Stress, deformation and structural measurement

Start from balance, compatibility and material response. Work through structural predictions with alternative load or support conditions and assess the measurements needed to test them.

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2. **Contact pressure between elastic cylinders**
   Two polished cylindrical rollers touch along a narrow strip. Dividing load by a nominal full face area gives little indication of the actual pressure. Use the local shape of the gap and elastic half-space compliance to obtain the pressure distribution, and decide whether a measured contact width can determine material stiffness.

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   A channel can twist even when a force passes through its centroid. Trace the longitudinal shear flow around its flanges, find the load line that produces bending without torsion, and design a loading experiment that separates the stiffness in bending from eccentricity relative to this shear centre.

5. **Where does a circular opening concentrate stress?**
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6. **Which members carry a load through a truss?**
   A weighing frame can have perfect external force balance and still contain a failed load path. Follow the reactions into individual bars, cut across the middle of the frame, and carry the same

reasoning into three dimensions.

7. Where does the neutral axis of an angle section lie?

A downward tip load can bend an angle section sideways. Construct its full area-moment matrix and use measured strains to distinguish coupling of bending axes from an unintended applied torque.

8. When does Euler’s column load cease to govern?

A stocky strut may yield while its elastic eigenvalue still looks comfortable. Separate a bifurcation of a straight member, stress amplification under eccentric loading and loss of tangent stiffness in a yielded section.

Stress, deformation and structural measurement · 1. Torsion in a thin-walled two-cell section

Two closed cells share one slender web. The applied couple gives only their combined torque, yet the common web can carry much less shear than either exterior wall. Derive the missing compatibility equation, calculate a twist and test which measurements distinguish the two circulations.

Obtain one torque equation from traction

Consider an initially straight prismatic member of length much greater than its section dimensions. The walls are thin relative to cell dimensions, shear modulus is uniform and Saint-Venant torsion is established away from ends. Define shear flow $q = \tau t$, the tangential force transmitted per unit member length along the wall midline. In a closed single cell without a longitudinal body load, equilibrium makes its circulatory shear flow constant. Each tangential force around the contour contributes a torque, and the line integral of the moment arm gives twice the enclosed midline area. Consequently $T = 2Aq$.

For two cells choose positive circulation with the same sense in both. Superposition gives

$$T = 2A_1 q_1 + 2A_2 q_2.$$

On the shared wall the two contour directions oppose each other, so its actual flow is $q_1 - q_2$ when viewed along cell one's circulation. Adding the two flows there would violate the chosen orientation. This torque equation alone cannot determine two unknown circulations. Treating each as proportional to area would be a guess about stiffness, not a derivation.

Enforce equal twist on both closed contours

The tangential shear strain integrated around a closed contour is related to axial twist rate θ'. Under the thin-wall Saint-Venant assumptions, warping displacement is single-valued after a closed circuit, leaving $2A\theta' = \oint q/(Gt)\,ds$. Let C_1, C_2 denote integrals of ds/t along the two exterior contours excluding their common wall and C_w its shared-wall integral. Compatibility then requires

$$2A_1 G\theta' = C_1 q_1 + C_w(q_1 - q_2),$$
$$2A_2 G\theta' = C_2 q_2 + C_w(q_2 - q_1).$$

The common twist is a geometrical condition because the cells belong to one connected section. With different materials replace each $ds/(Gt)$ term individually; pulling out one common G would then be invalid. For equal exterior geometry the equations give equal flows and zero net web shear, while the web can still stabilise shape and carry other load cases.

Take heights 40 mm and widths 60 and 40 mm, with 1 mm walls throughout, $G = 27$ GPa and $T = 10$ N m. The areas are 2400 and 1600 square millimetres. Exterior and common geometric coefficients are

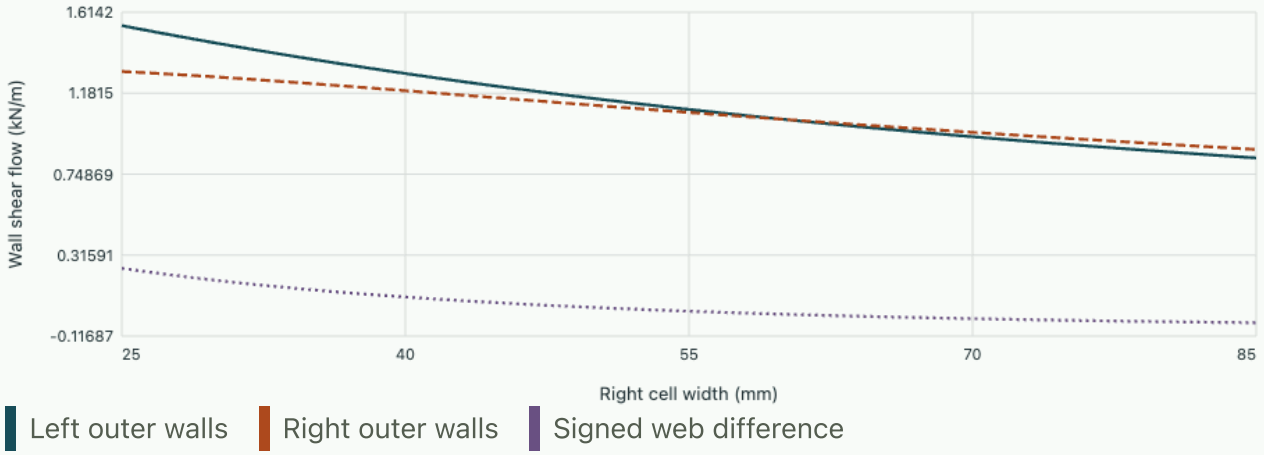
$$C_1 = 160, C_2 = 120, C_w = 40.$$

Solving gives flows 1.287 and 1.195 kN/m, a shared difference 0.092 kN/m, and twist rate 0.001617 rad/m. The exterior wall stresses equal their flow divided by thickness and are therefore 1.287 and 1.195 MPa. Over a one-metre span the ideal rotation is only 0.0926 degrees.

Open the live experiment

Try the calculation

Left cell width (mm)	60
Web thickness (mm)	1
Torque (N m)	10



At 40 mm right width the two circulations are 1.287 and 1.195 kN/m. The shared web carries their signed difference, 0.092 kN/m.

Background · Why does a shared wall carry the difference of the cell flows?

Trace a positive circulation counterclockwise around each closed cell. Along the common wall the two traces point in opposite directions. Give the common wall the positive direction of cell 1. Its physical shear flow is then $q_1 - q_2$; it is this flow divided by thickness that produces the local shear stress. For a common wall of length l and constant thickness t the stored shear energy per unit beam length is

$$U'_{\text{common}} = \frac{l}{2Gt}(q_1 - q_2)^2.$$

For identical cells under symmetric torsion, equal circulations leave the common wall unstressed in this ideal model even though the outer walls carry torque. If the common wall is very thin, its large compliance increasingly penalises unequal circulations. This energy check fixes the relative sign without depending on how the cross-section was drawn.

Which measurement fixes which resistance?

Sweep the right width through the fixed left width. The shared flow should pass smoothly through zero while the outer walls continue transmitting the global torque. Increase the web thickness. Its C_w decreases and it can transfer a given difference at smaller shear strain. The exterior walls and compatibility still control the complete response; the common wall need not carry a large flow simply because it is thick. Measure external twist and at least one wall strain if the individual circulations are the quantities of interest.

A diagonal strain-gauge pair on a thin wall gives an approximate shear strain when the gauges sample in-plane strain at plus and minus forty-five degrees. Their difference rejects common normal strain to first order. At a predicted 0.092 MPa common-wall shear, the strain is only about 3.4 microstrain for the stated modulus. A one-axis gauge zero error can therefore overwhelm the common-wall signal even while exterior torque and twist appear accurate. Confirm the sign by reversing the applied couple.

Check the wall flows by minimising strain energy

A second derivation makes an effective independent numerical test. Under a torque constraint choose the two cell flows so that the sum of their area moments gives the required torque. Their elastic energy per unit length is the sum of three terms, one from each group's exclusive walls and one from the shared wall:

$$U' = \frac{1}{2G}\left(C_1 q_1^2 + C_2 q_2^2 + C_w (q_1 - q_2)^2\right).$$

Perturb the first circulation while adjusting the second to keep torque constant. At the physical solution this variation cannot lower the energy. The resulting stationarity condition reproduces equal twist, including the minus sign on the common wall. A program that accidentally squares the sum on that wall would predict positive strain energy there even for two identical cells at equal flow. This unit test takes no measured twist as input and pinpoints a common implementation error. A second comparison differentiates the minimum energy with respect to applied torque and obtains twist per unit length. Because all wall stresses are linear in torque in this model, doubling torque doubles twist and quadruples stored energy. In a bench specimen, rapid departures from this scaling can reveal slip at a seam or local buckling before individual strain gauges identify which wall has changed.

Tutorial · Test a symmetry prediction numerically

Make both cells 50 mm wide with the same height and identical exterior walls. Find each circulating flow at 10 N m. Does changing only shared-web thickness affect ideal torsional twist in this exactly symmetric case?

▼ Use symmetry before solving two equations

Each area is 0.0020 square metres. Equal flows give

$$q = T/[2(A_1 + A_2)] = 1250\,\mathrm{N/m}.$$

The net web flow vanishes identically and hence its shear-compliance contribution drops out of both contour equations. With unchanged exterior geometry the ideal twist is unchanged by web thickness in this case. A measured change could reflect local distortion, end constraints, material changes during fabrication or other loads outside the thin-wall torsion model.

Tutorial · Infer torque from imperfect shear gauges

In the original unequal section calibrated exterior gauges indicate 1.30 and 1.18 MPa. Recover torque and predict shared-wall shear. Why is the torque result less sensitive to the gauge fluctuations than the difference?

▼ **Propagate the measured flows**

At one millimetre thickness the flows are 1300 and 1180 N/m. Torque is $2(0.0024 \cdot 1300 + 0.0016 \cdot 1180) = 10.016$ N m. Predicted shared flow is 120 N/m, or 0.120 MPa. Independent small errors affect a sum of two large positive contributions only modestly in relative terms, while the shared difference is a much smaller residual. Check twist compatibility as a separate residual instead of concluding from the accurate torque alone that both gauge readings are correct.

Tutorial · Identify a missing effect at a clamped end

A very short specimen twists less than the prediction and shows high longitudinal stresses near the clamp. Which model assumptions are questionable, and how could two specimen lengths test them without changing cross-section?

