Fields and the response of matter

Connect electric and magnetic measurements to relaxation times, carrier motion, thermal fluctuations and collective order. Use frequency, temperature and geometry to distinguish competing explanations.

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Send light along a plane boundary between a dielectric and a metal. Seek a wave that propagates parallel to the interface while decaying away on both sides. The signs in the magnetic and electric boundary conditions determine whether any such wave can exist.

10. Infer charge discreteness from fluctuations in a steady photocurrent

A detector can deliver a perfectly steady mean current over a long interval while its charge per short exposure fluctuates. Derive both a counting prediction and its spectral version, then use a bandwidth-limited measurement to separate partition fluctuations from thermal and readout noise.

11. Why a nearby gate changes the speed law of a two-dimensional plasma

A current pulse excites a travelling density modulation in a thin conducting sheet. Move a metal gate towards the sheet and the shape of the dispersion curve changes. Rebuild the restoring electric field from the surface charge before using any oscillator analogy.

12. Can a capacitance measurement reveal the fraction of embedded particles?

A sample contains well separated dielectric spheres in an insulating matrix. Its capacitance changes when the particle fraction changes. Derive a dilute estimate from one sphere and a self-consistent approximation from a local-field model. Use the inverse calculation to discover when the fraction becomes poorly determined.

13. Can two ohmic layers imitate a relaxing dielectric?

Model a film and its contact layer in series. Derive the frequency scale of interfacial charge accumulation and decide which features of a measured spectrum can arise from geometry alone.

14. From molecular dipoles to a dielectric susceptibility

Derive induced and orientational polarisation separately, then determine when temperature or local fields invalidate a constant dielectric coefficient.

Fields and the response of matter · 1. Measure how quickly a dielectric follows an electric field

Derive the time and frequency responses of a relaxing polarisation, locate the loss maximum and decide whether a measured sample contains more than one relaxation process.

From a delayed polarisation to a constitutive equation

A slab of insulating material is placed between parallel electrodes. At first the measured charge increases very quickly when a small voltage is switched on. It then continues to rise slowly even though the applied voltage is fixed. A constant real relative permittivity can describe the final charge but cannot predict this record. The experiment reveals at least two relevant time scales. For a measurement window that resolves only the slower process, treat the fast electronic response as instantaneous and describe the slower polarisation P by its approach to an equilibrium value.

At fixed weak field the equilibrium slow polarisation is $\epsilon_0 \Delta\epsilon E$, with $\Delta\epsilon > 0$. Assume the rate of change is proportional to the distance from that equilibrium, with one positive relaxation time τ. Then

$$\tau \dot{P} + P = \epsilon_0 \Delta\epsilon E(t).$$

This is a phenomenological material law. Its assumptions include linear response about the prepared state, a time-independent τ, and a sample whose constituents can be described by one dominant relaxation. Testing those assumptions is part of the experiment.

For $E(t) = E_s$ switched on at zero from an unpolarised state, solve using the integrating factor $e^{t/\tau}$. The result is

$$P = \epsilon_0 \Delta\epsilon E_s (1 - e^{-t/\tau}).$$

After one τ, 63.21% of the slow response has developed; after 3τ, 95.02%. Measuring an absolute fraction requires subtracting the fast initial jump and knowing the eventual plateau. A long baseline drift can look like a second relaxation if those references are not controlled.

Derive the complex dielectric response

Now use peak phasors with time dependence $e^{-i\omega t}$. Differentiation contributes $-i\omega$, so

$$P/E = \epsilon_0 \Delta\epsilon / (1 - i\omega\tau).$$

Include the high-frequency relative permittivity ϵ_∞, where “high” means above the slow process but below additional electronic resonances. The real and imaginary parts of relative permittivity are

$$\epsilon' = \epsilon_\infty + \frac{\Delta\epsilon}{1 + (\omega\tau)^2},$$

$$\epsilon'' = \frac{\Delta\epsilon\, \omega\tau}{1 + (\omega\tau)^2}.$$

At fixed field amplitude the polarisation current is

$$\dot{P} = -i\omega P.$$

The mean electrical work per unit time and volume is

$$\tfrac{1}{2}\mathrm{Re}(E^{*}\dot{P}) = \tfrac{1}{2}\omega\epsilon_0\epsilon''|E|^2.$$

Loss therefore has positive ϵ'' in the chosen time convention. Using $e^{+i\omega t}$ reverses the sign of the imaginary part of permittivity but leaves mean absorbed energy positive when the current convention is changed consistently.

Set $u = \omega\tau$. Differentiating $u/(1+u^2)$ gives $(1-u^2)/(1+u^2)^2$. Hence ϵ'' peaks at $u = 1$, with height $\Delta\epsilon/2$, and the real part lies halfway between its low- and high-frequency plateaux. The peak identified here is in the imaginary *permittivity*. A plot of dissipated power at fixed field includes an additional factor of frequency and need not have a peak at the same location.

Tie a frequency sweep to a step test

Suppose a material has $\epsilon_\infty = 2$, $\Delta\epsilon = 6$, and a measured loss peak at 1 kHz. Then $\tau = 1/(2\pi\,1000) \simeq 159\ \mu$s. At the peak the relative permittivity is $5 + 3i$ in our convention. A time-domain voltage step should show a slow charge transient reaching 63.2% of its added charge in roughly 159 µs. This comparison connects two experiments that weight electrode offsets and parasitic impedances differently.

An independent shape check follows by eliminating u. Let $X = \epsilon' - \epsilon_\infty$ and $Y = \epsilon''$. From the formulas $X^2 + Y^2 = \Delta\epsilon X$. Thus $(X - \Delta\epsilon/2)^2 + Y^2 = (\Delta\epsilon/2)^2$: a single Debye relaxation traces a semicircle in the complex plane. Departures from a semicircle can reflect several relaxation times, spatially inhomogeneous samples, leakage or imperfect fixture correction. They do not uniquely identify the microscopic mechanism.

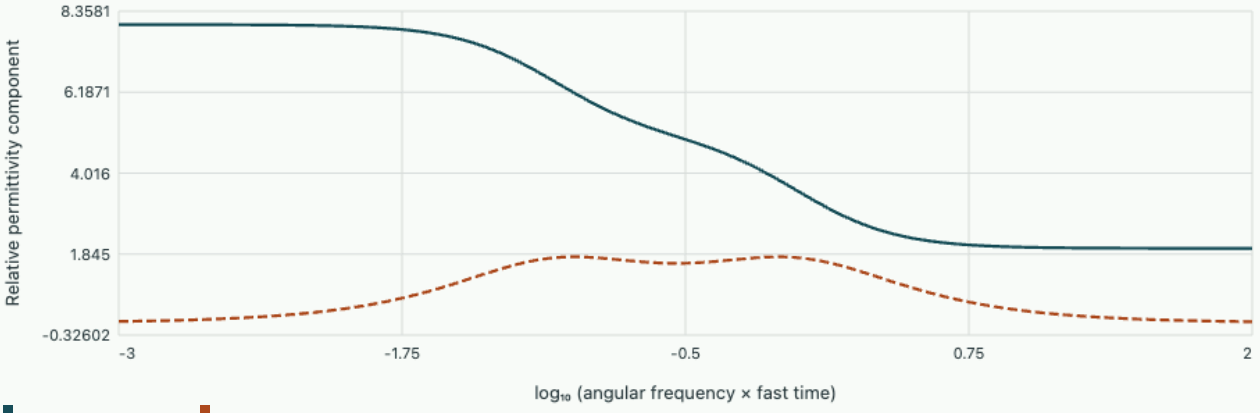
Resolve a second process only when the data require it

Open the live experiment

Try the calculation

Slow fraction of total dielectric strength **0.5**

Slow / fast relaxation time **12**



Largest sampled loss 1.7724 near ωτ_fast = 0.1. Relative permittivity tends to 8 at low frequency and 2 at high frequency.

The synthetic sample has two groups contributing a fixed total $\Delta\epsilon = 6$. Change the fraction in the slower group and its relaxation time relative to the faster group. At equal relaxation times the groups give exactly the single-process response whatever their fractions. When the times differ, the imaginary curves broaden and may separate. Observe the real dispersion steps as well. A broad lossy peak alone can be fitted by more than one pair of times and strengths.

In a practical measurement the low-frequency tail can contain DC conductivity. In the $e^{-i\omega t}$ convention,

$$\sigma E - i\omega\epsilon_0\epsilon E = -i\omega\epsilon_0[\epsilon + i\sigma/(\epsilon_0\omega)]E.$$

This added imaginary part is proportional to $1/\omega$, whereas the low-frequency side of an isolated Debye relaxation rises as ω. Change electrode separation and use a DC conduction measurement to challenge the interpretation. An electrode interface can add a further time scale, so an observed fit parameter deserves an uncertainty and a physical validation step.

Tutorial

The loss maximum is at 2 kHz and its height is 4 in relative-permittivity units. The high-frequency plateau is 3. Predict the low-frequency plateau and the slow response at $t = 100\ \mu$s after a step.

▼ **Work through the solution**

The single-time model gives $\Delta\epsilon = 8$ and the low-frequency plateau 11. The relaxation time is $1/[2\pi(2000)] \simeq 79.6\ \mu$s. The slow fraction after 100 µs is $1 - e^{-100/79.6} \simeq 0.715$. Repeat at another

step amplitude to check linearity.

At the loss peak, is the phase difference between the *total* displacement and the electric field always 45°?

▼ **Work through the solution**

The slow polarisation alone has a 45° phasor angle at $\omega\tau = 1$. The total displacement includes $\epsilon_\infty E$. Its angle is $\arctan[(\Delta\epsilon/2)/(\epsilon_\infty + \Delta\epsilon/2)]$, generally less than 45°.

Why does observing only a high-frequency tail constrain a ratio of parameters more tightly than the individual parameters?

▼ **Work through the solution**

For large $\omega\tau$,

$$\epsilon'' \simeq \Delta\epsilon/(\omega\tau).$$

Many values with the same ratio $\Delta\epsilon/\tau$ have very similar tails. Measure the peak or both plateaux to distinguish the two parameters.

Compare with material measurements

Inspect the orientational response and dielectric-loss treatment in MIT's material-properties course. Use the phasor convention of each source to predict the sign assigned to loss.

Fields and the response of matter · 2. Identify mobile charge with a transverse field

Derive a Hall-voltage measurement from transverse force balance, reverse current and field to reject offsets and explore what fails when two carrier types conduct.

Choose signs from a drawing

A rectangular conducting sample has its long axis along x, width w along y, and thickness t along z. A uniform magnetic field $B\hat{\mathbf{z}}$ is applied. An external circuit maintains conventional current $I > 0$ in $+x$. High-impedance transverse probes draw negligible current. Define the measured signal as $V_H = V(y = 0) - V(y = w)$. For uniform transverse field this convention gives

$$V_H = E_y w.$$

Reversing the voltmeter leads reverses the reported sign; writing a carrier sign without a lead convention is incomplete.

With one charge carrier species of number density n and signed charge q, the drift velocity satisfies

$$j_x = nqv_x = I/(wt).$$

If the transverse velocity vanishes, its magnetic force component is $-qv_xB$, opposed by qE_y. Thus $E_y = v_xB = j_xB/(nq)$. The resulting port measurement is

$$V_H = \frac{IB}{nqt}.$$

The width cancels for a uniform bar; changing the contact geometry or using a current source with a nonuniform injection profile can spoil that simplification near the terminals. The cancellation also exposes the sign. For electrons $q < 0$, positive conventional current means that the mobile charge drifts in $-x$. The inferred transverse field is negative for positive B in the chosen convention.

Reconstruct density and mobility from separate observations

Take $t = 0.50$ mm, $I = 5.0$ mA, $B = 0.20$ T and a transverse signal of -1.25 mV. The inferred signed Hall coefficient is

$$R_H = tV_H/(IB) = -6.25 \times 10^{-4}\ \mathrm{m^3/C}.$$

Using singly charged electrons gives

$$n = 1/(e|R_H|) \simeq 1.00 \times 10^{22}\ \mathrm{m^{-3}}.$$

A longitudinal voltage measurement, after correcting for probe separation, can yield the material conductivity. If $\sigma = 80$ S/m, the single-carrier mobility is

$$\mu = \sigma |R_H| = 0.050\ \mathrm{m^2/(V\,s)}.$$

These operations use different pieces of information. The longitudinal measurement alone gives only the product $n|q|\mu$. The transverse measurement supplies a sign and, in the one-carrier model, the density independently of mobility. This is an inverse experiment whose two observables have different sensitivities, not a more precise measurement of the same scalar.

One can derive the same result from a stationary Drude equation $0 = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - m\mathbf{v}/\tau$. The longitudinal drag gives

$$E_x = m v_x / (q\tau);$$

force balance in y gives the transverse field just obtained. In a homogeneous single-carrier sample, $E_y/E_x = q\tau B/m$, the signed Hall angle tangent. At fixed sample and field, doubling I doubles both voltages in this linear regime and leaves their ratio unchanged. Significant current-dependent apparent density can indicate heating or nonlinear contacts.

Isolate the transverse voltage from a misaligned contact

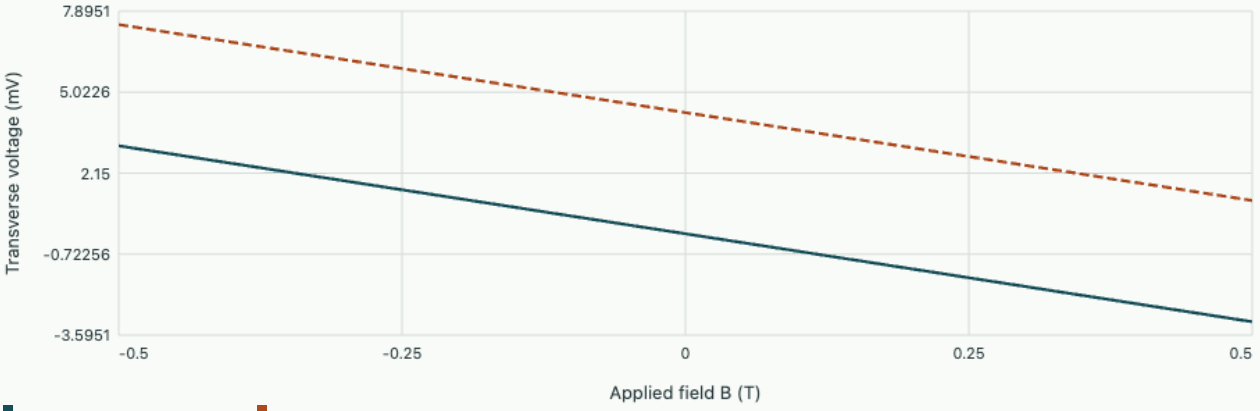
Practical side contacts seldom sit at exactly the same x. A fraction of the longitudinal voltage can leak into the transverse pair. Write the readout at a fixed current as $V_\mathrm{m}(B) = V_0 + \alpha V_{xx} + V_H(B)$. A background that is even in field cancels in $[V_\mathrm{m}(B) - V_\mathrm{m}(-B)]/2$. For example, readings of 3.05 mV and 5.55 mV at $+B, -B$ return -1.25 mV Hall signal and 4.30 mV of even background. Taking a single measured sign would have diagnosed the wrong carriers.

