

Take a straight uniform beam with small transverse deflection $w(x,t)$, density ρ, area A and second moment of area I about the bending axis. Euler–Bernoulli kinematics keeps cross-sections perpendicular to the centreline. The bending resistance is proportional to curvature EIw_{xx}. For a load-free part of the beam the gradient of moment supplies shear and the gradient of shear accelerates the mass. With restoring forces opposing deflection, the resulting equation is

$$\rho A\, w_{tt} + EI\, w_{xxxx} = 0.$$

The fourth spatial derivative calls for four boundary conditions. At a simple support the displacement vanishes and the bending moment is zero, hence

$$w = w_{xx} = 0.$$

A clamp instead fixes displacement and slope. Interchanging a pin and a clamp changes the allowed shape family even if the span and material remain fixed.

Obtain the simply supported series

Try $w = W\sin(kx)\cos(\omega t)$. Two spatial derivatives bring down $-k^2$, and four give $+k^4$. Substitution balances the inertia and stiffness terms. The conditions at x=0 hold for every k in this sine family. At x=L the sine must vanish, yielding

$$\omega^2 = \frac{EI}{\rho A}k^4,$$

$$k_n = \frac{n\pi}{L},$$

$$f_n = \frac{n^2\pi}{2L^2}\sqrt{\frac{EI}{\rho A}}.$$

A long travelling sinusoid has phase velocity $\omega/k = k\sqrt{EI/(\rho A)}$. The group velocity $d\omega/dk$ is twice that value in this model. A localized pulse contains many wavelengths, which can therefore separate as it propagates. Predictions at wavelengths near the thickness require accounting for rotary inertia and shear deformation.

Choose sensor positions as carefully as the frequency range

Use a 1 m steel strip 20 mm wide and 4 mm thick with $E = 200$ GPa and $\rho = 7800$ kg/m³. For bending about the thin direction $I = bh^3/12$ and $I/A = h^2/12$. The first three simply supported frequencies are roughly 9.18, 36.7 and 82.7 Hz. Doubling the width alone doubles stiffness and mass together and leaves these frequencies unchanged. Doubling thickness doubles each frequency. Shortening the span by 20% increases all three by $1/0.8^2 = 1.56$.

At the centre a displacement pickup sees $\sin(n\pi/2)$. It sees the first mode at its antinode and misses the second mode entirely. At the quarter point it sees that second mode strongly. An absent spectral peak can therefore result from excitation or sensing at a node. Move the measuring point before adjusting a fitted damping value to explain its absence.

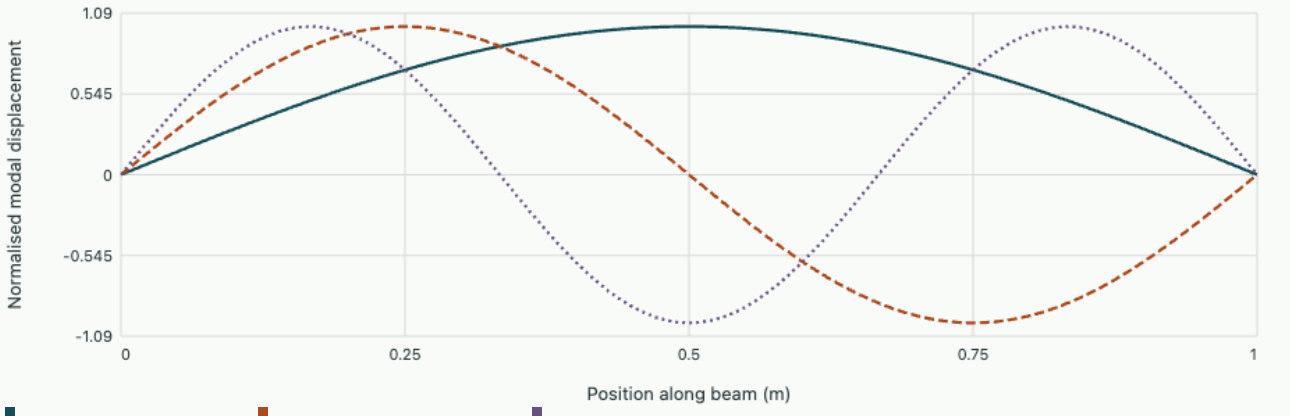
Find a common node of the first and third modes and a location that suppresses the second mode alone. Change the thickness and then the span. Use the resulting frequencies to test whether each scaling is linear or quadratic.

Open this experiment online

Try the calculation

Span (m) 1

Thickness (mm) 4



At midspan the second mode has a displacement node. The three frequencies are 9.18, 36.74, 82.66 Hz.

A related OneSimulation calculation

For a frame whose geometry and end payload preclude the simply supported sine family, compare the first three computed modes after changing the tip mass. Test whether all frequencies scale by the same factor.

Compare frame modes with OneSimulation

Tutorial · distinguish distributed bending from a tensioned string

A simply supported specimen has prominent peaks at 12 Hz and 108 Hz with the receiver at midspan. Infer which bending modes these could be and predict the frequency and receiver location for the missing mode. What would the third-to-first frequency ratio be for an ideal string?

▼ **Show the mode test**

The ratio $108/12 = 9$ agrees with n^2 for $n = 3$. The second bending mode should lie near $4(12) = 48$ Hz. Measure at one quarter span and excite away from its nodes. An ideal uniform string has $\omega = ck$ and a third-to-first ratio of 3. Checking both the midspan node and the shifted pickup spectrum distinguishes these models more effectively than a single peak.

Further reading · MIT 2.003SC · lecture 27 on vibration of continuous structures

Estimating a cantilever frequency from energy

Machine dynamics

A clamped beam carries a small sensor at its tip. Its precise mode shape is unknown. Use the static deflection shape to approximate the fundamental frequency and see why concentrating a small mass at the tip has a large effect.

Choose a displacement field that respects the support

Write $w(x,t) = q(t)\phi(s)$ with $s = x/L$, taking q as tip displacement. Choose the shape from a uniform cantilever under a static transverse tip force. The bending moment varies linearly from root to tip. Integrating curvature twice with $w(0) = w'(0) = 0$, then normalizing at $s = 1$, gives

$$\phi = s^2(3 - s)/2.$$

The shape satisfies both essential clamp conditions. Its free-tip shear need not match the exact unloaded vibration mode. The energy formulation can use such admissible trial shapes.

Background · Rayleigh quotient

$$R(u) = u^{\mathsf{T}} K u / (u^{\mathsf{T}} M u)$$

For symmetric stiffness and positive mass this estimates squared angular frequency. A kinematically admissible displacement is required for a useful structural estimate.

Compute effective stiffness and mass from the same field

The strain energy density of a slender beam in bending is $\frac{1}{2} E I w_{xx}^2$. Its translational kinetic energy is $\frac{1}{2}\rho A w_t^2$ per unit length. Changing variables from x to s introduces different powers of L into the two integrals. Include a point mass M at the tip, neglecting its rotary inertia.

$$U = \frac{EIq^2}{2L^3}\int_0^1 (\phi'')^2\, ds,$$

$$K = \frac{\dot{q}^2}{2}\left[\rho A L \int_0^1 \phi^2\, ds + M\right].$$

Since $\phi'' = 3(1 - s)$, the stiffness integral is 3. Squaring $s^2(3 - s)/2$ and integrating $(9s^4 - 6s^5 + s^6)/4$ gives $33/140$. The Euler–Lagrange equation for q is consequently a single oscillator equation with

$$\omega_R^2 = \frac{3EI/L^3}{(33/140)\rho A L + M}.$$

For M=0 this gives the dimensionless coefficient $\sqrt{140/11} = 3.5675$. Solving the exact unloaded cantilever condition $\cosh\beta\cos\beta = -1$ gives

$$\beta_1^2 = 3.5160.$$

The estimate is high by about 1.5%. The Rayleigh minimum principle explains that direction of error for conservative systems with admissible trial shapes. Restricting motion to a single shape restricts the set over which the energy quotient can attain its minimum.

Choose a sensor after estimating its dynamic effect

Take $EI = 10$ N m², $L = 0.50$ m and total beam mass 0.40 kg. The tip stiffness is 240 N/m and the distributed effective mass is only 0.0943 kg. The estimate gives 8.03 Hz without a sensor and 6.49 Hz with a 0.05 kg tip sensor. The sensor weighs one eighth of the beam, yet it has over half the effective mass associated with the chosen tip coordinate.

Place the same sensor at half-span instead. There $\phi(0.5) = 0.3125$. Its kinetic energy contributes $M\phi(0.5)^2 = 0.00488$ kg to the effective mass, so the estimate rises to 7.83 Hz. The sensor measures a smaller local motion as it perturbs the frequency less. Compare that trade-off at different attachment positions with the plot.

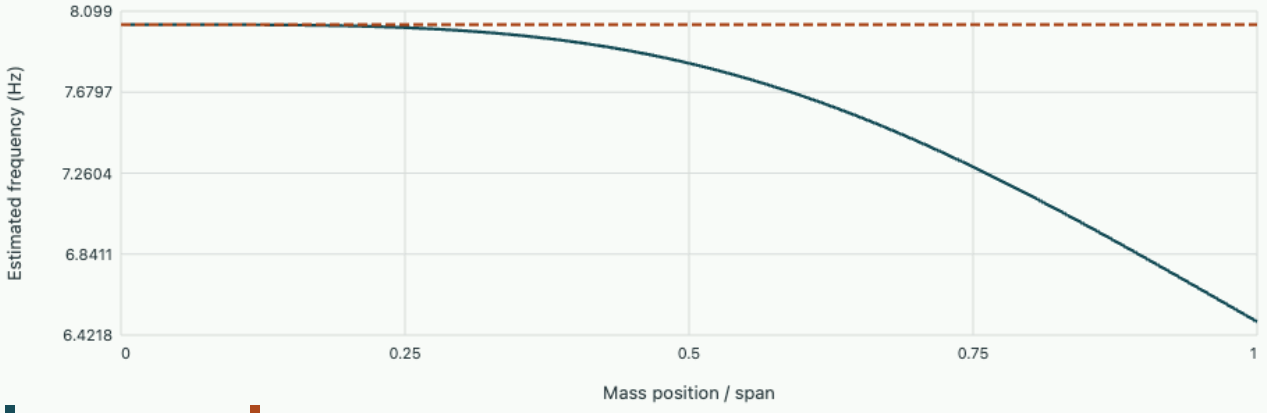
Move along the position axis with a fixed sensor mass. Identify the interval in which frequency is most sensitive to changing attachment position. Double the sensor mass and check whether the change in frequency doubles at the tip.