▼ **Compare distributed and end contributions**

Restrained warping, local load introduction and section distortion can create additional stresses and stiffness near the clamps. Fabricate geometrically similar fixtures at two or more lengths and compare differential rotation between interior stations per unit gauge length. An approximately constant interior slope matching the derived twist with endpoint departures supports a local end effect. If the slope itself varies throughout, check actual wall thickness, joint continuity, shear modulus and distortion. Increasing the number of contour integration points in an otherwise prismatic free-warping model does not recover a missing boundary mechanism.

MIT Ship Structural Analysis and Design · Saint-Venant torsion of multicell sections

Stress, deformation and structural measurement · 2. Contact pressure between elastic cylinders

Two polished cylindrical rollers touch along a narrow strip. Dividing load by a nominal full face area gives little indication of the actual pressure. Use the local shape of the gap and elastic half-space compliance to obtain the pressure distribution, and decide whether a measured contact width can determine material stiffness.

Replace the undeformed outlines by their local curvatures

Take two long parallel cylinders pressed together by total force F uniformly per engaged length L. Ignore tangential traction, adhesion, end effects and plastic deformation. In a cross-section perpendicular to the axes the local undeformed gap grows as $x^2/(2R^*)$, with $1/R^* = 1/R_1 + 1/R_2$ for two externally convex cylinders. A flat counterpart corresponds to infinite radius. For an internal conforming surface the signed curvature changes, but nearly conformal contact can violate the small-patch assumptions used here.

Normal deformations of both elastic bodies add. For isotropic half-spaces in line contact their combined modulus obeys $1/E^* = (1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2$. Replacing this by one body's modulus overestimates stiffness whenever the partner also deforms significantly. For two identical steels with $E = 210$ GPa and $\nu = 0.30$, $E^* = 115.4$ GPa. An elastically rigid counterpart would give roughly twice that value for the same first body.

Why the square-root distribution fits the quadratic gap

In two-dimensional half-space elasticity a line element of pressure produces a logarithmic normal surface displacement up to an arbitrary reference. Differentiating the displacement removes that reference and gives a singular kernel proportional to the inverse horizontal separation. In contact the two deformed faces conform, so the derivative of their combined displacement has magnitude x/R^*. Using a consistent compressive sign convention, the elastic equation for $|x| < a$ is

$$\frac{x}{R^*} = \frac{2}{\pi E^*}\,\mathrm{PV}\int_{-a}^{a} \frac{p(s)}{x-s}\,ds.$$

The trial shape $p(s) = p_0\sqrt{1 - s^2/a^2}$ gives a linear principal-value integral $\pi p_0 x/a$, exactly matching the gap slope. Normal force per unit length $W = F/L$ is the integral of this pressure, the area of a semiellipse: $W = \pi a p_0/2$. Combining the two conditions determines both quantities,

$$a = \sqrt{\frac{4WR^*}{\pi E^*}},$$

$$p_0 = \frac{2W}{\pi a}.$$

The full width is $2a$. Central pressure exceeds the average over the strip by $4/\pi$. The normal approach of ideal infinite cylinders has a logarithmic outer-length dependence, so absolute displacement from a

frame is less cleanly predicted by this local half-space construction than pressure or width. Include whole-body and fixture compliance before using crosshead displacement to infer a local contact modulus.

Use a width series and check its scaling

For $R^* = 10$ mm, $E^* = 115$ GPa and $W = 20{,}000$ N/m the predicted half-width is 47.1 micrometres and peak pressure about 271 MPa. Doubling force at constant length increases both width and pressure by a factor of $\sqrt{2}$. A nominal area built from the full millimetre-scale roller width would understate this pressure dramatically. Conversely, comparing peak hydrostatic or normal pressure directly with uniaxial yield stress is not a rigorous yield condition: evaluate the multiaxial subsurface stress and actual material strength for that separate question.

A pressure-sensitive film or optical contact image has finite resolution and can broaden the visible strip. Fit the square of measured half-width against W. A nonzero intercept, load-dependent slope or hysteresis on unloading can reveal optical threshold bias, initial crowning or irreversible deformation. Surface roughness can produce sparse microcontacts inside a broader nominal strip. Compare several axial positions to detect end loading or misalignment before interpreting the width as uniform line contact.

Compare another geometry-dependent elastic stress distribution in the thick-cylinder experiment

Compare a resolved pressure trace with total applied force

The pressure texture of a two-dimensional Hertz contact gives more information than one width measurement. Predict the integrated line load by integrating the measured pressure over transverse position and compare it with a load cell divided by contact length. The model semiellipse has mean pressure across its width equal to one quarter of pi times its peak. A sensor with limited spatial resolution tends to round the central peak and move the apparent edges. If its response kernel has unit area and the full footprint is recorded, however, convolution preserves the integrated force. Disagreement in area suggests gain error, incomplete coverage or a departure from uniform loading along the roller length.

Try the selected loading and then double it while keeping radius and modulus fixed. The recorded footprint broadens, so rescale the transverse axis by each run's own calculated half-width. Also divide pressure by its own peak. The two normalised shapes should collapse. That collapse tests shape and linear elasticity, while the scale factors separately test the force and length calibration. Repeating with the load applied closer to an end changes the axial uniformity assumption; a contact trace at the middle may then give a different inferred load from a trace close to the end.

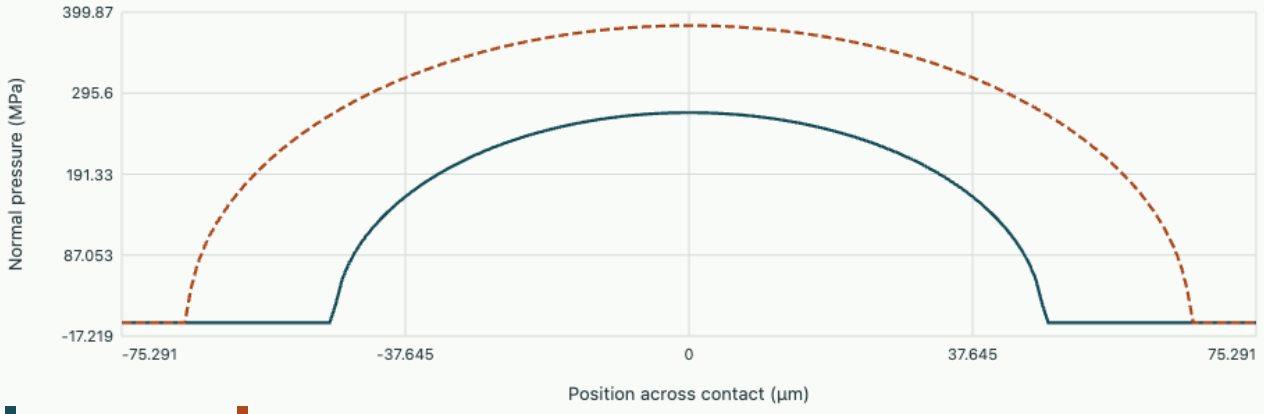
Open the live experiment

Try the calculation

Line load (kN/m) **20**

Reduced radius (mm) **10**

Reduced modulus (GPa) **115**



At the selected load the half-width is 47.1 µm and the peak pressure is 270.6 MPa.

Tutorial · Infer a modulus and estimate its systematic sensitivity

A roll pair with $R^* = 12.0$ mm is loaded at 15.0 kN/m. A camera gives a full width of 96.0 micrometres. Infer E^*. If optical blur makes the true full width five percent smaller than the reported value, by what factor does the inferred modulus change?

▼ **Convert full width before inverting**

Using $a = 48.0$ micrometres gives

$$E^* = 4WR^*/(\pi a^2) = 99.5\,\mathrm{GPa}.$$

Replacing a by $0.95a$ raises the inferred modulus by $1/0.95^2 = 1.108$, about eleven percent. Treating the quoted full width as a would instead underestimate modulus fourfold. Independent optical calibration and known line load both matter to a meaningful inverse measurement.

Tutorial · Change the engaged roller length

A fixed total force is spread uniformly over twice the original length. Predict changes in central pressure, half-width and total nominal contact area. Which change would reveal a failure of uniform load sharing?

▼ **Track load per length and the physical area**

The line load halves, so width and pressure each fall to $1/\sqrt{2}$ of their initial values. Total nominal area $2aL$ rises by $\sqrt{2}$, preserving total force after integrating pressure. If measured width at the centre remains nearly fixed while a large near-end maximum develops, the assumption that F/L is spatially uniform is suspect. Roller bending, tapered clearance or imposed misalignment can redistribute line load even at unchanged total force.

Tutorial · Can width identify both material moduli?

A laboratory wants to estimate E_1 and E_2 using the same pair at several loads. Is this possible from ideal widths alone if both Poisson ratios are known? Design a measurement that supplies additional information.

▼ **Identify the one combination present in the forward equation**

Every load yields the same effective reciprocal modulus combination. A load sweep tests the square-root law and improves precision but does not separate the two material moduli. Pair one unknown with a reference body of independently measured modulus and known curvature, then solve for the first unknown. A second arrangement constrains the other. Use multiple loads to verify elasticity and repeatability in each arrangement; a permanent flat left after unloading invalidates the reversible Hertzian inversion.

MIT Mechanics and Materials II · elastic deformation and Hertzian contact measurements

Stress, deformation and structural measurement · 3. Springback and residual stress after plastic bending

After a rectangular strip has been bent beyond first yield, removing its end couple does not return it to its original curvature. Work out the loading and unloading paths fibre by fibre. Then explain why an unloaded strip can contain a substantial stress field with zero net force and zero net bending moment.

Locate the elastic core and compute the applied moment

Consider a straight rectangular strip of width b, half-depth c and Young modulus E, in pure bending with plane sections remaining plane. Assume symmetric elastic-perfectly plastic response with yield magnitude σ_y, small strain, no hardening and no unloading-induced reverse yielding for the excursion being considered. Symmetry puts the neutral axis at mid-depth. Under increasing curvature κ the axial strain is κy; stress increases linearly until it reaches the signed yield plateau. First yield occurs at $\kappa_y = \sigma_y/(Ec)$.

At greater curvature define $y_e = \sigma_y/(E\kappa)$, the half-thickness of the remaining elastic core. Integrate stress times moment arm over the upper half and double. The core contributes $2bE\kappa y_e^3/3$ and the yielded layer contributes $b\sigma_y(c^2 - y_e^2)$. Substituting the definition of y_e gives

$$M = b\sigma_y(c^2 - y_e^2/3),$$

$$M_y = \frac{2}{3}b\sigma_y c^2,$$

$$M_p = b\sigma_y c^2.$$

The limiting fully plastic moment is one and a half times first-yield moment. It is approached as the elastic core shrinks; ideal plane-section theory does not justify arbitrary curvature once strains and geometry cease to be small. A load-controlled perfectly plastic model can also become indeterminate at a fully developed plastic hinge. A known curvature history is a cleaner basis for calculating springback in this ideal test.