Open the live experiment

Try the calculation

Electron density (10^{22} m^{-3}) 1

Current (mA) 5



At +0.2 T the instrument reads 3.0517 mV; at −0.2 T it reads 5.5483 mV. Half-difference -1.2483 mV.

The plot deliberately superposes a constant 4.3 mV pickup on a signed Hall signal. Vary density and current independently before reading the pair at $B = \pm 0.2$ T. Large density weakens the signal; current increases it. Verify that field reversal recovers the correct transverse voltage even when both instrument readings have the same sign. If the unwanted background changes between readings, reverse the field several times and check for repeatable drift.

With both current and field reversal, the double-odd combination $[V(+I,+B) - V(+I,-B) - V(-I,+B) + V(-I,-B)]/4$ isolates terms odd in both. A thermal emf that is independent of current cancels along with a stable contact offset. This symmetry decomposition is a physical assumption about the parasitics, so preserve all four readings and inspect the rejected channels too.

When a Hall density stops being a particle count

With electrons and holes, longitudinal conductivities add. In a weak magnetic field their transverse contributions also depend on squared mobilities. For positive mobilities μ_e, μ_h and electron and hole densities n, p, the two-carrier weak-field coefficient in the sign convention above is $R_H = (p\mu_h^2 - n\mu_e^2)/[e(p\mu_h + n\mu_e)^2]$. To see where the square appears, note that one factor of mobility converts the longitudinal field into drift; a second converts the resulting transverse magnetic force into a drift contribution. Summing species currents and imposing zero total transverse current gives the denominator management through the summed longitudinal conductivity.

Equal electron and hole densities with equal mobilities give zero Hall signal while current still flows readily. A single-species inversion would suggest unbounded density. At equal densities but unequal

mobilities, the faster species can set the sign even though neither species is more numerous. Additional field dependence, temperature variation and independent structural information are needed before treating a Hall number as a carrier census.

Tutorial

A measurement doubles the width w at fixed thickness, total current and field in a uniform sample. Predict both E_y and V_H.

▼ **Work through the solution**

The current density halves, so v_x and E_y halve. The voltage integrates that halved field over twice the width and stays the same. If fixed current density were held instead, the transverse voltage would double.

A nominal thickness is overestimated by 5%. What bias does this cause in the single-carrier density inferred from a fixed transverse signal?

▼ **Work through the solution**

$n = |IB|/(et|V_H|)$, so using $1.05t$ gives $n_{\mathrm{fit}} = n/1.05$, approximately 4.8% low. This systematic error is common to all repeated field sweeps.

Why might a metal yield a smaller Hall voltage than a semiconductor with the same dimensions at the same current?

▼ **Work through the solution**

In the one-carrier regime the signal is inversely proportional to carrier density. A large density allows the imposed current at a very small drift speed, requiring a correspondingly small transverse balancing field. Conductivity alone cannot determine this comparison without mobility information.

Follow the transport model

Compare the one-carrier assumptions with the treatment of mobile carriers in David Tong's solid-state physics course. Connect an independently measured conductivity with the signed transverse response before interpreting an effective mass or scattering time.

Fields and the response of matter · 3. Measure equilibrium field fluctuations with a capacitor

Derive an electrical noise measurement in two ways, determine what fraction reaches a finite-bandwidth detector and use agreement to calibrate an experiment.

A zero-average voltage can store measurable energy

Connect a capacitor C across a resistor R at a uniform absolute temperature T, with no external drive. Ideal circuit equations predict that any deliberately deposited charge decays on time scale RC. At thermal equilibrium, microscopic fluctuations continually exchange charge and energy with the capacitor. Symmetry gives zero mean voltage, but the stored energy involves V^2. In the classical limit a quadratic degree of freedom has mean energy $k_BT/2$, giving $\langle V^2\rangle = k_BT/C$. The resistor sets the time scale of equilibration; equilibrium mean-square voltage does not depend on resistance in this ideal model.

Check the dimensions: k_BT is an energy in joules and capacitance has units coulombs per volt. Their ratio is volts squared. At $T = 300$ K and $C = 100$ pF the RMS voltage is $\sqrt{k_BT/C} \simeq 6.44\ \mu\mathrm{V}$. A voltmeter that averages the signed voltage for a long time reports a value near zero. Measuring the noise requires an RMS, variance or appropriately calibrated spectral measurement.

Derive the same variance from a spectral measurement

Use a one-sided voltage power spectral density per hertz at positive frequencies. The classical Johnson–Nyquist open-circuit result for a resistor is $S_R(f) = 4k_BTR$ V²/Hz. Its square root is an amplitude spectral density in V/$\sqrt{\mathrm{Hz}}$, not a voltage variance. Use an equivalent series voltage noise source and the RC transfer function $H(f) = 1/[1 + 2\pi i f RC]$. A stationary linear system multiplies the voltage power spectrum by $|H|^2$, so

$$S_C(f) = \frac{4k_BTR}{1 + (2\pi f RC)^2},$$

\int_0^\infty S_C(f)\,df=\frac{k_BT}{C}.

The integral follows by $u = 2\pi RCf$ and $\int_0^\infty du/(1+u^2) = \pi/2$. It furnishes a detailed check on the factor four, the use of one-sided density, the conversion from hertz to angular frequency and the amplitude-versus-power distinction. If your result disagrees by a constant factor of two or 2π, audit these conventions before assigning a new noise temperature.

For $R = 10$ kΩ, $T = 300$ K, the low-frequency amplitude spectral density is approximately 12.9 nV/$\sqrt{\mathrm{Hz}}$. The RC corner at $C = 100$ pF is $f_c = 1/(2\pi RC) \simeq 159$ kHz. The open-circuit low-frequency density rises with R while the bandwidth shrinks in inverse proportion. Their integrated product gives the earlier resistance-independent variance.

Include a real observation bandwidth

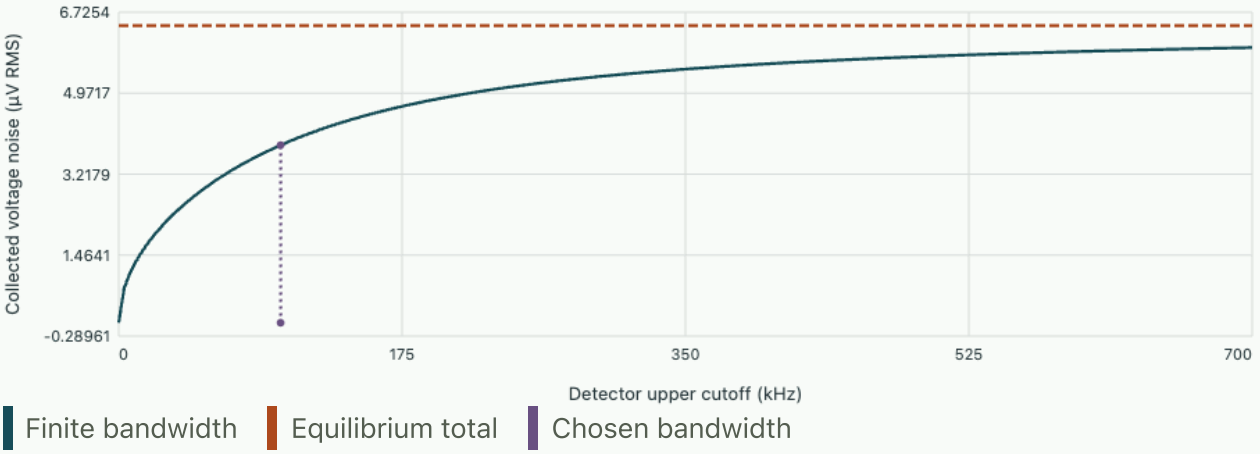
Suppose a first measurement adds an ideal low-pass cutoff B. The recorded fraction of equilibrium variance is $F = (2/\pi)\arctan(B/f_c)$. Half of the noise variance lies below the RC corner. That may be surprising if the corner was introduced as the frequency at which the *spectral density* reaches half its zero-frequency value. These two properties coincide for this single-pole spectrum for different mathematical reasons.

At $B = 0.1 f_c$, $F \simeq 0.06345$, so the recorded RMS is about 25.2% of the infinite-band RMS. At $B = 10 f_c$, $F \simeq 0.93655$, giving about 96.8% of the RMS. In an experiment, taking a square root one step too early changes this comparison. A real detector transfer function replaces the sharp cutoff by an integral of $S_C(f)|H_{\mathrm{det}}(f)|^2$.

Open the live experiment

Try the calculation

Resistance (kΩ)	**10**
Temperature (K)	**300**
Detector bandwidth (kHz)	**100**



RC corner 159.15 kHz. Detector reads 3.8461 µV RMS, collecting 35.713% of equilibrium variance.

Sweep the resistance by a factor ten. Observe how the corner moves along the frequency axis and how the fraction measured in a fixed bandwidth changes. Then vary temperature. In the classical regime spectral density and integrated variance are proportional to T, while RMS voltage grows only as $\sqrt{T}$. A plot labelled “noise” without units cannot distinguish these two dependencies.

Background · How does a longer voltage exposure alter the measured noise?

An equilibrium resistor–capacitor circuit has voltage covariance $C_v(s) = (k_B T/C)\exp(-|s|/\tau)$, where $\tau = RC$. Suppose a digitiser returns the mean voltage during an exposure T_m. Squaring a sample of this mean measures a different variance from averaging the instantaneous voltage squared. In the classical stationary model the two-time integral becomes

$$\operatorname{var}(\bar{v}) = \frac{2}{T_m^2}\int_0^{T_m}(T_m - s)C_v(s)\,ds = \frac{2k_BT}{C}\,\frac{u - 1 + e^{-u}}{u^2},$$

$$u = \frac{T_m}{\tau}.$$

Short exposures recover k_BT/C. At ten correlation times only about 18% of that variance remains. For very long exposures the variance decreases as $2\tau/T_m$. Taking more closely spaced samples of one exposure cannot undo that averaging.

A physical check and a calibration check

An RC time-domain experiment provides a second observable beyond integrated power. Under the same single-pole equilibrium model the voltage autocovariance is

$$\langle V(t)V(t+\tau)\rangle = (k_BT/C)e^{-|\tau|/(RC)}.$$

Its variance at zero lag agrees with equipartition, and its decay time is obtained from the homogeneous charge-relaxation equation. Repeating the measurement with a known extra parallel capacitance lowers the RMS and lengthens the correlation time. Seeing only one of those changes suggests that the observed noise may include substantial amplifier input voltage or an independent background.

The classical approximation assumes $hf \ll k_BT$ over the relevant band and a resistor that can be treated as frequency independent. A MHz room-temperature experiment is far inside this regime. At sufficiently high frequencies or cryogenic temperatures, a quantum noise model, an explicit definition of the measured ordering of the spectrum and the detector's energy exchange are needed. Extending an ideal classical white source to infinite frequency is only permissible as an approximation inside the convergent low-frequency RC calculation; it is not a microscopic model at arbitrarily high frequency.

Tutorial

At fixed C, T, doubling R in a detector with $B \ll f_c$ changes the measured RMS by what factor? Will the same factor apply when $B \gg f_c$?

▼ **Work through the solution**

At very small bandwidth, $S_C \simeq 4k_BTR$ is nearly constant, so variance doubles and RMS increases by $\sqrt{2}$. At large bandwidth the measurement approaches k_BT/C and becomes independent of R.

A two-sided spectrum in hertz is used instead. What density and integration limits preserve the same variance?

▼ **Work through the solution**

For this real voltage, the density for each sign of frequency is half the one-sided positive-frequency value. Integrate it over $(-\infty, \infty)$. Integrating the one-sided density over both signs would count the power twice.

A measured RMS is 10 µV, including independent amplifier noise of 6 µV RMS. Can 6 µV simply be subtracted to infer the sample contribution?

▼ **Work through the solution**

Independent zero-mean voltages add in variance. The inferred sample RMS is

$$\sqrt{10^2 - 6^2} = 8\ \mu\text{V}.$$

A deterministic offset or correlated noise would require a different treatment.

Compare with an actual laboratory

The MIT Junior Lab noise experiment treats the detector as part of the measurement. Identify its calibration observable before comparing the theoretical density with an instrument reading.

Fields and the response of matter · 4. When does a polar crystal change its preferred state?

Derive stable and unstable dielectric responses from a free-energy landscape. Separate a reversible small-signal capacitance from a switching experiment.

Why one voltage can support several measured polarisations

Some insulating crystals develop a spontaneous electric polarisation as temperature is lowered. The high-temperature phase may be symmetric under inversion so that states with polarisation $+P$ and $-P$ are equivalent in zero field. Near a continuous transition, treat one component of polarisation as an order parameter. A local free-energy density compatible with the symmetry is $f(P) = a(T - T_c)P^2/2 + bP^4/4$ with $a, b > 0$. The omitted terms limit how far in temperature or field this truncated expansion is reliable. The positive quartic term keeps the energy bounded when the quadratic coefficient is negative.

For a device whose applied electric field E is controlled, the appropriate potential in this simple model is $g(P; E) = f(P) - EP$. The linear term records external work; changing what is held fixed changes the energy to minimise. Differentiation gives the constitutive equation $E = a(T - T_c)P + bP^3$. An algebraic root is an equilibrium candidate. A stable local response also requires positive second derivative $a(T - T_c) + 3bP^2$. A very accurate rootfinder can return an unstable stationary point if this second test is omitted.

Find the equilibrium branches and their response

Introduce scaled polarisation p, field e and temperature parameter s such that $g = p^4/4 + sp^2/2 - ep$. Then $e = p^3 + sp$. For $s > 0$, the slope $3p^2 + s$ stays positive. Each field has one stable solution. At $e = 0$, $p = 0$ and the small-signal susceptibility $dp/de = 1/s$ increases when approaching the ideal critical point from above.

For $s = -1,\ e = 0$, stationary values are $-1, 0, +1$. The middle root has curvature –1 and is a local maximum. Each outer root has curvature 2 and is a minimum. The susceptibility about a selected outer state is $1/2$ in the scaled units. The fact that the two domains can carry opposite zero-field polarisation does not make the harmonic response negative. The mere existence of a negative-slope segment in the cubic constitutive curve is insufficient to infer a stable isolated small-signal device with that differential response.

Derive the limit of a metastable homogeneous state

Imagine preparing the negative-polarisation state at $s = -1$, then increasing the applied field slowly. A positive field lowers the energy of the positive state. The negative state can persist locally behind a barrier even after it ceases to be the global minimum. A model of perfectly homogeneous evolution without early nucleation follows this metastable branch until the curvature vanishes. Solve $3p^2 - 1 = 0$, choose $p = -1/\sqrt{3}$, and insert it into $e = p^3 - p$. The positive limiting field is

$$e = 2/(3\sqrt{3}) \simeq 0.3849.$$

At the opposite spinodal the signs reverse.

This is a limit of the particular energy model. Real switching can occur earlier through domain walls, nucleation, defects or thermally activated barrier crossing. Observing a smaller coercive field in a specimen is not evidence that the cubic equation was solved inaccurately. To compare an actual hysteresis loop with a calculation, state the sweep rate, boundary circuit, sample preparation and a dynamical or nucleation model in addition to equilibrium free energy.

