Magnetism, induction and field dynamics

Begin with a steady current and follow what changes when a source, a conductor or a field moves. Track energy and momentum with explicit system boundaries.

Contents

1. **Building a magnetic field from a measured current**
   A Hall probe beside a bent wire reports a direction and a magnitude. The current flowing past it is easy to measure, but a single current reading cannot specify the field unless the return path and geometry are also known. Starting with the local field equations, we will derive the integral law that makes those geometrical details calculable.
2. **The vector potential as an instrument for computing flux**
   An experimenter can measure a flux change with a coil without measuring the magnetic field over every square millimetre of the coil area. The vector potential gives a corresponding computational tool. Its line integral determines flux, while the freedom to choose its gauge can turn a difficult boundary problem into a manageable one.
3. **Which features of a current distribution survive at a distance?**
   A current loop just fits behind a probe mount. At a nearby point its size and placement matter. Far away the same winding behaves as though all that geometry were a single vector. Predicting when that replacement is accurate, and what remains when the vector cancels, requires an expansion of the source rather than an appeal to a field-line sketch.
4. **From microscopic moments to measured magnetisation**
   A magnetised sample can make a compass turn without a power supply attached. The macroscopic theory assigns it bound currents in its bulk and on its surface. These are calculated sources for the magnetic field. We will reconstruct them from magnetic moments, predict signals from finite samples, and ask what an exterior field measurement can identify.
5. **Interfaces, demagnetising fields and the work of magnetisation**
   Two samples made of the same material can respond very differently to an externally applied field if one is a long rod and the other is a thin disc. The induced moment depends on the internal field, which the magnetised sample helps create. The same distinction reappears when one tries to infer a force from stored magnetic energy.
6. **Motional emf and transformer emf in moving circuits**
   Two laboratory demonstrations produce the same voltmeter trace. In one, a stationary loop sees a changing magnetic field. In the other, a conductor moves through a stationary field. The charges

experience different local forces, yet under identifiable conditions both readings follow from the rate of change of magnetic flux.

7. Flux linkage, coupled coils and magnetic coenergy

A winding links flux through each turn. In two windings the current in either one can change the voltage in the other. A matrix of flux linkages describes this coupling, but predicting power or mechanical force requires specifying the current directions and the control conditions.

8. How a magnetic field relaxes into a finite conductor

A conducting sheet initially shields its interior from an applied magnetic step. Given time, a normal resistive metal admits a static tangential magnetic field. The same equations give a finite slab a discrete relaxation spectrum and a periodic drive a complex penetration depth. These are different initial and boundary conditions for one diffusion operator.

9. Displacement current and the charging capacitor

A wire feeds a capacitor whose dielectric blocks the transport of mobile charges between its plates. A magnetic probe near the gap still responds to charging. Resolving which surface a circulation law should use leads to displacement current, an electric field correction during a ramp, and an energy-flow test around the rim.

10. Energy flow and mechanical force from vacuum fields

An external support holds a charged capacitor plate or a current-carrying coil in place. One can infer the support force from fields just outside the body, without summing forces on every microscopic charge. A second surface integral tracks the energy feeding the same apparatus. Both methods start with Maxwell's equations, but they transport different conserved quantities.

11. Field momentum, momentum exchange and the quasistatic limits

A system can have static electric and magnetic fields with nonzero local electromagnetic momentum density. A circuit can also be electrically small while either induction or displacement current remains indispensable. Both puzzles become clearer when one follows the relevant conservation law and compares the scales of the discarded terms.

12. Posing an electromagnetic evolution problem

A solver cannot evolve 'a pulse in a box' from geometry and a time step alone. It needs material laws, compatible initial fields, a charge-conserving source, and enough boundary data to specify what enters the domain. The same bookkeeping is required when interpreting a real transient measurement.

13. Why does a Helmholtz pair have a useful uniform region?

A magnetometer averages the field over a finite sample. Derive the coil spacing that suppresses curvature and predict how rapidly the remaining calibration error grows away from the midpoint.

14. From atomic motion to magnetic susceptibility

Calculate the response of a bound orbit and of an ensemble of existing moments, then follow how interactions permit spontaneous magnetisation.

15. The electric field of a rotating magnet

Use charge balance and a surface matching calculation to predict a voltage around a rotating magnetic body, then close the mechanical and electrical power accounts.

Magnetism, induction and field dynamics · 1. Building a magnetic field from a measured current

A Hall probe beside a bent wire reports a direction and a magnitude. The current flowing past it is easy to measure, but a single current reading cannot specify the field unless the return path and geometry are also known. Starting with the local field equations, we will derive the integral law that makes those geometrical details calculable.

Decide what is steady

Imagine making two maps at successive times of the current density in a conducting ring. Charges move, but if the maps agree, the source can be stationary. The current through an oriented cross section is $I = \int \boldsymbol{J} \cdot d\boldsymbol{a}$. Charge balance in any fixed volume requires

$$\partial_t \rho + \nabla \cdot \boldsymbol{J} = 0.$$

A genuinely steady system has $\nabla \cdot \boldsymbol{J} = 0$, including across contacts where current changes direction. A single finite wire element carrying a constant current cannot be an entire isolated source: continuous current would have to appear at one end and disappear at the other. It can be a contribution to the field of an explicitly closed arrangement.

In vacuum the magnetostatic equations are

$$\nabla \cdot \boldsymbol{B} = 0,$$
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}.$$

Stokes's theorem turns the second into $\oint \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\mathrm{enc}}$. It gives one number for the whole closed path. For an infinite straight wire rotational and translational symmetry establish in advance that the field is azimuthal and constant along a coaxial circle. Only then does the integral determine a local field $B = \mu_0 I/(2\pi s)$. For a bent wire the same circulation equality holds. Extracting a pointwise magnitude from it is a separate inverse problem. A linear superposition integral resolves that problem directly.

Invert the local curl equation

Take the curl of Ampère's equation. In Cartesian coordinates the identity $\nabla \times \nabla \times \boldsymbol{B} = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B}$ applies componentwise. Since its divergence vanishes,

$$-\nabla^2 \boldsymbol{B} = \mu_0 \nabla \times \boldsymbol{J}.$$

Assume first that the sources occupy a bounded region of space and the field decays at large distance. For each component the inverse of $-\nabla^2$ is the Newton kernel $G(\boldsymbol{r} - \boldsymbol{r}') = 1/(4\pi|\boldsymbol{r} - \boldsymbol{r}'|)$. Its normalisation follows from integrating $-\nabla^2 G$ over a little sphere. The outward derivative contributes precisely one.

Thus the field is $\mu_0 \int G\, \nabla' \times \boldsymbol{J}\, d^3 r'$. Move the source derivative off the current by integrating the product $\nabla' \times (G\boldsymbol{J})$. Its surface integral at infinity vanishes for localised currents. The remaining term

is $-\mu_0 \int (\nabla' G) \times \boldsymbol{J}\, d^3r'$. Writing $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$,

$$\nabla' G = +\boldsymbol{R}/(4\pi R^3).$$

Interchanging the two factors in the cross product supplies another minus sign.

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}\, d^3r'.$$

The integrand tells us where direction comes from. It is perpendicular both to the local current and to the vector pointing from source to probe. A segment aimed straight at the probe contributes zero there. A very distant transverse segment contributes as its length divided by squared distance. The special $1/s$ field of an infinite wire results from summing contributions along an extended length. A free-space field could still contain an additional homogeneous contribution if large coils elsewhere impose a background; the assumed boundary condition fixed that possibility. In a cavity bounded by magnetic material one needs the corresponding boundary solution or all equivalent sources.

Collapse a slender wire into a curve

Within a narrow conducting filament take arc length u, unit tangent $\boldsymbol{t}(u)$ in the current direction, and transverse section $S(u)$. If the observation distance and the local radius of curvature both greatly exceed the wire radius, the kernel changes little across a section. The volume element contributes its area times du, while the current density integrated over that area gives I. Their product becomes

$$I\boldsymbol{t}\, du = I\, d\boldsymbol{\ell}'.$$

The resulting Biot–Savart integral is a geometric idealisation with units $\mu_0 I/\mathrm{m} = \mathrm{T}$.

For a straight section on the z-axis from $-a$ to $+a$, observe on the perpendicular bisector at $(s, 0, 0)$, $s > 0$. A current directed towards $+z$ gives

$$\boldsymbol{e}_z \times (s\boldsymbol{e}_x - z'\boldsymbol{e}_z) = s\boldsymbol{e}_y.$$

The only surviving component requires the elementary integral $\int s(s^2 + z'^2)^{-3/2} dz' = z'/[s\sqrt{s^2 + z'^2}]$, hence

$$B_y = \frac{\mu_0 I}{2\pi s} \frac{a}{\sqrt{a^2 + s^2}}.$$

With $a = 0.10\,\mathrm{m},\ s = 0.040\,\mathrm{m},\ I = 3.0\,\mathrm{A}$, the infinite-wire value is $15.0\,\mu\mathrm{T}$ and the finite-section contribution is $13.93\,\mu\mathrm{T}$. A detector with a one-percent calibration would resolve the difference. Moving the probe twice as far changes the finite-length factor as well as $1/s$, so simply dividing the field by two is insufficient. To predict the experiment one must add the field from the return conductor.

Tilting that return to point approximately towards the probe can reduce its contribution even when its nearest distance stays similar.

Sum a loop before invoking its symmetry

Put a circular loop of radius a in the xy-plane with counterclockwise current as seen from $+z$. A source element $a\,d\varphi\,\boldsymbol{e}_\varphi$ sees the axial point $z\boldsymbol{e}_z$ along $z\boldsymbol{e}_z - a\boldsymbol{e}_s$. Its cross product is $a\,d\varphi(z\boldsymbol{e}_s + a\boldsymbol{e}_z)$. The radial terms integrate to zero because their directions rotate around the ring. Every axial term has the same direction and distance factor. Integrating the factor around the circle gives

$$B_z(z) = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}}.$$

At the centre the field is $\mu_0 I/(2a)$. For $a = 0.050\,\mathrm{m}$ and $I = 2.0\,\mathrm{A}$, this is $25.13\,\mu\mathrm{T}$, comparable to ordinary ambient geomagnetic field scales. At $z = a$ it is only $2^{-3/2} = 0.3536$ of the central reading, or $8.89\,\mu\mathrm{T}$. A practical isolation procedure measures once with positive current and once with negative current. Half the reading difference removes a background stable between the two runs. An intensity-only sensor that records $|\boldsymbol{B}|$ instead of a signed axial component needs a separate procedure, since magnitudes of vectors do not add linearly.