Subtract an elastic stress field to satisfy moment release

During a purely elastic unloading of moment magnitude M, every fibre responds with slope E. The curvature decreases by $M/(EI)$, where $I = 2bc^3/3$. The remaining curvature and signed residual stress profile are

$$\kappa_r = \kappa - \frac{M}{EI},$$

$$\sigma_r(y) = \sigma_{\mathrm{load}}(y) - \frac{M}{I}y.$$

The antisymmetric residual stresses have zero net force. Their bending resultant is zero because the second term removes exactly the original moment. A free strip can therefore remain curved even though its internal stress has neither an axial nor a bending resultant. Its plastic strain varies through

the thickness and makes the unstrained reference of each fibre incompatible with one common straight shape.

Take $E = 200$ GPa, $\sigma_y = 250$ MPa, $b = 20$ mm and $c = 2.0$ mm. The yield curvature is 0.625 per metre, first moment 13.33 N m and plastic limit 20.0 N m. At curvature 1.25 per metre, $y_e = 1.0$ mm and applied moment 18.33 N m. Bending stiffness is 21.33 N square metres and elastic curvature recovery 0.859 per metre. The unloaded curvature is about 0.391 per metre. The surface formerly in tension ends up in compression of about 93.75 MPa, well below reverse yield under this idealised law.

Compare elastic-core growth and plastic moment in the course's bending chapter

Measure curvature and residual stress with different observables

A photograph of the unloaded centreline can determine residual curvature but cannot recover the full through-thickness stress profile. A single strain gauge bonded before loading measures total strain change at its location; separating plastic strain from elastic residual strain requires a constitutive model or an independent release measurement. If a thin surface layer is removed, the remaining section can change curvature as its previously self-equilibrated stress resultants are unbalanced. That change contains information about the removed residual stress but needs a recalculated section geometry.

Measure moment and curvature through loading and initial unloading as separate data sets. The small unloading slope gives an independent estimate of EI, while the first slope departure during loading constrains yield. Friction at fixture rollers, nonuniform moment and axial tension during forming can produce an asymmetric neutral axis that the simple pure-couple solution cannot fit. Reversing the strip or comparing strains on both faces is an inexpensive diagnostic.

Look inside the strip after the applied moment vanishes

In the plot compare the stress at the surface with that on the boundary of the surviving elastic core. The applied stress there reaches yield but the subsequent elastic unloading stress is proportional to distance from the middle. The surface therefore experiences a greater stress change during unloading than the point at the edge of the core. This permits a tensile residual just inside a compressive outer skin on one side. Reflection about the middle reverses all the signs. A surface strain reading alone cannot reveal that self-equilibrated pattern.

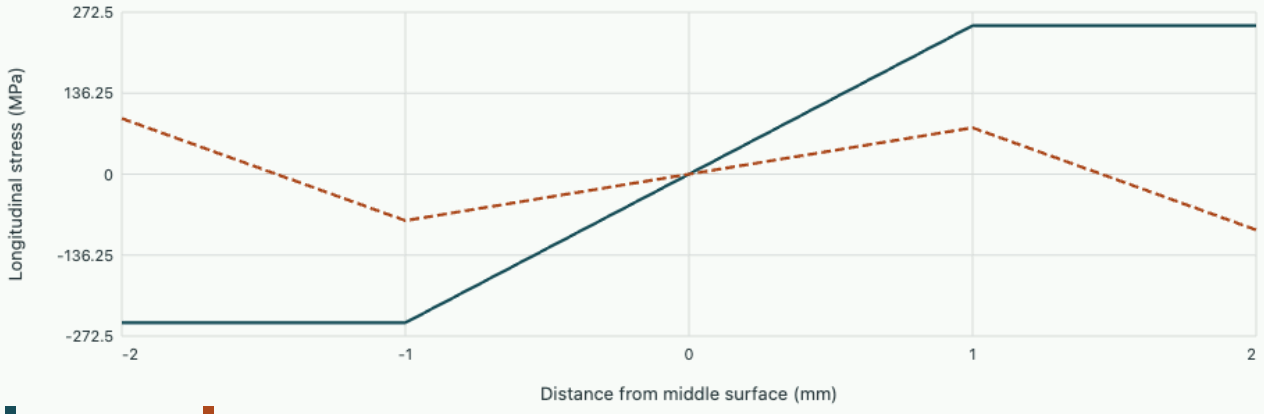
Open the live experiment

Try the calculation

Applied curvature / first-yield curvature **2**

Yield stress (MPa) **250**

Half-thickness (mm) **2**



Residual curvature is 0.391 per metre. The previously tensile surface has residual stress -93.75 MPa.

Check the area under the residual stress over the entire depth. Antisymmetry makes the axial resultant vanish. The moment also vanishes, which follows because subtracting the elastic unloading distribution subtracts exactly the applied moment. The two cancellation tests are stronger than checking the stress at one point. They can be applied to a numerical fibre calculation even when hardening changes the shape of each branch. If an experiment instead clamps the strip straight after bending, the imposed curvature differs from free springback and a reaction moment appears. Compare the same unloading boundary condition before fitting a hardening coefficient.

Tutorial · Predict a residual profile at the elastic-core edge

For the numerical strip above calculate the residual stress at $y = +1.0$ mm after unloading, and compare its sign with the outer surface. Where in the originally tensile outer layer is residual stress zero?

▼ **Use the loading plateau and the unloading slope**

The point at the core boundary reached plus 250 MPa. Its elastic unloading change is $E(M/EI)y = 171.875$ MPa, so residual is plus 78.1 MPa. At the outer surface the unloading change is twice that and residual negative. Within the former tensile plateau the zero occurs at $y = \sigma_y/[E(M/EI)] = 1.455$ mm. The sign change is necessary for the residual field to carry no net moment, although the exact value follows from its detailed profile.

Tutorial · Recover a yield stress from first moment and depth

A second strip of width 15 mm and total thickness 3.0 mm first deviates reproducibly from linear bending near 6.00 N m. Infer yield stress assuming the same model, and estimate the consequence of a two-percent thickness calibration error.

▼ **Use the section modulus and its sensitivity**

With $c = 1.5$ mm, $\sigma_y = 3M_y/(2bc^2) = 267$ MPa. Yield stress scales inversely with squared thickness, so a two-percent overestimate of thickness biases stress downward by nearly four percent. A small proportionality deviation can also reflect fixture or material nonlinearity before conventional yield. A tensile specimen or repeated unload–reload cycles provide independent evidence of irreversibility.

Tutorial · Test whether elastic unloading remains admissible

How would you detect reverse yielding or a Bauschinger effect when applying this springback calculation to a previously prestrained alloy? What can and cannot be repaired by substituting a fitted effective Young modulus?

▼ **Check the stress path before reusing the formula**

Compute the signed unloaded stresses throughout the section and compare the full unloading path against measured reverse yield behaviour. A reverse-yield threshold lower than the virgin value can be reached before the nominal moment is fully removed. Observe changes of unloading slope and hysteresis in cyclic bending or in a tension–compression test on similarly prestrained material. A fitted scalar modulus might reproduce one final curvature yet fail residual stresses and subsequent reverse loading. A history-dependent constitutive law with an appropriate hardening rule is then required for transferable predictions.

MIT Techniques for Structural Analysis and Design · material response and plastic deformation

Stress, deformation and structural measurement · 4. Why an open channel twists under a transverse load

A channel can twist even when a force passes through its centroid. Trace the longitudinal shear flow around its flanges, find the load line that produces bending without torsion, and design a loading experiment that separates the stiffness in bending from eccentricity relative to this shear centre.

Find the internal horizontal force in each flange

Use a thin channel made of a vertical web of midline height h and two horizontal flanges of midline width b projecting to the same side. Take uniform thickness $t \ll h, b$, ignore corner radii and flange through-thickness bending, and consider a vertical shear resultant V. Symmetry places the shear centre on the horizontal axis through mid-height but does not place it on the web or centroid. Set the web midline at $x = 0$ and both flanges at positive x.

For bending about the horizontal centroidal axis, the thin-wall second moment is

$$I_x \simeq th^3/12 + 2bt(h/2)^2.$$

The usual beam shear-flow relation $q = VQ/I_x$ follows from balancing the longitudinal stress change of a cut-off strip over an incremental beam length, using $dM/dz = V$ with a consistent beam-axis orientation. At a free flange edge the cut-off area vanishes and the flow is zero. A distance s inwards from that free edge, $Q \simeq tsh/2$. The flow increases linearly toward the web.

Integrating the flow over one flange gives a horizontal resultant magnitude $F_f = Vthb^2/(4I_x)$. Top and bottom flange forces are opposing, so they contribute zero net horizontal force but a couple $F_f h$ about the web line. The web carries the net vertical shear and its contribution has zero moment about its own midline in this thin-wall representation. Balance the internal couple with the moment of the external vertical force about the web. The shear centre is on the side opposite the flanges, at

$$x_{\mathrm{sc}} = -e,$$

$$e = \frac{th^2b^2}{4I_x} = \frac{3b^2}{h + 6b}.$$

The sign describes the stated channel opening direction. Reflecting the channel reverses the sign in a fixed apparatus. The result is independent of uniform thickness in this thin-wall limit because both shear flow first moment and bending second moment scale with thickness. It does not imply that the torsional stiffness or stress is independent of thickness.

Measure a load line outside the material

Take $h = 60$ mm, $b = 30$ mm and $t = 2$ mm. The shear centre lies 11.25 mm behind the web. The thin-wall centroid is at $x_c = b^2/(h + 2b) = 7.50$ mm on the opening side. A vertical load of 100 N applied through the centroid therefore introduces a torsional moment magnitude $100(0.01875) = 1.875$ N m about the shear centre. A cantilever channel clamped against warping can show both Saint-Venant twist and additional restrained-warping stresses, especially near the clamp.

Mount an adjustable loading arm across the tip and apply several known vertical forces at signed positions x. In a sufficiently linear regime the measured twist under fixed load is proportional to $x - x_{sc}$. Fit its zero crossing; this eliminates the need to know absolute torsional rigidity in the first estimate. A force applied through an external bracket can reach a line outside the actual material. Ensure the bracket supplies a force and no unintended direct couple, and take readings away from local load introduction.