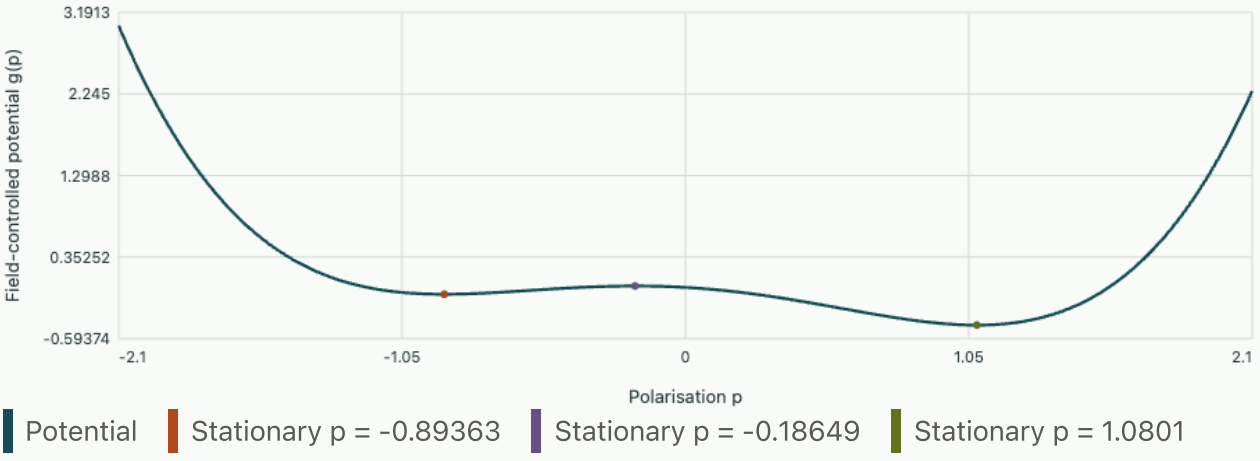
Read a landscape rather than tracing an unlabelled loop

Open the live experiment

Try the calculation

Temperature coefficient s **-1**

Applied field e **0.18**



Stationary polarisations: -0.89363 (minimum), -0.18649 (maximum), 1.0801 (minimum).

Set $s = -1$. At zero applied field, locate the two equal wells. Increase positive e gradually and watch the negative well become shallower. The browser locates real stationary points of the cubic, classifies them by curvature and reports their positions. Its plotted potential shows the energy barrier separating basins. Choose $s > 0$ and repeat to see why fitting the same S-shaped hysteresis loop above a continuous transition would be inappropriate.

A below-transition sample can also form spatially varying domains. Adding a positive gradient term $K(\nabla P)^2/2$ penalises interfaces, while depolarising fields can favour a mixture with reduced net surface charge. This competition sets new length scales absent from the homogeneous model. An electrode measuring total charge can return near-zero mean polarisation from oppositely oriented domains even

though each domain is strongly polarised. A local probe and a bulk capacitance measurement therefore constrain different parts of the model.

Turn a capacitance measurement into a consistency test

For a small harmonic voltage imposed about a prepared local minimum, the differential response is $\delta P/\delta E = 1/[a(T - T_c) + 3bP^2]$, provided the driving frequency is slow enough for this local variable to respond and the amplitude is small enough to stay in the same well. Add the vacuum and fast polarisation terms when converting to total displacement D. In a parallel-plate geometry capacitance per area then follows from $\delta D/\delta E$ divided by thickness.

A large-signal charge–voltage measurement crosses different branches and may dissipate work around a loop. The energy per volume over a closed polarisation cycle is $\oint E\,dP$ with the sign convention for electrical work. A single equilibrium graph of $f(P)$ returns to its initial state with no prescribed loop area. Measured hysteresis thus carries dynamical information that a polynomial fit to one reversible branch cannot uniquely reconstruct.

Tutorial

For $s = -4$ in the scaled model, find zero-field minima and the positive homogeneous spinodal field.

▼ **Work through the solution**

Zero-field minima lie at $p = \pm 2$. The negative spinodal is

$$p = -\sqrt{4/3} = -2/\sqrt{3}.$$

Substitution gives positive field $16/(3\sqrt{3}) \simeq 3.08$. The $|s|^{3/2}$ dependence follows from the cubic.

A Newton solver initialised at $p = 0$ reports exact convergence for $s = -1, e = 0$. What additional observation rejects that result as the prepared stable state?

▼ **Work through the solution**

Its residual is exactly zero but the curvature $3p^2 + s = -1$ is negative. A small perturbation lowers the potential. Check a measured weak-field slope or the relaxation after a perturbation, and compare the energy of the two minima.

Why can the homogeneous coercive field be much larger than the measured coercive field?

▼ **Work through the solution**

The calculation removes a local barrier only when a spatially uniform state ceases to exist. A real material can switch by creating a small favourable domain and moving a wall; electrodes and defects can lower the needed activation energy. The relevant material and time scales differ.

Compare material descriptions

Use MIT's ferroelectrics and polarisation material to distinguish a microscopic source of displacement from the macroscopic free-energy model used here.

Fields and the response of matter · 5. Distinguish magnetic screening from electrical diffusion

Compare the frequency dependence, phase and thickness dependence of field penetration in a normal metal and a superconducting slab.

Perfect conductivity and magnetic expulsion are different statements

A normal conductor has current density $\mathbf{J} = \sigma \mathbf{E}$. Combining Faraday's and low-frequency Ampère laws in a homogeneous medium gives

$$\partial_t \mathbf{B} = (\mu\sigma)^{-1} \nabla^2 \mathbf{B}.$$

A step in field diffuses inwards over a length of order $\sqrt{t/(\mu\sigma)}$. Under sustained sinusoidal drive the skin depth is $\delta = \sqrt{2/(\mu\sigma\omega)}$. As ω tends to zero, this skin depth tends to infinity. An ordinary nonmagnetic conductor in its final static equilibrium generally admits a uniform applied magnetic field.

The London description of an isotropic superconductor in a Meissner state makes a distinct, equilibrium constitutive assumption:

$$\nabla \times \mathbf{J}_s = -\mathbf{B}/(\mu_0 \lambda_L^2),$$

with penetration length λ_L. In a homogeneous specimen carrying supercurrent and at frequencies where displacement current and normal losses may be neglected,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_s.$$

Curl once more and use $\nabla \cdot \mathbf{B} = 0$. The resulting field satisfies

$$\nabla^2 \mathbf{B} = \mathbf{B}/\lambda_L^2.$$

Unlike the diffusive skin depth, the equilibrium length remains finite for a static applied field.

Taking the formal limit $\sigma \to \infty$ in a normal conductor suppresses the electric field for a finite current and can freeze a pre-existing flux during changes. That argument does not determine which static field a specimen adopts when cooled through its transition. Magnetic expulsion is extra thermodynamic physics. The comparison below assumes a weak enough field and a suitable state, omitting flux vortices, trapped flux and nonlinear or nonlocal effects.

Solve a slab with equal applied fields on both faces

Let a broad slab fill $-d/2 < x < d/2$, and impose identical tangential magnetic fields $B_a \hat{\mathbf{z}}$ on its two faces. Symmetry requires $B'(0) = 0$. Applying both boundaries to the London equation gives

$$B(x) = B_a \frac{\cosh(x/\lambda_L)}{\cosh[d/(2\lambda_L)]},$$

\frac{\overline B}{B_a}=\frac{2\lambda_L}{d}\tanh\frac{d}{2\lambda_L}.

Differentiating gives the screening current $J_y = -B_z'/\mu_0$. Currents flow in opposite directions at opposite faces; their net transport current across the thickness is zero. For a thick slab each face approaches a half-space with exponential decay. For $d \ll \lambda_L$, the two surface layers overlap. Expanding the average gives $1 - d^2/(12\lambda_L^2) + \cdots$, so even a superconductor can have a substantial mean internal field in a sufficiently thin specimen under this parallel-field geometry.

Take $d = 100$ nm and $\lambda_L = 50$ nm. The centre field is $B_a/\cosh 1 \simeq 0.6481 B_a$; the mean is $\tanh 1 \simeq 0.7616 B_a$. The distinction between centre value and volume average is already larger than many instrumental error bars. A measurement of magnetic moment in a finite non-slab geometry additionally requires its demagnetising field. Do not directly substitute the infinite-slab mean into an arbitrary sample-shape calibration.

Obtain the equilibrium length as an energy minimum

The same static profile follows from minimising $\int_{-d/2}^{d/2} [B^2 + \lambda_L^2 (B')^2]\, dx/(2\mu_0)$ with boundary values imposed. The first term is magnetic field energy; the second is the energy associated with the screening current in this effective description. Vary B by a function η vanishing at the faces. Integrating $B'\eta'$ by parts leaves an interior factor $B - \lambda_L^2 B''$. Its vanishing returns the London equation. This route makes the physical tradeoff clear: complete expulsion would require extremely sharp gradients and costly currents, while uniform penetration pays magnetic energy throughout the bulk.

The reduction assumes a fixed superconducting condensate amplitude. Near a critical temperature or critical field, allowing that amplitude to change introduces an additional coherence length and may alter both the field state and the validity of London theory. A numerical fit giving a stable value of λ_L at one temperature does not prove this approximation over an entire phase diagram.

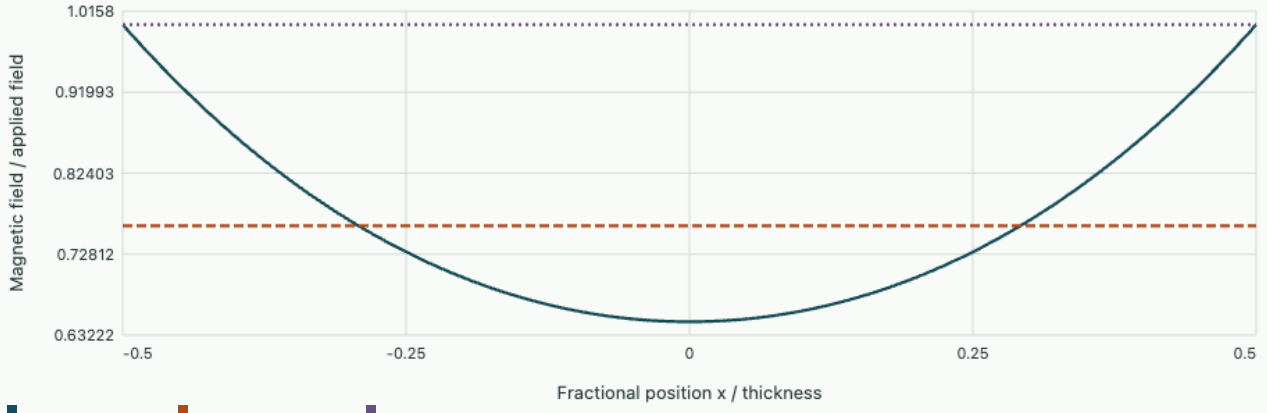
Compare two experimental controls

Open the live experiment

Try the calculation

Slab thickness d (nm) **100**

London length λ (nm) **50**



At d/λ = 2 the centre retains 64.805% of the applied field and the thickness average 76.159%.

Change the thickness and the penetration length independently. Find two different specimens with the same d/λ_L; their normalised profiles against fractional thickness match, even though their local physical gradients and current densities can differ. If all you can measure is a single normalised profile, geometry calibration is part of the inference of λ_L.

A frequency sweep supplies another distinction. In a normal sheet the complex field has both attenuation and phase lag with depth, and δ changes as $f^{-1/2}$. The static London profile above is real and determined by a material length. At higher frequencies superconductors can also exhibit complex response and dissipation, so measuring a single attenuation length in one frequency band cannot by itself establish static flux expulsion. Use the normal-film impedance experiment for a controlled comparison with its different rear boundary condition.

Tutorial

In a thick specimen, how far from one isolated face must a probe be to see 1% of the applied static field? How does the answer depend on field amplitude in this model?

▼ **Work through the solution**

The half-space profile is e^{-x/λ_L}. Set it to 0.01 to obtain

$$x = \lambda_L \ln 100 \simeq 4.605\lambda_L.$$

Linearity makes this scaled distance independent of B_a, until the assumed Meissner and weak-field conditions fail.

For a symmetric slab with $d = 4\lambda_L$, compute the centre and mean fractions.

▼ **Work through the solution**

The centre gives $1/\cosh 2 \simeq 0.266$. The mean is $\tanh 2/2 \simeq 0.482$. Both surface layers contribute substantially to the average even when the centre field is much smaller.

A phase-sensitive measurement detects a substantial depth-dependent phase lag in a slowly driven slab. Does the static London solution predict this?

▼ **Work through the solution**

The static real solution has no phase lag. Assess normal-fluid losses, vortex motion, measurement phase reference and the finite drive frequency. Comparing only magnitudes can conceal these distinct mechanisms.

Read the equilibrium model

The MIT applied-superconductivity course places London's laws alongside microscopic and order-parameter treatments. Use that ordering to identify the assumptions needed when taking a static limit.

Fields and the response of matter · 6. When a periodic dielectric has no propagating Bloch wave

Derive a band gap using two lossless transparent materials and distinguish evanescent Bloch propagation from absorption of optical power.

Construct a unit cell whose waves interfere

Stack two transparent, lossless, nonmagnetic dielectrics periodically in one dimension. Their refractive indices are n_1, n_2, their physical thicknesses d_1, d_2, and the period $a = d_1 + d_2$. Work at normal incidence, so polarisation directions are equivalent for isotropic layers. Each separate material admits a propagating wave at the chosen optical frequency, yet the periodic stack may not. The effect comes from coherent interface reflections, not from a negative refractive index or an absorptive constituent.

Describe the local field by its tangential electric and magnetic components. Propagation through one homogeneous layer connects them by a 2×2 transfer matrix with diagonal $\cos\phi_j$, off-diagonal terms proportional to $iZ_j \sin\phi_j$ and $i\sin\phi_j / Z_j$, where $\phi_j = \omega n_j d_j / c$ and $Z_j = Z_0 / n_j$. Changing the chosen propagation direction or time convention may reverse the two imaginary signs together. The real trace and unit determinant of the cell matrix below are unaffected.

Read the dispersion relation from the trace

At an interface both tangential components are continuous in the absence of imposed sheet currents. Multiply the two propagation matrices to form one cell. Its determinant is one because each layer's determinant is

$$\cos^2\phi + \sin^2\phi = 1.$$

Thus the two eigenvalues are reciprocal. For a Bloch mode they are $e^{\pm i k_B a}$ on a pass band. Taking half the trace gives

$$\cos(k_B a) = \cos\phi_1 \cos\phi_2 - \frac{1}{2}\left(\frac{n_1}{n_2} + \frac{n_2}{n_1}\right)\sin\phi_1 \sin\phi_2.$$

When the right side lies between –1 and 1 a real Bloch phase exists. If its magnitude exceeds one, a decaying and a growing Bloch factor replace a pair of unit-modulus factors. For a bounded excitation of a semi-infinite stack choose the decaying direction. The attenuation per cell is $\operatorname{arcosh}\left|\operatorname{trace}(M)/2\right|$. A lossless semi-infinite crystal in a stop band reflects incident power. Spatial attenuation of a Bloch amplitude in this region is compatible with zero volumetric heat production.