The loop also supplies an internal test of rough field approximations. At $|z| \gg a$, expanding $(1 + a^2/z^2)^{-3/2}$ gives

$$B_z = \mu_0 I a^2/(2|z|^3)[1 - 3a^2/(2z^2) + \cdots].$$

In the median plane a dipole moment points opposite to the local external field, whereas on either extension of its axis the field points in the moment direction. A scalar drawing that simply marks a plus sign for $z > 0$ and a minus sign for $z < 0$ would confuse the direction of the observation position with the direction of the field.

Give a singular source a physical cross section

The filament field diverges as the mathematical distance to an ideal line tends to zero. For a long cylindrical conductor of radius a carrying uniform steady current I, the field is regular. An Amperian circle at $s < a$ encloses Is^2/a^2. Its circulation yields $B_\varphi(s) = \mu_0 I s/(2\pi a^2)$, tending to zero at the centre. Outside it joins the filament result continuously. If all the current instead flowed in an infinitesimal surface sheet, the interior field would vanish and a tangential jump would appear at the surface. The outside measurement of a perfectly symmetric straight wire would be identical in these two ideal models.

For a $2.0\,\mathrm{mm}$ radius wire with $I = 4.0\,\mathrm{A}$, the surface value is $0.400\,\mathrm{mT}$. Halfway out, uniform bulk flow gives $0.200\,\mathrm{mT}$; a thin outer sheet gives zero there. The same reasoning underlies an inverse estimate of current density from measured interior field:

$$J_z = (1/\mu_0 s) d(sB_\varphi)/ds.$$

Differentiating noisy measurements can amplify high spatial frequencies, so one can instead compare flux integrals over annuli. The difference $2\pi[s_2 B(s_2) - s_1 B(s_1)]/\mu_0$ gives the current through an annulus without numerical differentiation.

Set up a finite-current computation

For a polygonal coil, sum each oriented segment contribution as a three-component vector. Split a curved path into short sections, evaluate geometry at each midpoint, and compare with a second run using twice as many sections. For a smooth integrand at a fixed point away from the winding one expects second-order midpoint convergence. Near a filament the kernel varies on the source-to-probe separation. Refining until the segment length is small relative to that separation can require much finer sampling than refining the plotted contour itself. At a point inside the conductor, improving a line approximation cannot replace a cross-sectional current model.

An independent quality test is symmetry. The two transverse axial components of a centred circular coil must cancel separately to roundoff for a symmetric discretisation. An axial field may nonetheless be inaccurate if all segment lengths systematically approximate the wrong radius. Check the absolute axial benchmark as well as the zero components. A second useful test reverses every edge orientation while keeping the probe and axes fixed; all three field components must reverse. These checks detect mistakes in geometry, integration and sign that a visually plausible field-line plot can hide.

Compare source geometry in the steady-current experiment

Design a nearly uniform calibration field

A single loop has a local maximum of axial field at its centre, with appreciable curvature. A sensor of finite size averages across positions and can read a different value from a nominal point probe. Two identical coaxial loops with equal current permit the curvature to be removed at a chosen point. Put their centres at $z = \pm d/2$. Let $b(z) = K(a^2 + z^2)^{-3/2}$, $K = \mu_0 N I a^2/2$, be the field from one N-turn loop centred at the origin. By superposition,

$$B(z) = b(z - d/2) + b(z + d/2).$$

This is even in z for every spacing, so its first derivative at the midpoint vanishes. A flat-looking midpoint on a coarse graph consequently gives no information about whether the separation was set correctly.

Differentiate once to obtain

$$b' = -3Kz(a^2 + z^2)^{-5/2}.$$

The next derivative is

$$b'' = 3K(4z^2 - a^2)(a^2 + z^2)^{-7/2}.$$

The pair therefore has $B''(0) = 0$ when $4(d/2)^2 = a^2$: the separation equals the radius. At that setting the central value is

$$B(0) = (4/5)^{3/2}\mu_0 N I / a.$$

With $a = d = 0.10\,\mathrm{m}$, 50 turns on each winding and 0.20 A, it is $89.9\,\mu\mathrm{T}$. A thermal expansion increasing both radius and separation by one percent preserves their ratio yet lowers the field calibration nearly one percent at fixed current. Geometrical uniformity and absolute calibration are distinct tolerances.

There is a useful way to estimate the size of the flat region without trusting the plot scale. Write $u = z/a$ and expand each factor $[1 + (u \pm 1/2)^2]^{-3/2}$ in the exact two-coil formula. Odd terms cancel in the sum, and the quadratic term was eliminated by design. The first surviving correction is

$$B(z)/B(0) = 1 - (144/125)u^4 + O(u^6).$$

At one tenth of a radius it predicts a fractional axial fall of 1.15×10^{-4}, roughly 0.012 percent. At one quarter radius the quartic estimate is 0.45 percent; comparison with the unexpanded expression is more appropriate if a tight tolerance is specified. With a single loop, by contrast, $B(z)/B(0) = 1 - 3u^2/2 + O(u^4)$, already losing about 1.5 percent at the first of those positions.

Is the field also uniform slightly off the axis? In a current-free axisymmetric region, smooth fields obey $\nabla \times \boldsymbol{B} = 0$ and $\nabla \cdot \boldsymbol{B} = 0$. Write $B_z(s, z) = B_0(z) + c(z)s^2 + \cdots$ and $B_s = d(z)s + \cdots$; odd radial axial terms would fail smoothness at the axis. From the divergence $2d + B_0' = 0$. The azimuthal curl gives $d' = 2c$, hence $B_s = -sB_0'/2 + \cdots$ and $B_z = B_0 - s^2 B_0''/4 + \cdots$. Eliminating axial curvature removes the leading radial variation too. It does not remove the field of an asymmetrically routed supply lead, which breaks the expansion's assumed symmetry. Twisting a distant supply pair and measuring a reversed-current map test that separate contribution.

Separate the feed from the return in a field measurement

Consider two long straight parallel conductors separated by $2a$. The right conductor carries $+I$ along z; the left carries $-I$. On the line of centres at position $x > a$, their azimuthal fields oppose. Adding signed magnitudes gives

$$B_y = \frac{\mu_0 I}{2\pi}\left(\frac{1}{x - a} - \frac{1}{x + a}\right) = \frac{\mu_0 I a}{\pi(x^2 - a^2)}.$$

At $a = 5.0\,\mathrm{mm}$, $x = 50\,\mathrm{mm}$, $I = 1.0\,\mathrm{A}$, the field is $0.808\,\mu\mathrm{T}$. The nearer conductor alone would contribute $4.44\,\mu\mathrm{T}$. Omitting the return because it is 'the other lead' would overpredict the signal by more than a factor of five. Well outside the wire spacing but far from the ends, the leading transverse

field falls as $1/x^2$; at still greater distances than the full length of a finite closed rectangular winding, the complete assembly ultimately behaves as a three-dimensional dipole and falls as $1/r^3$. The apparent power law therefore depends on which distances are large relative to which parts of the apparatus.

Now move the probe between the conductors to $x = 0$. Both fields point in the same transverse direction there, with magnitude $\mu_0 I/(\pi a) = 80.0\,\mu\mathrm{T}$ for the same parameters. A probe displaced laterally by a fraction of the spacing can therefore see a large change even at fixed current. Reversing both lead currents swaps every sign. Reversing just the nominal return current would create parallel currents and describe a new unseen return elsewhere, illustrating why 'individual wire fields' require their circuit context. If the readings are used to infer a small current, fit the two known conductor positions and the background rather than using the distance to the nearest visible piece of copper as the only geometry input.

The two examples provide complementary ways to verify a coil calculation experimentally. For a calibration pair, motion over a known traverse tests a deliberately small spatial derivative, while reversing current rejects an additive background. For opposed leads, the sharp transition between addition and cancellation tests probe alignment and whether the return has been modelled. A single calibration at the midpoint could succeed in either geometry with the wrong sign convention or an erroneously rescaled current. A signed scan over a measured range of positions exposes those degeneracies.

Recover a magnetic field from a steady current

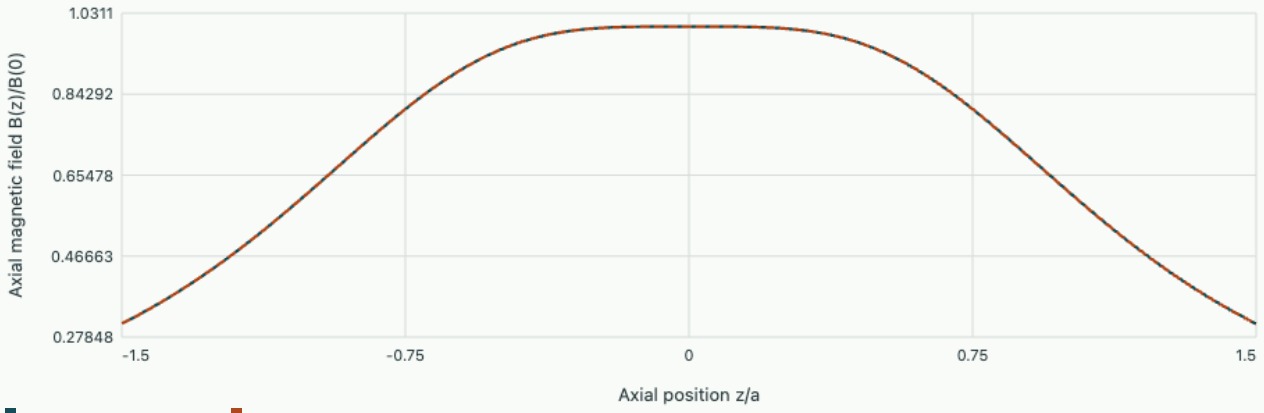
Compare the on-axis contributions of current loops before and after changing their spacing. Check the sign and spatial uniformity near the centre.