Compare the open-channel result with twist compatibility in a closed multicell member

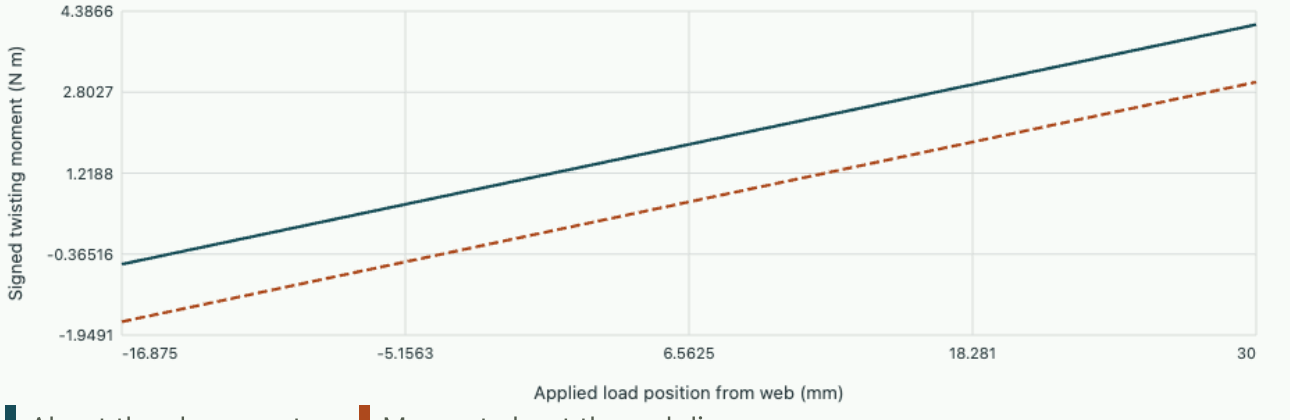
Find the same shear centre with a signed load-position sweep

Keep the vertical force fixed and move its line of application laterally while recording twist at a section a known distance from a clamp. In a member with constant torsional compliance in the test range, the slope of twist versus lateral position contains both the force and compliance. Its zero crossing locates the shear centre without requiring a separate torsional stiffness measurement. The second curve in the model shows the moment computed about the web. Its zero crossing is set by the coordinate origin. Only the other crossing predicts a vanishing twist of the section in the simple beam idealisation.

Open the live experiment

Try the calculation

Midline web height (mm)	60
Flange width (mm)	30
Vertical load (N)	100



The zero-twist line lies 11.25 mm behind the web. A centroidal load produces 1.875 N m of torque.

Repeat the sweep with the force reversed and the same attachment bracket. A constant apparatus torque shifts the two inferred zero positions in opposite directions because the applied torque per unit position changes sign. Their mean can remove a load-independent offset. A bracket whose compliance

changes asymmetrically under reversal requires a separate correction. Measure twist with two gauges spanning a known width so that common vertical translation cancels. If the two gauges are far apart along the beam instead of across the same section, differing bending slopes can contaminate the apparent twist and shift the crossing.

Tutorial · Recover the shear-centre location from two angular readings

At fixed load an adjustable fixture gives twist minus 0.10 degrees when the load lies at $x = -15.0$ mm and plus 0.30 degrees at $x = 0$. Infer the shear-centre coordinate and predict twist when the force passes through the centroid of the example.

▼ **Use signed interpolation and check the geometric result**

The slope is $0.40/15 = 0.0267$ degrees per millimetre. The zero crossing is $-15.0 + 0.10/0.0267 = -11.25$ mm, matching the predicted shear centre. At $x = 7.50$ mm the lever arm is 18.75 mm and the predicted twist 0.500 degrees. This inference assumes no independent angular offset. Reversing load lifts out an offset unchanged by force sign, while moving the bracket with no force checks alignment.

Tutorial · Predict the effect of making the flanges shorter

At unchanged web height and thickness, halve flange width to 15 mm. Predict both centroid and shear-centre location. Is their separation simply half its old value?

▼ **Recompute both first moments**

The centroid is $225/(60 + 30) = 2.50$ mm, while $e = 3(225)/(60 + 90) = 4.50$ mm. Their distance becomes 7.00 mm instead of 18.75 mm. Both formulas contain a squared flange width and a changing perimeter or second moment, so halving width does not simply halve the eccentricity. In the formal limit of vanishing flange width channel torsional resistance also becomes very small, requiring careful control of imperfection and fixture compliance.

Tutorial · Distinguish eccentric bending from torsional warping

A short channel shows nonzero mean twist even at the predicted load line and opposite longitudinal strains on the free flange tips near its clamp. Could changing shear modulus alone move the predicted shear centre? Suggest two specific measurement changes.

▼ **Test the assumptions that control the zero crossing**

For this homogeneous thin elastic channel the centre follows force and moment equilibrium of bending shear and ideal geometry; changing one common shear modulus principally changes torsional compliance rather than moving that ideal geometric point. Compare two specimen lengths and measure twist slopes at several interior stations to identify restrained end warping. Reverse channel orientation while keeping the apparatus fixed to separate a channel-related eccentricity from a fixed bracket couple. If the discrepancy follows the specimen, measure actual corner radii, wall thickness asymmetry and local distortion and use a suitable open-section beam or shell model before attributing every measured longitudinal strain to pure flexure.

MIT Structural Analysis and Control · torsion–flexure and the shear centre

Stress, deformation and structural measurement · 5. Where does a circular opening concentrate stress?

Make a hole in a flat sheet pulled along one direction. Derive the conditions on the empty edge, compute the circumferential stress and check how quickly the disturbance decays as a strain gauge moves into the sheet.

Keep the free edge and the distant traction in one solution

Take an isotropic linear-elastic sheet, long and wide relative to a circular through-hole of radius a. At a large distance apply uniaxial tension S along x. The empty edge can carry neither normal nor tangential traction, so $\sigma_{rr} = \tau_{r\theta} = 0$ at $r = a$. The circumferential stress $\sigma_{\theta\theta}$ is parallel to this edge and need not vanish. Writing uniaxial tension in polar coordinates introduces only a constant angular sector and a sector in $\cos(2\theta)$ and $\sin(2\theta)$.

In plane elasticity an Airy stress function satisfies equilibrium automatically and must be biharmonic for a homogeneous isotropic solid without body forces. The circular-hole solution can be built by adding decaying terms in those same two angular sectors to the distant uniform stress. One convenient potential is

$$\Phi = \frac{S}{4}\left[r^2 - 2a^2\log(r/a) - (r^2 - 2a^2 + a^4/r^2)\cos(2\theta)\right].$$

Differentiate twice with respect to radius to obtain the hoop stress. Writing $u = (a/r)^2$ gives

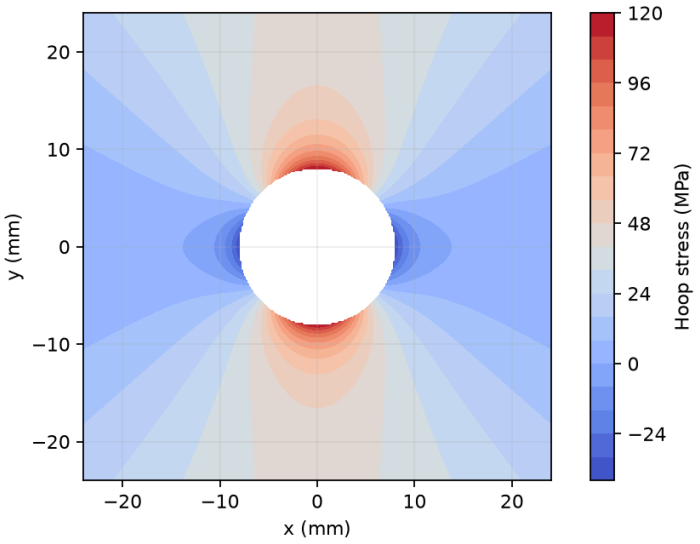
$$\sigma_{\theta\theta} = \frac{S}{2}[1 + u - (1 + 3u^2)\cos(2\theta)].$$

The radial and shear stress expressions in the calculation cell follow from $\sigma_{rr} = \Phi_r/r + \Phi_{\theta\theta}/r^2$ and $\tau_{r\theta} = -\partial_r(\Phi_\theta/r)$. Check both edge tractions explicitly; imposing zero hoop stress instead would erase the stress concentration we aim to measure.

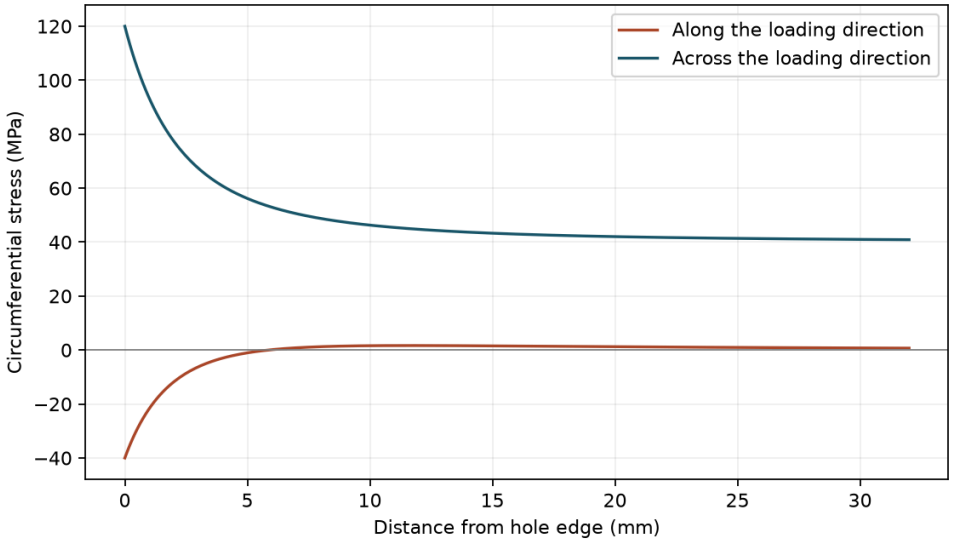
Locate a gauge and interpret its reading

On the edge $u = 1$, hence $\sigma_{\theta\theta} = S[1 - 2\cos(2\theta)]$. With $S = 40$ MPa, gauges at the top and bottom of the opening align with tensile hoop stress of 120 MPa. At the point facing along the remote pulling direction the hoop stress is compressive at −40 MPa. The nominal far-field stress refers to the gross, undisturbed section. A factor based on a finite specimen's reduced net section would use a different reference.

Along the transverse radius, moving from $r = a$ to $r = 2a$ reduces the ratio to $1 + u/2 + 3u^2/2 = 1.21875$. A gauge whose active length samples a substantial fraction of the hole radius therefore averages a gradient. Failure to read a threefold concentration at a distant or finite-area gauge does not by itself invalidate the elastic solution.



The white interior represents removed material. The colour field is one stress component expressed relative to the local circumferential direction.