Open this experiment online

Try the calculation

Attached mass (kg) **0.05**

Bending rigidity (N m²) **10**



The estimate is 6.49 Hz with the sensor at the tip and 7.83 Hz at half-span. The bare-beam estimate is 8.03 Hz.

A related OneSimulation calculation

The course frame model has several modes and an independently variable payload rotary inertia. Compare a pure translation-mass change with a rotary-inertia change; the scalar point-mass estimate above addresses the first mechanism.

Open the OneSimulation frame eigenvalue experiment

Tutorial · estimate an unknown attached mass

The same bare beam is fitted with a small tip part and resonates at 6.0 Hz. Estimate the attached mass. Give one reason the inferred mass could differ from a scale reading even when the frequency measurement is accurate.

▼ **Show the mass estimate**

The required effective mass is

$$240/(2\pi \cdot 6)^2 = 0.1689\,\mathrm{kg}.$$

Subtract 0.0943 kg for the beam to obtain about 0.0746 kg. Clamp compliance would reduce the effective stiffness and frequency, which this fit would interpret as extra tip mass. The trial shape and omission of rotary inertia also limit the estimate for a part with appreciable extent.

Further reading · Bower, Applied Mechanics of Solids · Rayleigh–Ritz methods

Weak forms and finite elements

Numerical partial differential equations

Derive a variational statement by integration by parts, assemble element matrices for a one-dimensional rod and explain what the Galerkin solution minimises.

FROM FIRST PRINCIPLES

Obtain the weak diffusion problem by moving one derivative

For a stationary diffusion problem $-\nabla \cdot (k\nabla u) = f$, suppose temperature is prescribed on part of the boundary and the outward flux $q_n = -k\nabla u \cdot \mathbf{n}$ on the remainder. A classical second derivative may not exist across a material interface. Instead multiply the conservative equation by a smooth test function v that vanishes where u is prescribed. The divergence theorem moves one derivative from the flux onto v:

$$\int_\Omega k\nabla u \cdot \nabla v \, d\Omega = \int_\Omega f v \, d\Omega - \int_{\Gamma_N} v q_n \, d\Gamma.$$

Approximate u with continuous element shape functions N_j that are differentiable inside each element. Choose each allowable shape function as the test function. Linearity then yields element contributions

$$K_{ij}^{(e)} = \int_e k\nabla N_i \cdot \nabla N_j \, d\Omega,$$

$$f_i^{(e)} = \int_e f N_i \, d\Omega.$$

Neighbouring elements contribute to the same global unknown at a shared node. Adding their contributions enforces weak equilibrium. The flux sign in the boundary term follows from an outward-positive definition; switching to prescribed inward heat supply reverses that sign in the load vector.

Background · Weak form

$$\int k\nabla u \cdot \nabla v \, dV = \int f v \, dV + \text{boundary terms}$$

Integration by parts lowers the derivative requirement. Essential and natural boundary data enter the discrete system differently.

DERIVATION AND APPLICATIONS

How a Robin boundary enters both the stiffness matrix and the load

A convecting boundary couples an unknown surface temperature to an ambient temperature. Unlike a prescribed heat source it contributes to both sides of the assembled equations. Consider steady diffusion $-\nabla \cdot (k \nabla T) = s$ in a two-dimensional region of unit thickness, with $k > 0$. On a straight boundary segment impose the empirical film law $-k \nabla T \cdot \mathbf{n} = h(T - T_\infty)$ using the outward normal. Multiply the field equation by a test function v vanishing on prescribed-temperature boundaries, integrate, and apply the divergence theorem:

$$\int_\Omega k \nabla v \cdot \nabla T \, dA - \int_{\partial\Omega} v k \nabla T \cdot \mathbf{n} \, ds = \int_\Omega v s \, dA.$$

Substitution on the film boundary Γ places the unknown surface term on the left and the ambient contribution on the right:

$$\int_\Omega k \nabla v \cdot \nabla T \, dA + \int_\Gamma h v T \, ds = \int_\Omega v s \, dA + \int_\Gamma h v T_\infty \, ds$$

with any other natural boundary terms added separately. To assemble one linear edge of length ℓ, use edge functions $N_1 = 1 - \xi$, $N_2 = \xi$, $\xi \in [0, 1]$. Insert the nodal interpolation for T and choose each N_i as test function. Constant h, T_∞ give

$$K_\Gamma = \frac{h \ell}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad f_\Gamma = \frac{h \ell T_\infty}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The off-diagonal term is present because both nodes influence temperature across the same edge. In the limit $h = 0$ the boundary is insulated. A very large film coefficient constrains the temperature toward ambient but can introduce a large stiffness contrast.

WORKED EXAMPLE

Assemble a convecting edge by hand

Use a one-metre out-of-plane thickness, $\ell = 0.300$ m, $h = 20.0$ W/(m² K) and $T_\infty = 290$ K. Then K_Γ has diagonals 2.00 W/K and off-diagonals 1.00 W/K. The load is $(870, 870)$ W. At nodal temperatures 300 and 310 K the film residual $K_\Gamma T - f_\Gamma$ is $(40, 50)$ W. The sum 90 W agrees with $h \ell [(300 + 310)/2 - 290]$.

SECOND APPLICATION

Test a constant solution before using the boundary routine

Set both temperatures to 290 K, the ambient value. Each row has sum 3.00 W/K, so every residual is $3(290) - 870 = 0$. If the code includes the ambient load but forgets the matrix, it predicts 1740 W of heating at thermal equilibrium. This constant-field patch test detects sign errors and omission of the surface matrix independently of the volume mesh.

Reduce the differentiability required by equilibrium

A rod is fixed at $x = 0$, pushed by a distributed load f and by an end force t at $x = L$. Axial displacement u satisfies $-(EAu')' = f$, $u(0) = 0$ and $EAu'(L) = t$. The product EA is positive. A trial profile assembled from linear pieces has jumps in its derivative, so differentiating it twice pointwise is unsuitable. Instead multiply the strong equation by a test displacement v vanishing at the fixed end and integrate by parts once.

The resulting equation is

$$\int_0^L EA\,u'v'\,dx = \int_0^L fv\,dx + tv(L).$$

Only first derivatives are required. The fixed displacement is imposed directly on trial and test spaces; the end traction appears in the load functional. This clarifies why prescribing both force and displacement independently at the same end can overconstrain a second-order problem. The test functions represent allowable virtual displacements, and the integral identity expresses equality of internal and external virtual work.

Background · Why can the derivative jump while the displacement stays continuous?

At a bonded interface with no concentrated force, displacement and axial force are continuous. If EA jumps, the strain u' can jump inversely even though the product EAu' is continuous. For a continuous piecewise linear displacement, the weak first derivative is the ordinary piecewise constant slope. The squared-gradient energy is therefore finite.

A displacement jump is different. Replacing a jump of size d by a straight ramp over width δ costs strain energy $EAd^2/(2\delta)$ for constant positive stiffness, which diverges as the ramp shrinks. A standard conforming displacement space encodes this distinction. If the mechanics really allow slip, the interface needs its own model and a space which permits the two traces to differ.

Assemble one element and connect it to its neighbour

On an element from x_i to x_{i+1} of length h, use shape functions $N_1 = 1 - s,\ N_2 = s$ with $s = (x - x_i)/h$. Their physical derivatives are $-1/h,\ 1/h$. For constant EA, integrating products

$EAN_a'N_b'$ gives the matrix $(EA/h)\left(\begin{smallmatrix}1 & -1\\ -1 & 1\end{smallmatrix}\right)$. A constant distributed load contributes $fh(1,1)^T/2$. Summing element contributions for a shared global node enforces balance there without prescribing the internal traction manually.

Take $L=1,\ EA=1,\ f=2,\ t=0$ in consistent units and use two equal elements. After eliminating the fixed displacement, the stiffness matrix is $\left(\begin{smallmatrix}4 & -2\\ -2 & 2\end{smallmatrix}\right)$ and the load is $(1,0.5)^T$. Solving gives

$$u(0.5)=0.75,\ u(1)=1.$$

The exact solution is $2x-x^2$ and agrees at both nodes. Inside the first element at $x=0.25$, the linear interpolant gives 0.375, whereas the exact value is 0.4375. Nodal agreement can coexist with a substantial interior error.

Assemble a physical triangle from a reference element

A line element has one length scale. A triangle can stretch, rotate and shear, so the element area alone cannot determine its diffusion matrix. Start with the reference vertices $(0,0),(1,0),(0,1)$ and the linear shape functions $1-\xi-\eta,\xi,\eta$. Send the reference triangle to a physical cell with vertices P_0,P_1,P_2 by $x=P_0+J(\xi,\eta)^T$, where the columns of J are P_1-P_0 and P_2-P_0. For a nondegenerate triangle, the chain rule and area transformation give

$$\nabla_x N_i = J^{-T}\nabla_{\xi,\eta}\widehat{N}_i,$$
$$dx\,dy = |\det J|\,d\xi\,d\eta.$$

To see why the inverse transpose occurs, take an arbitrary displacement $d\widehat{x}$. Both gradients must describe the same change in a shape function. Thus $(\nabla_x N_i)^T J\,d\widehat{x} = (\nabla\widehat{N}_i)^T d\widehat{x}$ for every such displacement. For unit isotropic conductivity, the transformed gradients are constant and the physical stiffness entries reduce to

$$K_{ij} = |T|\nabla N_i\cdot\nabla N_j,$$

with $|T|=|\det J|/2$. A constant cell load f contributes $f|T|/3$ at each vertex because each linear shape function has mean $1/3$.

A sheared cell and an exact affine-field check

Take $P_0=(0,0),P_1=(2,0),P_2=(1,1)$. The columns of J are $(2,0)$ and $(1,1)$, so the cell area is 1. Applying J^{-T} to the three reference gradients gives $(-1/2,-1/2),(1/2,-1/2),(0,1)$. Their dot products and the load $f=3$ yield

$$K=\begin{pmatrix}1/2 & 0 & -1/2\\ 0 & 1/2 & -1/2\\ -1/2 & -1/2 & 1\end{pmatrix},\qquad b=(1,1,1)^T.$$

Every row sums to zero because a constant field has zero gradient. For a more discriminating test use $u = x + 2y$, with vertex values $U = (0, 2, 3)^T$. Weighting the three gradients by these values gives $(1, 2)$. The physical integral of its squared gradient equals $1(1^2 + 2^2) = 5$, and direct multiplication gives $U^T K U = 5$. A transposed mapping error can pass the constant test and fail this second one.