Open this experiment online

Try the calculation

Coil separation / radius **1**

Axial test position / radius **0.25**



Selected pair Helmholtz spacing d/a = 1

At z/a = 0.25 the selected field differs from the central field by -0.42%. The sign distinguishes a central peak from a central dip close to the origin.

Resolve a current strip from its measured field

A long conducting strip occupies the region between $x = -a$ and $x = a$ in the plane $z = 0$. Conventional current flows in positive y. Suppose leads far from the observation region distribute the current uniformly across the width. Calling the surface current K fixes the current in the narrow ribbon between x' and $x' + dx'$ as $K\,dx'$. The total current is therefore $I = 2aK$. A wide trace on a board and a narrow round wire carrying the same current look similar at a distance. An observer close enough to resolve the width should be able to tell them apart. We can calculate exactly how close the observer has to come in this ideal geometry.

Each ribbon is an infinitely long straight source. Its contribution lies on a circle around the y axis through x'. At an observation point $(x, 0, z)$, its direction is proportional to $\hat{y} \times [(x - x')\hat{x} + z\hat{z}]$. The field vector is the transverse displacement rotated through a right angle and divided by its squared length. The familiar inverse distance of the straight wire is still present because the numerator has the magnitude of one distance. On the centre line the contributions to B_z cancel in pairs. All contributions to B_x add with the same sign above the plane. For positive z,

$$B_x(0, z) = \frac{\mu_0 K}{2\pi} \int_{-a}^{a} \frac{z\,dx'}{x'^2 + z^2} = \frac{\mu_0 K}{\pi} \arctan \frac{a}{z}.$$

At heights much smaller than the half-width the arctangent approaches $\pi/2$. A locally infinite sheet gives the same field, $\mu_0 K/2$. At heights much greater than the half-width, replacing the arctangent by its argument gives $\mu_0 I/(2\pi z)$. The first correction to this wire prediction has relative size $-a^2/(3z^2)$. A single formula thus shows when an idealisation loses the length scale that distinguishes different

sources. At height $z = a$ the strip gives exactly one half of its value immediately above the centre of the sheet.

Take a strip of width $40\,\mathrm{mm}$ with current $8.0\,\mathrm{A}$. The surface current is $200\,\mathrm{A/m}$, and the limiting centre field is $125.7\,\mu\mathrm{T}$. At $z = 5\,\mathrm{mm}$, the exact field is about $106.1\,\mu\mathrm{T}$. At $20\,\mathrm{mm}$, it is $62.8\,\mu\mathrm{T}$. A line-current estimate at this second position gives $80.0\,\mu\mathrm{T}$. This discrepancy occurs even when the source is perfectly long. Extending the source in the y direction cannot cure an approximation that discarded the width in the x direction.

A scan at fixed height reveals another part of the vector field. Integrating the signed z component gives

$$B_z(x, z) = \frac{\mu_0 K}{4\pi} \ln \frac{(x - a)^2 + z^2}{(x + a)^2 + z^2}.$$

The logarithm vanishes at the centre, changes sign across it and is negative on the positive x side. Just above an edge the ideal formula grows in magnitude as the height decreases. The singular geometry assumed zero thickness and terminated a uniformly flowing sheet abruptly at a perfectly sharp boundary. A probe of finite active size measures a spatial average. A model of a real edge also needs its thickness and actual current distribution. Neither complication changes the centre-line limiting fields at heights well above the thickness and well inside the long region of the strip.

A student measures a short-distance plateau and records a centre field equal to half the plateau at a nominal height of $12\,\mathrm{mm}$. What width follows, and how could a sensor-position offset corrupt the inference?

▼ Work through the solution

The half-height condition requires an arctangent equal to $\pi/4$, so the actual separation equals the half-width. The inferred full width is $24\,\mathrm{mm}$ if the sensitive element lies at the point used to define the nominal height. Suppose instead that the sensitive element sits $2\,\mathrm{mm}$ behind the face of its package and the nominal reading gives the face height above the strip. The field probe was then $14\,\mathrm{mm}$ from the current sheet, and the inferred full width becomes $28\,\mathrm{mm}$. A fit at several heights can estimate an unknown offset together with the width. A remote return wire adds an approximately constant background over a short scan, so reverse the source current and measure the field of the return geometry separately when the desired precision demands it.

What happens to a far-field and a near-field reading if the same total current is spread over twice the width?

▼ Work through the solution

In the far field the leading term depends only on total current, so it is insensitive to the change provided the observation height is still much larger than the new width. On the near-field centre plateau the field halves because $K = I/(2a)$ halves. At an intermediate height use the arctangent with the new half-

width and the new value of K. Doubling the width while keeping the surface current fixed describes a different experiment because the supply then delivers twice as much total current.

David Tong, Magnetostatics, on deriving a field from its current distribution.

Tutorial on recovering fields from geometry

A square loop has side $2a$ and current I. Find the field at its centre by using the finite-section result. For $a = 0.050\,\mathrm{m},\ I = 2.0\,\mathrm{A}$, compare it with a circular loop of equal enclosed area.

▼ **Work through the solution**

Each straight side is at perpendicular distance a, and its half-length is a, giving $\mu_0 I/(2\pi a\sqrt{2})$ per side. All four point along the same normal. The result is

$$\sqrt{2}\mu_0 I/(\pi a) = 22.63\,\mu\mathrm{T}.$$

Equal area requires a circular radius $2a/\sqrt{\pi} = 0.0564\,\mathrm{m}$; its field is $22.27\,\mu\mathrm{T}$. Equal enclosed areas imply equal dipole moments far away, but a central probe distinguishes the two windings.

A cylindrical wire with azimuthal field $B_\varphi = bs^3$ inside radius a carries an unknown axial current distribution. Reconstruct it and compare the current inside $a/2$ with the current inferred at a.

▼ **Work through the solution**

Ampère gives

$$I_{\mathrm{enc}} = 2\pi b s^4/\mu_0.$$

The differential current through an annulus is $2\pi s J_z\,ds$, yielding $J_z = 4bs^2/\mu_0$. The half-radius encloses $1/16$ of the total. A uniform-density wire would enclose $1/4$, so a single exterior reading cannot determine the distribution while several interior readings could do so.

At what ratio a/s does the infinite-wire approximation overestimate the finite straight section on its perpendicular bisector by less than one percent, measured against the exact contribution?

▼ **Work through the solution**

The ratio of approximate to exact is $\sqrt{1+(s/a)^2}$. Requiring it below 1.01 gives

$$a/s > 1/\sqrt{1.01^2 - 1} = 7.05.$$

The full length must exceed roughly 14 times the perpendicular distance. Even a section meeting this condition still requires an estimate of the return path field.

Magnetism, induction and field dynamics · 2. The vector potential as an instrument for computing flux

An experimenter can measure a flux change with a coil without measuring the magnetic field over every square millimetre of the coil area. The vector potential gives a corresponding computational tool. Its line integral determines flux, while the freedom to choose its gauge can turn a difficult boundary problem into a manageable one.

Separate local representation from global information

In a small simply connected patch with no holes a sufficiently smooth divergence-free magnetic field has a potential $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Taking a divergence gives zero automatically because mixed Cartesian derivatives commute. For a closed curve bounding an oriented surface within the patch, Stokes's theorem gives

$$\oint \boldsymbol{A} \cdot d\boldsymbol{\ell} = \int \boldsymbol{B} \cdot d\boldsymbol{a}.$$

The dimensions are useful: A has units of tesla metres or webers per metre; A integrated along a curve has units of webers, whereas B is flux per area. Neither a single component nor the pointwise magnitude of $\boldsymbol{A}$ equals a force per charge.

Add the gradient of any smooth single-valued scalar χ to $\boldsymbol{A}$. Its curl vanishes and its integral around any closed loop is zero, so this changes neither flux nor local force. In a time-dependent problem one must at the same time change ϕ by $-\partial_t \chi$, since $\boldsymbol{E} = -\nabla\phi - \partial_t \boldsymbol{A}$. This coupled transformation cancels the two derivatives of χ. It permits moving an electric field contribution between the scalar and vector formulas while leaving their sum intact. A gauge choice simplifies the equations for the potentials; grounding a conductor instead imposes a physical condition on a voltage measurement.

Derive the free-space Coulomb-gauge potential

Substitute $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ into Ampère's equation. The double curl gives

$$\nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}.$$

Choose $\nabla \cdot \boldsymbol{A} = 0$, called the Coulomb gauge. Starting from any initial $\boldsymbol{A}$ one can impose this condition by solving $\nabla^2 \chi = -\nabla \cdot \boldsymbol{A}$ with appropriate boundary behaviour. A further gauge addition is possible if χ is harmonic and preserves those conditions. With localised steady currents and decay at infinity the resulting solution is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\, d^3 r'.$$

The scalar-looking kernel acts separately on the Cartesian components of the vector current. Verify the gauge explicitly by taking the divergence under the integral. Because $\nabla G = -\nabla' G$, integrating by

parts leaves a term proportional to $\int G\,\nabla'\cdot\boldsymbol{J}\,d^3r'$ and a surface current term at infinity. Both vanish for the specified steady localised source. Taking the curl instead gives $\nabla G\times\boldsymbol{J}$, which is precisely the Biot–Savart integrand after reversing the cross-product order. For a curved wire, integrating its directed elements with the scalar kernel can be simpler than integrating three different projected field components.