The horizontal coordinate measures actual distance outward from the wall of the hole. Moving the gauge outward changes the inferred peak.

Exercise · rotate the sensor

At what angle between zero and 90 degrees does the edge hoop stress change sign?

▼ Work through the solution

Set $1 - 2\cos(2\theta) = 0$. Thus $2\theta = 60^\circ$ and $\theta = 30^\circ$. This is a zero of the tangential normal stress on a boundary already free of radial and shear traction.

Exercise · change the radius

Double the hole radius at the same remote stress in an effectively infinite sheet. What changes in the peak and the stress gradient in MPa per millimetre?

▼ **Work through the solution**

The peak remains $3S$ because the stress field depends only on r/a and angle. The physical length scale doubles, hence an otherwise corresponding spatial gradient halves. Finite edge distances, plastic yielding and rounded specimen boundaries can introduce new scales and change these conclusions.

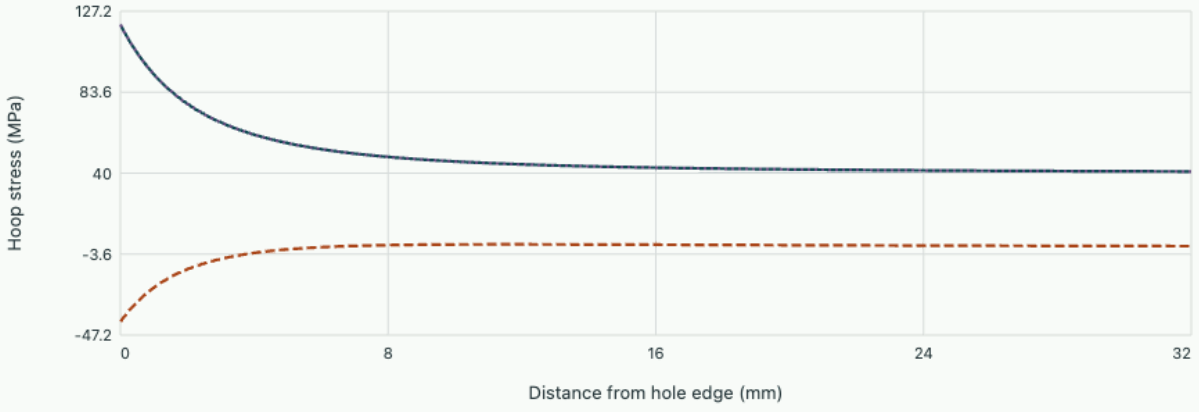
Interactive calculation · Hole stress

Change the physical and numerical inputs of the notebook model. Compare the updated curves and balance checks with the recorded notebook figures. Each plot uses the same equations and assumptions documented in that notebook; its readout names any analytical or reduced-model reference.

Open this live calculation

Try the calculation

Hole radius (mm)	8
Remote tension (MPa)	40
Radial-ray angle (degrees)	90



Selected edge hoop stress 120.000 MPa; peak around the hole is 120.000 MPa. Infinite isotropic plate, circular traction-free hole, remote uniaxial stress.

Download this example's editable notebook

Open editable notebook Download .ipynb

Stress, deformation and structural measurement · 6. Which members carry a load through a truss?

A weighing frame can have perfect external force balance and still contain a failed load path. Follow the reactions into individual bars, cut across the middle of the frame, and carry the same reasoning into three dimensions.

Earn the two-force assumption

Ideal pins transmit forces through their centres. Apply loads at joints, neglect member self-weight or lump it at joints, and treat each initially straight bar as hinged at both ends. Moment balance on that bar requires its two end forces to be collinear. A single scalar N now describes each member. Positive N pulls away from the joint. For a bar joining positions x_i and x_j, use $e_{ij} = (x_j - x_i)/|x_j - x_i|$. Reversing the joint reverses its column direction.

$$\sum_j N_{ij} e_{ij} + R_i + P_i = 0.$$

At an unloaded joint with two noncollinear bars their coefficients must both vanish. With three bars, two collinear and no applied load, project normal to their shared line to eliminate both and identify the third as a zero-force member. A brace classified this way may still resist the next load case or restrain buckling.

Read a two-panel bridge from both ends

Place bottom joints A, B, C at (0,0), (2,0), (4,0) m, and D, E at (1,1), (3,1). Connect AB, BC, AD, DB, BE, EC and DE. Pin A, put C on a vertical-reaction roller. Apply 6 kN down at D and 2 kN down at E. Taking moments about A gives $4C_y = 6(1) + 2(3)$, hence $C_y = 3$ and $A_y = 5$ kN. At A, the vertical projection gives $N_{AD} = -5\sqrt{2}$ kN, and horizontal equilibrium gives $N_{AB} = 5$ kN. At C the corresponding forces are $N_{EC} = -3\sqrt{2}$ and $N_{BC} = 3$ kN.

Cut AB, DB and DE immediately left of B and take moments on the left fragment about B. AB and DB intersect there. The reaction at A contributes –10 kN m, the downward load at D contributes +6 kN m, and the horizontal force in DE contributes $-N_{DE}(1\,\mathrm{m})$. Thus $N_{DE} = -4$ kN. Solving at D gives $N_{DB} = -\sqrt{2}$ kN. At E the corresponding diagonal BE is in tension at $+\sqrt{2}$ kN. The opposite signs are physically measurable under this asymmetric loading.

When a section cut can determine three forces

Three equilibrium equations suffice only if the three cut-member force lines supply independent force and moment conditions. If all three lines meet at one point, their moments about that point vanish together. Choose another cut or a second free body.

Lift the joint into three dimensions

Place three feet at equal azimuthal intervals on a circle of radius 1 m. Pin the apex 1 m above its centre. Measure the horizontal 0° direction toward foot 1. The directions from apex to feet are $(\cos\theta, \sin\theta, -1)/\sqrt{2}$. A downward load W and horizontal load H give $N_2 = N_3$ by symmetry. Taking horizontal and vertical projections yields

$$N_1 = -\frac{\sqrt{2}}{3}(W + 2H),$$

$$N_2 = N_3 = \frac{\sqrt{2}}{3}(H - W).$$

At $W = 6$ and $H = 3$ kN, all three bars are in compression. At $H = 6$ kN two members unload simultaneously. Pins capable of taking tension and unanchored compression-only feet then predict different admissible states. The notebook resolves all three joint equations and plots that boundary.

Interactive experiment · Rotate a loaded tripod and change its geometry

Move the height, foot radius and horizontal load independently, then turn the load through a full circle. The browser solves the three joint-equilibrium equations for every setting. The plot sweeps horizontal load while the spatial view shows the selected operating point. Each bar points from the apex to its own foot, so a positive signed force pulls the apex toward that foot.

$$L = \sqrt{r^2 + h^2},$$

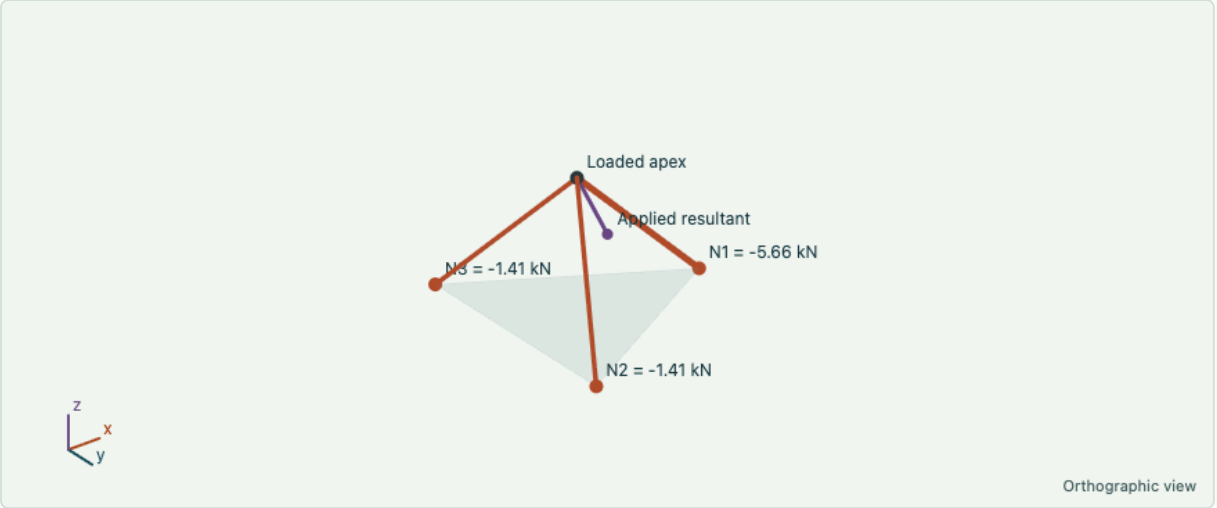
$$N_i = -\frac{WL}{3h} - \frac{2HL}{3r}\cos(\theta_i - \alpha).$$

First keep *h = r = 1 m*, *W = 6 kN* and azimuth zero to recover the notebook. Then lower the apex: the vertical load requires increasingly large compression. Rotate the horizontal load by 60° and identify the first member that would require an anchored foot. Rotate the 3D view to inspect the distinct force directions; bar colour records tension or compression, not a stress contour. This ideal pin-jointed model omits deformation and buckling.

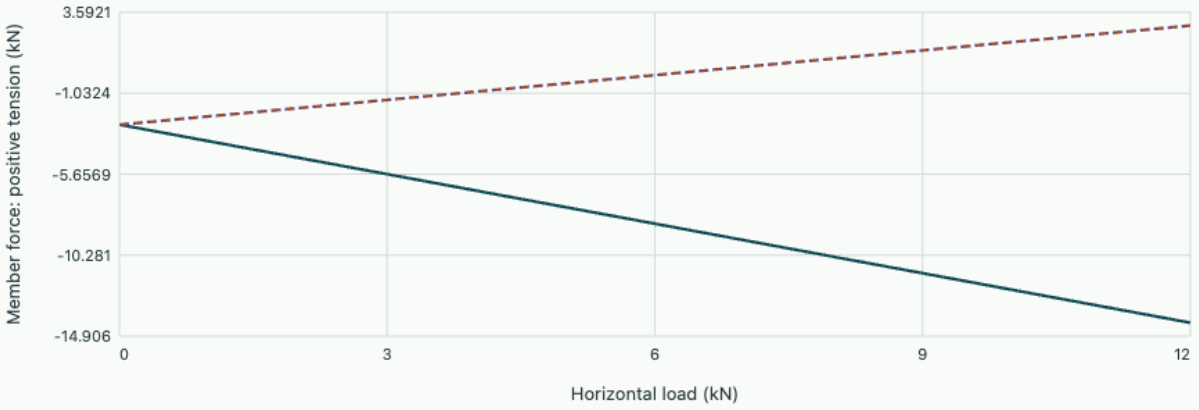
Open this experiment online

Try the calculation

Apex height (m)	1
Foot radius (m)	1
Downward load (kN)	6
Horizontal load (kN)	3
Horizontal load azimuth (degrees)	0



Three-dimensional pin-jointed tripod at the selected load. Orange members carry compression; teal members carry tension; grey means unloaded. The purple segment shows the applied force direction with a symbolic length. Members retain their undeformed geometry. Axes: x = x (m); y = y (m); z = z (m).