Obtain a quantitative stop-band width

Choose $n_1 d_1 = n_2 d_2 = \lambda_0/4$. Writing $s = \omega/\omega_0$, both layer phases are $\phi = \pi s/2$. At $s = 1$ the half trace is $-\frac{1}{2}(n_2/n_1 + n_1/n_2)$. Recognising $\cosh(\log r) = (r + r^{-1})/2$ gives centre attenuation $|\log(n_2/n_1)|$ per cell. For $n_1 = 1.5, n_2 = 2.0$ this is 0.2877, so a decaying eigencomponent has

amplitude factor 0.75 per cell. This statement about an eigencomponent should not be confused with the exact transmittance of a specific finite stack, which also depends on entrance and exit impedances.

The first gap edges obey half trace –1. With $\cos^2\phi = 1 - \sin^2\phi$, solve for the edge sine or cosine. The two edge phases are symmetric about $\pi/2$ and satisfy

$$|\cos\phi| = |n_2 - n_1|/(n_1 + n_2).$$

Hence the fractional angular-frequency width is $(4/\pi)\arcsin(|n_2 - n_1|/(n_1 + n_2))$, about 0.183 for the example. The lowest gap width tends continuously to zero as index contrast vanishes. A gap in a plotted numerical spectrum at $n_1 = n_2$ would indicate an implementation or plotting problem.

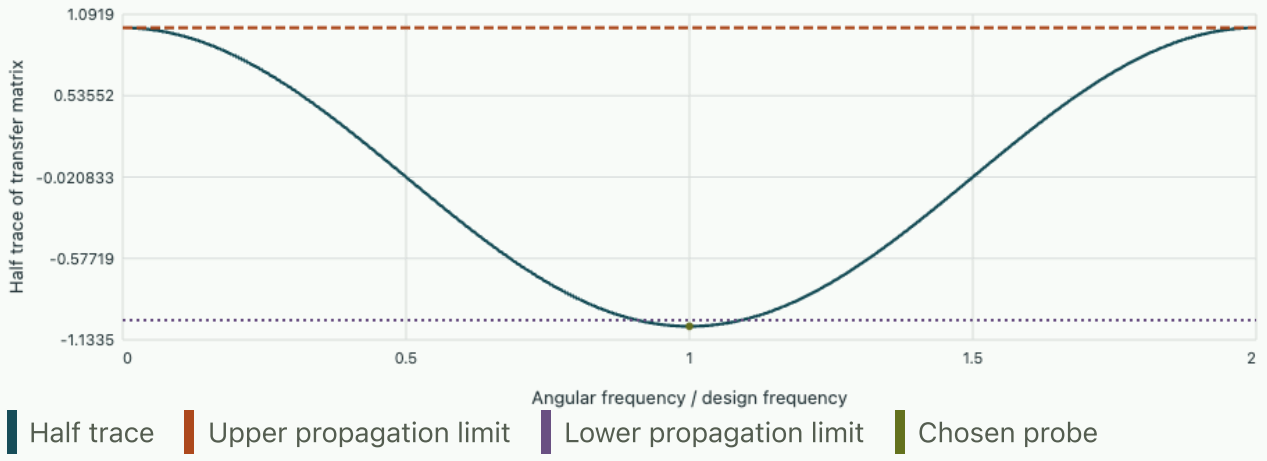
Compare a frequency scan with a finite mirror

Open the live experiment

Try the calculation

High index n_2 (n_1 = 1.5) 2

Probe frequency / design frequency 1



First gap width / design frequency 0.18252. Probe half trace -1.0417. Attenuation per cell 0.28768.

The graph plots half the transfer-matrix trace against normalised frequency and marks the admissible interval for a real Bloch phase. Change the index contrast first, keeping the layers quarter wave at the same reference frequency. The physical layer thickness adjusts in this family; it is a different experiment from changing a material while manufacturing all thicknesses fixed. Use the frequency control to read an attenuation exponent at the centre and just outside the gap.

At low frequency a wavelength spans many cells and a homogenised description can predict the long-wave speed. At the first gap the field changes phase by order one across a cell, exactly where the homogenisation scale separation fails. Conversely, an electromagnetic calculation showing multiple

bands at very low frequency should be checked against a uniform-medium or long-wavelength effective-index limit. Compare a finite stack's reflection with the Bragg-mirror experiment. The bulk trace determines whether a propagating state can exist. A finite sample adds interference between boundaries and possible surface or defect effects.

Tutorial

If the two refractive indices become equal, simplify the trace formula for arbitrary thicknesses.

▼ **Work through the solution**

The impedance factor becomes one, so the half trace is $\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2 = \cos(\phi_1+\phi_2)$. It never exceeds unit magnitude. Labelling a uniform medium by an artificial period folds the dispersion into zones but does not create a physical gap.

At the central quarter-wave frequency for index ratio 2, how many periods reduce a decaying Bloch component's amplitude by a factor of 100?

▼ **Work through the solution**

The amplitude factor per cell is

$$e^{-\log 2} = 1/2.$$

Require $2^{-N} \le 0.01$, giving $N \ge 7$. A prediction for transmitted optical power also needs termination and impedance matching.

A measured stack exhibits both a transmission minimum and measurable heating. Can the minimum be attributed entirely to a band gap?

▼ **Work through the solution**

A lossless band gap can reflect power without heating. Measure reflected and transmitted flux under consistent normalisation, then infer absorption from $1 - R - T$ after accounting for diffuse scattering and other exit channels. The bulk stop-band model and material absorption can coexist.

Compare with band calculations

MIT's mathematical nanophotonics course computes full periodic Maxwell modes. The one-dimensional trace above is a small benchmark with known pass and stop boundaries that a numerical eigenproblem should reproduce.

Fields and the response of matter · 7. Why rotating a layered dielectric changes a capacitance measurement

Derive the two principal permittivities of a fine laminate, rotate the constitutive tensor and distinguish the applied electrode condition from an assumed field direction.

Compare the same material in two orientations

Build a specimen by repeating layers with scalar permittivities $\epsilon_1, \epsilon_2 > 0$, occupying fractions f and $1 - f$. The period is much smaller than both the specimen scale and the spatial variation of the applied field. Suppose both materials are linear, free of mobile charge and effectively lossless in the measurement band. An observer using a large capacitor wants one law relating averaged displacement and electric field. A scalar average cannot describe both arrangements of the same layers relative to the plates.

Place the field tangential to all internal interfaces first. A narrow rectangular electrostatic loop straddling an interface gives continuity of tangential E. Both layers see the same tangential field, but their displacements differ in proportion to permittivity. Averaging gives

$$\overline{D}_t = [f\epsilon_1 + (1 - f)\epsilon_2]\overline{E}_t.$$

The tangent-direction effective permittivity is an arithmetic mean. This is a parallel connection of capacitances at shared voltage in the special capacitor geometry.

Apply a purely normal displacement instead. No free charge lies at the internal interfaces, so their normal displacement is continuous. Normal fields differ as $E_i = D_n/\epsilon_i$. Averaging the voltage drops gives

$$\overline{E}_n = [f/\epsilon_1 + (1 - f)/\epsilon_2]\overline{D}_n,$$

with a harmonic-mean effective permittivity. The local continuity condition changes which field is shared between constituents and thereby changes the relevant averaging operation.

Check the amount of anisotropy from field energy

For $\epsilon_1 = 2\epsilon_0, \epsilon_2 = 10\epsilon_0$ at equal fractions,

$$\epsilon_t = 6\epsilon_0,$$
$$\epsilon_n = (10/3)\epsilon_0.$$

The arithmetic-to-harmonic ratio is 1.8. A constituent does not need to be crystalline or anisotropic for the composite to show distinct principal responses. The geometry of the internal boundaries is enough.

Average the energy density for the tangential experiment,

$$u = f\epsilon_1 E_t^2/2 + (1 - f)\epsilon_2 E_t^2/2 = \epsilon_t E_t^2/2.$$

For the normal experiment,

$$u = D_n^2[f/\epsilon_1 + (1-f)/\epsilon_2]/2 = D_n^2/(2\epsilon_n).$$

These expressions supply a check independent of the capacitor-circuit analogy and identify which energy should be used when field or free charge is controlled. Energy positivity follows directly from the positive constituent permittivities. It does not require guessing an effective law from the largest measured component.

Rotate the law, then specify what the electrodes control

Use principal axes $\hat{n}, \hat{t}$ and suppose a macro-field makes angle θ to the layer normal. If $\overline{\mathbf{E}} = E(\cos\theta\,\hat{n} + \sin\theta\,\hat{t})$, then $\overline{\mathbf{D}} = E(\epsilon_n \cos\theta\,\hat{n} + \epsilon_t \sin\theta\,\hat{t})$. Project along the imposed field and divide by E:

$$\epsilon_E(\theta) = \epsilon_n \cos^2\theta + \epsilon_t \sin^2\theta.$$

The perpendicular displacement is $E(\epsilon_t - \epsilon_n)\sin\theta\cos\theta$. It is generally nonzero. A device that instead fixes the direction of averaged displacement while allowing a transverse field measures $\epsilon_D(\theta) = [\cos^2\theta/\epsilon_n + \sin^2\theta/\epsilon_t]^{-1}$. These two directional responses correspond to different constraints. They agree at principal axes and for isotropic material. A pair of unspecified “angled capacitor” measurements cannot be interpreted uniquely without describing the contacts and edge conditions.

At 45° in the example, the field-controlled projection gives about $4.667\epsilon_0$, while the displacement-controlled projection gives $4.286\epsilon_0$. These are both legitimate scalar ratios from one symmetric positive-definite tensor. Neither supplies a universal scalar permittivity for the rotated material. The complete vector response also predicts displacement tilt and field energy for any orientation.

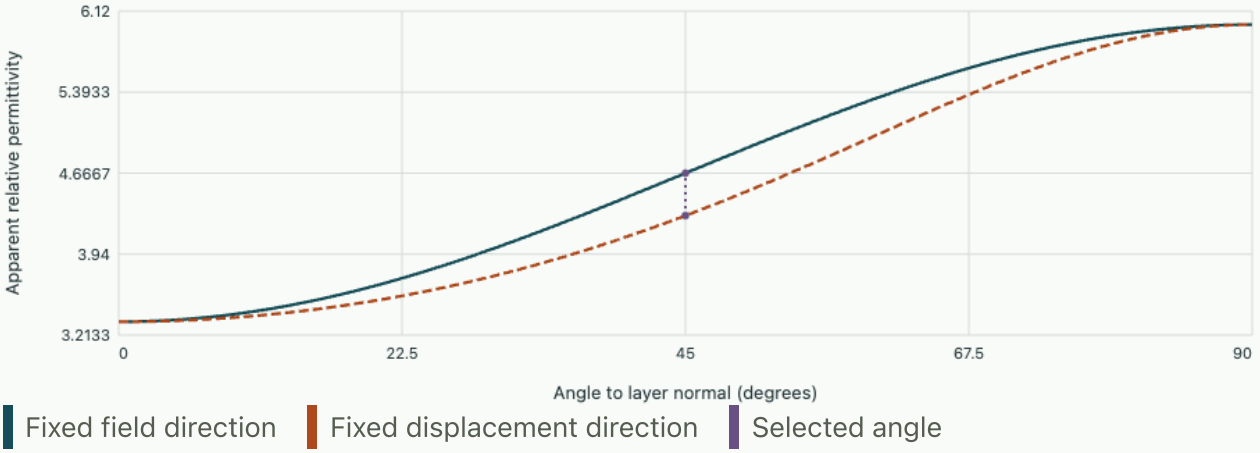
Test the measurement against the unit-cell model

Open the live experiment

Try the calculation

Fraction of low-permittivity phase **0.5**

Measurement angle (degrees) **45**



Fixed field direction Fixed displacement direction Selected angle

Principal relative permittivities: normal 3.3333, tangential 6. At 45° the two responses are 4.6667 and 4.2857.

Change the volume fraction while keeping the constituent values fixed. At a pure phase all orientations and both control conditions must collapse to a single value. At intermediate fractions, find the angle where the transverse component would be largest. It occurs where $|\sin\theta\cos\theta|$ is largest, but its magnitude also reflects the material contrast. Now reduce the contrast mentally to one and predict all curves without recomputing a single cell.

A numerical field model offers an independent homogenisation experiment. Use periodic tiling in the material directions and impose a macroscopic voltage difference plus periodic fluctuations on opposite faces. For a unit imposed field along each coordinate direction, average the resulting displacement over the cell. The resulting vectors supply the tensor columns. A grounded box on every side is a different boundary problem and can suppress the tangential response. Increase the number of material periods at fixed specimen scale and refine the individual period separately. These two limits test homogenisation and numerical representation, respectively.

How far can the effective law travel?

Electrostatic homogenisation assumes the cell fields respond quasistatically. In a wave experiment near the Bragg gap, phase changes substantially across one period and a local constant tensor is inadequate. A finely layered capacitor and a dielectric mirror can be made from the same constituents while requiring different models at different frequencies. Test the Bloch spectrum to identify the regime where layering begins selecting propagation bands. A low-frequency effective index alone cannot predict that gap.

Large spatial contrast also places a practical demand on mesh and interface handling. A normal gradient can jump strongly even when potential and displacement are continuous. A smooth-looking interpolated

voltage surface is weak evidence for an accurate energy density. Compare constituent fields, total flux and energy consistency before accepting the extracted effective tensor.

Tutorial

Show that $\epsilon_t - \epsilon_n$ is nonnegative for every positive pair of constituents without appealing to a named mean inequality.

▼ Work through the solution

Put both terms over the positive denominator $f\epsilon_2 + (1-f)\epsilon_1$. The numerator after cancellation is $f(1-f)(\epsilon_1 - \epsilon_2)^2$, so the difference is nonnegative and vanishes for identical materials or a pure phase.

Equal fractions give measured relative values 6 tangentially and $10/3$ normally. Can you infer both constituents if the laminate assumptions hold?

▼ Work through the solution

Their sum is 12. The harmonic mean is $2ab/(a+b) = 10/3$, giving product 20. The positive roots of $x^2 - 12x + 20 = 0$ are 2 and 10. The data do not identify which label belongs to which phase without another observation.

Why can a constituent occupy very little volume yet strongly affect the normal response?

▼ Work through the solution

Every normal field line in this lamination crosses each layer. The inverse response weights f/ϵ; even a thin, low-permittivity layer can contribute a large voltage drop. A field along the layers distributes displacement differently and may be much less sensitive to that same phase.

Fields and the response of matter · 8. How a magnetic field rotates the polarisation of transmitted light

Resolve a linearly polarised wave into circular eigenmodes, infer a refractive-index difference from an analyser and distinguish nonreciprocal rotation from a simple change of reference axes.

Derive rotation from propagation rather than drawing an arrow

Let monochromatic light propagate along $+z$ through a transparent magneto-optic sample, with an externally prepared axial magnetic field. Define fixed laboratory vectors $\mathbf{e}_\pm = (\hat{x} \pm i\hat{y})/\sqrt{2}$ and use phasors $e^{i(kz-\omega t)}$. These definitions specify the handedness algebraically. Calling a vector simply "right circular" without stating the viewing and time conventions can reverse a reported rotation sign.