Begin with a uniform field and two different potentials

For $\boldsymbol{B}=B_0\hat{\boldsymbol{z}}$, the choice $\boldsymbol{A}_1=B_0x\hat{\boldsymbol{y}}$ immediately yields

$$B_z=\partial_x A_y.$$

A symmetric construction uses $\boldsymbol{A}_2=\boldsymbol{B}\times\boldsymbol{r}/2=B_0(-y\hat{\boldsymbol{x}}+x\hat{\boldsymbol{y}})/2$. Both have zero divergence, so the Coulomb condition alone has not fixed them. Their difference is $\nabla(-B_0xy/2)$. A uniform imposed field does not decay at infinity, so no contradiction with the free-space uniqueness statement arises.

A rectangle has vertices $(0,0),(a,0),(a,b),(0,b)$. In the first gauge only the right edge contributes B_0ab to a counterclockwise line integral. In the symmetric gauge that edge contributes half and the upper edge, traversed leftward, contributes the other half. For $a=0.20\,\mathrm{m}, b=0.080\,\mathrm{m}, B_0=0.030\,\mathrm{T}$, both return $4.80\times10^{-4}\,\mathrm{Wb}$. If the field doubles smoothly over 0.040 s, average emf around the stationary loop is $-0.0120\,\mathrm{V}$. Trying to assign the measured emf exclusively to the right edge just because one static gauge makes its line integral special would confuse a chosen potential representation with a physical local electric field.

Work around the axis of a long solenoid

Consider an ideal long solenoid of radius a, with axial field B_0 inside and negligible external field. Rotational symmetry suggests $\boldsymbol{A}=A_\varphi(s)\hat{\boldsymbol{\varphi}}$. A circle of radius $s<a$ encloses flux πs^2B_0; a circle outside encloses the entire πa^2B_0. Dividing by its circumference gives

$$A_\varphi(s)=\begin{cases}B_0 s/2, & s<a,\\ B_0a^2/(2s), & s>a.\end{cases}$$

The two expressions agree at $s=a$. In cylindrical coordinates $B_z=s^{-1}\partial_s(sA_\varphi)$; it gives the desired constant inside and zero outside. Nevertheless every exterior loop encircling the solenoid has a nonzero integral. One cannot remove this exterior potential by a globally single-valued scalar in that punctured domain. Locally A_φ equals a constant times the gradient of the polar angle, but the angle changes by 2π upon one circuit. A small loop entirely outside and not surrounding the axis has zero integral. These two experiments distinguish a local zero curl from a global zero circulation.

At $B_0=12\,\mathrm{mT}$, $a=10\,\mathrm{mm}$, the enclosed flux is $3.77\,\mu\mathrm{Wb}$. At $s=20\,\mathrm{mm}$, $A_\varphi=3.00\times10^{-5}\,\mathrm{T\,m}$; at twice the radius it is half as large but its loop length is twice as large. If B_0 changes in time, the exterior loop can carry a nonzero induced emf even where its wire sits in nearly zero magnetic field. The change of enclosed flux determines it. For a finite solenoid the returning flux eventually changes

the large-radius behaviour, so extending the ideal $1/s$ result to all distances would describe the wrong source.

Calculate the coupling of two separate circuits

Drive a slender closed loop 1 with steady current I_1. Its vector potential at loop 2 is $\mu_0 I_1/(4\pi)\oint_1 d\boldsymbol{\ell}_1/R$. Integrating along the second loop yields the Neumann integral $M = \Phi_{21}/I_1 = [\mu_0/(4\pi)]\oint_1\oint_2 d\boldsymbol{\ell}_1\cdot d\boldsymbol{\ell}_2/R$. The dot product and distance are symmetric under interchange, so $M_{21} = M_{12}$ in this reciprocal vacuum setting. Reversing either chosen winding orientation changes the sign; reversing both restores it. The two loops must stay distinct for this simple filament integral. Sending loop 2 onto loop 1 creates a singular self-term. A finite wire radius and the interior current distribution enter a physical self-inductance.

Approximate a small pickup as sampling a uniform axial field from a large drive loop. Let $a_1 = 50\,\mathrm{mm}$, pickup radius $a_2 = 5\,\mathrm{mm}$, and separation $z = a_1$. The drive produces

$$B_z/I_1 = \mu_0/(2a_1\,2^{3/2}) = 4.44\,\mu\mathrm{T/A}.$$

The flux per ampere through a single pickup turn is area πa_2^2 times this field, or $0.349\,\mathrm{nH}$. Fifty tightly wound pickup turns give about $17.45\,\mathrm{nH}$. A changing drive of $2.0\times10^4\,\mathrm{A/s}$ induces about $0.349\,\mathrm{mV}$ in magnitude. The approximation uses the small pickup radius relative to the scale on which the drive field varies. Increasing the pickup radius tenfold calls for actual area integration.

Compare linkage, reciprocity and stored energy for coupled coils

Calculate internal and external inductance in a coaxial return

Flux linkage can be subtle even for the simplest straight conductor once the current has a finite cross section. Consider a long coaxial pair with inner radius a and an ideal thin return shell at b. Work well away from the ends. For equal and opposite currents the exterior field vanishes. In the annulus $a < s < b$,

$$B_\varphi = \mu_0 I/(2\pi s).$$

A longitudinal rectangular surface spanning the two conductors and length ℓ has flux $\Phi = \ell\int_a^b B_\varphi\,ds$. All the inner current is enclosed at every radius in that annulus, so its external flux linkage per unit length is $(\mu_0 I/2\pi)\ln(b/a)$. The corresponding inductance per unit length has units H/m and is finite because the return sets b.

In terms of the potential $B_\varphi = -dA_z/ds$, so the same answer is $\ell[A_z(a) - A_z(b)]$ for the oriented rectangle. A constant added to A_z cancels between the two longitudinal edges. There are no end contributions for a purely longitudinal potential along radial closing segments. Only the potential difference matters. This example displays which pieces of the earlier infinite-wire result are measurable. Extending the return to ever larger radius makes the linkage grow logarithmically and

eventually invalidates the approximation of infinitely long straight sections, since real end connections cannot be neglected at arbitrarily large radii.

If the inner conductor carries uniform steady current, $B_\varphi(s) = \mu_0 I s/(2\pi a^2)$ inside it. The internal magnetic energy per length follows directly by integrating over rings:

$$U_{\mathrm{int}}/\ell = \int_0^a \frac{B_\varphi^2}{2\mu_0}\, 2\pi s\, ds = \frac{\mu_0 I^2}{16\pi},$$

$$L_{\mathrm{int}}/\ell = \frac{\mu_0}{8\pi}.$$

The unexpectedly radius-independent internal result is an instructive scaling check. At fixed current a smaller radius raises the typical internal magnetic-field magnitude inversely with radius, but the area in which energy is stored falls as the square of radius. Those factors cancel. This coefficient applies to a uniformly conducting solid inner wire and an ideal sheet return. A solid outer conductor with finite thickness has its own internal energy, and at high frequency skin effect progressively removes the approximately uniform low-frequency current distribution.

A naive internal flux integral $\int_0^a B_\varphi ds$ would give twice the correct linkage. The reason appears by following current filaments: flux in a shell at radius s links only the fraction $I_{\mathrm{enc}}/I = s^2/a^2$ of the inner current on the relevant side. Weighting by that fraction yields

$$\int_0^a (s^2/a^2) B_\varphi ds = \mu_0 I/(8\pi).$$

This is also the answer from the energy derivative and gives an independent interpretation of partial internal linkages. For $a = 1.0\,\mathrm{mm}, b = 4.0\,\mathrm{mm}$, the external contribution is $0.2773\,\mu\mathrm{H/m}$ and the inner contribution $0.0500\,\mu\mathrm{H/m}$. In a 2.0 m low-frequency section carrying 1.0 A, about $0.327\,\mu\mathrm{J}$ of magnetic energy is stored with these approximations. Inferring the geometry from a measured value without subtracting the internal part would bias the inferred logarithmic radius ratio.

Recover the transverse part of a computed potential

A numerical solution may represent an otherwise exact magnetic field by a potential containing substantial gradient contributions. An error estimate based on comparing two raw vector-potential arrays can therefore be misleading. For a periodic calculation choose a nonzero Fourier wavevector $\boldsymbol{k}$. The curl depends only on the component of $\widehat{\boldsymbol{A}}$ perpendicular to $\boldsymbol{k}$, since $i\boldsymbol{k} \times \widehat{\boldsymbol{A}}_{\parallel} = 0$. A Coulomb-gauge representative is obtained by subtracting the projection $\boldsymbol{k}(\boldsymbol{k}\cdot\widehat{\boldsymbol{A}})/k^2$. This is precisely the gradient that the scalar Poisson gauge-fixing problem removes. A constant zero mode needs its own convention since both the derivative and this formula vanish or become undefined at $\boldsymbol{k} = 0$.

Take coordinates in metres, $\boldsymbol{k} = (2, 1, 0)\,\mathrm{m}^{-1}$, and a real potential $\boldsymbol{A} = (3, 1, 0)\cos(2x + y)\,\mathrm{mT\,m}$. The projection amplitude is $(2.8, 1.4, 0)\,\mathrm{mT\,m}$. Removing it leaves $(0.2, -0.4, 0)\,\mathrm{mT\,m}$. Directly differentiating in either representation gives

$$B_z = \partial_x A_y - \partial_y A_x = +\sin(2x + y)\,\mathrm{mT}.$$

The potential arrays differ by a component more than six times as large in norm as the retained one, but the predicted magnetic field agrees. A solver that tracks flux through a physical pickup instead of one gauge component would recognise this equivalence.

This Fourier argument does not turn every longitudinal component of the physical electric field into a gauge artifact. Physical $\boldsymbol{E}$ combines both potentials and its longitudinal projection is constrained by charge. A shift of $\boldsymbol{A}$ in time must be accompanied by the scalar shift derived earlier. An independent check on a computed transient is to form both $-\nabla\phi - \partial_t \boldsymbol{A}$ and $\nabla \times \boldsymbol{A}$, then apply exactly the same discrete differentiation conventions as were used in the field solver. Discrete gradient, curl and divergence operators built on incompatible grids need not cancel as their continuum identities do. If two gauges yield slightly different measured voltages after discretisation, refining the mesh and comparing closed-loop fluxes helps separate differentiation error from genuinely different imposed boundary data.