At the selected load: N1 = -5.657 kN; N2 = -1.414 kN; N3 = -1.414 kN. Joint force residual 1.3e-15 kN. All members are in compression or unloaded.

Continue in the OneSimulation notebook · Download its Python model

OneSimulation experiment · Two legs unload as the horizontal apex force reaches the vertical load

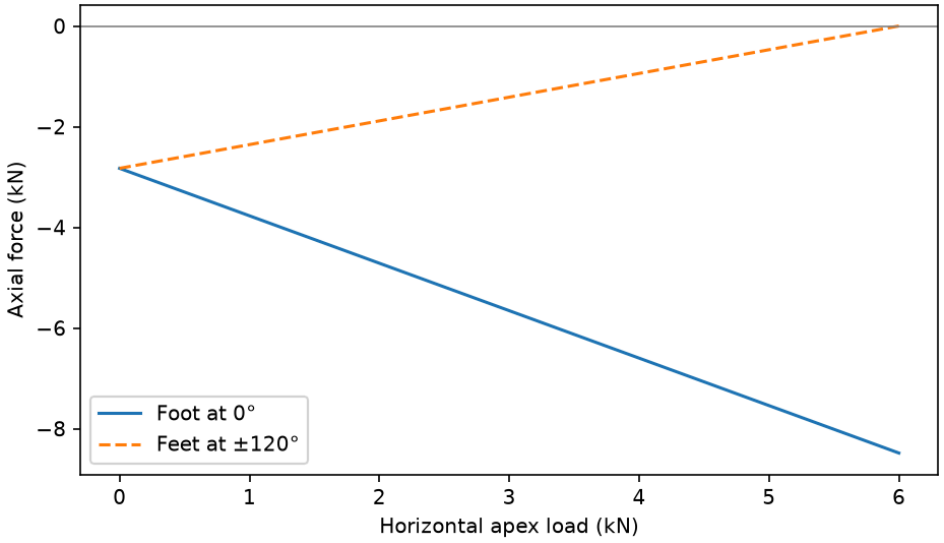
Run the notebook with OneSimulation 0.2.0, NumPy, SciPy and Matplotlib. Change the input data and compare your run with the recorded result.

```
"""Resolve measured loads on a three-dimensional tripod into signed member forces."""
import numpy as np
import matplotlib.pyplot as plt
from onesim.numerics.nonlinear import LeastSquares

angles = np.deg2rad([0, 120, 240])
# Columns point from the loaded apex toward a foot; positive force is tension.
directions = np.array([np.cos(angles), np.sin(angles), -np.ones(3)]) / np.sqrt(2)
horizontal = np.linspace(0, 6, 61)  # kN
forces = []
for load in horizontal:
    external = np.array([load, 0.0, -6.0])
    fit = LeastSquares().solve(lambda force: directions @ force + external, [-2.0]*3)
    assert np.linalg.norm(directions @ fit.x + external) < 1e-6
    forces.append(fit.x)
forces = np.array(forces)
print("H=3 kN: signed member forces [kN]", np.round(forces[30], 3))
print("H=6 kN: signed member forces [kN]", np.round(forces[-1], 3))
```

Reference run

```
H=3 kN: signed member forces [kN] [-5.657 -1.414 -1.414]
H=6 kN: signed member forces [kN] [-8.485  0.     0.   ]
```



Two legs unload as the horizontal apex force reaches the vertical load

Download editable notebook · Download Python script

Exercises on force paths

Move both bridge loads to E while retaining their combined magnitude. Which support reaction and which top-chord force change?

▼ **Worked solution**

The right reaction is $8(3)/4 = 6$ kN and the left is 2 kN. Taking moments about B on the same left fragment gives $-2(2) - N_{DE} = 0$, so DE still carries –4 kN. An unchanged chord force does not imply unchanged diagonals.

Remove BE from the original frame. Does external equilibrium establish rigidity?

▼ **Worked solution**

The pin and roller provide three independent reaction components. Five planar joints admit ten equilibrium rows against six members and three reactions. There cannot be full rank. The original load may no longer be supported without motion. For geometries satisfying the count, independence must still be checked.

What happens when the apex height tends to zero at fixed radius and vertical load?

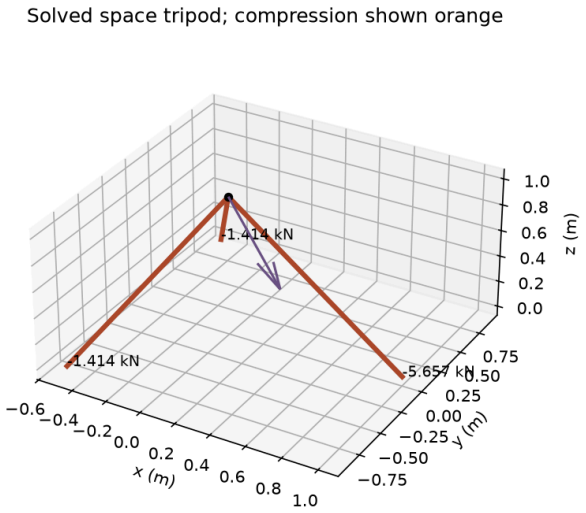
▼ **Worked solution**

Every vertical component of a member direction tends to zero. Maintaining finite upward resultant requires unbounded compression before the idealised matrix loses rank at zero height. Strength, deformation and buckling constrain the real frame much earlier.

Further reading · Hibbeler, Engineering Mechanics: Statics in SI Units, 15th ed., Chapter 6, especially sections 6.1–6.5.

Spatial notebook output · inspect the same solved model in3D

The downloadable notebook now also draws the solved member forces or discrete beam eigenvectors in three dimensions. Its spatial figure below is produced by the actual OneSimulation run.



Recorded spatial output; the beam view enlarges transverse dimensions to make its mode shape legible.

Download the executed notebook with3D output

Stress, deformation and structural measurement · 7. Where does the neutral axis of an angle section lie?

A downward tip load can bend an angle section sideways. Construct its full area-moment matrix and use measured strains to distinguish coupling of bending axes from an unintended applied torque.

Balance two bending moments with one stress plane

Consider a straight, homogeneous prismatic beam in small-deflection Euler–Bernoulli bending. Coordinates x, y lie in the section, with their origin at the area centroid. Plane sections remain plane and zero axial resultant eliminates the constant part of the stress. Write $\sigma = ax + by$. Fix signs by defining $M_x = \int y\sigma \, dA$ and $M_y = -\int x\sigma \, dA$. These signs follow from the cross product of position with axial traction. Then

$$\begin{bmatrix} I_{yy} & I_{xy} \\ I_{xy} & I_{xx} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -M_y \\ M_x \end{bmatrix}, \quad I_{xy} = \int_A xy \, dA.$$

Rotating coordinates to an orthogonal eigenbasis diagonalises this matrix. Its positive determinant is the Cauchy–Schwarz inequality applied to the coordinates. Under pure M_x, $a/b = -I_{xy}/I_{yy}$, so the neutral line $y = -(a/b)x$ generally slopes. A moment aligned with a principal axis recovers ordinary uniaxial bending.

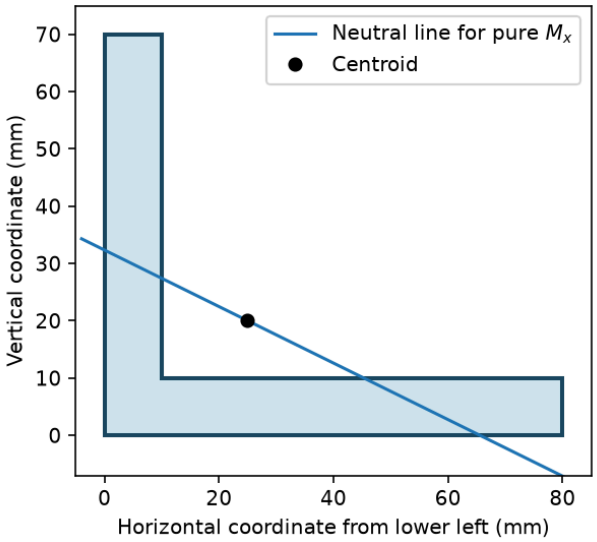
Build the section before inverting the moments

Use an 80 by 10 mm horizontal rectangle and a 10 by 60 mm upright above its left edge. The centroid measured from the lower left corner is (25,20) mm. Translating each rectangle to the centroid gives $I_{xx} = 606667$, $I_{yy} = 851667$, and $I_{xy} = -420000$ in mm⁴. For instance, the horizontal part contributes $800(40-25)(5-20) = -180000$ mm⁴ to the product. Adding independent values of $bh^3/12$ would omit these offsets.

Take $M_x = 1$ kN m and $M_y = 0$. With moments in N mm, the result is $a = 1.2343$ and $b = 2.5029$ MPa/mm. The neutral line has slope −0.4932. At the horizontal flange corner (80,10) the predicted longitudinal tension is 42.86 MPa. The two principal area moments are approximately 291667 and 1166667 mm⁴. These are useful geometry checks because a rigid rotation cannot change them.

If the tip load misses the shear centre

An eccentric transverse force can supply both a bending moment and a torque. The formula above predicts longitudinal bending stress in a section away from load introduction. A separate twist measurement or a loading couple with no net transverse force helps isolate the source of a sideways signal.



The L section and its neutral line under positive bending about the horizontal axis. The centroid is outside the material.