Identify the energy that is best approximated

Let $a(u, v) = \int EA\, u'v'$ and denote the external work by $\ell(v)$. The discrete solution u_h satisfies $a(u_h, v_h) = \ell(v_h)$ for every test function from its finite-dimensional space. The continuous solution satisfies the same equation for every such test, so $a(u - u_h, v_h) = 0$. This Galerkin orthogonality means the error is perpendicular to the trial space in the energy inner product. For any other admissible discrete w_h, write $u - w_h = (u - u_h) + (u_h - w_h)$ and expand its squared energy norm. The cross term vanishes and $\|u - w_h\|_a^2 = \|u - u_h\|_a^2 + \|u_h - w_h\|_a^2$.

Thus the conforming Galerkin solution is the best energy approximation in its space for this symmetric linear problem and exact integration. The statement concerns a specific norm and model. It does not guarantee pointwise optimal stresses or carry unchanged into nonsymmetric transport problems. In the rod example the computed strain is constant on each element, replacing a linearly varying exact strain. Refinement improves this derivative field even when a table of nodal displacements already looks exact.

Recover a support reaction

The reaction at a prescribed-displacement node is hidden if one inspects only the reduced interior system. Assemble all element internal forces Ku_h before discarding constrained rows, then subtract the full external load vector. The entries at free degrees of freedom should be near zero; the entry at a prescribed degree of freedom gives the reaction in the chosen sign convention.

In the uniformly loaded rod, the first element strain is

$$(0.75 - 0)/0.5 = 1.5.$$

Its internal nodal force at the left is -1.5, while the first node receives an external distributed-load share of $+0.5$. Their difference -2 is the support reaction. It balances the total applied load $2 \cdot 1 = 2$. Looking only at the derivative in the first element would have reported magnitude 1.5, missing that load share. A reaction from assembled balance and a stress recovered by pointwise differentiation are related but distinct quantities on a finite mesh.

Obtain actual norm errors from the Galerkin property

For $-u'' = f$ on the unit interval with homogeneous Dirichlet values, the elementary energy inner product is $a(u, v) = \int u'v'$. Let $I_h u$ be the continuous piecewise-linear interpolant of a sufficiently smooth exact solution at mesh nodes. On each element its slope is the average of u'. Every linear finite-element test function v_h has constant derivative on that element. Consequently

$$\int_E (u - I_h u)' v_h' = v_h' [u - I_h u]_{\partial E} = 0.$$

Summing proves global energy orthogonality.

The Galerkin solution satisfies the same orthogonality, and positive definiteness gives uniqueness. Thus $u_h = I_h u$ in this one-dimensional, constant-coefficient problem with exactly integrated load and matching boundary values. Nodal errors vanish, although the field and derivative between nodes can still be inaccurate. Variable coefficients inside an element generally destroy the simple telescoping argument because the factor multiplying v_h' is no longer constant. The result also should not be transferred to multidimensional triangulations without proof.

Compare nodal, field and flux accuracy for one solution

Choose $u = x(1 - x)$, so $f = 2$. On an element starting at a write $s = x - a \in [0, h]$. Subtract the straight interpolant from the parabola to get $u - u_h = s(h - s)$, vanishing at both nodes and attaining $h^2/4$ at the element centre. Its derivative is $h - 2s$; the exact and numerical slopes agree at the midpoint but differ by magnitude h at each endpoint taken from that element. Testing only the midpoint slope or the nodal values would entirely conceal these other discrepancies.

Compute the squared energy error on this element:

$$\int_0^h (h - 2s)^2 ds = h^3/3.$$

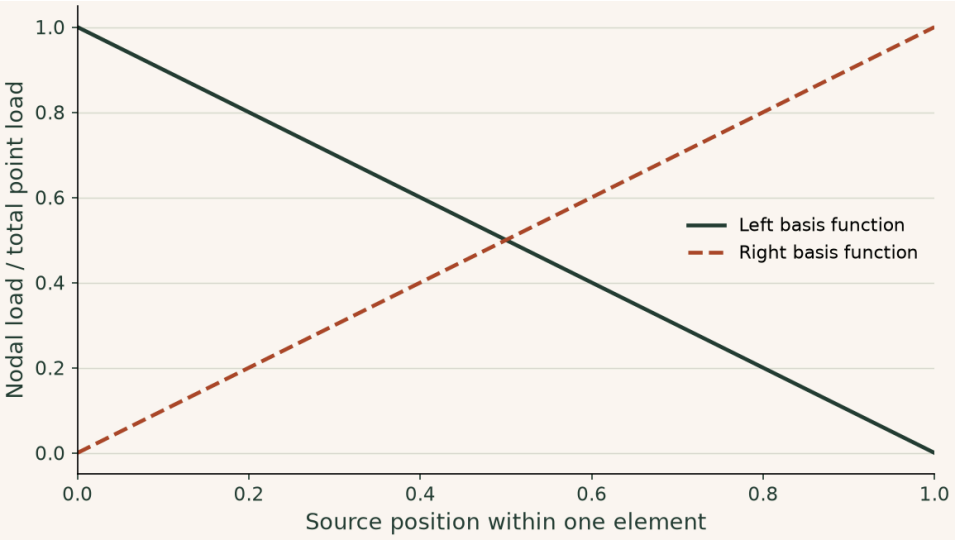
With $1/h$ equal elements, the global energy error is $h/\sqrt{3}$. Similarly the element squared field error is $\int_0^h s^2 (h - s)^2 ds = h^5/30$, giving global L^2 error $h^2/\sqrt{30}$. For $h = 1/4$, the two errors are approximately 0.14434 and 0.01141; the maximum field error is 0.015625. Halving the mesh width halves the energy error but divides the field error by four. All these measurements concern the very same numerical solution whose nodal error is exactly zero.

For a physical diffusion flux with constant conductivity one, flux error equals minus derivative error. A cell-averaged flux test is also special here: the numerical slope equals the exact average slope on each element for the same reason used in the proof. To verify local flux resolution, compare at more than a midpoint or a cell average and retain the measure used for convergence. A reported order is a statement about an observable and a norm as well as a finite-element space.

These formulas also identify a misleading manufactured test: if an automated comparison stores only nodal differences, the estimated order is undefined because both coarse and fine errors vanish. Adding independently evaluated quadrature points inside elements exposes the field error immediately. To check a flux implementation, use a quadrature rule capable of resolving the squared linear derivative error; a one-point midpoint rule would again report zero despite the nonzero exact energy norm.

Distribute a moving point source in a weak form

A one-dimensional Poisson problem has a concentrated source $Q\delta(x-a)$. Its weak load against a linear hat function is $Q\phi_i(a)$, so the two adjacent degrees of freedom share it in proportion to their shape functions.



At 30% of the element length the split is 70% left, 30% right. The loads sum to Q and reproduce the correct first moment. Placing the whole source at the nearest node preserves total load but jumps abruptly at mid-element.

▶ **Read values from the plot**

Change parameters · Distribute a moving point source in a weak form

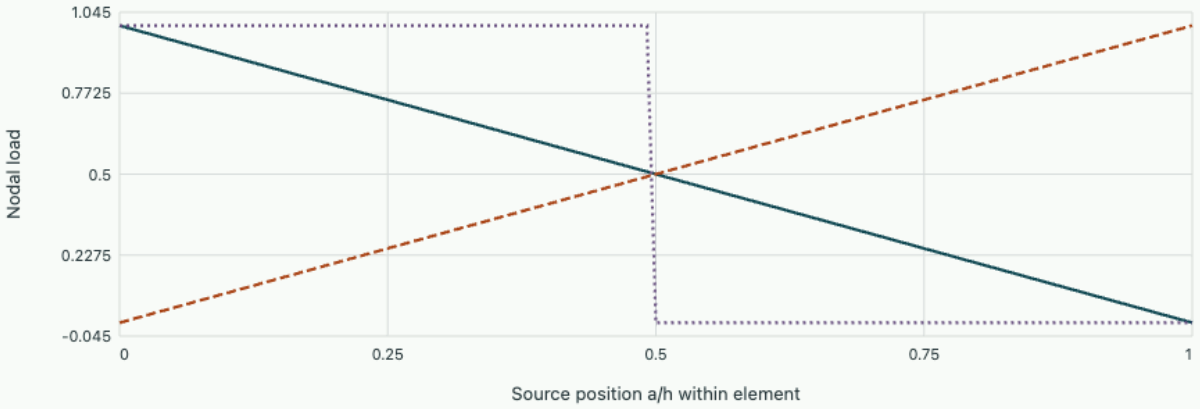
Change one input at a time and compare the recalculated curves with the argument above. The browser evaluates this small model locally; the accompanying numerical values state its assumptions and reference checks. Reset restores the reference inputs, and the data can be downloaded as CSV.

Open this calculation with adjustable inputs

Try the calculation

Source strength Q **1**

Selected element coordinate a/h **0.3**



Left hat-function load Right hat-function load Nearest-node left load

Selected loads 0.7 and 0.3 sum to Q = 1 and reproduce the source first moment. Nearest-node placement preserves only the total.

Building a local residual estimator and refining where it matters

Numerical partial differential equations

A finite-element solution can satisfy the assembled equations to machine precision while still missing curvature between nodes. Integrate the residual of the computed field by parts on each element and use the resulting cell and interface terms to decide where to spend the next degrees of freedom.

Ask what the discrete solution does to an unresolved test function

Consider $-u'' = f$ on a unit interval with zero Dirichlet data. Use continuous piecewise-linear finite elements with exact integration on a possibly nonuniform mesh. The weak solution satisfies $a(u, v) = \int u'v' = \int fv$ for every test function vanishing at the endpoints. For $e = u - u_h$, subtract the computed field to obtain

$$a(e, v) = R(v) = \int fv - \int u_h' v'.$$

The algebraic system makes $R(v_h) = 0$ for all functions in the chosen finite-element space. Tests with more local structure expose what is missing.

$$R(v) = \sum_K \int_K (f + u_h'')v \, dx + \sum_{i \text{ interior}} J_i v(x_i),$$

$$J_i = u_h'(x_i^+) - u_h'(x_i^-).$$

The sign follows by adding the right boundary term of the left cell and the left boundary term of the right cell. On a linear cell $u_h'' = 0$ classically. Its piecewise constant gradient jumps at interior nodes. In diffusion with varying coefficient k the corresponding jump is in ku_h', since continuity of normal flux is the conservation condition. A discontinuous material coefficient can cause an exact gradient jump without any flux imbalance.