An input linearly polarised in x is $(\mathbf{e}_+ + \mathbf{e}_-)/\sqrt{2}$. Suppose the medium supports wave numbers $k_\pm = 2\pi n_\pm/\lambda_0$ for the two bases, with negligible absorption. After a distance L, factor out the common phase $e^{i(k_+ + k_-)L/2}$. The remaining Cartesian phasor is $\hat{x}\cos\Delta - \hat{y}\sin\Delta$, where $\Delta = (k_+ - k_-)L/2$. Thus the polarisation angle in the fixed x, y plane is

$$\theta = \frac{\pi L}{\lambda_0}(n_- - n_+).$$

Equal transmissions of the two components preserve a linearly polarised output. Unequal absorption makes the circular amplitudes unequal and generally produces an ellipse. Rotating a linear analyser then finds a finite minimum, even for a perfectly coherent field. Residual intensity at the nominal extinction angle can also arise from stray light or imperfect polariser extinction, which must be measured independently.

Derive a magneto-optic eigenbasis

For axial magnetisation, a simple lossless transverse relative-permittivity block is $\begin{pmatrix} \epsilon & ig \\ -ig & \epsilon \end{pmatrix}$ with real g at the frequency of observation. Acting on $(1, i)$ gives $(\epsilon - g)(1, i)$; acting on $(1, -i)$ gives $(\epsilon + g)(1, -i)$. For relative permeability near one, $n_+^2 = \epsilon - g$, $n_-^2 = \epsilon + g$. At small g their difference $n_- - n_+ \simeq g/\sqrt{\epsilon}$. Reversing magnetisation reverses the sign of the off-diagonal coefficient and interchanges the two eigenvalues. Both the existence of circular modes and the sign-reversal test now follow from a constitutive model.

If $n_- - n_+$ is proportional to magnetic field over a specified range, write $\theta = VBL$, with Verdet constant V in rad/(T m). This is a measurable material and wavelength dependent coefficient. It is not determined by the mean refractive index alone. A strong nearby absorption line can give substantial frequency dependence and dichroism, invalidating a lossless narrow-band extrapolation.

Work through an analyser calibration

For $\lambda_0 = 600$ nm, $L = 0.20$ mm and measured index difference 5.0×10^{-4}, the rotation is $\pi(2.0 \times 10^{-4})(5.0 \times 10^{-4})/(6.0 \times 10^{-7}) = \pi/6$, or 30°. An ideal analyser at angle α transmits

$I/I_{\mathrm{in}} = \cos^2(\alpha - \theta)$. Its maxima occur at 30° and 210°; an intensity measurement determines a linear polarisation angle modulo 180°. For a sample thick enough to rotate several half turns, an intensity maximum alone cannot give an unwrapped θ. Vary field or path length continuously from a known reference.

If the applied field is independently 0.40 T in that example, $V = \theta/(BL) \simeq 6.54 \times 10^3$ rad/(T m). The arithmetic is dimensionally correct, but the large value warrants a physical check of thickness, field calibration, units and material spectrum before reporting it as an ordinary glass coefficient. Degrees substituted for radians would make the inferred material coefficient too large by $180/\pi$.

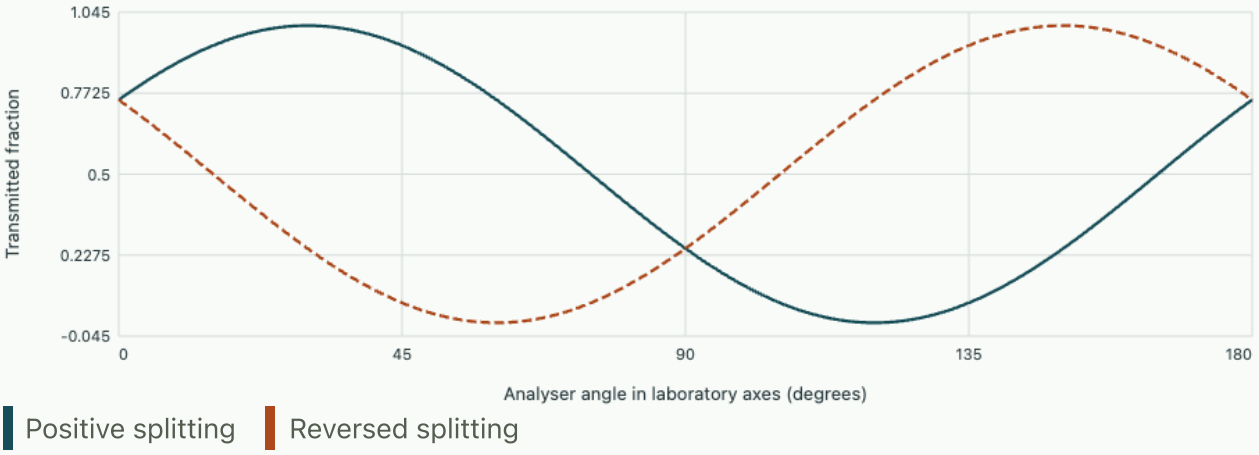
Use field reversal to reject alignment errors

Open the live experiment

Try the calculation

Index difference (10^{-4}) **5**

Sample length (mm) **0.2**



Unwrapped single-pass rotation 30°. At a 45° analyser the two transmitted fractions are 0.93301 and 0.066987.

The experiment shows the predicted analyser traces for opposite index splitting, with the same input polariser. Increase sample thickness to see why tracking the maximum by one measurement can select the wrong branch. Reversing the splitting reverses the tilt in fixed laboratory axes. A fixed analyser offset shifts both inferred angles equally and cancels in half their difference, provided the branch has been unwrapped consistently.

For small rotation, an analyser exactly at a crossed position transmits intensity proportional to θ^2. Its response has no sign sensitivity close to zero. Set the analyser at 45° instead:

$$\cos^2(\pi/4 - \theta) = [1 + \sin(2\theta)]/2 \simeq 1/2 + \theta.$$

The difference between fields $+B$ and $-B$ then measures a signed signal linear in small rotation. This trades extinction of the background for sensitivity to sign. Balance the two orthogonal analyser outputs if source intensity fluctuates appreciably during a sweep.

Ask what happens on a return pass

For the ideal Faraday medium at fixed magnetisation, the circular eigenmodes relative to a fixed laboratory basis are determined by its transverse material tensor. On propagation back through the sample, the accumulated positive propagation path again distinguishes the same laboratory circular basis after accounting for the reflection boundary. With a suitable simple mirror at normal incidence, the resulting material rotation doubles on a round trip in fixed transverse coordinates. In a reciprocal optically active material, the eigenvalue assignment with respect to the laboratory basis reverses on reversing propagation, and the single-pass rotation cancels on the ideal return. Write both transformations in one fixed basis before comparing signs.

A real double-pass setup includes mirror phase shifts, stress birefringence of windows and possible ellipticity. Reversing magnetic field, removing the sample and recording both analyser quadratures provide separate checks. A nonzero rotation in one transmitted beam proves only that its polarisation changed; establishing nonreciprocity requires comparing a reversed propagation experiment with a stated polarisation convention.

Tutorial

Two analyser maxima occur at 47° and 39° for opposite weak fields. If the response is linear and the same branch is tracked, what field-induced rotation and static offset do they imply?

▼ **Work through the solution**

Half the difference gives 4 degrees of signed rotation for the first field relative to zero. The mean is 43 degrees, the combined input polarisation and analyser zero. Conversion to a Verdet constant additionally needs the applied field and path length.

In the 600 nm example, what thickness uncertainty corresponds to 1 degree of rotation uncertainty at fixed index difference?

▼ **Work through the solution**

Rotation is proportional to thickness. Relative uncertainty $1/30$ therefore gives about $0.20/30 = 0.0067$ mm, or 6.7 µm. Uncertainty in wavelength or field would add additional contributions.

An ideal analyser gives a minimum 4% of its maximum. Can this be explained solely by a rotation of an ideal linearly polarised monochromatic input in a lossless circularly birefringent sample?

▼ **Work through the solution**

Pure rotation preserves a linear state, so an ideal analyser should extinguish it completely. Differential loss, ellipticity, depolarisation, finite bandwidth or a detector background are possible alternatives. Use circular-analysis or full Stokes measurements and an empty-path baseline to distinguish them.

Compare with a full polarisation measurement

Use the Stokes reconstruction experiment to determine which intensities reveal a circular component. A fitted linear rotation alone cannot measure that missing component.

Fields and the response of matter · 9. Can a flat interface guide light without mirrors?

Send light along a plane boundary between a dielectric and a metal. Seek a wave that propagates parallel to the interface while decaying away on both sides. The signs in the magnetic and electric boundary conditions determine whether any such wave can exist.

Decide what confinement requires

Put a homogeneous dielectric with positive relative permittivity ϵ_d above $z = 0$, a lossless dispersive metal of relative permittivity $\epsilon_m(\omega)$ below it and take both permeabilities to be μ_0. Use phasors with $e^{i\beta x - i\omega t}$. For the transverse magnetic polarisation write $H_y = H_0 e^{-\kappa_d z}$ in the dielectric and $H_y = H_0 e^{+\kappa_m z}$ in the metal. Both κ must be positive real numbers for this bound, lossless mode. Substitution into the homogeneous wave equation gives

$$\kappa_j^2 = \beta^2 - \epsilon_j k_0^2,$$

$$k_0 = \omega/c.$$

Continuity of tangential magnetic field without an externally imposed surface current identifies the two amplitudes H_0. The metal's mobile charges are already included in its bulk permittivity. Introducing the same current a second time as an arbitrary sheet would describe another boundary. Continuity of normal electric displacement follows here from the same magnetic amplitude and common tangential wavenumber.

The sign that selects a material pair

Use $\nabla \times \mathbf{H} = -i\omega\epsilon_0\epsilon_j \mathbf{E}$. Its x component is $-\partial_z H_y$. Hence $E_x = -i\partial_z H_y/(\omega\epsilon_0\epsilon_j)$. Continuity of this tangential electric field gives

$$\kappa_d/\epsilon_d = -\kappa_m/\epsilon_m.$$

No two positive real permittivities can satisfy that equation with two positive decay constants. Squaring and eliminating κ_j gives a candidate propagation constant.

$$\beta^2 = k_0^2 \frac{\epsilon_m \epsilon_d}{\epsilon_m + \epsilon_d}.$$

Check it in the unsquared equation. With positive dielectric permittivity the physical branch requires $\epsilon_m < -\epsilon_d$. On this branch $\beta > \sqrt{\epsilon_d}\, k_0$, so the mode lies outside the propagation cone of a plane wave in the dielectric. For transverse electric polarisation the corresponding condition replaces permittivities by permeabilities. With two ordinary positive permeabilities the sign requirement fails. The wave exists because of a coupled electric field and charge response at a surface.

Work an example before using the dispersion diagram

Let $\epsilon_d = 1$, $\epsilon_m = -4$ at the chosen frequency. Then $\beta/k_0 = \sqrt{4/3} = 1.155$,

$$\kappa_d/k_0 = 1/\sqrt{3},$$

$$\kappa_m/k_0 = 4/\sqrt{3}.$$

At vacuum wavelength 600 nm the electric and magnetic field amplitudes in the dielectric fall by a factor e after 165 nm. The corresponding distance in the metal is 41 nm. An intensity measurement falls by a factor e at half these distances. A wave sent directly through air has a parallel wavevector no larger than k_0. An index-1.50 prism could match the parallel component at $\theta = \arcsin(1.155/1.50)$, about 50.4 degrees. A practical evanescent coupler also needs finite overlap with the surface field.

Identify the surface resonance without assuming an infinite measured signal

Take the lossless Drude law $\epsilon_m = 1 - \omega_p^2/\omega^2$. The guided branch ends at $\omega_{\mathrm{sp}} = \omega_p/\sqrt{1+\epsilon_d}$, where $\epsilon_m + \epsilon_d$ tends to zero. Approaching this frequency from below increases β at finite ω. In the electrostatic limit the restoring electric field and inertia of mobile charge select the limiting frequency. At small ω, by contrast, β approaches the dielectric light line and the wave spreads far into the dielectric. A measured resonance has a finite width. Collisions, surface roughness, microscopic nonlocal response and the optical coupling geometry matter before any divergence could be observed.

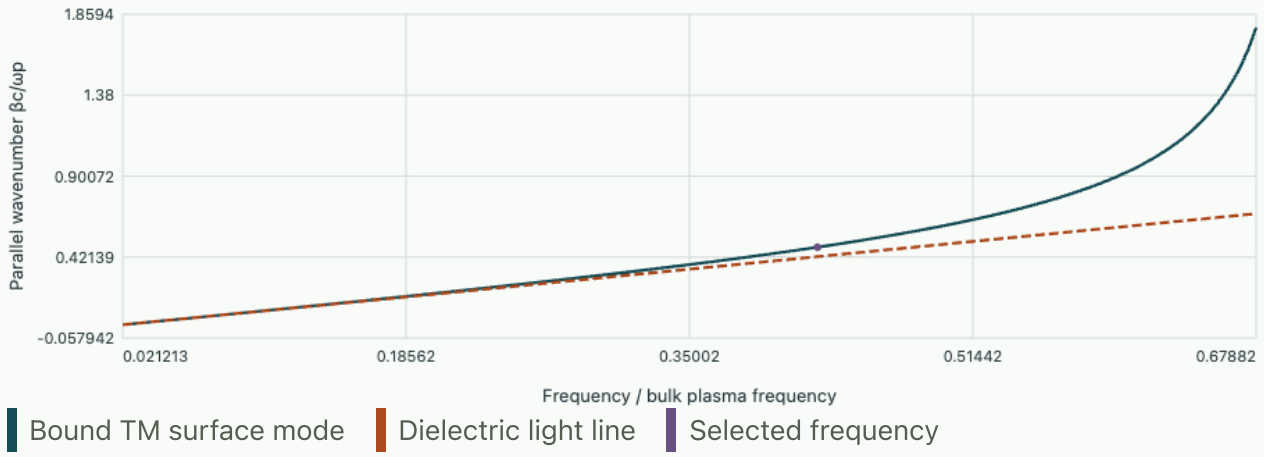
Change the dielectric permittivity. Predict separately the light-line slope and the frequency approached by the surface mode. The plot shows the propagation and dielectric light lines against frequency in units of the bulk plasma frequency, with a marker at the chosen fraction of the surface resonance.

Open the live experiment

Try the calculation

Dielectric relative permittivity **1**

Selected frequency / surface resonance **0.6**



Surface resonance ωsp/ωp = 0.70711. At the selected frequency β/k₀ = 1.1319.

Tutorial

If the dielectric permittivity becomes 2.25 with the same ideal metal, what bulk-normalised frequency does the surface mode approach? Does an air beam now have enough tangential momentum near the resonance?

▼ **Work through the solution**

The asymptote moves from $1/\sqrt{2} = 0.7071$ to $1/\sqrt{3.25} = 0.5547$. The parallel wavenumber exceeds the dielectric light line $1.5k_0$, which already exceeds the air limit. Direct incidence from air cannot provide the necessary parallel phase gradient.

Two near-field probes lie 120 nm apart normal to an interface at the 600 nm example. Their measured intensities have ratio 0.48. Is that consistent with sampling only the bound mode in the dielectric?

▼ **Work through the solution**

Its predicted intensity ratio is

$$\exp[-2(120/165)] = 0.234.$$

A ratio near 0.48 is close to the amplitude ratio. Check whether the detector is actually a linear heterodyne field measurement. Background light would also flatten an intensity decay.

Suppose a calculation squares the boundary equation and reports a propagating surface wave for two positive dielectrics. How can it be rejected without repeating the whole derivation?