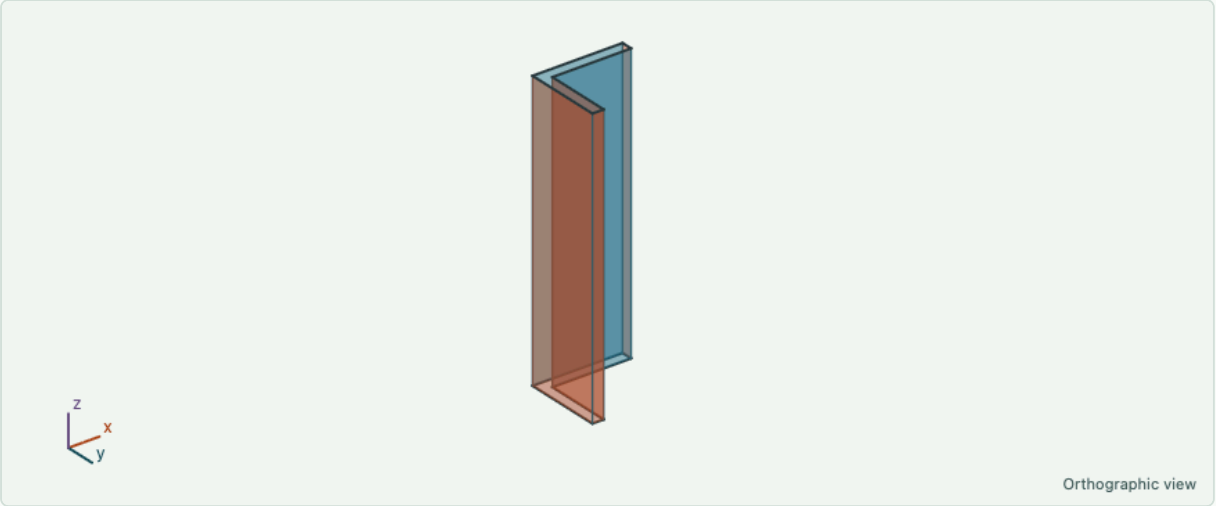
Interactive calculation · Turn the bending couple on an asymmetric section

Change Mx and My independently. The stress plane, perimeter stresses and spatially coloured beam faces are recomputed from the coupled area-moment matrix. A pure moment about a geometric axis need not bend about that axis.

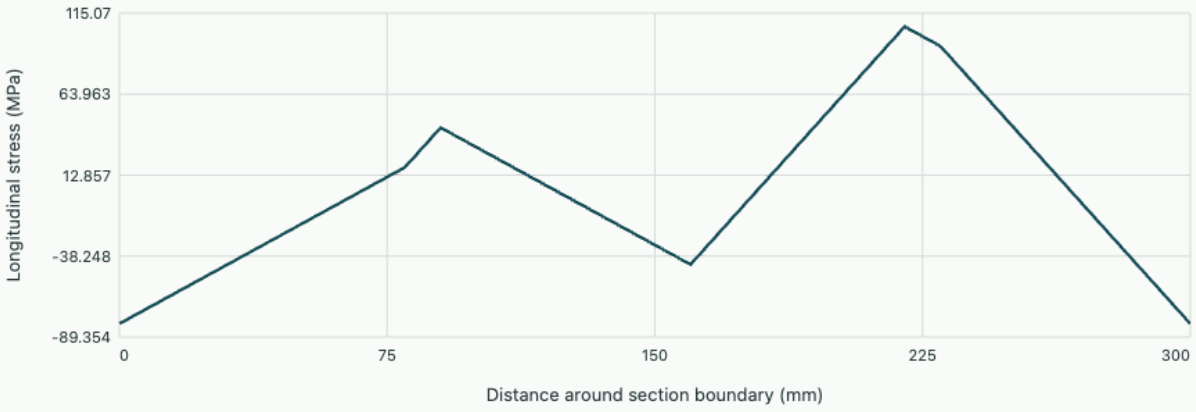
Open this live calculation

Try the calculation

Bending moment Mx (kN m)	1
Bending moment My (kN m)	0



Three-dimensional 250 mm segment of the L section under a pure end-moment couple. Orange surface patches have tensile longitudinal stress and blue patches compressive stress at their midpoints; colours are qualitative. Geometry is undeformed. The full area-moment coupling is solved, as in the linked OneSimulation notebook. Axes: x = section x (mm); y = section y (mm); z = beam axis (mm).



Signed boundary stress

Stress coefficients a=1.2343 and b=2.5029 MPa/mm. At (80,10) mm, stress 42.857 MPa. Bending moments are constant along this prismatic segment.

Exercises on signed strain readings

A strain gauge is glued at (0,0) mm. What signal should it give for steel with $E = 200$ GPa?

▼ **Worked solution**

Its stress is $1.2343(-25) + 2.5029(-20) = -80.9$ MPa. Its axial strain is −405 microstrain. A wrong choice of corner origin would falsely predict zero.

Can two axial gauges recover both bending moments?

▼ **Worked solution**

Write their two rows as (x_i, y_i) relative to the centroid and solve for a, b from $E\epsilon_i$. Recovery requires $x_1 y_2 - x_2 y_1 \neq 0$. Two gauges on the same centroidal line give proportional information even if they lie on opposite sides of the section.

Further reading · Hibbeler, Mechanics of Materials, 11th ed., Section 6.5 on unsymmetric bending.

OneSimulation notebook · solved spatial stresses, code and output

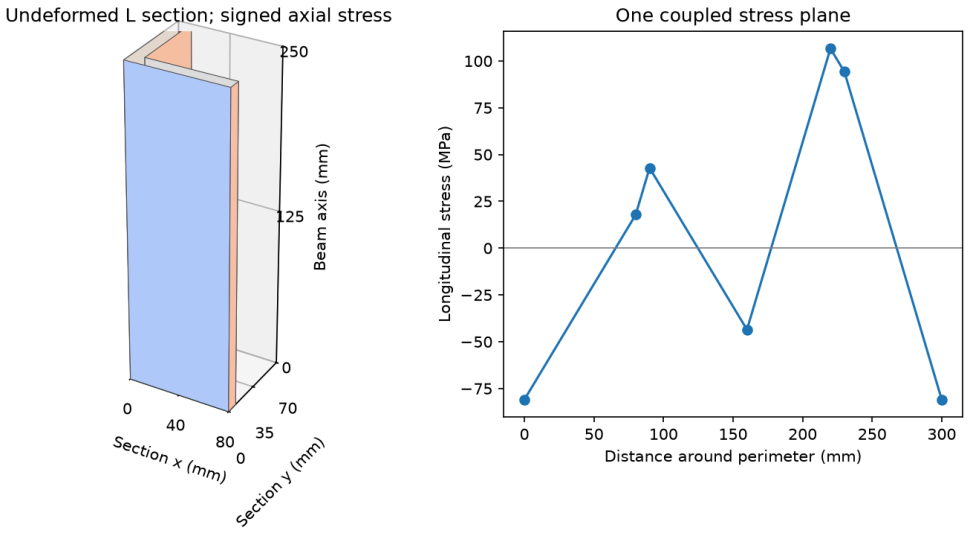
The code solves the illustrated stress model with OneSimulation and checks its equilibrium or eigensystem. The figure is the output of that code.

```
"""Resolve a biaxial moment with OneSimulation and inspect a 3D prismatic L section."""
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d.art3d import Poly3DCollection
from onesim.numerics.linear.matrices.dense_linear_algebra import solve_dense, symmetric_eigensystem

# mm, N, MPa. Two nonoverlapping rectangles; moments follow the chapter signs.
rectangles = [(800., 40., 5., 80., 10.), (600., 5., 40., 10., 60.)]
area = sum(row[0] for row in rectangles)
centroid = np.array([sum(A*x for A,x,y,w,h in rectangles),
                     sum(A*y for A,x,y,w,h in rectangles)])/area
Ixx = sum(A*h*h/12+A*(y-centroid[1])**2 for A,x,y,w,h in rectangles)
Iyy = sum(A*w*w/12+A*(x-centroid[0])**2 for A,x,y,w,h in rectangles)
Ixy = sum(A*(x-centroid[0])*(y-centroid[1]) for A,x,y,w,h in rectangles)
matrix = np.array([[Iyy,Ixy],[Ixy,Ixx]])
Mx, My = 1e6, 0.0  # N mm; change these two inputs as in the live view
coefficients = solve_dense(matrix,np.array([-My,Mx]))
values, axes = symmetric_eigensystem(matrix)
assert np.allclose(matrix@coefficients,[-My,Mx])
assert np.allclose(axes@np.diag(values)@axes.T,matrix)
outline = np.array([[0,0],[80,0],[80,10],[10,10],[10,70],[0,70],[0,0]],dtype=float)
stress = (outline-centroid)@coefficients
print('Centroid [mm]:',centroid)
print('Stress-plane coefficients [MPa/mm]:',np.round(coefficients,6))
print('Principal area moments [mm^4]:',np.round(values,3))
print('Stress at (80,10) mm [MPa]:',round(stress[2],6))

fig = plt.figure(figsize=(9,4.2),layout='constrained')
ax = fig.add_subplot(121,projection='3d')
faces=[];colors=[]
scale=max(abs(stress))
for i in range(len(outline)-1):
    p,q=outline[i:i+2]
```

```
    faces.append([[*p,0],[*q,0],[*q,250],[*p,250]])
    colors.append(plt.cm.coolwarm(.5+.5*(stress[i]+stress[i+1])/2/scale))
ax.add_collection3d(Poly3DCollection(faces,facecolors=colors,edgecolors='0.3',linewidths=.5))
ax.set(xlim=(0,80),ylim=(0,70),zlim=(0,250),xlabel='Section x (mm)',ylabel='Section y
(mm)',zlabel='Beam axis (mm)',title='Undeformed L section; signed axial stress')
ax.set(xticks=[0,40,80], yticks=[0,35,70], zticks=[0,125,250])
ax.tick_params(axis='x', pad=1)
ax.tick_params(axis='y', pad=8)
ax.tick_params(axis='z', pad=1)
ax.zaxis.labelpad = 16
ax.yaxis.labelpad = 18
ax.set_box_aspect((80,70,250))
ax2=fig.add_subplot(122)
arc=np.r_[0,np.cumsum(np.linalg.norm(np.diff(outline,axis=0),axis=1))]
ax2.plot(arc,stress,'o-')
ax2.axhline(0,color='0.5',lw=.8)
ax2.set(xlabel='Distance around perimeter (mm)',ylabel='Longitudinal stress (MPa)',title='One
coupled stress plane')
fig.savefig('complete-engineering-unsymmetric3d.png',dpi=170)
plt.show()
```



Recorded output from the executed OneSimulation notebook; change its inputs to reproduce the browser study.

Download the executed editable notebook · Download the Python model

Stress, deformation and structural measurement · 8. When does Euler’s column load cease to govern?

A stocky strut may yield while its elastic eigenvalue still looks comfortable. Separate a bifurcation of a straight member, stress amplification under eccentric loading and loss of tangent stiffness in a yielded section.