Scale the cell and node terms before marking

Subtract the finite-element interpolant or a local quasi-interpolant from v inside the residual. Galerkin orthogonality permits this subtraction and bounds the remaining test magnitude by local length times its gradient. This leads to a computable residual indicator. One common one-dimensional choice with a symmetric edge length h_i equal here to the smaller neighbouring cell is

$$\eta_K^2 = h_K^2 \int_K f^2 \, dx + \frac{1}{2} \sum_{i \in \partial K \text{ interior}} h_i J_i^2,$$

$$\eta^2 = \sum_K \eta_K^2.$$

Sharing the node contribution prevents double counting. Under the usual mesh and coefficient assumptions such estimates bound the energy error up to constants; their absolute numbers need not equal the measured error. Data oscillation can matter if the source is approximated or oscillates within a cell. To mark by bulk contribution, sort η_K^2 in descending order and take cells until their sum exceeds a chosen fraction of the global squared estimator. Bisect marked cells, transfer the coefficient data, assemble and solve again. Reusing the old gradient jumps after refinement would skip the effect of the new trial space.

Work through the first two-element decision

Let $f = 1$ on the left half and $f = 9$ on the right. With two equal cells and zero end values there is a single interior hat unknown. Diagonal stiffness is 4 and the integrated hat load is

$$1(0.25) + 9(0.25) = 2.5.$$

The middle value is therefore 0.625. The slopes are 1.25 and –1.25, giving $J = -2.5$. Since $h = 0.5$, the two cell terms are 0.125 and 10.125. Each side receives $(1/2)(0.5)(2.5)^2 = 1.5625$ from the node. The right cell contributes $11.6875/(1.6875 + 11.6875) = 87.38\%$ of the squared indicator. A 50% bulk mark selects only that cell.

For this particular problem the finite-element solution equals the piecewise-linear interpolant of the true nodal values. This follows by testing the derivative of the interpolation error against a constant basis gradient on each element. For constant f on each element the derivative error is therefore a line of slope –f with zero mean. Its exact squared energy norm is $\sum f_K^2 h_K^3/12$. Splitting the right half into two equal cells reduces that half's contribution by a factor of four. Total energy error falls from 0.924 to 0.470 with one extra degree of freedom. Splitting only the left cell leaves about 0.920. Run the loop and compare the indicator's ranking with this independent verification quantity.

Run the mesh from zero to three refinement rounds with a 50% bulk mark. Record error against free values. Repeat with a 90% mark and compare equal computational costs where available. Which choice targets the source localization most strongly?

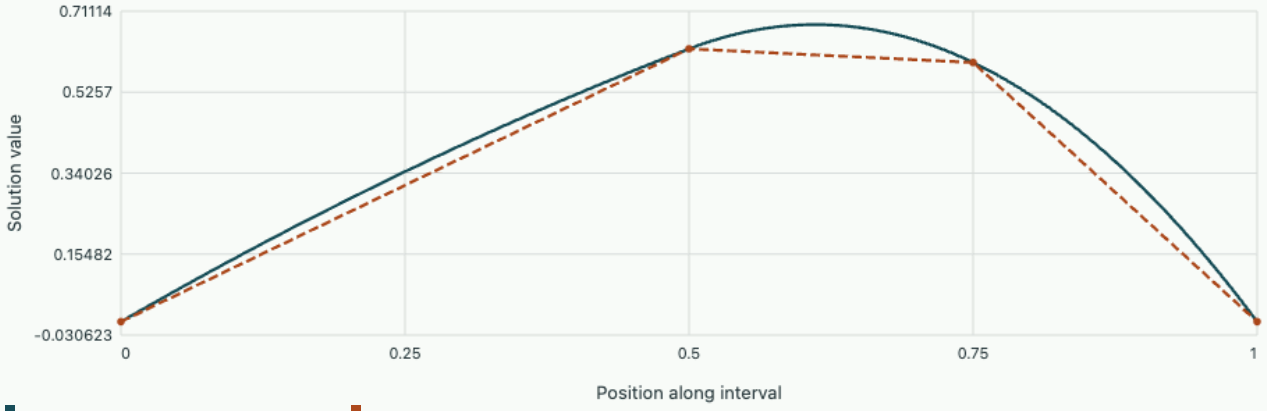
Open this experiment online

Try the calculation

Refinement rounds **1**

Right-hand source **9**

Bulk marking fraction **0.5**



Exact quadratic pieces Adaptive linear elements

After 1 refinement round there are 3 cells and 2 free values. Exact energy error is 0.4705. The global residual indicator is 2.0963.

Tutorial · predict the benefit of two candidate splits

Consider a left cell of length 0.50 with $f = 1$ and a right cell of length 0.125 with $f = 9$. Which single split would remove more exact squared energy error? Show how stiffness residual and source localization answer distinct questions.

▼ **Show the local comparison**

Bisection removes three quarters of that cell's $f^2 h^3/12$. Before division by 12 the left weight is $1(0.5)^3 = 0.125$; the right weight is

$$81(0.125)^3 = 0.1582.$$

Refining the small right cell still yields about 27% more improvement. Conversely the assembled nodal residual could already be zero on either mesh after a direct solve. Reducing the algebraic tolerance further would improve neither piecewise-linear profile. This calculation is a verification benchmark made possible by the manufactured piecewise-constant data.

Further reading · Joachim Schöberl, Interactive Finite Elements · a posteriori error estimation by residuals

Why two finite-element wave solvers propagate the same mode at different speeds

Numerical partial differential equations

Build the element mass matrix by integrating shape functions and track the effect of diagonal lumping on the wave spectrum.

From a continuum wave to nodal masses

For $u_{tt} = c^2 u_{xx}$, multiply by a test function and integrate by parts. On a periodic line the boundary contributions cancel. A piecewise-linear element of width h uses shape functions $1-\xi, \xi$, with $\xi \in [0,1]$. Integrating products of shape functions gives

$$M_e = (h/6)\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Integrating derivative products gives

$$K_e = (c^2/h)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Row-sum lumping replaces each mass row by its sum on the corresponding diagonal, $h/2$. This preserves the mass associated with a uniform velocity but changes the response to a velocity that varies within an element.

Background · Numerical dispersion

$$c_{\mathrm{ph,h}} = \omega_h(k)/k$$

A discretised wave can retain its amplitude and still propagate at the wrong speed. Compare phase and group errors over the actual resolved wavelengths.

Diagonalise the assembled periodic problem

A periodic nodal mode has $u_j = e^{ij\theta}$. Each neighbour contributes $e^{\pm i\theta}$ relative to node j. Thus K multiplies by $2c^2(1-\cos\theta)/h$, consistent M by $h(2+\cos\theta)/3$, and lumped M by h. Dividing stiffness by mass gives the discrete squared frequency. Relative to the exact frequency $c\theta/h$,

$$\frac{\omega_L}{c\theta/h} = \frac{2\sin(\theta/2)}{\theta},$$

$$\frac{\omega_C}{c\theta/h} = \frac{2\sin(\theta/2)}{\theta}\sqrt{\frac{3}{2+\cos\theta}}.$$

Both tend to one for long waves; the leading quadratic errors have opposite signs. This is a spatial effect, present even with exact time integration.

Two wavelengths in one mesh

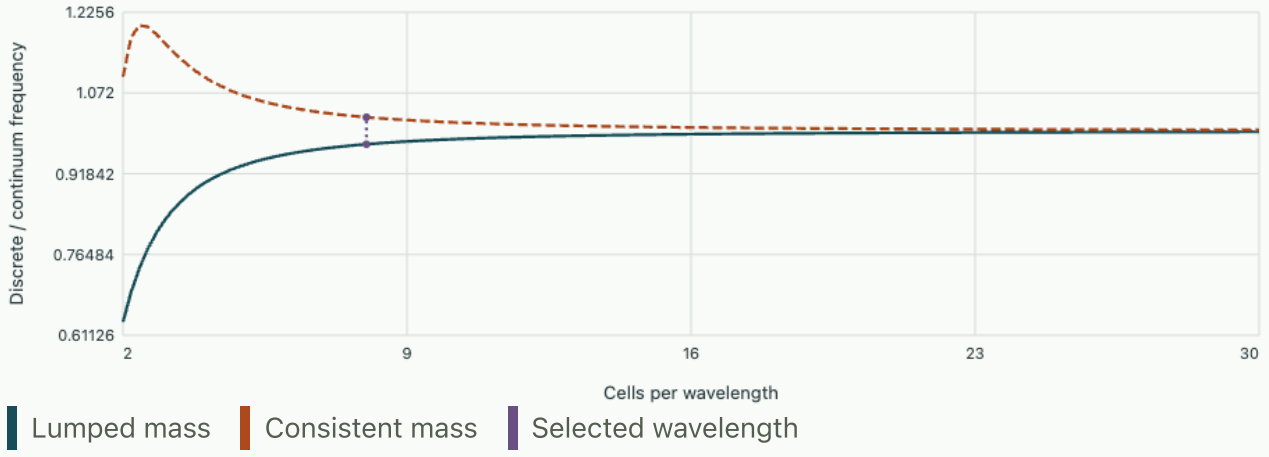
At four cells per wavelength $\theta = \pi/2$. The lumped ratio is $2\sqrt{2}/\pi \approx 0.9003$; the consistent ratio is about 1.1027. One wave solver lags by roughly ten per cent and the other leads by roughly ten per cent. At eight cells per wavelength the ratios move much closer to one, but repeated propagation over many wavelengths can still accumulate visible phase error.

At the shortest grid mode $\theta = \pi$, the consistent frequency is $\sqrt{12}c/h$ and the lumped frequency $2c/h$. A central-difference time integrator stable for $h_t\omega \leq 2$ therefore requires a smaller step for the consistent mass. Accuracy at one wavelength and stability at the grid scale are different design criteria.

Open this experiment online

Try the calculation

Cells per selected wavelength **8**



At 8 cells per wavelength the lumped ratio is 0.9745 and the consistent ratio 1.0259.