▼ **Work through the solution**

Compute both positive normal decay constants and substitute them into $\kappa_d/\epsilon_d + \kappa_m/\epsilon_m = 0$. The sum cannot vanish. Even a real positive β^2 alone does not establish decay plus the correct boundary sign.

Compare a single-interface solution with the two-interface standing-wave condition in the dielectric slab chapter.

Fields and the response of matter · 10. Infer charge discreteness from fluctuations in a steady photocurrent

A detector can deliver a perfectly steady mean current over a long interval while its charge per short exposure fluctuates. Derive both a counting prediction and its spectral version, then use a bandwidth-limited measurement to separate partition fluctuations from thermal and readout noise.

Start with what the counter actually reports

Let statistically independent detection events bring charge q at mean rate ν. Count events during a rectangular exposure of duration T. Subdivide it into small bins of length δt in which a single event has probability $\nu\delta t$ and two events have negligible probability compared with δt. Independent Bernoulli bins have additive means and variances. Taking δt towards zero while T is fixed gives equal count mean and variance, both νT. The integrated charge is $Q = qN$, and the indicated average current is Q/T.

$$I = q\nu,$$

$$\operatorname{var}(Q) = qIT,$$

$$\operatorname{var}(\bar{I}_T) = qI/T.$$

The relative counting uncertainty is $1/\sqrt{\nu T}$, which falls with exposure. The variance of the integrated charge itself grows with exposure. Confusing the sum and its average reverses the predicted trend with integration time.

Derive the spectral density from a known measurement window

Define the one-sided current spectrum $S_I(f)$ for $f > 0$, so that current variance at an instrument output equals its integral weighted by the squared gain. Averaging over the rectangular exposure uses impulse response $h(t) = 1/T$ between 0 and T, zero elsewhere. Its Fourier amplitude is a sinc times a phase factor. Parseval's identity, applied to the real impulse response, gives the one-sided equivalent noise bandwidth $\int_0^\infty |H(f)|^2 df = 1/(2T)$. Equate the counting result to $S_I/(2T)$.

$$S_I = 2qI,$$

$$[S_I] = \mathrm{A}^2/\mathrm{Hz}.$$

The factor of two belongs to the one-sided convention. The square root is an amplitude spectral density, measured in $\mathrm{A}/\sqrt{\mathrm{Hz}}$. A one-pole filter with −3 dB frequency f_c instead has gain squared $1/[1 + (f/f_c)^2]$. Its equivalent noise bandwidth is $\pi f_c/2$. Replacing that integral by f_c silently changes a noise prediction by a measurable factor.

Separate uncorrelated sources by changing the mean

A photodiode feeds an ideal current-measuring amplifier with a $1.0\ \mathrm{M\Omega}$ feedback resistor at 300 K. Assume photon arrivals have Poisson statistics, unity internal charge gain, negligible dark-current fluctuations and an otherwise ideal amplifier. Refer the resistor noise to the input as a parallel current

source, with density $4k_B T_\mathrm{R}/R$. Independent variances add. At $I = 0.80$ microampere and equivalent bandwidth $B = 10.0$ kHz the photocurrent gives 50.63 pA rms and the resistor contributes 12.87 pA. The total current noise is their quadrature sum, about 52.24 pA. Adding their rms amplitudes would predict 63.50 pA despite the assumed independence.

Keep the resistor temperature and filter fixed while sweeping the photocurrent. Plot measured variance divided by equivalent bandwidth against the actual mean current. The slope is $2q$ and the intercept measures the current-independent floor. In the more general model $\mathrm{var}(N) = F\langle N\rangle$ for long exposures, the low-frequency slope is $2qF$. A measured slope alone cannot separate charge from a Fano factor. Establishing independent electrons and unity gain is part of inferring the elementary charge.

Run the measurement with controlled non-Poisson counting

The graph plots variance across a current sweep, quoted as spectral density in $\mathrm{pA}^2/\mathrm{Hz}$. Vary the Fano factor and the readout resistance independently. Predict which operation affects the intercept. Read the current at which the signal fluctuations equal the readout fluctuations and check whether increasing the detection bandwidth changes this crossover for two white sources.

Open the live experiment

Try the calculation

Fano factor 1

Feedback resistance (MΩ) 1



The two noise contributions are equal at 51.704 nA. The zero-current density is 0.016568 pA²/Hz.

A detector impulse $i_p(t)$ with finite transit time has integrated area q. Poisson arrivals give a low-frequency plateau because the current pulses look like point charges when averaged slowly. At frequencies comparable with the inverse transit time the spectrum gains a factor $|\tilde{i}_p(f)/q|^2$. A curved

spectrum can arise from that single-event shape even with independent arrival times. Compare the curve with a calibrated impulse response before attributing it to correlations of separate events.

Tutorial

An exposure of 20 microseconds collects a mean of 200 electrons. Predict the standard deviation in electrons and the indicated mean current. What happens after averaging four independent exposures?

▼ **Work through the solution**

The standard deviation is $\sqrt{200} = 14.14$ electrons and the mean current is

$$200e/(20\,\mu\mathrm{s}) = 1.60\,\mathrm{pA}.$$

Averaging four independent readings halves the rms fluctuation in indicated current. Adding their raw counts gives a different observable with mean 800 and standard deviation 28.28.

For $R = 1\ \mathrm{M\Omega}$, $T = 300$ K and $F = 1$, at what current do resistor and shot noise have equal variance?

▼ **Work through the solution**

Equate $2eI$ and $4k_BT/R$ to obtain

$$I = 2k_BT/(eR) = 51.7\,\mathrm{nA}.$$

Both variances scale with a common equivalent noise bandwidth, so it cancels in this white-noise model.

A colleague multiplies a measured noise density by the optical carrier frequency to obtain the receiver noise. Which frequency interval belongs in this calculation?

▼ **Work through the solution**

Integrate the electrical noise spectrum over the measurement filter's squared gain. The optical field creates charge carriers, but the transimpedance circuit and estimator determine which fluctuations in their arrival rate reach the output. A carrier frequency and an electrical equivalent noise bandwidth have the same dimension and different operational meanings.

Use the RC noise experiment to derive the equilibrium resistor contribution independently.

Fields and the response of matter · 11. Why a nearby gate changes the speed law of a two-dimensional plasma

A current pulse excites a travelling density modulation in a thin conducting sheet. Move a metal gate towards the sheet and the shape of the dispersion curve changes. Rebuild the restoring electric field from the surface charge before using any oscillator analogy.

Solve the electrostatic problem for one Fourier mode

Place a two-dimensional electron sheet at $z = 0$ in a homogeneous dielectric of absolute permittivity ϵ. The equilibrium positive background is fixed. Let the electron number perturbation be $\delta n\, e^{iqx - i\omega t}$ with $q > 0$, so the perturbation of charge per area is $\sigma = -e\,\delta n$. Away from the sheet the potential solves $(\partial_z^2 - q^2)\phi = 0$. Decay on both sides selects $Ae^{-q|z|}$. Across zero, the normal field discontinuity gives

$$2\epsilon q A = \sigma.$$

Consequently the bare sheet has $\phi(0) = \sigma/(2\epsilon q)$. The inverse power of q reflects the field extending a distance of order $1/q$ away from the sheet.

Ground a parallel, perfectly conducting plane at $z = d$. Place an image of opposite surface charge at $2d$ to enforce zero potential at the conductor. The transverse exponential suppresses the contribution that travels from this image to the physical sheet. Linear superposition therefore gives

$$\phi(0) = \frac{-e\,\delta n}{2\epsilon q}\left(1 - e^{-2qd}\right).$$

This result assumes that charge is free to flow into the gate from a ground reservoir and that the dielectric is homogeneous throughout the gap. A floating finite plate would require an additional global charge condition. The zero mode also needs its own electrostatic reference and is excluded from this derivation.

Combine continuity with the acceleration equation

Writing u for the sheet electron velocity along x, number conservation gives

$$-i\omega\delta n + iqn_0 u = 0.$$

The electric force on a single electron is $-eE_x = +ieq\phi(0)$, and the acceleration equation is

$$-i\omega m^* u = ieq\phi(0).$$

Solving for u, substituting the potential and cancelling the common density perturbation gives

$$\omega^2(q) = \frac{n_0 e^2 q}{2\epsilon m^*}\left(1 - e^{-2qd}\right).$$

Check dimensions using n_0 as a number per square metre. A three-dimensional density in this formula would leave a spurious inverse length. In an ungated plane the frequency scales as $\sqrt{q}$. Its group velocity $d\omega/dq$ is half the phase velocity. When qd is small, the exponential has first-order difference $2qd$ and the field is largely confined in the gate gap. Then $\omega \simeq sq$ with $s = \sqrt{n_0 e^2 d/(\epsilon m^*)}$. The field and inertia now behave like a distributed capacitance and kinetic inductance. Both waves are collective charge oscillations even though one has the same long-wavelength dispersion exponent as sound.

Put numbers on the length-scale assumption

Use sheet density $2.0 \times 10^{15}\,\mathrm{m}^{-2}$, effective mass $0.067 m_e$ and relative permittivity 12.9. For a 5.0 micrometre density wavelength,

$$q = 2\pi/\lambda = 1.257 \times 10^{6}\,\mathrm{m}^{-1}.$$

A 50 nm gate separation gives $qd = 0.0628$. Taking a square root after evaluating the electrostatics gives 342.4 GHz without the gate and 117.6 GHz with it. The small-qd linear speed $6.07 \times 10^{5}\,\mathrm{m\,s}^{-1}$ would predict roughly 121.4 GHz. The few-percent difference can be seen well before the field has ceased to be substantially gated.

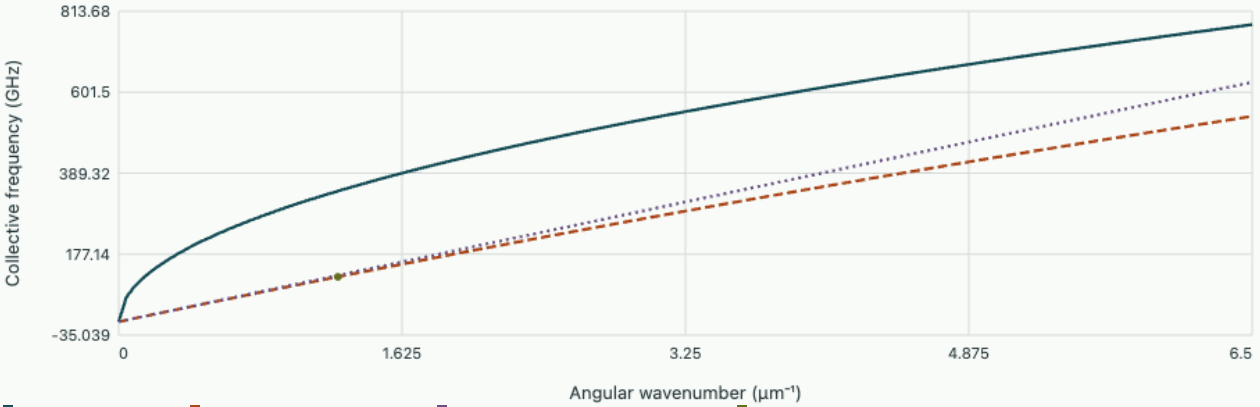
The experiment lets you vary the distance and the selected wavelength. Compare the full dispersion with its gate-screened tangent. The horizontal coordinate is inverse micrometres for the angular wavenumber. Predict where $qd = 1$ lies before interpreting a bend. Check the reported selected frequency by converting q to an ordinary wavelength with the 2π included.

Open the live experiment

Try the calculation

Gate separation (nm) 50

Selected wavelength (µm) 5



At the chosen wavelength qd = 0.062832. Gated mode 117.65 GHz. Free sheet 342.36 GHz.

Infer a parameter and test whether it is identifiable

A low-wavenumber velocity measurement determines $n_0 d/(\epsilon m^*)$. Doubling n_0 and halving d gives the same leading speed, so one such measurement cannot separate density and spacing. At sufficiently large qd, ω^2/q becomes independent of d and determines $n_0/(\epsilon m^*)$. Combining low and high wavenumber patterns can therefore identify d separately. In a real sheet, collisions, scattering, electron pressure and a nonlocal conductivity can invalidate the simple acceleration law. An inferred gate dimension is credible only if a model fitted in one regime predicts data in the other.

Tutorial

Derive the ratio of exact group to phase velocity from the dispersion and state both limiting values.

▼ Work through the solution

Differentiate $\ln \omega$ with respect to $\ln q$. The answer is

$$v_g/v_p = \tfrac{1}{2}[1 + 2qd/(\exp(2qd) - 1)].$$

It tends to one for a closely screened long wave and to one half when the gate is far from the evanescent field.

A measured long-wave speed increases by ten percent with unchanged density and permittivity. Under the ideal gate model, how much must the spacing have changed?

▼ Work through the solution

Since $s^2 \propto d$, the inferred spacing increases by $1.10^2 - 1 = 0.21$, or 21 percent. A simultaneous change of effective mass would invalidate this inference. Check the large-q coefficient independently.

Explain why an electromagnetic calculation of waves propagating only at $c/\sqrt{\epsilon_r}$ misses this mode, and why the present electrostatic derivation cannot be extrapolated to arbitrary phase speeds.

▼ Work through the solution

The moving surface charge supplies a dynamic constitutive response with its own inertia. A constant bulk permittivity alone omits that degree of freedom. Conversely our use of Laplace's equation neglects propagation delay. If the mode approaches the dielectric light line one must keep the retarded Maxwell fields and solve for both charge motion and electromagnetic propagation.

Compare the bulk charge-density experiment and the electrostatic image construction.

Fields and the response of matter · 12. Can a capacitance measurement reveal the fraction of embedded particles?

A sample contains well separated dielectric spheres in an insulating matrix. Its capacitance changes when the particle fraction changes. Derive a dilute estimate from one sphere and a self-consistent approximation from a local-field model. Use the inverse calculation to discover when the fraction becomes poorly determined.

What does a single particle actually contribute?

Let the host and sphere have positive, frequency-independent relative permittivities h and s and let each sphere have radius a. The imposed field near an isolated sphere is approximately uniform. The exterior potential is the sum of a uniform-field term and a dipole term proportional to $\cos\theta/r^2$; the regular interior solution is linear in $r\cos\theta$. Impose continuity of potential and of normal electric displacement at $r = a$. The internal field then equals $3hE_{\mathrm{loc}}/(s+2h)$. The induced dipole is its integrated excess polarisation relative to the host:

$$p = \frac{4\pi a^3}{3}\epsilon_0(s-h)E_{\mathrm{in}} = 4\pi\epsilon_0 h a^3 \frac{s-h}{s+2h}E_{\mathrm{loc}}.$$

A low-permittivity inclusion can have a dipole opposite to its local field. It has removed more polarisable host material. This sign need not imply any negative permittivity. Define the contrast $b = (s-h)/(s+2h)$. The denominator includes two components of the host contribution from the exterior field. Simply averaging permittivities as if every particle experienced the applied field would ignore this boundary calculation.