Compare slenderness against the material stress scale

For a pin-ended straight column with effective length L_e, radius of gyration $r = \sqrt{I/A}$ and uniform modulus E, the homogeneous transverse balance admits a nonzero mode when $P_E = \pi^2 E I / L_e^2$. The corresponding average stress falls as the inverse square of $\lambda = L_e / r$. If this value exceeds the proportional limit, part of the constitutive relation assumed by the eigenproblem is already invalid.

One engineering interpolation for intermediate steel columns is a Johnson parabola starting at yield stress S_y. Write $\sigma_J = S_y - c\lambda^2$ and require value and first derivative to match Euler’s curve at the join. Eliminating c and the joining slenderness gives

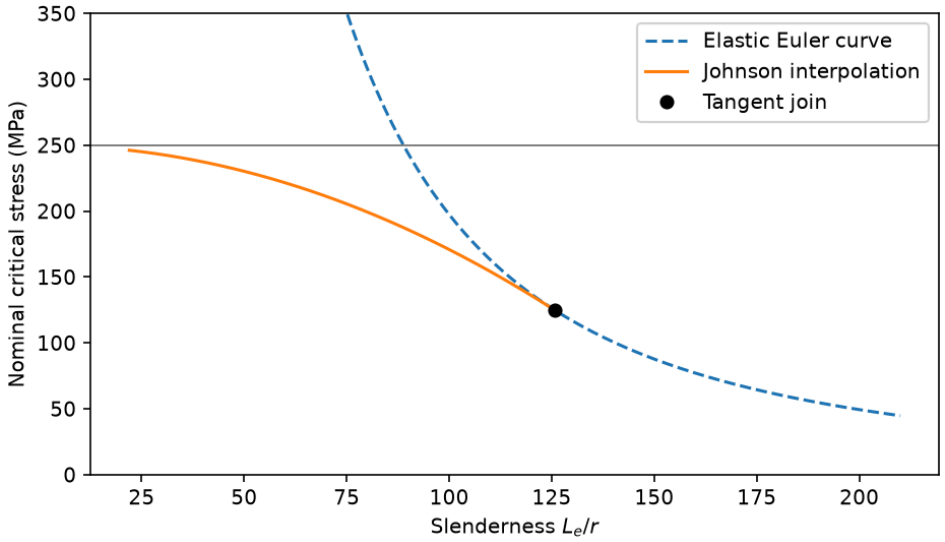
$$c = \frac{S_y^2}{4\pi^2 E},$$

$$\lambda_c = \sqrt{\frac{2\pi^2 E}{S_y}}.$$

For $E = 200000$ MPa and $S_y = 250$ MPa the join is 125.7. At $\lambda = 70$ the interpolation predicts 211.2 MPa whereas elastic Euler predicts 402.8 MPa. The parabola is a material/design model. Measured residual stress, end restraint and a prescribed column curve can change the permissible design load.

Why a tangent modulus appears

Linearise a measured compressive material curve about the current stress. An incremental strain changes stress by $E_t d\epsilon$. Under uniform continued loading, replacing E by $E_t(\sigma)$ in the incremental bending resistance leads to an implicit tangent-modulus estimate. Unloading fibres may respond with a different modulus. Check which constitutive path the test actually follows.



Elastic buckling overpredicts the stress of the intermediate column. The two model curves meet with a common tangent at slenderness 125.7.

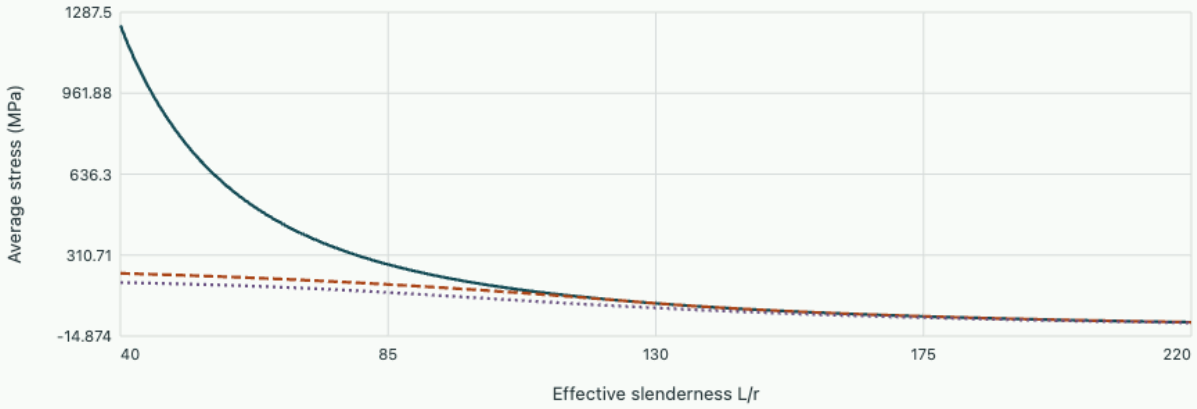
Interactive calculation · Compare yielding, eccentricity and an elastic eigenvalue

Change elastic stiffness and yield stress before adding eccentricity. The secant branch solves the first-fibre-yield equation below its elastic pole; it is not a post-buckling strength model.

Open this live calculation

Try the calculation

Elastic modulus (GPa)	200
Yield stress (MPa)	250
Eccentricity parameter ec/r²	0.2



Smooth Johnson/Euler join at slenderness 125.664. At L/r=100 eccentric first yield occurs at 138.945 MPa. These are conditional model estimates, not an industrial column rating.

Derive the secant amplification under eccentric load

Apply equal end eccentricities e on the same side of an ideal pin-ended member of length L. A section at lateral deflection v sees moment magnitude $P(e+v)$. Taking positive v toward the load line gives $EIv'' = -P(e+v)$. Set $k^2 = P/EI$, use symmetry $v'(0) = 0$ at midspan, and enforce $v(L/2) = 0$ at a pin. Thus $e + v(x) = e\cos(kx)/\cos(kL/2)$. At the outer fibre of distance c_f,

$$\sigma_{\max} = p\left[1 + \frac{ec_f}{r^2}\sec\left(\frac{\pi}{2}\sqrt{p/p_E}\right)\right],$$

$$p = P/A,$$

$$p_E = P_E/A.$$

For $\lambda = 100$ and $ec_f/r^2 = 0.20$, $p_E = 197.39$ MPa. Solving the monotone pre-buckling branch for first yield at 250 MPa gives $p = 138.94$ MPa. The formula locates first fibre yield while the member remains nominally elastic. Post-yield redistribution needs an updated material and section model.

Combine an initial bow, locked-in stress and incremental stiffness

Make the constitutive assumption explicit with four equal-area longitudinal fibres at $y = \pm a$ and $\pm a/3$. Use compression positive. At zero external load, both outer fibres have residual compression +60 MPa and both inner fibres have residual tension –60 MPa. The initial resultant force and bending moment vanish, but subsequent yielding starts at different applied loads. Take $E = 200000$ MPa, compressive knee $S_y = 250$ MPa and continued-loading tangent $E_t = 4000$ MPa. Plane sections remain plane. The numerical model below linearises lateral response around the uniformly compressed reference path.

For axial strain increment $d\epsilon_0$ and curvature increment $d\kappa$, every fibre sees $d\sigma_i = E_i(d\epsilon_0 + y_i d\kappa)$. Summing force and moment gives the tangent matrix with $A_t = \sum E_i A_i$, $B_t = \sum E_i A_i y_i$ and $D_t = \sum E_i A_i y_i^2$. Holding incremental axial force fixed and eliminating $d\epsilon_0$ yields effective bending rigidity $D_b = D_t - B_t^2/A_t$. Here equal opposite fibres have the same tangent on the straight reference path, so $B_t = 0$. The outer and inner fibre pairs contribute 90% and 10% of geometric I, respectively.

The outer pair first reaches 250 MPa when $E\epsilon_0 = 190$ MPa. The inner fibres then carry 130 MPa and the mean applied stress is 190 MPa. Before this point $D_b/I = 200000$ MPa. Afterwards $D_b/I = 0.9(4000) + 0.1(200000) = 23600$ MPa until the inner pair reaches its knee at mean stress 251.2 MPa. With $a = 12$ mm, $r = 8.944$ mm and $L/r = 32$, the elastic and second-stage tangent eigenstresses $p_t = \pi^2 D_b/(AL^2)$ are 1927.66 and 227.46 MPa. The latter lies inside its assumed reference-load interval.

Now prescribe a small initial half-sine bow of amplitude $w_0 = 0.050$ mm, without load eccentricity. Approximate the additional lateral displacement by the same half-sine. At midspan the moment is Pw where w denotes the total amplitude, including the initial bow. Differentiating equilibrium gives $D_b(\pi/L)^2 dw = w\,dP + P\,dw$. Thus the amplification law for a branch of constant tangent is

$$(p_t - p)dw = w\,dp,$$

$$w(p) = w(p_a)\frac{p_t - p_a}{p_t - p}.$$

Carry the actual displacement across each change in stiffness. At $p = 190$ MPa the first branch gives $w = 0.05547$ mm. On the second branch
$w(210) = 0.05547(227.46 - 190)/(227.46 - 210) = 0.1190$ mm. Replacing E by E_t at zero load would lose the prior history and use the wrong fraction of yielded material.

Where the tangent estimate stops being reliable

An initial bow makes yielding start at slightly different loads on individual fibres and vary along the span. This small-imperfection model evaluates tangents on the symmetric reference path and loses accuracy near its asymptote. For a finite imperfection, update fibres and unloading moduli station by station and follow the equilibrium path. The calculated 227.46 MPa is a tangent-reference threshold, not a measured collapse strength.

Keeping all reference data, estimate deflection at 220 MPa and identify which measurement best tests this model.

▼ **Worked solution**

The second branch gives $w = 0.05547(37.46)/(7.46) = 0.278$ mm. Opposed axial strain gauges at midspan, a lateral displacement reading and the initial unloaded shape separate compression, curvature and imperfection. Divergence from the predicted growth can reveal nonuniform yielding before the tangent asymptote.

Exercises on competing limits

At the join λ_c, what fraction of yield stress do both laws give?

▼ **Worked solution**

Substitute the expression for λ_c into either branch. Both give $S_y/2 = 125$ MPa and have the same negative slope, a direct check on the interpolation.

A 400 mm² pin-ended specimen has the geometry of the eccentric example. What maximum force first produces a nominally yielded fibre? What if the eccentricity doubles?

▼ **Worked solution**

The first force is $138.94 \times 400 = 55.58$ kN. In the secant bracket the coefficient increases from 0.2 to 0.4. Holding p fixed raises peak stress, so the new admissible root is lower. Re-solve before using it as a load rating.

Further reading · Hibbeler, Mechanics of Materials, 11th ed., Chapter 13, especially sections 13.4–13.7.