Tutorial on interpreting a convergence test

Which mass gives the exact kinetic energy for a linear velocity varying from v to $-v$ on one element?

▼ **Worked solution**

The continuum energy with unit density is $hv^2/6$. The consistent matrix reproduces it by exact integration. The lumped kinetic energy is $hv^2/2$, because it assigns both end masses the full endpoint speed.

How would you isolate spatial dispersion from time stepping in a measurement of a cavity eigenfrequency?

▼ **Worked solution**

Solve the generalised eigenproblem $Kq = \omega^2 Mq$ directly for increasingly fine meshes or converge time steps on each fixed mesh. Then compare with the continuum frequency and report both h and the time tolerance.

Action density, stress waves and the boundary that reflects them

Analytical mechanics from constraints to invariants

A chain of coupled masses becomes an elastic field when the wavelength is long compared with the spacing. Derive the field equation and its boundary traction from the same variation, follow energy and stress across a junction, and decide whether a measured pulse is limited by material waves or by numerical dispersion.

Pass from a discrete rod to an action density

Let equal masses of spacing a move longitudinally by $u_j(t)$, with nearest-neighbour spring constant k_s and mass m_s per site. Their Lagrangian is $\sum_j [m_s \dot{u}_j^2/2 - k_s(u_{j+1}-u_j)^2/2]$. For variations smooth on the spacing, $(u_{j+1}-u_j)/a \to u_x$ and $\sum_j a\, f_j \to \int f(x)\,dx$. Hold the line density $\rho A = m_s/a$ and axial rigidity $EA = k_s a$ fixed. The continuum Lagrangian density per unit reference length is

$$\mathcal{L} = \rho A\, u_t^2/2 - EA\, u_x^2/2,$$

with A area, ρ volume density and E Young modulus. Both terms have units of joules per metre. This is a slender-rod model with small strain, primarily uniaxial stress and wavelengths long compared with the cross-sectional dimensions.

Vary the field in the action over $[0, L] \times [t_1, t_2]$. Integration of $\rho A\, u_t \delta u_t$ by parts in time, with fixed endpoint fields, produces $-\rho A\, u_{tt}\delta u$. Integration of $-EAu_x\delta u_x$ by parts in space produces $+\partial_x(EAu_x)\delta u$ and a remaining term $-[EAu_x\delta u]_0^L$. Thus the interior equation is $\rho A u_{tt} = \partial_x(EAu_x)$, and the end force conjugate to displacement is axial force $N = EAu_x$. Adding external work $F_L(t)\delta u(L)$ at the right face yields

$$N(L) = F_L.$$

A prescribed displacement instead imposes $\delta u(L) = 0$, leaving its reaction as an output. Keeping this boundary term prevents prescribing displacement and independent traction simultaneously at the same end of a second-order rod.

Determine the energy flux and characteristic impedance

Assume time-independent coefficients. Multiply the field equation by u_t and rearrange to obtain

$$\partial_t[\rho A u_t^2/2 + EAu_x^2/2] + \partial_x[-EAu_x u_t] = 0.$$

The energy flux is $-Nu_t$, with positive sign denoting energy moving towards increasing x. The minus sign follows the convention of positive tensile N: positive mechanical power entering through a right face is Nu_t, which is inward with respect to the rod on its left. This identity controls energy transport across both smooth material changes and sharp joints.

For a homogeneous right-moving wave $u = F(x - ct)$, $c = \sqrt{E/\rho}$, particle velocity $v = u_t = -cF'$ and axial force $N = EAF' = -Zv$, where $Z = \rho A c = EA/c$. A left-moving wave has $N = +Zv$. The impedance is force divided by particle velocity and has units $\mathrm{kg/s}$. If one instead uses stress, impedance is ρc per unit area. Doubling the area of the same material keeps wave speed unchanged but doubles the force–velocity impedance. A transit-time measurement and a reflection measurement therefore provide independent material or geometry information.

For a rod with $E = 70\,\mathrm{GPa}$, $\rho = 2700\,\mathrm{kg/m^3}$ and $A = 1.0 \times 10^{-4}\,\mathrm{m^2}$, $c \approx 5092\,\mathrm{m/s}$ and $Z \approx 1375\,\mathrm{kg/s}$. A right-going compressive pulse with particle speed $0.020\,\mathrm{m/s}$ carries axial force $-27.5\,\mathrm{N}$, compressive stress $0.275\,\mathrm{MPa}$, and instantaneous power $Zv^2 = 0.550\,\mathrm{W}$ across a section. A rectangular approximation lasting $100\,\mu\mathrm{s}$ carries about $55\,\mu\mathrm{J}$ and occupies $0.509\,\mathrm{m}$. The pulse speed is kilometres per second; the material particles move only millimetres per second. Treating these as the same velocity would spoil force, impulse and energy estimates.

Match a bonded interface and distinguish amplitude from power

At a perfect massless bond displacement and axial force are continuous. For an incident right-going velocity amplitude v_i in rod 1, let v_r be reflected left-going velocity and v_t transmitted right-going velocity. Velocity continuity gives

$$v_i + v_r = v_t.$$

Force continuity gives

$$-Z_1 v_i + Z_1 v_r = -Z_2 v_t.$$

Thus

$$r_v = \frac{Z_1 - Z_2}{Z_1 + Z_2},$$

$$t_v = \frac{2Z_1}{Z_1 + Z_2},$$

$$\mathcal{R} = r_v^2,$$

$$\mathcal{T} = \frac{Z_2}{Z_1} t_v^2 = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2}.$$

The power fractions add to one for real positive impedances. The reflected force coefficient has opposite sign to the reflected velocity coefficient because the wave direction reverses. At a free end $Z_2 \to 0$, reflected velocity equals incident velocity, so end velocity doubles while total force vanishes. At a rigid end $Z_2 \to \infty$, velocity reverses and force doubles. The sign a strain gauge reports can therefore differ from that of a velocity-sensitive optical measurement at the same reflection.

A step from the worked rod into one of the same material with twice the cross-sectional area gives $Z_2 = 2Z_1$, $r_v = -1/3$, $t_v = 2/3$, and power fractions $1/9$ and $8/9$. A reduced transmitted velocity amplitude does not imply that most energy is reflected. If the interface includes a concentrated mass

M_b, force jumps instead obey $N_2 - N_1 = M_b \dot{v}_b$ with signs determined by forces exerted on that mass from either side. At high frequency the reaction of that mass can mimic a much stiffer termination. Frequency-dependent reflection is then an expected boundary effect, rather than evidence that the bulk modulus varies with frequency.

Connect continuum modes to their lattice predecessor

For a periodic chain the exact discrete equation is

$$m_s \ddot{u}_j = k_s(u_{j+1} - 2u_j + u_{j-1}).$$

Substitute $u_j = U e^{i(kja - \omega t)}$ to obtain

$$\omega = 2\sqrt{k_s/m_s}\,|\sin(ka/2)|.$$

In the first positive Brillouin zone, $\omega/(ck) = \sin(ka/2)/(ka/2)$ and group velocity $d\omega/dk = c\cos(ka/2)$, where $c = a\sqrt{k_s/m_s}$. The continuum rod retains only the leading low-ka term. A wave with eight sites per wavelength has phase speed about $0.9745c$ and group speed $0.9239c$. A narrowband packet's peak can arrive later than the continuum prediction even when a single crest is timed fairly accurately.

A finite-difference code approximating the continuum rod can produce the same spatial dispersion with its mesh spacing in place of a physical lattice spacing. Distinguish them experimentally or numerically by changing the computational mesh while holding the physical structure fixed. Lattice effects persist as the numerical grid is refined; artificial spatial dispersion diminishes. A separate time-step study is needed if a time integrator also disperses the wave. Comparing both peak arrival and phase at two frequencies tests more of the dynamics than fitting a single transit time.

Derive transverse beam waves from curvature energy

A slender beam bending laterally is governed by a different elastic energy from the axial rod. Let $w(x,t)$ denote small transverse deflection of its neutral axis. Under the Euler–Bernoulli assumption that cross sections remain plane and normal to that axis, the axial strain a distance z from the neutral axis is approximately $-z w_{xx}$. Integrating the local elastic energy $E\epsilon_{xx}^2/2$ across the section gives $EI_A w_{xx}^2/2$ per unit length, where $I_A = \int z^2\,dA$ is a geometrical second moment of area, with units m^4. It is distinct from a mass moment of inertia. The kinetic contribution uses line density ρA as before.

Varying $\mathcal{L} = \rho A w_t^2/2 - EI_A w_{xx}^2/2$ requires integrating twice by parts in space. With constant coefficients the interior variation gives

$$\rho A w_{tt} + EI_A w_{xxxx} = 0.$$

The boundary terms are $-[EI_A w_{xx}\delta w_x] + [(EI_A w_{xx})_x \delta w]$. Thus slope variations pair with bending moments and displacement variations with shear forces, with signs set by the chosen face orientation. A

clamped end prescribes both deflection and slope; a free end sets moment and shear to zero. A simply supported end prescribes deflection and zero bending moment. Using only one boundary condition per end, as for the axial rod, leaves the fourth-order beam problem underdetermined.

A sinusoid $e^{i(kx-\omega t)}$ gives

$$\omega = \sqrt{EI_A/(\rho A)}\, k^2.$$

Phase speed grows proportionally to wavenumber and group speed is twice phase speed in this model. A narrow transverse pulse in a long beam consequently spreads even in a perfectly homogeneous non-dissipative continuum. By contrast the axial rod was nondispersive in its long-wavelength continuum approximation. At high enough frequency the simple bending model omits rotary inertia, shear deformation and eventually three-dimensional elastic behaviour. Extrapolating its unbounded speed to arbitrary k would use the slender, low-frequency approximation far beyond its derivation.

For a simply supported beam with length L, the sine modes satisfy both end conditions and give

$$\omega_n = (n\pi/L)^2 \sqrt{EI_A/(\rho A)}.$$

For a rectangular section of width w_b and thickness d in the bending direction, $I_A = w_b d^3/12$ and $A = w_b d$. Doubling thickness doubles all the natural angular frequencies at fixed L, E, ρ, whereas doubling width alone leaves the ideal frequencies unchanged. The stiffness increases and so does the inertia. A beam with $L = 0.50\,\mathrm{m}$, $d = 5.0\,\mathrm{mm}$ and the earlier aluminium properties has $\sqrt{EI_A/(\rho A)} \approx 7.35\,\mathrm{m^2/s}$, giving fundamental $46.2\,\mathrm{Hz}$. The second bending mode is near four times that frequency, a clear difference from the approximately harmonic axial mode sequence.