Leading density and a local-field closure

For number density n, the inclusion fraction is $f = 4\pi a^3 n/3$. To first order in f, replace E_{loc} by the macroscopic field and add the dipoles. The predicted effective relative permittivity is $h(1+3fb)$, up to terms of second and higher order. To obtain the Maxwell–Garnett approximation, imagine a spherical exclusion region containing a representative particle. Approximate the far distribution of particles by its smooth excess polarisation $P = np$. Removing the polarised sphere produces a local field increment $P/(3\epsilon_0 h)$ along P. Substituting $E_{\mathrm{loc}} = E + P/(3\epsilon_0 h)$ into $P = n\alpha E_{\mathrm{loc}}$ gives

$$\frac{\epsilon_{\mathrm{eff}} - h}{\epsilon_{\mathrm{eff}} + 2h} = fb,$$

$$\epsilon_{\mathrm{eff}} = h\,\frac{1+2fb}{1-fb}.$$

The closure resums some higher-order terms. It does not solve the exact many-particle boundary-value problem at arbitrary density. Short-range correlations, connected clusters, nonspherical particles and anisotropy can matter. Identifying a distinguished host and separated inclusions is an experimental assumption. Interchanging host and particle labels while preserving their volume fractions generally gives another microstructure and another prediction.

Infer a fraction from a measured capacitor

Take $h = 2$, $s = 8$ so $b = 0.5$. At a fraction of 0.15, Maxwell–Garnett gives an effective relative permittivity $2(1.15)/(0.925) = 2.4865$. The first-order estimate gives 2.45. For electrodes of area $100\,\mathrm{mm}^2$ separated by $0.5\,\mathrm{mm}$, and negligible edge fields, these numbers give 4.40 pF and 4.34 pF. A stray capacitance of 0.1 pF is larger than their difference. A regression reporting many fractional digits cannot decide between these approximations until the empty fixture and lead capacitances are known.

Invert the ratio before inserting a measured value. If $\epsilon_{\mathrm{eff}} = 2.50$ the inferred fraction is

$$(2.50 - 2)/[(2.50 + 4)0.5] = 0.154.$$

The derivative of inferred fraction with respect to the measured effective value is $3h/[b(\epsilon_{\mathrm{eff}} + 2h)^2]$. An absolute uncertainty of 0.03 on the effective permittivity corresponds to approximately 0.0085 on the fraction for this example. If s approaches h, b approaches zero and the same measurement loses sensitivity. Increasing the number of data points only helps random scatter. It cannot create constituent contrast.

Compare geometry-dependent predictions at the same volume fraction

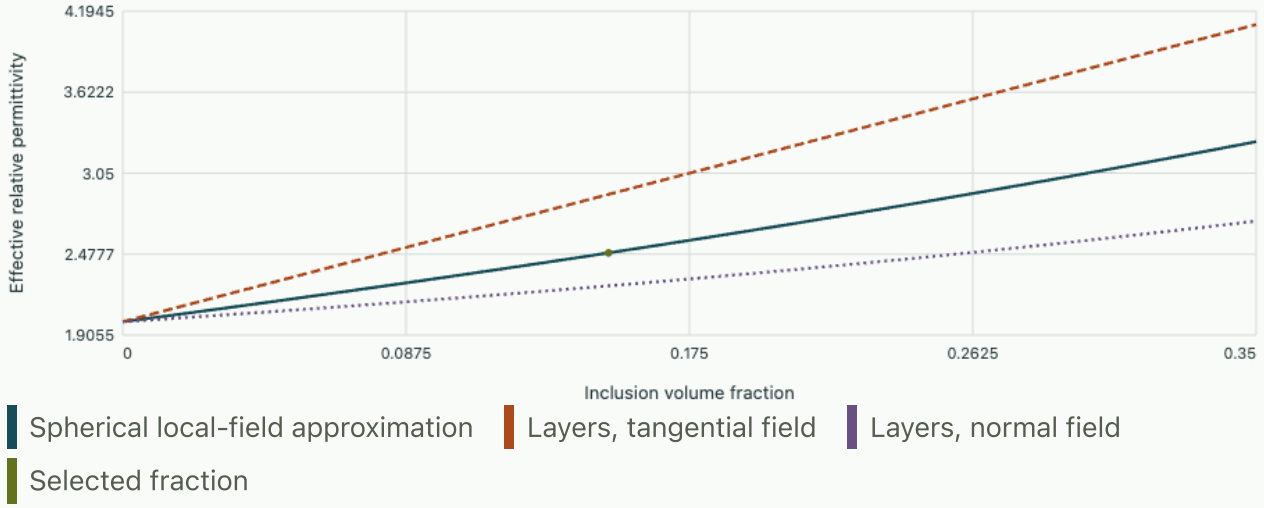
The graph uses a host permittivity of two and lets the particle permittivity and selected volume fraction vary independently. Compare the sphere estimate to the exact arithmetic and harmonic values for a laminate with fields parallel and perpendicular to its layers. All curves coincide for equal constituents and pure host. Separated spheres have no preferred direction under isotropic sampling, whereas a laminate does. Agreement of one measured scalar with a mixing formula cannot establish which geometry was present.

Open the live experiment

Try the calculation

Particle relative permittivity **8**

Selected volume fraction **0.15**



At the selected fraction εeff = 2.4865. The first-order dilute estimate gives 2.45.

Tutorial

If the particle radius is halved at fixed dilute volume fraction, what happens to the predicted effective permittivity? What unmodelled effect could reveal the smaller radius?

▼ **Work through the solution**

There are eight times more particles per volume and each ideal dipole has one eighth of its original polarizability. Their product is unchanged in quasistatic volume mixing. A finite-thickness interfacial layer occupies an increasingly large fraction as radius shrinks. Dynamic scattering at non-negligible size can also distinguish the samples.

What does the local-field formula predict for cavities of relative permittivity one embedded in a host of four at $f = 0.10$?

▼ **Work through the solution**

Here $b = (1 - 4)/(1 + 8) = -1/3$. The effective result is

$$4(1 - 2/30)/(1 + 1/30) = 3.613.$$

The sign and magnitude lie between the pure constituent values. A value larger than four from this lossless model would require rechecking the sign of the excess dipole.

A series of samples has the same measured capacitance in one direction. Propose a measurement that can separate isotropically dispersed spheres from a stack of layers.

▼ **Work through the solution**

Reorient the electrodes or the sample and measure normal and tangential components at fixed geometry and frequency. A laminate generally exhibits different arithmetic and harmonic responses. An isotropic distribution of dilute spheres has a scalar effective response. Repeat with a second fixture orientation to exclude direction-dependent stray capacitance.

Work through the single-particle calculation in the dielectric sphere chapter and compare the planar geometry in the layered material experiment. The University of Utah electromagnetic materials course gives the local-field construction.

Fields and the response of matter · 13. Can two ohmic layers imitate a relaxing dielectric?

Model a film and its contact layer in series. Derive the frequency scale of interfacial charge accumulation and decide which features of a measured spectrum can arise from geometry alone.

Match total current at a leaky interface

Take harmonic fields proportional to $e^{i\omega t}$. In layer j, Ohm's law and Maxwell's displacement current give $J = Y_j E_j$ with $Y_j = \sigma_j + i\omega\epsilon_j$. Use absolute permittivity in SI units. Total current is continuous across the interface. If the first layer occupies thickness fraction f, the imposed mean field is

$$\bar{E} = fE_1 + (1-f)E_2.$$

Eliminating the layer fields yields

$$Y_{\text{eff}} = \frac{Y_1 Y_2}{fY_2 + (1-f)Y_1},$$

$$\tau = \frac{f\epsilon_2 + (1-f)\epsilon_1}{f\sigma_2 + (1-f)\sigma_1}.$$

The denominator is proportional to $1 + i\omega\tau$. A pole has appeared even though each material has frequency-independent conductivity and permittivity. The free interfacial charge per unit area is $\epsilon_2 E_2 - \epsilon_1 E_1$ with the normal from layer 1 to layer 2. At low frequency the less conducting layer takes a larger field. At high frequency the displacement currents determine the field sharing.

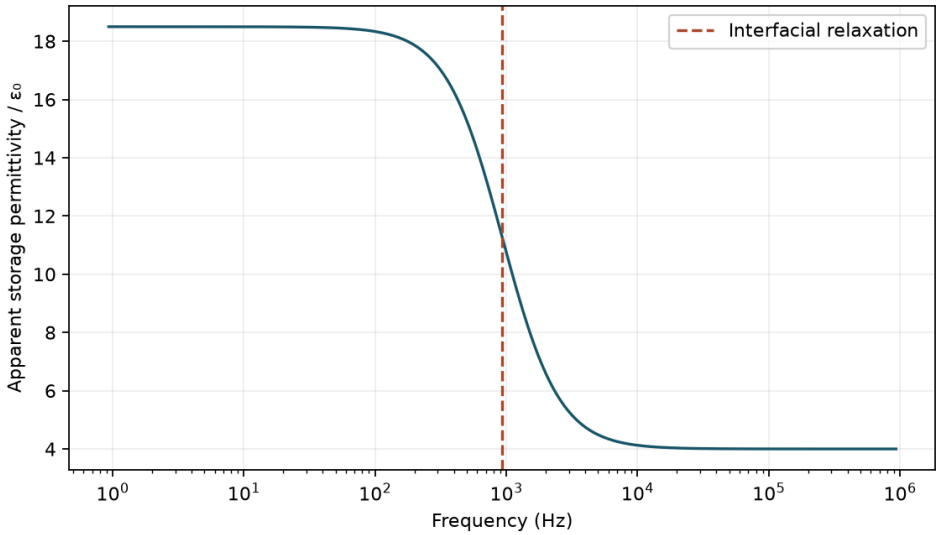
Separate dielectric storage from direct conduction

Infer the relative storage permittivity as $\mathrm{Im}(Y_{\text{eff}})/(\omega\epsilon_0)$. A DC current is real and drops out of this quantity. For layers with the same relative permittivity, their high-frequency effective value must equal that common value. In the DC limit the effective conductivity is the thickness-weighted harmonic mean. These two checks are independent. The interface can nevertheless produce a large rise in apparent storage on the low-frequency side. Use a first layer that fills 20% of the gap and has conductivity 10^{-8} S/m. Give the other layer conductivity 10^{-6} S/m and both layers relative permittivity 4. The DC conductivity is 4.81×10^{-8} S/m. An applied mean field of 1000 V/m therefore drives 4.81×10^{-5} A/m². The weakly conducting layer needs about 4808 V/m to carry it and the second layer only 48.1 V/m. Their thickness-weighted mean recovers the imposed field.

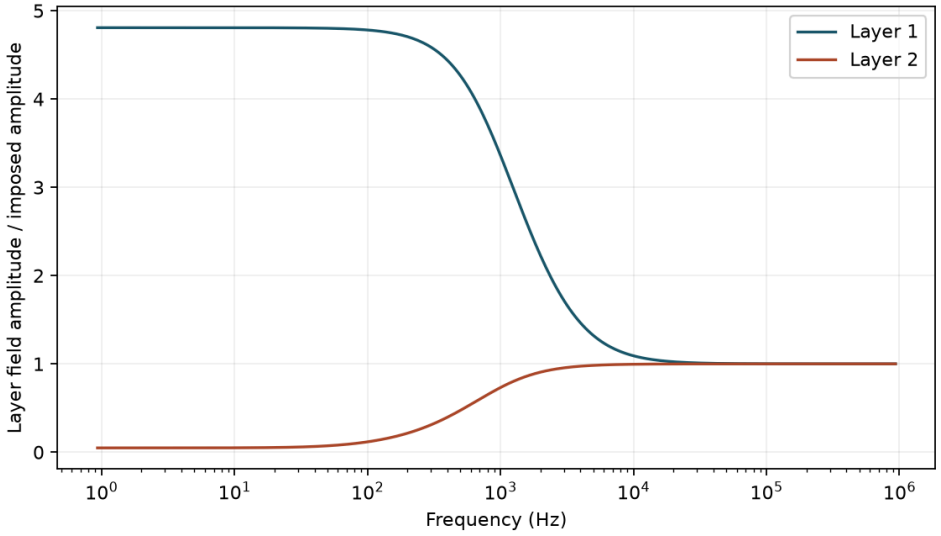
The time constant is 0.170 ms, corresponding to 935 Hz. Expanding the two complex conductivities to first order in frequency gives the low-frequency relative storage permittivity

$$4[f\sigma_2^2 + (1-f)\sigma_1^2]/[f\sigma_2 + (1-f)\sigma_1]^2 = 18.50.$$

At high frequency it falls to 4. Make the conductivities equal in the notebook and predict the interfacial charge before running it. The denominator still defines a time scale algebraically but its pole cancels out of the measured admittance.



Use the measured transition frequency and a thickness series to distinguish interfacial storage from an intrinsic molecular relaxation.



Each curve is an amplitude. Add the complex fields with their phases before checking that the thickness-weighted average equals the applied field.

Exercise · remove the relaxation

Which choice of layer properties makes the interfacial charge vanish for every frequency?

▼ Work through the solution

The charge vanishes if $\epsilon_2/Y_2 = \epsilon_1/Y_1$. The imaginary terms cancel identically after cross multiplication, leaving $\epsilon_2\sigma_1 = \epsilon_1\sigma_2$. The two dielectric relaxation times are then equal. Check the limit by making both layers identical.

Exercise · exchange the materials

Relabel the layers and their thickness fractions. Which measurements can distinguish the order of this stack?

▼ **Work through the solution**

The two-terminal admittance is unchanged under $(1, 2, f) \mapsto (2, 1, 1 - f)$. It cannot identify which material is nearer the upper electrode. A local field measurement at a known depth or a contact-specific measurement can distinguish the configurations.

Try it · Leaky layers

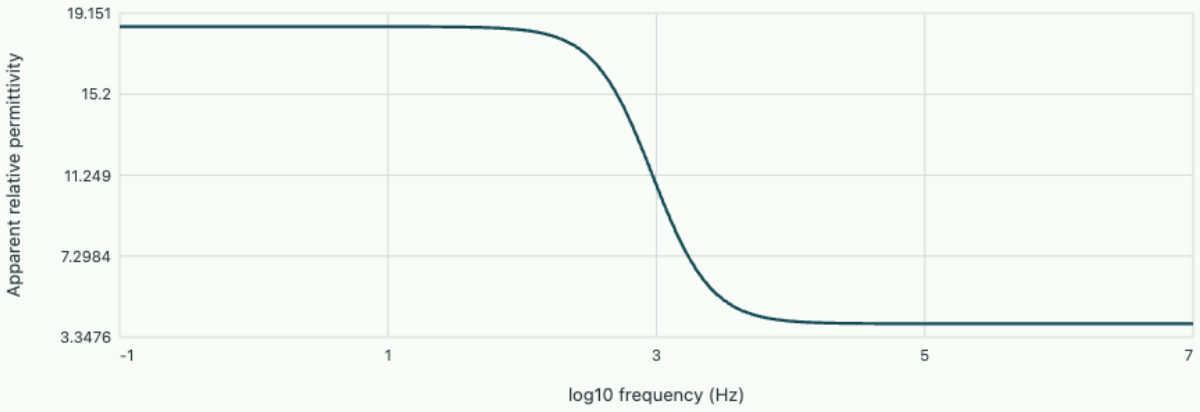
Change the inputs and compare the recalculated curves with the preceding derivation and baseline figure. The axes give the units; the numerical readout states the model assumptions and any limiting cases. Reset the inputs to return to the reference setting.

Open this live calculation

Try the calculation

Fraction occupied by layer 1 **0.2**

Conductivity ratio σ2/σ1 **100**



Effective storage permittivity

Both intrinsic relative permittivities equal 4. Interfacial relaxation frequency 934.71 Hz. A large low-frequency apparent permittivity can arise from conductivity contrast.