Follow a charged particle through a time-dependent change of gauge

An electric field in the central region between large closely spaced electrodes can be approximately uniform over a particle trajectory. Let its measured value be $E_x(t) = E_0 \cos \omega t$, and neglect magnetic deflection over the short central path. A scalar-potential description takes $\phi = -xE_0 \cos \omega t$ and $\mathbf{A} = 0$. The electric force then appears to arise entirely from a spatial derivative. Can the same experiment be described with a vanishing scalar potential? The gauge rule answers this question constructively. To make $\phi' = \phi - \partial_t \chi = 0$, choose a function whose time derivative is the given scalar potential. Integrating at fixed position gives

$$\chi(x,t) = -\frac{xE_0}{\omega}\sin \omega t,$$

$$A'_x = -\frac{E_0}{\omega}\sin \omega t,$$

$$\phi' = 0.$$

The curl of this spatially uniform vector potential vanishes. Its negative time derivative gives precisely the measured electric field. A spatially uniform oscillating electric field with no magnetic field everywhere in unbounded source-free space would violate the Maxwell–Ampère law. The electrode example uses a central-region approximation whose accuracy depends on dimensions and frequency. Changing gauge inside that approximation leaves its range of validity exactly where it was. Retaining the small induction fields of a real device also allows gauge transformations of the more complete potentials.

A particle of charge q and mass m, released from rest at $t = 0$, has kinetic momentum $mv_x = (qE_0/\omega)\sin \omega t$. Obtain this result first by integrating the physical force. In the new potentials

the canonical momentum is $p_x = m v_x + q A'_x$, which vanishes at all times for this initial condition. The Lagrangian $L = m v^2/2 + q \mathbf{v}\cdot\mathbf{A} - q\phi$ in these variables has no explicit x dependence. Its Euler–Lagrange equation indeed says that p_x is conserved. In the original potentials $p_x = m v_x$ changes because the scalar potential depends on x. Both descriptions assign the same speed, trajectory and work to the particle, even though the conserved canonical expression looks different.

This agreement follows for any path, without assuming the particle followed the physical trajectory. Substitute $\mathbf{A} + \nabla\chi$ and $\phi - \partial_t\chi$ into the Lagrangian. Its change is $q(\mathbf{v}\cdot\nabla\chi + \partial_t\chi) = q\, d\chi/dt$. Integrated between fixed spacetime endpoints it contributes a fixed boundary term to the action. Varying the path between those endpoints leaves the equations of motion unchanged. The extra canonical momentum $q\nabla\chi$ is the derivative of that boundary term with respect to the endpoint position. Keeping the canonical momentum numerically fixed when comparing two gauges would accidentally assign different physical initial velocities.

For a singly charged ion of mass $6.64 \times 10^{-27}\,\mathrm{kg}$, take $E_0 = 2.0\,\mathrm{V/m}$ and $f = 100\,\mathrm{kHz}$. At one quarter cycle the impulse is $5.10 \times 10^{-25}\,\mathrm{kg\,m/s}$ and the speed is about $76.8\,\mathrm{m/s}$. The displacement follows by a second integration,

$$x(t) - x(0) = q E_0 [1 - \cos(\omega t)]/(m\omega^2).$$

It is about $0.122\,\mathrm{mm}$ at the same instant. The geometry ought therefore to supply an approximately uniform region comfortably longer than a few tenths of a millimetre. At half a cycle the velocity is again zero. The particle has still moved because its velocity remained positive throughout that interval.

Change a vector potential while holding the magnetic field fixed

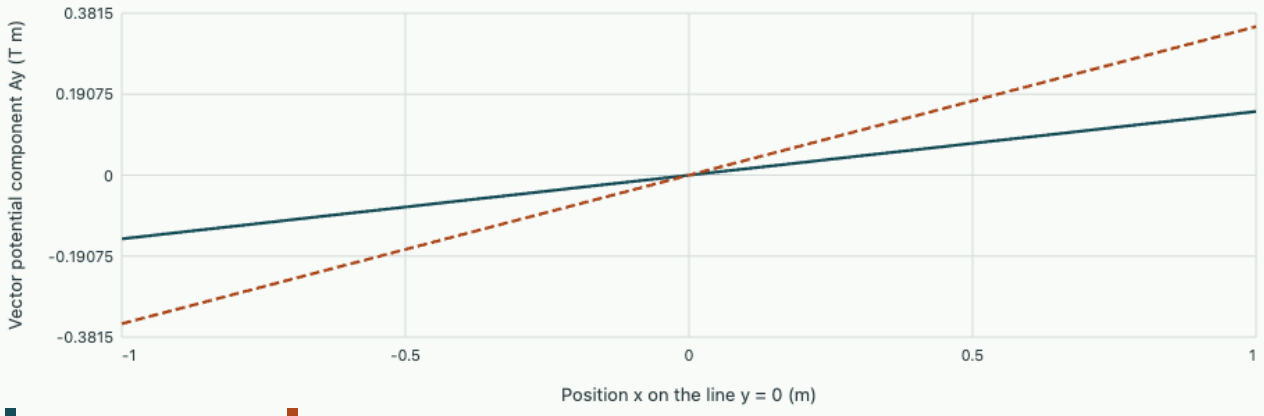
Add a gradient to the potential of a uniform magnetic field. Compare the change of a local component with the circulation around a closed rectangle.

Open this experiment online

Try the calculation

Uniform magnetic field B (T) **0.3**

Gauge coefficient g (T) **0.2**



∂xAy = 0.35 T, ∂yAx = 0.05 T. Curl = 0.3 T. Both loop integrals around 0.30 × 0.20 m are 0.018 Wb.

Preserve gauge invariance in a numerical circulation

One can check gauge invariance without differentiating noisy data twice. Store each oriented edge integral $a_{ij} = \int_i^j \mathbf{A} \cdot d\mathbf{l}$ on a mesh. Adding a scalar gauge function specified at the vertices changes an edge integral by precisely $\chi_j - \chi_i$. Summing around a face gives four such differences on a rectangular grid. Each vertex occurs once as an end and once as a beginning. The terms cancel exactly, including on irregularly shaped faces, provided that the last edge returns to the first vertex and shared edges use opposite signs for adjacent faces. The result is a simple test for a discrete curl operator that should survive grid refinement unchanged.

A rectangular face has consecutive gauge values $0, 2, 5, -1$ in units of $\mu\mathrm{Wb}$. A program stores initially zero edge integrals. What should it write after the gauge transformation, and what does the sum measure?

▼ **Work through the solution**

Walking around the oriented boundary yields changes of $+2, +3, -6, +1\,\mu\mathrm{Wb}$. Each value is allowed to change independently, but their signed sum is zero. That sum is the magnetic flux through the face by Stokes' theorem. Reversing the boundary should negate every contribution and still give zero. If a neighbouring face uses the same stored value with the same sign for their shared edge, interior contributions will survive in the total flux of a patch. Such a result indicates a mismatch of edge orientation even if some perfectly symmetric test configurations accidentally give zero.

Why does division by the edge length matter when using vector components exported by another code?

▼ **Work through the solution**

The discrete cancellation applies directly to line integrals. A stored component tangential to an edge has units of potential per unit length in this representation and must first be multiplied by its edge length, or integrated with a suitable quadrature if it varies. Adding raw component values around a stretched cell weights physically different lengths equally. A gauge test on a uniform mesh can conceal this error. Repeat it on cells of unequal lengths and compare fluxes in webers.

David Tong, Electromagnetism and Relativity, on potentials and gauge transformations.

Tutorial on gauge and flux

An infinite axial wire yields

$$B_\varphi = \mu_0 I/(2\pi s).$$

Choose a vector potential of the form $A_z(s)\hat{\boldsymbol{z}}$. Determine the difference $A_z(2s) - A_z(s)$, and explain why $A_z(\infty) = 0$ cannot be imposed.

▼ **Work through the solution**

The cylindrical curl gives $B_\varphi = -dA_z/ds$, so $A_z(s) = -(\mu_0 I/2\pi)\ln(s/s_0)$. The difference is $-(\mu_0 I/2\pi)\ln 2$, independent of s_0. An infinite wire is not a localised current with a remote return. Its logarithm cannot both fit the field and tend to zero at infinity. A coaxial return supplies a physically meaningful outer length.

A closed circle outside the ideal solenoid is deformed into an elongated loop without crossing the solenoid. The computed circulation changes by 8 percent. What three physical or numerical hypotheses would you test?

▼ **Work through the solution**

Under the ideal conditions the enclosed flux stays fixed for deformations entirely through field-free space. Check whether the solenoid has significant end fields, whether the computed geometry changes the winding number, and whether the line quadrature resolves the parts passing closest to the axis. A change of a smooth single-valued gauge cannot change that closed integral.

Take $\chi = \alpha x t$ in a field-free configuration initially represented by $\phi = 0, \boldsymbol{A} = 0$. Recover the fields after transformation.

▼ **Work through the solution**

The new potentials are

$$\phi' = -\alpha x,\ \boldsymbol{A}' = \alpha t \hat{\boldsymbol{x}}.$$

The static-looking scalar gradient supplies $+\alpha\hat{\boldsymbol{x}}$ and the negative vector time derivative supplies $-\alpha\hat{\boldsymbol{x}}$, cancelling. The curl of the spatially uniform vector potential vanishes. A scalar voltage difference alone need not be the complete physical electric field in a time-dependent gauge.

Magnetism, induction and field dynamics · 3. Which features of a current distribution survive at a distance?

A current loop just fits behind a probe mount. At a nearby point its size and placement matter. Far away the same winding behaves as though all that geometry were a single vector. Predicting when that replacement is accurate, and what remains when the vector cancels, requires an expansion of the source rather than an appeal to a field-line sketch.