Match a force pulse to a support spring and dashpot

A right termination exerts force on the rod through a dashpot with mechanical impedance Z_L. For a real positive dashpot impedance the reflected velocity fraction is $r = (Z - Z_L)/(Z + Z_L)$. Taking $Z_L = Z$ gives zero reflection at the termination even though energy leaves a finite rod: the dashpot dissipates the arriving power. With $Z_L = 3Z$ one obtains a reflected velocity of $-1/2$ of the incident value, a reflected energy fraction one quarter and absorbed power fraction three quarters. A free or rigid end also reflects energy, but it absorbs no net power in the ideal limit. A measured weak reflection alone is therefore not proof of weak damping inside the rod.

For a pure end spring of stiffness k_L attached to a rigid support, the velocity impedance is $Z_L = k_L/(i\omega)$ under $e^{i\omega t}$. It is imaginary, so the reflection coefficient has unit magnitude at every frequency and a frequency-dependent phase. At low frequency the spring impedance is large and the end looks nearly fixed. At high frequency its impedance is small and the end looks nearly free within the lumped approximation. An impulsive signal contains many frequencies and can return with a changed waveform even without any transmitted or dissipated energy. A support combining a resistor-like dashpot and a spring gives both amplitude loss and phase variation.

A measurement with strain gauges at two positions can separate forward and backward waves if their path delays are resolved. In a narrow frequency band the two records are linear combinations of advancing and returning phase factors. If the gauge spacing corresponds to an integer number of half wavelengths, these combinations become linearly dependent and the separation is ill-conditioned. Change the spacing, use a broadband pulse, or combine a velocity measurement with a strain measurement at one station so that the two characteristic directions enter with opposite signs. The relation $v = (v_+ + v_-)$, $N = Z(-v_+ + v_-)$ then gives a direct local decomposition.

Derive sound propagation from mass and momentum conservation

The displacement-action argument has a fluid analogue. Take an initially uniform fluid of density ρ_0 and pressure p_0, moving little from equilibrium in one spatial dimension. Write small perturbations ρ', p', v. Linearised mass conservation is

$$\partial_t \rho' + \rho_0 \partial_x v = 0.$$

Momentum balance for a short fluid parcel is $\rho_0 \partial_t v = -\partial_x p'$. If compression over an acoustic cycle is approximately isentropic, $p' = c_s^2 \rho'$ with $c_s^2 = (\partial p / \partial \rho)_s$. Differentiate the first equation in time, eliminate the velocity derivative with momentum balance, and obtain $\partial_t^2 p' = c_s^2 \partial_x^2 p'$.

The thermodynamic derivative matters because pressure can respond differently when compression is slow enough for heat exchange. For a perfect gas with ratio of specific heats γ, isentropic perturbations give

$$c_s^2 = \gamma p_0 / \rho_0.$$

Taking an isothermal derivative would give p_0/ρ_0 instead. For $p_0 = 101\,\mathrm{kPa}$, $\rho_0 = 1.20\,\mathrm{kg/m^3}$, and $\gamma = 1.40$, the isentropic speed is about $343\,\mathrm{m/s}$, while the isothermal calculation gives $290\,\mathrm{m/s}$. A measured transit time can discriminate between these constitutive conditions. Wall heat exchange in a very narrow tube can alter which approximation is appropriate over a chosen frequency band.

For a right-moving sound wave $p' = \rho_0 c_s v$, with the pressure positive in compression. This differs in sign from the tensile-positive axial force used for the rod, so match physical force conventions before borrowing formulas across the two examples. The acoustic power per area is $p' v$. For sinusoidal pressure with rms amplitude $1.0\,\mathrm{Pa}$ in the stated gas, the mean intensity is

$$p_{\mathrm{rms}}^2 / (\rho_0 c_s) = 2.43 \times 10^{-3}\,\mathrm{W/m^2}.$$

A peak amplitude of $1.0\,\mathrm{Pa}$ would yield half this number. Writing rms or peak explicitly prevents a routine factor-of-two error when comparing a microphone measurement with an energy flux.

Determine whether a measured transit reveals a modulus or a geometry

Suppose two axially loaded slender rods have measured wave speeds agreeing within their uncertainties. That alone determines only the ratio E/ρ under the slender-rod assumptions. Measuring force–velocity impedance $Z = \rho A c$ and cross section as well gives $\rho = Z/(Ac)$ and $E = Zc/A$. With $Z = 1375\,\mathrm{kg/s}$, $c = 5092\,\mathrm{m/s}$ and $A = 1.0 \times 10^{-4}\,\mathrm{m^2}$, the inversion gives

$$\rho = 2700\,\mathrm{kg/m^3},$$

$$E = 70.0\,\mathrm{GPa}.$$

If area were overestimated by five percent, both inferred quantities would be underestimated by almost five percent while their ratio remained correct.

Now examine a thick elastic solid under a narrow compressional disturbance with wavelengths short compared with its transverse dimensions. Lateral deformation can be constrained by surrounding material. In an isotropic three-dimensional continuum the longitudinal plane-wave modulus is $\lambda + 2\mu_s$, where λ, μ_s are Lamé coefficients, rather than the uniaxial-stress Young modulus of a slender rod. The shear-wave speed satisfies $c_T^2 = \mu_s/\rho$ and longitudinal speed $c_L^2 = (\lambda + 2\mu_s)/\rho$. Measuring both, together with density, can recover the two independent elastic moduli. Using only one longitudinal speed to infer Young modulus without specifying the deformation geometry can give a physically incorrect value.

For $\rho = 2500\,\mathrm{kg/m^3}$, $c_L = 5000\,\mathrm{m/s}$ and $c_T = 3000\,\mathrm{m/s}$, these relations give

$$\mu_s = 22.5\,\mathrm{GPa},$$

$$\lambda = 17.5\,\mathrm{GPa}.$$

The implied Young modulus $E = \mu_s(3\lambda + 2\mu_s)/(\lambda + \mu_s) = 54.8\,\mathrm{GPa}$, while $\rho c_L^2 = 62.5\,\mathrm{GPa}$. Both numbers represent material stiffness, but they enter different mechanical boundary conditions. A long thin specimen of the same material should have an axial speed closer to $\sqrt{E/\rho} = 4680\,\mathrm{m/s}$. Comparing the bulk and rod experiments tests the constitutive and geometrical assumptions behind the measured speed.

Derive orthogonality when a rod changes material along its length

For a rod with positive varying density $\rho(x)$, area $A(x)$ and modulus $E(x)$, a separated free oscillation $u(x,t) = \phi(x)\cos\omega t$ satisfies

$$-[EA\phi']' = \omega^2 \rho A \phi.$$

Suppose two modes ϕ_n, ϕ_m satisfy the same homogeneous fixed or free boundary conditions and the correct continuity of displacement and axial force across every perfectly bonded interface. Multiply the first equation by ϕ_m, the second by ϕ_n, subtract and integrate. Interior terms produced by integration by parts cancel pairwise, including the interface flux terms, and the endpoint terms vanish. Thus

$$(\omega_n^2 - \omega_m^2) \int \rho A \phi_n \phi_m \, dx = 0.$$

Mode shapes with distinct frequencies are orthogonal under the mass-weighted inner product, not generally under a simple average of displacements measured at uniformly spaced stations. An initial velocity distribution $v_0(x)$ projects onto mode n with coefficient $\int \rho A v_0 \phi_n \, dx / \int \rho A \phi_n^2 \, dx$. In a rod with a heavy section, the same imposed velocity there stores more energy and carries more modal weight. An experiment interpreting raw camera amplitudes without line-density weighting can misassign energy among modes. Normalising each measured mode shape to unit maximum does not fix the mass weighting needed for energies.

Multiplying one eigenvalue equation by its own shape gives

$$\omega_n^2 = \int EA(\phi_n')^2 dx / \int \rho A \phi_n^2 dx.$$

For an admissible trial function satisfying prescribed displacement conditions, the same quotient bounds the fundamental squared frequency from above when rigid translations are excluded. Consider a uniform fixed–free rod and trial $\phi = x/L$. The numerator is EA/L and denominator $\rho A L/3$, giving $\omega_{\mathrm{trial}} = \sqrt{3}\, c/L$. The exact fundamental is $\pi c/(2L)$, so this one-parameter shape overestimates frequency by about 10.3%. It need not satisfy the free-end traction condition itself because that is a natural condition of the variational minimiser; it does need to satisfy the essential fixed-end displacement.

For a free–free rod the constant displacement mode has zero frequency. Using an arbitrary positive trial function without removing its mass-weighted mean can drive a quotient towards zero by adding a large constant. To estimate the first deforming mode impose orthogonality to the rigid translation, $\int \rho A \phi \, dx = 0$. This has an immediate mechanical interpretation: the deformation mode should not translate the centre of mass of an isolated rod. The null mode and its mass-weighted constraint reappear in numerical eigenproblems built from discretised stiffness and mass matrices.

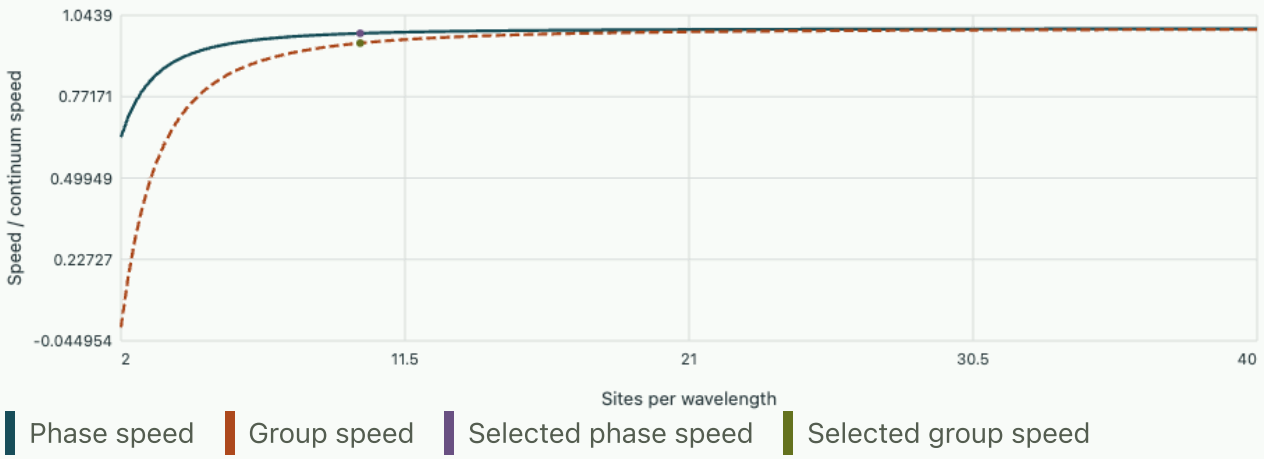
Find where a discrete elastic chain ceases to look continuous

Shorten the wavelength relative to the lattice spacing. Compare the phase and group velocities with those inferred from a continuum action.