Download the corresponding editable notebook. The browser recalculates the reduced equations directly; the notebook exposes the full calculation and its checks.

Open editable notebook Download .ipynb

Fields and the response of matter · 14. From molecular dipoles to a dielectric susceptibility

Derive induced and orientational polarisation separately, then determine when temperature or local fields invalidate a constant dielectric coefficient.

Displace a bound charge before defining a material coefficient

Consider a neutral molecule whose negative charge cloud can move a small distance relative to its positive centre. Represent the restoring force along the applied field by a spring constant k and the displaced charge magnitude by q. Static balance gives kx=qE. The dipole moment is p=qx, so its proportionality coefficient is fixed by the restoring physics.

$$p = \alpha E,$$

$$\alpha = q^2/k.$$

The energy includes both deformation and the coupling to the external field. Minimising kx²/2−qxE gives the equilibrium displacement and the equilibrium energy −αE²/2. The factor one half follows from creating the moment while raising the field. A permanent, already established dipole instead has orientation energy −pE cosθ. The difference matters in a force gradient and in a thermodynamic partition sum.

> **Recall · Dipole moment, torque and field energy**
>
> Two charges ±q separated by vector d have p=qd directed toward positive charge. Expanding their potential energy in a slowly varying external potential gives U=p·∇V=−p·E. Vary the angle to obtain torque p×E. Vary position to obtain the gradient force. This expansion requires the charge separation to be small compared with the scale on which the external field changes.

For N independently responding molecules per unit volume, P=NαE if the local and macroscopic fields agree. Combining D=ε₀E+P with D=εE gives $\chi_e = N\alpha/\varepsilon_0$. Each equality has an assumption. A dense solid can change the local field, and a large displacement can change the spring constant.

Count orientations with the correct angular measure

Now take a freely rotating permanent dipole of fixed magnitude p in thermal equilibrium at temperature T. Put u=cosθ and x=pE/(kBT). A thin ring on the orientation sphere has area 2π sinθ dθ. Consequently equal intervals in u have equal orientation measure. Boltzmann weighting gives the normalised density below.

$$Z_u(x) = \int_{-1}^{1} e^{xu}\, du = \frac{2\sinh x}{x},$$

$$f(u) = \frac{e^{xu}}{Z_u(x)}.$$

Differentiate the integral with respect to x. The derivative inserts u in the numerator, so the alignment is the logarithmic derivative of Z. Differentiating a second time gives a variance and therefore a nonnegative differential response.

$$\langle u\rangle = \frac{d\log Z_u}{dx} = \coth x - \frac{1}{x} = L(x),$$

$$\frac{d\langle u\rangle}{dx} = \langle u^2\rangle - \langle u\rangle^2.$$

Series expansion of sinh and cosh gives L(x)=x/3−x³/45+… . Thus the orientational susceptibility of a dilute gas is Np²/(3ε₀kBT). At large x the alignment approaches 1−1/x, saturating at P=Np. A linear law cannot remain valid in that limit.

$$P = N\left(\alpha + \frac{p^2}{3k_BT}\right)E + O(E^3).$$

For p=3.0×10⁻³⁰ C m, T=300 K and E=10⁶ V/m, x=7.24×10⁻⁴. The mean orientation is only 2.41×10⁻⁴. A visibly nonzero macroscopic polarisation can therefore coexist with almost random individual molecular directions. Doubling E doubles the weak-field response, while doubling T halves its orientational part. The induced contribution in this elementary model stays fixed.

OneSimulation · When does the linear orientation law fail?

Try alignment and the angular measure

Change pE/(kBT) to concentrate the probability near the field direction, then lower the integration resolution. Equal intervals in u = cosθ represent equal solid-angle bands. The zero-field density is therefore 1/2 on this axis. The reported mean compares a trapezoidal integral with the exact Langevin law and its weak-field expansion.

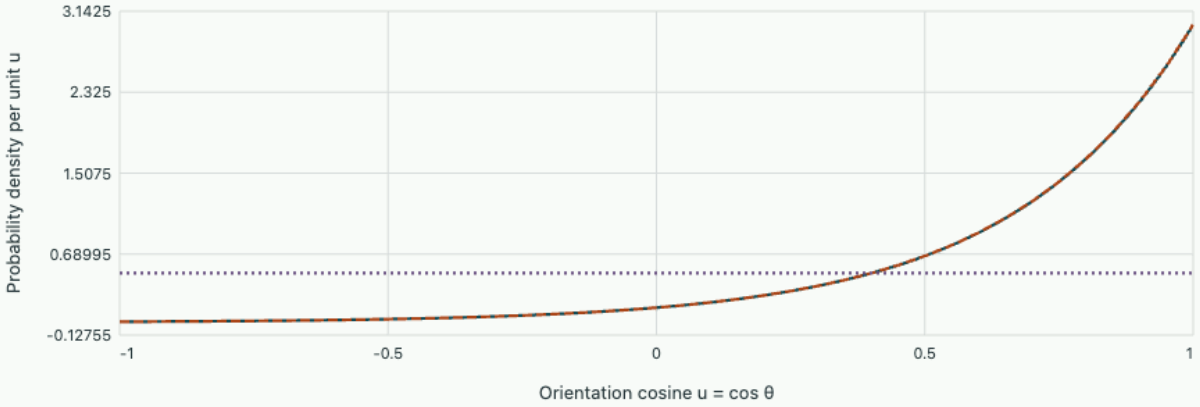
The browser uses a simple composite trapezoidal rule so that resolution error is visible. The OneSimulation notebook below obtains Gaussian quadrature nodes from an eigenproblem and provides an independent, higher-order calculation of the same integral. Increase `order` there and compare convergence at fixed field energy. The model assumes freely rotating, noninteracting permanent dipoles in thermal equilibrium.

Open this experiment online

Try the calculation

Dipole energy / thermal energy pE/(kBT) **3**

Trapezoidal integration intervals **40**



Boltzmann orientation density | Integration samples | Zero-field isotropic density

Exact mean cos θ = 0.671636; trapezoidal mean = 0.672883; weak-field estimate = 1. Absolute quadrature error = 1.25e-3. Density normalisation from this grid = 1.00187. The downloadable notebook independently uses Gaussian quadrature through OneSimulation.

Reproduce with OneSimulation

The code obtains Gaussian integration nodes and weights from a symmetric eigenproblem using OneSimulation. It then integrates the probability density derived above. The analytic Langevin law supplies an independent check. Run this locally with OneSimulation, NumPy and Matplotlib installed.

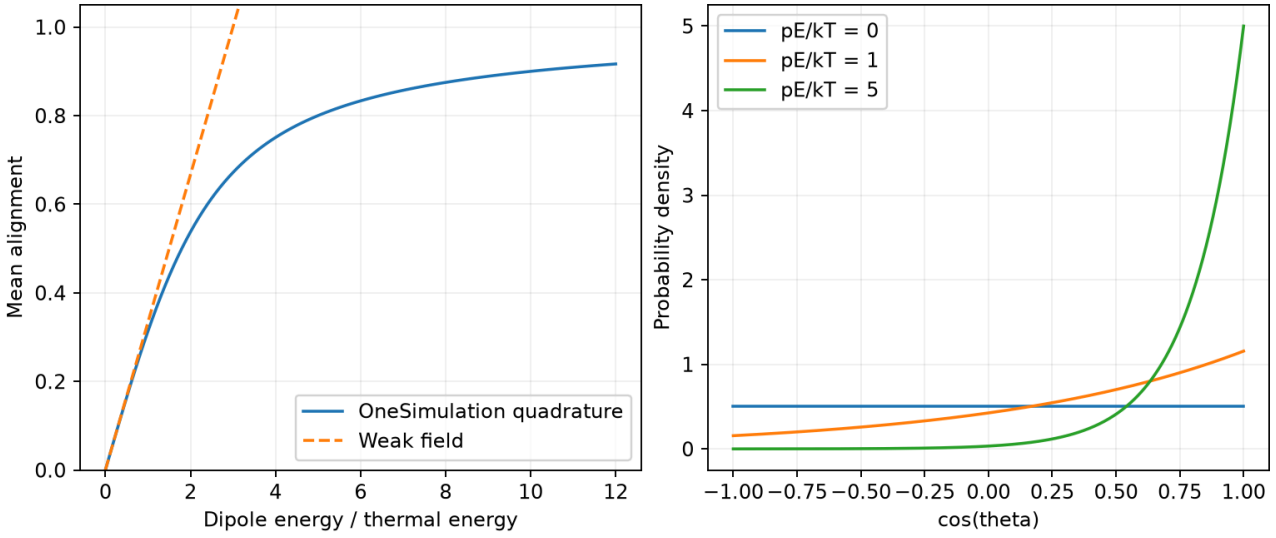
```
"""Orient freely rotating permanent dipoles and compare quadrature with the exact law."""
import numpy as np
import matplotlib.pyplot as plt
from onesim.numerics.linear.matrices.dense_linear_algebra import symmetric_eigensystem

# The eigenvectors give Gaussian weights for the integral over u = cos(theta).
order = 32
indices = np.arange(1, order, dtype=float)
off_diagonal = indices / np.sqrt(4 * indices**2 - 1)
jacobi = np.diag(off_diagonal, 1) + np.diag(off_diagonal, -1)
nodes, vectors = symmetric_eigensystem(jacobi)
weights = 2 * vectors[0, :]**2
field_energy = np.linspace(0, 12, 121)  # x = p E / (k_B T)
mean_alignment = []
for x in field_energy:
    probability = weights * np.exp(x * (nodes - 1))
    mean_alignment.append(np.dot(nodes, probability) / probability.sum())
mean_alignment = np.asarray(mean_alignment)
exact = np.zeros_like(field_energy)
exact[1:] = 1 / np.tanh(field_energy[1:]) - 1 / field_energy[1:]
print(f"Maximum quadrature error: {np.max(abs(mean_alignment - exact)):.2e}")
for x in (0.2, 1.0, 3.0, 10.0):
    index = np.argmin(abs(field_energy - x))
```

```
    print(f"pE/kT = {x:4.1f}, <cos(theta)> = {mean_alignment[index]:.6f}, linear = {x/3:.6f}")
assert np.max(abs(mean_alignment - exact)) < 1e-12
fig, axes = plt.subplots(1, 2, figsize=(9, 3.8), layout="constrained")
axes[0].plot(field_energy, mean_alignment, label="OneSimulation quadrature")
axes[0].plot(field_energy, field_energy / 3, "--", label="Weak field")
axes[0].set(xlabel="Dipole energy / thermal energy", ylabel="Mean alignment", ylim=(0, 1.05))
axes[0].legend()
for x in (0, 1, 5):
    u = np.linspace(-1, 1, 301)
    normalizer = 2 if x == 0 else 2*np.sinh(x)/x
    axes[1].plot(u, np.exp(x*u)/normalizer, label=f"pE/kT = {x}")
axes[1].set(xlabel="cos(theta)", ylabel="Probability density")
axes[1].legend()
for axis in axes:
    axis.grid(alpha=.2)
fig.savefig("electrodynamics-orientation.png", dpi=170)
```

Output from the supplied parameters

```
Maximum quadrature error: 8.88e-16
pE/kT =  0.2, <cos(theta)> = 0.066490, linear = 0.066667
pE/kT =  1.0, <cos(theta)> = 0.313035, linear = 0.333333
pE/kT =  3.0, <cos(theta)> = 0.671636, linear = 1.000000
pE/kT = 10.0, <cos(theta)> = 0.900000, linear = 3.333333
```



The probability concentrates near the field direction. At x=3 the weak-field law already predicts complete alignment, while the integral gives 0.672. Change the quadrature order and find when numerical error becomes visible separately from the linear-model error.

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Close the local-field problem

In an isotropic, approximately cubic environment imagine a spherical cavity around the chosen molecule. Let n point from the centre into the material. The material normal at the cavity wall is –n, giving bound charge $\sigma_b = -P\cdot n$. A surface patch contributes $dE(0) = -\sigma_b\, n\, d\Omega/(4\pi\varepsilon_0)$. Integrating with $\int n_i n_j\, d\Omega = 4\pi\delta_{ij}/3$ gives the cavity contribution $P/(3\varepsilon_0)$ at its centre. Under the Lorentz-cavity approximation, which

neglects the remaining discrete near-neighbour correction, E_loc=E+P/(3ε₀). Write α_eff=α+p²/(3kBT) in the weak-field regime and solve P=Nα_eff E_loc for P.

$$\frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{N\alpha_{\text{eff}}}{3\epsilon_0}.$$

For Nα_eff/(3ε₀)=0.10 this gives εr=1.3333. The independent-molecule estimate gives 1.30. The discrepancy increases with density. The divergence predicted as that parameter approaches unity indicates the breakdown or instability of this simple constitutive closure. Molecular correlations, anisotropy, or nonlinear ordering must then be considered before assigning a physical permittivity.

Remember · A cavity field is a model of the surroundings

The factor one third belongs to a spherical cavity in the isotropic construction. The macroscopic depolarising factor of the entire specimen is a different geometric quantity. The local-field calculation addresses what one molecule feels. The boundary-value calculation addresses how the specimen and electrodes determine the macroscopic E.

Follow the ambiguity of the unit cell in a polar crystal

Take an infinite alternating chain of charges +q at na and –q at na+a/2, with one chain per cross-sectional area A. A neutral cell containing these two sites has moment –qa/2 and polarisation –q/(2A). Choose the equivalent negative ion at na–a/2 instead, keeping the same infinite crystal. That cell has moment +qa/2 and polarisation +q/(2A). The two assignments differ by q/A although every actual charge position is unchanged.

A finite sample supplies the missing information. Its end charges depend on how it is cut and on whether charges are supplied at the surfaces. Reassigning the cell changes the book-keeping of bound and free end charge together. The measurable total surface charge and external field remain determined by the physical termination. A bulk dipole-per-cell convention alone therefore does not determine the surface charge of an ionic crystal. In three dimensions the corresponding polarisation increment is qR/Ω for lattice vector R and cell volume Ω.

Exercises from response to inference

1. At the same number density, the measured dilute susceptibility is 0.010 at 300 K and 0.007 at 600 K. Separate a temperature-independent contribution from a contribution proportional to 1/T.

▼ **Work through the solution**

Write $\chi = A + B/T$. Subtraction gives $0.003 = B/600$, hence $B = 1.8$ K and $A = 0.004$. At 300 K the orientational contribution is 0.006. To infer p from B also requires N. If density changes with temperature, correct that effect before fitting this law.

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2. A linear extrapolation predicts $P = 1.5$ Np. Determine what assumption has failed and estimate the large-field response for $x = 10$.

▼ **Work through the solution**

A mean cosine cannot exceed unity. The small-x expansion has been used outside its domain. $L(10) = 0.900000004\ldots$, giving $P \approx 0.900$ Np for freely rotating permanent dipoles. The induced contribution is a separate term and can change the total moment magnitude.

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3. Repeat the unit-cell construction after replacing the cross-sectional area by $2A$. Explain which quantity changes and what measurement must specify the crystal termination.

▼ **Work through the solution**

The polarisation branch spacing becomes $q/(2A)$. Surface charge per area and the external field are intensive quantities, so the chain density matters. An end-charge measurement with specified electrodes and termination determines the physical surface preparation.