Expand the kernel of a localised source

Take a bounded steady current distribution of size a near the origin, with observation distance $r \gg a$. The Coulomb-gauge potential is $\boldsymbol{A} = (\mu_0/4\pi)\int \boldsymbol{J}(\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|\,dV'$. Taylor expand the scalar kernel about the observation point while holding the source coordinates fixed. Since $\partial_i(1/r) = -r_i/r^3$, the first two terms are $1/r + \boldsymbol{r}'\cdot\boldsymbol{r}/r^3$. Terms one order farther involve quadratic products $r'_i r'_j$ and second derivatives of $1/r$. For a source entirely inside a sphere of radius a the expansion is systematically useful beyond that sphere; a probe near a wire can violate the required scale separation even at an apparently large distance from the centre of the apparatus.

The leading vector coefficient would be $\int J_i\,dV'$. But $J_i = \partial'_j(r'_i J_j)$ for a divergence-free steady source. Integrating converts it to an outer surface term that vanishes for localised currents. For a line loop the same fact reads $\oint d\boldsymbol{\ell}' = 0$. A finite source with a remote return omitted can give a spurious nonzero leading term. The disappearance of the $1/r$ magnetostatic potential here follows from charge conservation and closed currents. A radiating time-dependent source is governed by a retarded integral with a different far-field expansion.

Identify the magnetic moment without assuming a circular loop

Define $T_{ij} = \int r'_i J_j\,dV'$. The integral of $\partial'_k(r'_i r'_j J_k)$ is zero at large distance, giving $T_{ij} + T_{ji} = 0$. Thus only its antisymmetric part survives, and three independent components remain. Define $\boldsymbol{m} = \frac{1}{2}\int \boldsymbol{r}'\times\boldsymbol{J}\,dV'$. Contracting its components with the observation vector converts the first surviving potential into

$$\boldsymbol{A}_{\mathrm{dip}} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3},$$

$$\boldsymbol{B}_{\mathrm{dip}} = \frac{\mu_0}{4\pi r^3}[3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}] \quad (r>0).$$

To take this curl, apply the product rule to $r^{-3}(\boldsymbol{m}\times\boldsymbol{r})$. The curl of the second factor is $2\boldsymbol{m}$, while $\nabla r^{-3} = -3\boldsymbol{r}/r^5$. The vector triple product in the other term leaves the radial projection and subtracts three copies of the moment term, giving the expression above. It describes exterior points of the actual source; replacing the source by an ideal point dipole at its own location needs distributional terms. The moment is origin-independent for a closed current because shifting the origin adds a constant vector crossed with $\int \boldsymbol{J}\,dV' = 0$.

For a planar loop $\boldsymbol{m} = (I/2)\oint \boldsymbol{r}'\times d\boldsymbol{\ell}' = I\boldsymbol{a}$, where the oriented area follows from projecting onto each Cartesian plane. For a polygon one may sum oriented triangles from any chosen origin; the edges

to the origin cancel between adjacent triangles. A nonplanar closed wire still has an oriented projected area vector, although it has no unique scalar enclosed area. Stacking N identical turns multiplies the vector by N. This definition makes a measured moment a current times a geometrical area, with units $\mathrm{A\,m^2} = \mathrm{J/T}$.

Use distance and orientation to determine a moment

A 40-turn circular winding of radius $15\,\mathrm{mm}$ carries $0.20\,\mathrm{A}$. Its moment is

$$40(0.20)\pi(0.015)^2 = 5.65 \times 10^{-3}\,\mathrm{A\,m^2}.$$

At $r = 0.15\,\mathrm{m}$ the axial point-dipole field is

$$2(\mu_0/4\pi)m/r^3 = 0.335\,\mu\mathrm{T};$$

at an equatorial point the same distance away it is $0.168\,\mu\mathrm{T}$ directed opposite the moment. A scalar probe looking only at magnitudes would find a two-to-one change. A three-axis probe can use the sign reversal of the axial component to distinguish a dipole from a residual uniform background.

The exact axial field of the circular loop is $B_z = (\mu_0 N I a^2/2)(a^2+z^2)^{-3/2}$. Divide it by the dipole value to obtain $[1+(a/z)^2]^{-3/2}$. At ten radii it is 0.9852 of the dipole prediction, giving an error near 1.5 percent despite the large separation. Reducing that error below one percent relative to the exact value requires $[1+(a/z)^2]^{3/2} - 1 < 0.01$, or $z/a > 12.3$. A far-field fit to the moment using several sensor distances can examine how the fitted $z^3 B_z$ approaches a constant. It also exposes uncertainty in the assumed source centre: a small offset in z produces a relative distance error amplified roughly threefold in the inferred moment.

Arrange two loops to cancel the leading term

Place two identical coaxial loops at $z = \pm d/2$ with moments $+m\hat{\boldsymbol{z}}$ on the positive side and $-m\hat{\boldsymbol{z}}$ on the negative side. At a far positive point each small loop contributes its axial dipole field with its own signed moment. Define $K = \mu_0/(2\pi)$. Their sum, $Km[(z-d/2)^{-3} - (z+d/2)^{-3}]$, has no z^{-3} term. Taylor expansion to first order in d/z gives $3Kmd/z^4$. This quadrupolar decay is steeper because probing from farther away makes the spatial separation of the opposed moments harder to resolve. The result applies when both d/z and individual loop radius divided by z are small.

Near the midpoint a different approximation is needed. The exact axial fields of the two finite loops cancel at $z = 0$ but their derivatives add. For single-turn loops of radius a, differentiating the exact expression gives $B_z'(0) = 3\mu_0 I a^2 d/[2(a^2 + d^2/4)^{5/2}]$ in the stated orientation. With $a = d = 0.040\,\mathrm{m}$, $I = 1.0\,\mathrm{A}$, this is $6.74 \times 10^{-4}\,\mathrm{T/m}$. A 2.0 mm displacement from the centre then produces approximately $1.35\,\mu\mathrm{T}$. A gradiometer uses differences like these to suppress an approximately uniform background. A slight current imbalance gives a residual ordinary dipole, which eventually dominates at sufficiently large distance because it falls more slowly.

Measure torque and work as an independent calibration

A small loop in a uniform external field has forces $I\,d\boldsymbol{\ell}\times\boldsymbol{B}$ on its sides. Their net force vanishes since $\oint d\boldsymbol{\ell}=0$. The net torque survives: reducing $\oint \boldsymbol{r}\times(I\,d\boldsymbol{\ell}\times\boldsymbol{B})$ with the same antisymmetric geometrical tensor yields

$$\boldsymbol{\tau}=\boldsymbol{m}\times\boldsymbol{B}.$$

With a fixed permanent moment or a maintained current in an imposed field, the mechanical potential for orientation is $-\boldsymbol{m}\cdot\boldsymbol{B}$ up to a constant. If an external source maintains the current its electrical work matters in the total energy balance.

The winding above placed in $B=0.030\,\mathrm{T}$ at 30° experiences $8.48\times10^{-5}\,\mathrm{N\,m}$. Turning it quasistatically from alignment to perpendicular increases this orientation energy by $mB=1.70\times10^{-4}\,\mathrm{J}$. A force measurement in a slowly varying field samples the gradient: along an axis for a fixed aligned moment $F_z\simeq m\,dB_z/dz$. In a truly uniform field the dipole has torque but no translational force. A convenient test of an inferred field gradient is therefore to compare the force at the same point with moment parallel and antiparallel. Their difference $2mB_z'$ rejects a constant support force. Magnetic susceptibility, moment reorientation or a large source spanning appreciable changes in gradient invalidate the rigid-point formula.

Calibrate a gradiometer without assuming the background vanishes

A paired pickup consists of two equal small sensing loops connected in opposition. Put their centres at $z=z_0\pm l/2$, both normal to the same axial field component $B(z)$, and use N_p turns of area A_p for each. In the small-loop limit their net linkage along the series path is $N_pA_p[B(z_0+l/2)-B(z_0-l/2)]$. The spatially uniform part cancels algebraically. Expanding about the midpoint leaves $N_pA_pl[B'(z_0)+l^2B'''(z_0)/24+\cdots]$. A finite separation therefore estimates the derivative averaged over the baseline. The cubic derivative, rather than the second derivative, sets its first truncation error because the measurement is centred.

Suppose a calibration winding produces a sinusoidal gradient of peak amplitude $2.0\,\mathrm{mT/m}$ at 100 Hz across that baseline. With $N_p=50,\ A_p=2.0\times10^{-4}\,\mathrm{m^2}, l=0.030\,\mathrm{m}$, the peak linkage is $6.0\times10^{-7}\,\mathrm{Wb}$. Its time derivative gives a peak emf $0.377\,\mathrm{mV}$. Changing frequency to 200 Hz while keeping gradient amplitude unchanged should double this emf if pickup impedance and input instrumentation do not distort the measurement. A DC gradient alone produces no voltage in a stationary inductive pickup; a calibrated DC Hall pair or a deliberately moved sample implements a different measurement.

A small fractional area-turn imbalance ε couples a uniform field back into the differenced signal. For example take the positive pickup to have factor $N_pA_p(1+\varepsilon/2)$ and the negative one $N_pA_p(1-\varepsilon/2)$. Expansion now contains both lB' and εB. If a sinusoidal background of amplitude $20\,\mu\mathrm{T}$ shares the drive frequency and $\varepsilon=0.01$, its apparent gradient is

$$\varepsilon B/l = 6.67\,\mu\mathrm{T/m}.$$

This is only a third of one percent of the calibration gradient but could dominate a much smaller sample signal. Reversing the sample while leaving the pickup fixed tests which contribution reverses. Rotating the whole assembly might also change its coupling to external wiring and would implement a less selective test.

For an axial dipole field $B(z) = C/z^3$ sufficiently far from the loop, $B' = -3C/z^4$ and $B''' = -60C/z^6$. The relative leading baseline error is therefore $5l^2/(6z_0^2)$. At $l/z_0 = 0.30$, the differential reading overestimates the pointwise derivative magnitude by about 7.5 percent before accounting for finite sensing-loop area. A gradient device may be well calibrated in a nearly linear reference field and still give a biased point-gradient interpretation near a sample. Comparing measurements at two baselines tests this spatial averaging while retaining rejection of the common background.