Open this experiment online

Try the calculation

Sites per wavelength 10



Frequency and phase speed are 98.36% of the continuum prediction. Group speed is 95.11%.

Track work and impulse in a one-dimensional pulse experiment

A finite displacement pulse provides a useful warning about integrating strain and velocity records. A right-going strain pulse of one sign can carry net momentum $\int \rho A v \, dx$. Because $v = -c u_x$, its integrated material momentum is $-\rho A c[u(+\infty) - u(-\infty)]$ at a fixed time. A localised displacement bump that returns to the same displacement on both sides has zero net ordinary material momentum, even though its energy is positive and travels to the right. Its strain and particle-velocity records contain positive and negative parts. A unipolar velocity pulse leaves behind a displacement step, produced by a source that has supplied net impulse.

For a right-moving rectangular velocity pulse of amplitude $0.020\,\mathrm{m/s}$, duration $100\,\mu\mathrm{s}$ in the worked rod, the local displacement jump measured in time is $2.0\,\mu\mathrm{m}$. Its impulse magnitude across a section is $Z v \Delta t = 2.75 \times 10^{-3}\,\mathrm{N\,s}$, while its transmitted energy is

$$Z v^2 \Delta t = 55.0\,\mu\mathrm{J}.$$

A second equal pulse of opposite velocity sign launched later cancels the net displacement and material momentum but doubles the transmitted energy. Thus integrating a signed velocity signal and integrating its square distinguish different properties of the same experimental record.

A displacement sensor has a small linear drift during a burst. Which inferred observables are particularly sensitive?

▼ **Work through the solution**

Differentiating adds a constant offset to particle velocity. Integrating that apparent velocity over a long window produces a growing spurious displacement jump and momentum estimate. Squaring it adds

both a cross term with the physical velocity and a positive baseline energy. Measure a pre-pulse baseline, choose an integration interval covering the physical pulse, and test that the result is stable when the window is moderately extended. An independent strain gauge gives another characteristic combination; comparison with $N = -Zv$ can expose drift or an overlapping reflection.

Can a finite-element rod model with a plausible first eigenfrequency still fail an impulse test?

▼ Work through the solution

The first frequency mostly probes a long-wavelength weighted stiffness-to-mass ratio. A brief pulse contains higher modes and is sensitive to temporal resolution, dispersion, boundary tractions and the way discrete source work enters. Verify modal frequencies at several wavelengths, check the pulse's positive energy and signed impulse separately, and compare the measured incoming and reflected characteristic fluxes. A single standing-wave eigenvalue cannot validate all of those physically independent balances.

Recover a jump condition at an interface with a point mass

For a concentrated mass M_b joining two rods include $M_b u_t(x_b)^2/2$ as a separate contribution to the Lagrangian. Its time variation contributes $-M_b u_{tt}\delta u$ at the joint. The left and right rod boundary terms contribute respectively $-N_1\delta u$ and $+N_2\delta u$. Requiring stationarity for an independently varying shared displacement gives exactly $N_2 - N_1 = M_b u_{tt}$. This derivation identifies both the sign and the constitutive assumption of a perfectly bonded, nonslipping joint.

For identical rods and a sinusoidal drive at angular frequency ω, the reflected velocity coefficient becomes $r = -i\omega M_b/(2Z + i\omega M_b)$ under the stated phasor convention. At $\omega M_b = 2Z$ it reflects half the power and transmits half. The mass itself absorbs no cycle-averaged energy, though it stores kinetic energy during a cycle. For $Z = 100\,\mathrm{kg/s}$ and $M_b = 0.10\,\mathrm{kg}$, the equal-power frequency is about $318\,\mathrm{Hz}$. A real joint that loses appreciable power at that frequency needs a damping or radiation term in addition to its measured mass.

Tutorial on stress waves

A gauge $0.80\,\mathrm{m}$ from a free end records the echo of a sharp disturbance in the worked rod. What echo delay and strain sign should it expect?

▼ Work through the solution

The extra path from gauge to end and back is $1.60\,\mathrm{m}$, giving $1.60/5092 = 0.314\,\mathrm{ms}$. A free end has zero net axial force, so the reflected force and strain have opposite signs to the incident ones. The velocity reflection keeps its sign. Finite pulse width and any change of material between gauge and end can obscure that simple result.

A bonded step has measured reflected velocity coefficient -0.20 and no detectable dissipation. Infer its impedance ratio and transmitted energy fraction.

▼ **Work through the solution**

Rearranging $r_v = (1 - r)/(1 + r)$ gives

$$r = Z_2/Z_1 = (1 - r_v)/(1 + r_v) = 1.50.$$

The energy reflection is 0.04 and transmission 0.96. The velocity transmission is 0.80, whose square alone is only 0.64. Multiplying by the transmitted-to-incident impedance ratio recovers the correct energy fraction.

A rod model with a prescribed displacement at its right end reports nonzero work there, even though the variational boundary term vanished in deriving its equation. Is there a contradiction?

▼ **Work through the solution**

Vanishing of $N\delta u$ follows because the allowed virtual displacement at a prescribed boundary is zero. Actual boundary velocity can be nonzero when the prescribed displacement varies in time. The actuator supplies power Nu_t at the right boundary. Compare the time-integrated input with the change of stored energy plus any losses or outgoing energy rather than identifying an admissible variation with actual displacement.

Compare a finite chain with its long-wavelength continuum.

Fließbach includes string vibrations, beam bending and sound in his continuum mechanics treatment.

Applied study

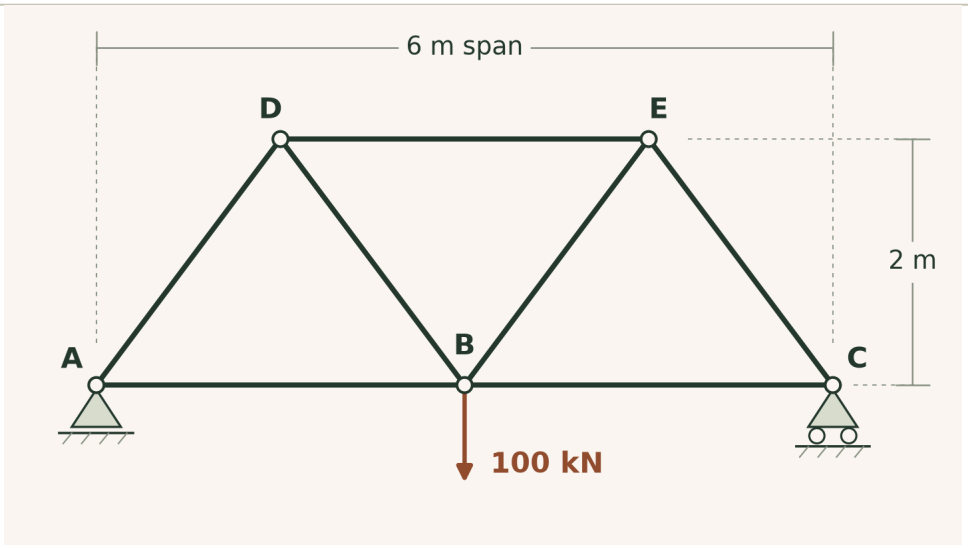
Trace loads through a structure and distinguish equilibrium, stress and deformation.

Where does a joint load go?

A lifting frame spans 6 m between a pin and a roller. A machine hanging from the middle applies a downward 100 kN load. The frame must carry that load without yielding, buckling or deflecting too far. Each check asks for a different property of the structure.

For the first calculation, idealise the joints as frictionless pins, the members as straight elastic bars and the load as a joint force. Neglect self-weight and out-of-plane motion. Each bar is then a two-force member. Its end forces lie along its axis. Let positive member force mean tension.

Place bottom joints A, B and C at $x = 0, 3, 6$ m and $z = 0$. Top joints D and E sit at $x = 1.5, 4.5$ m and $z = 2$ m. Join AB, BC and DE horizontally, then connect the zigzag A–D–B–E–C. Support A is pinned, C is a horizontal roller, and the load acts at B. Horizontal and vertical force balance and moments about A give $A_x = 0$ and $A_z = C_z = F/2 = 50$ kN.



The pin at A provides horizontal and vertical reactions. The roller at C permits horizontal movement and provides a vertical reaction.

Use a joint free-body diagram

Each diagonal rises 2 m over 1.5 m, so its length is 2.5 m and $\sin\theta = 0.8,\ \cos\theta = 0.6$. Assume that every unknown force pulls away from its joint. Vertical balance at A then gives $50 + 0.8N_{AD} = 0$, hence $N_{AD} = -62.5$ kN. Horizontal balance gives

$$N_{AB} = -0.6N_{AD} = 37.5\,\mathrm{kN}.$$

The minus sign identifies a compressed diagonal.

At B the two inner diagonals share the downward load, each supplying $F/2$ vertically. Each therefore carries 62.5 kN in tension. At D the horizontal components of AD and DB both point to the right and equal 37.5 kN. Member DE carries 75 kN in compression to close the joint balance. Reflection supplies the remaining forces. The horizontal support reactions vanish, yet the chords carry substantial horizontal forces inside the frame.

From member strain to a displacement

With $E = 200$ GPa and area $A = 1000\,\mathrm{mm}^2$ throughout, the top chord carries a nominal compressive stress of 75 MPa. To find the movement of B, store the elastic strain energy of all seven bars. A bar of length ℓ_i extends by $N_i \ell_i / (EA)$ and stores $N_i^2 \ell_i / (2EA)$. Ramp the load slowly from zero. The external work is $F\delta/2$ because the displacement grows linearly with load. Equating the two gives

$$\delta = \frac{1}{F} \sum_i \frac{N_i^2 \ell_i}{EA}.$$

Use two 3 m bottom chords at 37.5 kN, one 3 m top chord at 75 kN and four 2.5 m diagonals at 62.5 kN. These groups contribute 0.422 mm, 0.844 mm and 1.953 mm respectively. Joint B moves down by 3.219 mm. Compression stores positive energy just as tension does. The largest individual force and the largest group contribution to displacement occur in different members.