Distinguish finite pickup area from multipole structure

Even a single coil usually records surface flux rather than the field at one mathematical point. For a dipole $m\hat{\boldsymbol{z}}$ on the symmetry axis of a coaxial circular pickup of radius b, one can calculate the flux exactly within the point-dipole approximation by integrating the vector potential along its boundary. The azimuthal potential is $A_\varphi = \mu_0 m s/[4\pi(s^2+z^2)^{3/2}]$. Multiplication by $2\pi b$ gives

$$\Phi(z) = \frac{\mu_0 m b^2}{2(b^2+z^2)^{3/2}}.$$

Compare with the point-field estimate $\pi b^2 B_z(0,z) = \mu_0 m b^2/(2|z|^3)$. The exact flux is smaller by the factor $[1+(b/z)^2]^{-3/2}$. This has exactly the same mathematical form as the correction for a finite drive loop and point probe. There is a physical reason: the mutual inductance is reciprocal. A measured deviation from a dipole fit can result from the source size, the pickup size, or both. A measurement at one distance without knowledge of the pickup geometry cannot assign the correction uniquely.

For $m = 1.0\times10^{-3}\,\mathrm{A\,m^2}$, $b = 10\,\mathrm{mm}$, $z = 30\,\mathrm{mm}$, the point-field estimate is $2.33\,\mathrm{nWb}$ per turn. The actual dipole flux through that circle is $1.99\,\mathrm{nWb}$, about 15 percent lower. If the pickup is doubled in radius, it does not simply collect four times as much flux at the same distance. In the formal limit $b\to\infty$ the explicit expression falls as $1/b$, tending to zero: the very distant return flux of the dipolar field then cancels the central flux in the whole plane. For fixed nonzero z, differentiating the expression with respect to b gives an extremum at $b = \sqrt{2}|z|$; collecting flux with an ever wider loop eventually reduces the signal.

One can turn that observation into a geometry choice for a weak-signal experiment. First choose a drive frequency slow enough for the moment to remain quasistatic and for the pickup circuit response to be calibrated. Select a pickup radius near the theoretical optimum for the fixed stand-off, then compare an intentionally smaller pickup whose field is more local. Agreement of the source moments inferred with each correct area formula provides evidence for the dipole model over the distances

used. Disagreement that persists after correcting alignment and finite pickup radii motivates resolving higher moments or modelling the actual source current, rather than assigning a different magnetic moment to the same sample at each probe distance.

Build a current dipole from a rotating charged shell

Consider a thin spherical shell of radius a carrying charge Q uniformly painted onto its insulating surface. A motor rotates the shell with angular speed Ω about z. Every little patch moves, yet the charge density observed at a fixed point in space stays constant. The steady current density is a surface current $\mathbf{K} = \sigma \Omega a \sin\theta \, \hat{\boldsymbol{\phi}}$, where $\sigma = Q/(4\pi a^2)$. This model assumes that charge remains attached to the shell and that the rim speed is small compared with the speed of light. A conducting shell allowed to rearrange its charges introduces another physical question.

Slice the sphere into narrow latitude bands. The distance across a band perpendicular to its current is $a\,d\theta$, so the circulating current associated with the band is $dI = K_\phi a \, d\theta$. Its loop area is $\pi a^2 \sin^2\theta$. All the band moments point along the rotation axis for positive charge. Summing them gives

$$m_z = \pi \sigma \Omega a^4 \int_0^\pi \sin^3\theta \, d\theta = \frac{4\pi\sigma\Omega a^4}{3} = \frac{Q\Omega a^2}{3}.$$

The integral can be checked by substituting $u = \cos\theta$, which changes $\sin^3\theta\,d\theta$ into $-(1-u^2)\,du$. The equatorial bands dominate much more strongly than their charge alone suggests. Each patch there travels faster and encloses a larger area. Using the full charge as though it were on an equatorial ring would give $Q\Omega a^2/2$, an overestimate by a factor of three halves. Both models have the same radius and total charge, so the difference tests precisely what information a magnetic dipole moment contains.

There is a useful local check. For the latitude ring at angle θ, its radius squared is $a^2\sin^2\theta$ and the distance squared from every source point to the sphere centre is a^2. Insert these into the exact axial field of a loop. The result is

$$dB_z = \mu_0 dI \sin^2\theta/(2a).$$

Integrating gives

$$B_z(0) = 2\mu_0\sigma\Omega a/3 = \mu_0 Q\Omega/(6\pi a).$$

This finite answer at the origin cannot be obtained by extrapolating a point-dipole field inwards. A uniformly magnetised sphere with $\mathbf{M} = \sigma\Omega a\,\hat{z}$ has exactly this surface current. Its solution gives a uniform interior field and an exact dipolar exterior because its angular dependence is a single degree-one harmonic.

For $a = 50\,\mathrm{mm}$, $Q = 1.0\,\mu\mathrm{C}$ and $\Omega = 100\,\mathrm{rad/s}$, the dipole moment is $8.33 \times 10^{-8}\,\mathrm{A\,m^2}$. The central field is about $0.133\,\mathrm{nT}$. On axis at $200\,\mathrm{mm}$, the field is only $2.08\,\mathrm{pT}$. Separating a field that weak from an ambient steady field requires a mechanism such as reversing the rotation or the charge

repeatedly and averaging synchronous differences. Rotor vibration in a spatially varying background can generate a signal correlated with rotation, too. An uncharged control run tests that contribution without changing the motor geometry.

Distinguish a local current from its far-field moment

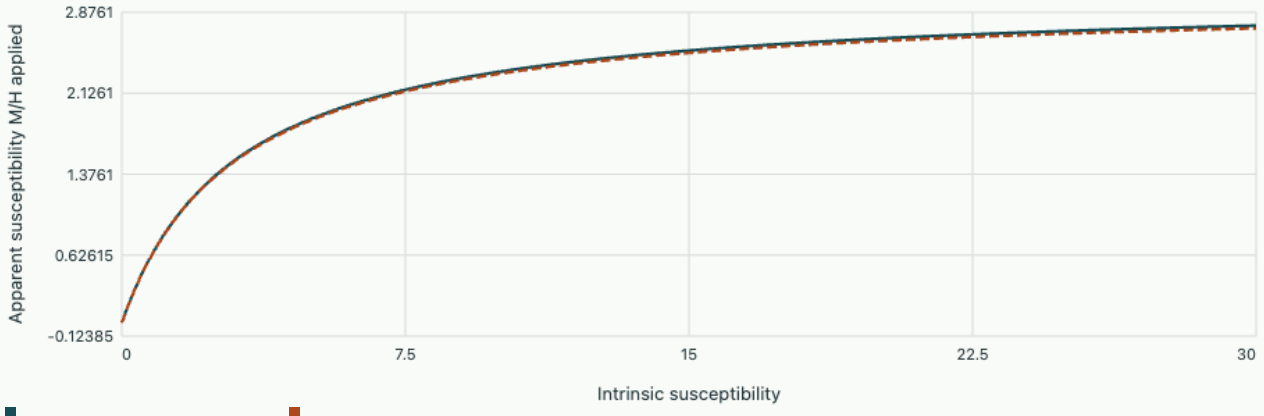
Explore the field generated by a magnetised body. Identify which exterior measurements are still sensitive to the detailed current distribution.

Open this experiment online

Try the calculation

Demagnetising factor N **0.33**

Intrinsic susceptibility **9**



Selected ellipsoid Sphere N = 1/3

At the selected susceptibility the apparent value is 2.27; the internal H field is 25.2% of the applied H field.

Compare magnetic moment and angular momentum

Suppose the mass M is distributed over the same shell in the same proportion as its charge. A mass element dm contributes axial angular momentum $dL_z = \Omega a^2 \sin^2\theta\, dm$. A transported charge element dq returns after the period $2\pi/\Omega$, so its contribution to the current moment is

$$dm_z = \Omega a^2 \sin^2\theta\, dq/2.$$

If every element has $dq/dm = Q/M$, these relations give $m_z/L_z = Q/(2M)$ on any rigid axisymmetric body, independent of whether the matter fills a solid ball or only a shell. The simple proportionality is classical and assumes the same spatial distributions. It supplies no derivation of the intrinsic spin moment of an electron.

A solid insulating sphere of radius a carries a uniformly distributed volume charge instead. Recover its moment by adding thin spherical shells and compare the angular momentum ratio.

▼ **Work through the solution**

The charge inside a radial shell is $dQ = 3Qr^2dr/a^3$. Each shell contributes $\Omega r^2 dQ/3$ from the latitude-band result. Integration from zero to a yields $Q\Omega a^2/5$. More charge now occupies small cylindrical radii, which reduces the moment at fixed total charge. If mass is also uniform in volume, the moment of inertia is $2Ma^2/5$. Their ratio is again $Q/(2M)$. If instead essentially all the mass resides in a solid core while the charge remains on an insulating outer shell, inserting the shell charge result and solid inertia gives $5Q/(6M)$, illustrating why the distribution hypothesis matters.

What mechanical measurement independently determines the moment of the shell without resolving its field in space?

▼ **Work through the solution**

Place the rotor in a calibrated approximately uniform external magnetic field at right angles to its axis. The torque has magnitude mB, obtained by summing the loop torques. With the numerical shell above and $B = 0.10\,\mathrm{T}$ it is $8.33 \times 10^{-9}\,\mathrm{N\,m}$. Reverse the charge and repeat the torque measurement to isolate the contribution odd in charge. Maintaining the axis fixed requires a compensating support torque. A nonuniform imposed field would also exert a net force and could couple a spurious lever-arm torque into the measuring fixture.

David Tong, Magnetostatics, on dipole moments, fields and forces.

Tutorial on testing a multipole approximation

Two nominally opposed loops have moment difference $0.01m$ and separation d. Estimate the positive axial distance at which their residual dipole equals their leading quadrupole in magnitude.

▼ **Work through the solution**

The respective magnitudes are $K(0.01m)/z^3$ and $3Kmd/z^4$. Equating them gives $z = 300d$. An imbalance of one percent can therefore determine the most distant field even though a closer scan shows an excellent quadrupolar law.