For a member that carries a transverse load between joints, the bending moment also contributes to strain energy. Study bending in a clamped beam.

Transfer problem · A shaft that carries torque

A drive shaft transmits 5.0 kW at 600 rpm to a lifting drum. Its internal load is a torque, even when its axial force is zero. In a uniform solid circular shaft of radius r_o, cross-sections rotate relative to their neighbours. A fibre at radius r acquires shear strain $\gamma = r\,d\phi/dx$. Using $\tau = G\gamma$ and taking moments of the shear stresses gives

$$T = GJ\,d\phi/dx,$$

with $J = \int_A r^2 dA = \pi r_o^4 / 2$.

Take $r_o = 15$ mm, length 0.80 m and $G = 80$ GPa. The angular speed is $600(2\pi/60) = 62.8$ rad/s, so the transmitted torque is

$$T = P/\omega = 79.6\,\mathrm{N\,m}.$$

Here $J = 7.95 \times 10^{-8}\,\mathrm{m}^4$. The surface stress is $Tr_o/J = 15.0$ MPa, and the twist angle is $TL/(GJ) = 0.0100$ rad or 0.57°. Doubling the radius at the same power and speed reduces stress eightfold and twist sixteenfold. An area check alone would capture neither dependence.

Exercises

1. Stiffen only the top chord

Double the area of DE and leave the other bars as they are. Find the new top stress and the displacement at B. Why do the diagonal forces stay the same?

▼ **Show the force and flexibility calculation**

All member forces in this statically determinate geometry follow from equilibrium, which contains no area or modulus. Top stress halves to 37.5 MPa. Only the 0.844 mm energy contribution halves, so the displacement becomes

$$3.219 - 0.422 = 2.797\,\mathrm{mm}.$$

Stiffening one bar has a much smaller effect than doubling every area.

---

2. Check the compression member

Suppose DE has a weak-axis second moment of area $I = 3.0 \times 10^{-7}\,\mathrm{m}^4$ and pin ends 3 m apart. Use the ideal elastic buckling load $P_{\mathrm{cr}} = \pi^2 EI/\ell^2$ to test the axial solution.

▼ **Show the stability check**

The ideal buckling load is 65.8 kN, below the 75 kN compressive demand. Even if the nominal stress is below yield, the assumed straight configuration can lose stability. Setting $P_{\mathrm{cr}} = 75$ kN gives $I = 3.42 \times 10^{-7}\,\mathrm{m}^4$ at the theoretical threshold. An axial truss calculation alone cannot detect this bending instability.

ONESIMULATION LABORATORY

Distinguish force from deflection in a truss

Compare recorded OneSimulation runs on this page. To test your own inputs, download the notebook and model files and run the model in an activated OneSimulation environment.

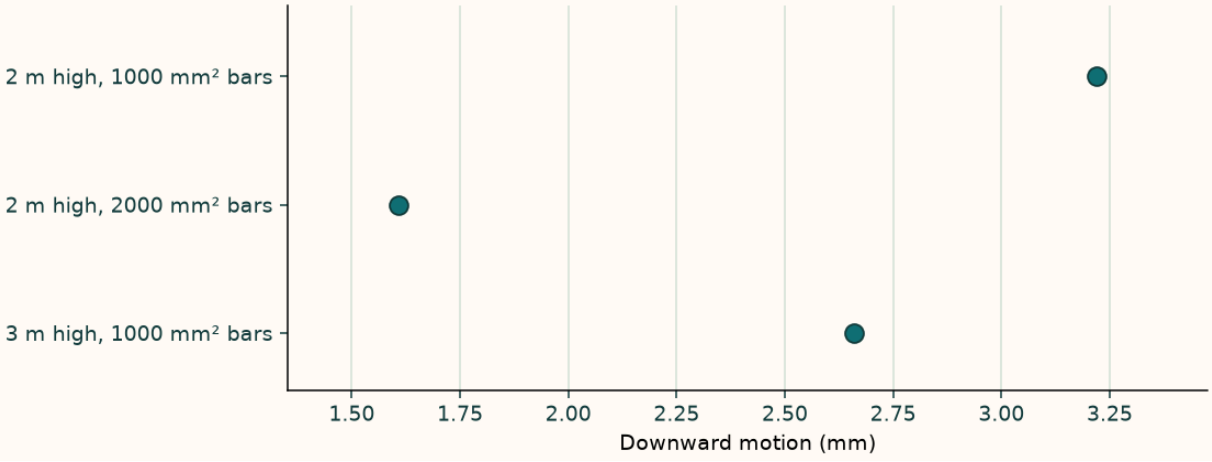
Run the example once and record the *plane truss* results for member “top 3-4” and the vertical displacement of bottom joint 1. At `a=3.0`, `h=2.0`, `area=10e-4` and `force=100e3`, expect −75.0 kN and 3.219 mm downward. The same script also prints a separate space-truss solution.

1. Change `area` from `10e-4` to `20e-4` m² for every bar. Rerun from the model setup. Predict unchanged forces and 1.609 mm deflection. Check both outputs.

2. Restore the area and change `h` from `2.0` to `3.0` m. The steeper diagonals need less axial force for the same vertical component. Expect −50.0 kN in the top chord and 2.659 mm of deflection. Compare the gain in stiffness with the reduced top stress.

Plane truss under a 100 kN centre load

Truss configuration	Top chord (kN)	Downward motion (mm)
2 m high, 1000 mm² bars	−75.0	3.219
2 m high, 2000 mm² bars	−75.0	1.609
3 m high, 1000 mm² bars	−50.0	2.659



Doubling every bar area halves the calculated joint deflection. Increasing truss height also reduces deflection, and changes the chord force as shown in the table.

View the OneSimulation truss model and its member-force comparison.

Download experiment files (ZIP)

Extract the files together and open `plane_and_space_truss.ipynb` in a local OneSimulation environment.

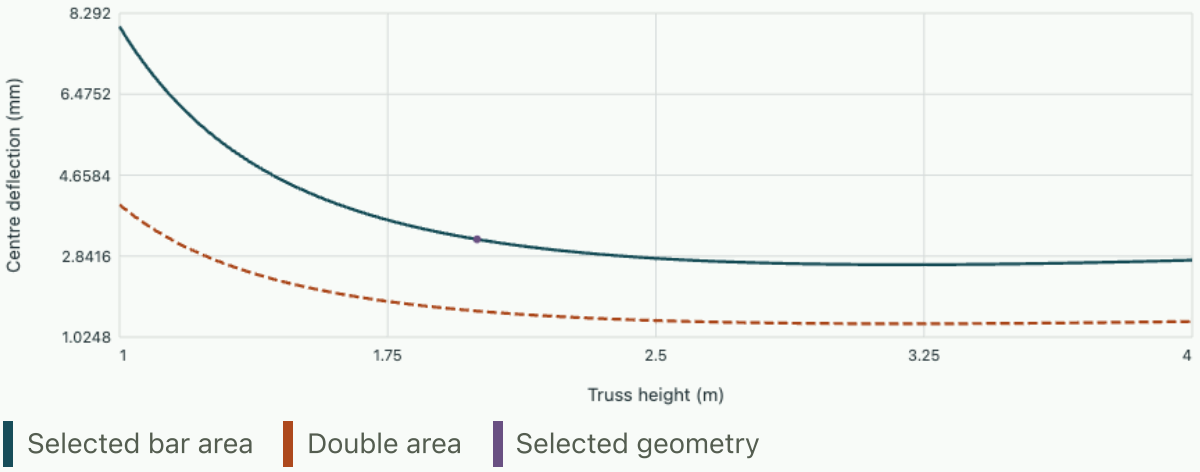
Interactive reference for the OneSimulation laboratory

Change truss height, bar area and centre load. The browser recomputes member forces by the method of joints and deflection by virtual work for the exact plane truss in the linked OneSimulation laboratory.

Open the live laboratory reference

Try the calculation

Truss height (m)	2
Area of each bar (mm²)	1000
Centre load (kN)	100



Method of joints: top chord -75.000 kN; unit-load deflection 3.219 mm. Panel length3 m, E200 GPa, ideal pinned bars. Doubling uniform area halves displacement without changing forces. These are the exact analytical references for the linked OneSimulation plane truss.

Problem sheet 1 · Infer a truss area without mistaking it for a height change

Notation and assumptions

Read each defining equation with its assumptions and units, then follow the link to the argument that gives the equation its meaning.

Axial strain

$$\varepsilon = du/dx$$

This is engineering strain for small displacement gradients in a straight bar. A measured extension is an integral over the gauge length.

Derivation · Axial stress, strain and compatibility

Euler load

$$P_{\mathrm{cr}} = \pi^2 EI/L_{\mathrm{eff}}^2$$

The ideal slender elastic column buckles in its lowest admissible mode. Support conditions determine effective length and change the result quadratically.

Derivation · Elastic stability of compression members

Second moment of area

$$I = \int_A y^2 \, dA$$

Measure the transverse coordinate from the stated bending axis. This section property has units of length to the fourth power and enters the flexural rigidity.

Derivation · Elastic curves and virtual work

Shear flow

$$q = VQ/I$$

In elementary beam theory the first moment Q belongs to the cut portion of the section. Shear flow is force per length. Divide by the local width to obtain shear stress.

Derivation · Transverse shear and connections in built-up beams

Stress intensity

$$K_I = Y\sigma\sqrt{\pi a}$$

Opening-mode intensity has units of stress times the square root of length. Geometry and the definition of crack length are part of the calibration.

Derivation · Crack driving force and fracture assessment

Traction

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma}\mathbf{n}$$

Specify the plane normal before projecting the force per area into normal and shear parts. A stress component is not a net force until integrated over an area.

Derivation · From cut forces to local equilibrium

Further reading

- Work through axial and bending examples by drawing a free body before choosing a stress distribution. Test each deformation against compatibility and the support conditions. MIT · Mechanics and Materials I
- Continue from elementary stress states to material behaviour and structural response when simple elastic estimates cease to be adequate. MIT · Mechanics and Materials II

Compare specific topics with the university sources