A nonplanar wire follows three sides of the unit square in the xy-plane, then returns to its start via a raised corner. Can its dipole moment have an in-plane component? Give a construction that determines it without surface integration.

▼ **Work through the solution**

Sum $\frac{I}{2}\sum_j \boldsymbol{r}_j \times \boldsymbol{r}_{j+1}$ over its vertices in traversal order, including the closing edge. Its projections onto the vertical coordinate planes can span nonzero signed area, generating in-plane moment

components. A rigid translation of every vertex adds a telescoping zero, providing a check.

Opposed windings produce fields that scale nearly as r^{-4}. By what factor does a small transverse pickup's induced voltage amplitude change if source-to-pickup distance doubles while the whole source current is driven sinusoidally at the same sufficiently low frequency?

▼ Work through the solution

If the pickup area and orientation remain fixed, its linked flux scales as its normal field. The time derivative supplies the same factor ω in both runs. Its voltage amplitude therefore falls by about 16, provided the magnetostatic spatial approximation and quadrupolar regime remain valid. A dipole-dominated residual would instead give about eight.

Compare a static cancellation with cancellation of the leading radiating multipole

Magnetism, induction and field dynamics · 4. From microscopic moments to measured magnetisation

A magnetised sample can make a compass turn without a power supply attached. The macroscopic theory assigns it bound currents in its bulk and on its surface. These are calculated sources for the magnetic field. We will reconstruct them from magnetic moments, predict signals from finite samples, and ask what an exterior field measurement can identify.

Define a material field at an intermediate scale

Choose a volume large enough to average over microscopic moments and small enough to resolve the sample's macroscopic variation. Let $\boldsymbol{M}(\boldsymbol{r})$ be the vector sum of moments within it divided by its volume. Its units are $\mathrm{A/m}$. A permanent aligned collection may maintain $\boldsymbol{M}$ without an applied field. An induced collection may point with or against the applied field. Electric charges need not traverse the entire sample in a closed macroscopic wire for either mechanism. Electron orbital moments and spin contribute to the measured moment; for a classical exterior field calculation any small magnetic dipole has the same leading field once its moment is specified.

A bar of volume $8.0\,\mathrm{cm}^3$ with uniform $M = 2.0 \times 10^4\,\mathrm{A/m}$ has total moment $m = 0.160\,\mathrm{A\,m}^2$. A torsion balance in a uniform $10\,\mathrm{mT}$ transverse field would measure a maximum torque $1.60 \times 10^{-3}\,\mathrm{N\,m}$ if its moment stayed fixed. Dividing a measured torque by field and sample volume infers the component normal to the external field. Alignment of a previously soft material could instead alter the very moment being measured. Distinguishing fixed-source and induced-response experiments will matter when we calculate mechanical energy.

Derive the bound sources by integration by parts

Sum the dipole vector potentials over a material region V. With observation point outside the matter, a volume element contributes $(\mu_0/4\pi)\boldsymbol{M}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')dV'/R^3$, where $R = |\boldsymbol{r} - \boldsymbol{r}'|$. Note that $\nabla'(1/R) = (\boldsymbol{r} - \boldsymbol{r}')/R^3$. Apply the identity $\nabla' \times (\boldsymbol{M}/R) = (\nabla' 1/R) \times \boldsymbol{M} + (\nabla' \times \boldsymbol{M})/R$. The desired first cross product is minus the first term on the right. A volume integral of a curl is the surface integral of outward normal crossed with the vector. This yields

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\left[\int_V \frac{\nabla' \times \boldsymbol{M}}{R}\,dV' + \oint_{\partial V} \frac{\boldsymbol{M} \times \boldsymbol{n}'}{R}\,dA'\right].$$

Comparing with the ordinary current potential identifies $\boldsymbol{J}_b = \nabla \times \boldsymbol{M}$, in A/m², and $\boldsymbol{K}_b = \boldsymbol{M} \times \boldsymbol{n}$, in A/m. The first can vanish throughout a uniformly magnetised object, while the second closes around its exterior. A literal set of microscopic loops illustrates the sign. Neighbouring loop-edge currents cancel between equal adjacent moments, whereas their edges on the outer boundary remain uncompensated. When the moments vary slowly, interior cancellation is incomplete and yields the curl.

An alternative treatment extends $\boldsymbol{M}$ by zero outside. Its distributional curl already includes the surface sheet at the discontinuity. Use either this full derivative or the separate bulk and surface integrals; including both descriptions simultaneously doubles the boundary source. At an internal interface with

chosen normal from material 1 to 2 the two surface contributions combine to $(\boldsymbol{M}_1 - \boldsymbol{M}_2) \times \boldsymbol{n}$. This expression is unchanged by interchange of the two labels accompanied by reversal of the normal.

Work through a uniformly magnetised cylinder

Let a finite circular cylinder of radius a and length $2l$ have $\boldsymbol{M} = M\hat{\boldsymbol{z}}$. On the curved side $\hat{\boldsymbol{z}} \times \hat{\boldsymbol{s}} = \hat{\boldsymbol{\varphi}}$, hence $K_\varphi = M$; on its end faces the cross product vanishes. The cylinder is equivalent, for its magnetic field, to a finite solenoid having turn-current per unit length M. A strip of sidewall of height dz' carries ring current $M\,dz'$. Insert this into the axial finite-loop field from the first chapter and integrate from $-l$ to l.

$$B_z(z) = \frac{\mu_0 M}{2}\left[\frac{z+l}{\sqrt{a^2+(z+l)^2}} - \frac{z-l}{\sqrt{a^2+(z-l)^2}}\right].$$

At the centre $B_z = \mu_0 M l/\sqrt{a^2+l^2}$. For $a = 10\,\mathrm{mm}$, $l = 20\,\mathrm{mm}$ and $M = 1.0\times10^4\,\mathrm{A/m}$, the value is $11.24\,\mathrm{mT}$, below the long-cylinder limit $12.57\,\mathrm{mT}$. At the end centre $z = l$, only the first term contributes, giving $6.10\,\mathrm{mT}$. A probe just outside the end and one just inside see the same normal $\boldsymbol{B}$ in the limit, even though the auxiliary normal $\boldsymbol{H}$ can jump because $\boldsymbol{M}$ ends abruptly. The field near an edge is not uniform and can vary strongly with the sensor's position. Stating the axis and position is therefore essential when inferring magnetisation from a reading.

Use a nonuniform sample to separate bulk and surface effects

For an infinitely long cylinder set $\boldsymbol{M} = M_0(s/a)\hat{\boldsymbol{\varphi}}$ for $s < a$, zero outside. The cylindrical curl gives $J_{b,z} = s^{-1}\partial_s(sM_\varphi) = 2M_0/a$ in the bulk. The side normal is $\hat{\boldsymbol{s}}$, so $K_{b,z} = -M_0$. Integrated over a cross section the bulk current is $2\pi a M_0$; the axial sheet supplies $-2\pi a M_0$. Their cancellation means an exterior Amperian circle sees zero enclosed current. Its zero field follows from symmetry and the absence of externally imposed fields.

Inside, a circle at s encloses $2M_0\pi s^2/a$. Dividing its circulation by $2\pi s$ gives

$$B_\varphi = \mu_0 M_0 s/a.$$

Thus $\boldsymbol{H} = \boldsymbol{B}/\mu_0 - \boldsymbol{M} = 0$ both inside and outside here, despite nonzero interior $\boldsymbol{B}$. For $a = 20\,\mathrm{mm}$, $M_0 = 500\,\mathrm{A/m}$, the bulk current density is $5.0\times10^4\,\mathrm{A/m^2}$, total bulk current $62.8\,\mathrm{A}$, and field just inside the rim $0.628\,\mathrm{mT}$. A magnetometer outside reports zero within this ideal geometry. Such a null measurement establishes neither zero $\boldsymbol{M}$ nor absence of all microscopic moments.

Compare a field map and a pickup-coil measurement

A stationary inductive pickup near a stationary permanent magnet has no emf from that static flux alone. An experiment can move the sample, rotate it or modulate a known magnetising field to produce

$-N\,d\Phi/dt$. A Hall sensor or calibrated magnetometer can instead respond to a static field. To compare methods, place a small pickup near the axis of the preceding finite cylinder and oscillate its axial position as $z = z_0 + \zeta\cos\omega t$. For ζ small, the flux per turn is approximately $A_p[B_z(z_0) + \zeta B_z'(z_0)\cos\omega t]$. The voltage amplitude is $NA_p\zeta\omega|B_z'(z_0)|$. It measures a gradient multiplied by displacement amplitude, whereas a static Hall value measures the field itself.

At the symmetric cylinder centre $B_z'(0) = 0$, so the first harmonic vanishes even though the field is strong. Expanding one more order gives a constant shift and a second harmonic proportional to $B_z''(0)\zeta^2$, since $\cos^2\omega t = (1 + \cos 2\omega t)/2$. Finding a very small first harmonic at the centre is therefore a check on positioning and symmetry. It is not a calibration of zero magnetisation. Far from the ends, a measurement of the whole sample's dipole field instead yields $\int \boldsymbol{M}\,dV$. Different internal distributions can share that integral while producing different nearer fields. Report the geometry, the actual observable and the assumed uniformity whenever quoting a single recovered value of M.

Examine bound currents and field measurements in a magnetised sample

Use fictitious poles to compute a measurable field

There is a second useful representation of a given magnetisation. In a simply connected region without free current, $\nabla \times \boldsymbol{H} = 0$, so choose $\boldsymbol{H} = -\nabla\psi$. Substitution into $\nabla\cdot(\boldsymbol{H} + \boldsymbol{M}) = 0$ gives

$$\nabla^2\psi = \nabla\cdot\boldsymbol{M}.$$

The potential therefore behaves mathematically as if it were produced by volume density $\rho_m = -\nabla\cdot\boldsymbol{M}$ and surface density $\sigma_m = \boldsymbol{M}\cdot\boldsymbol{n}$. The surface term follows by extending the magnetisation by zero across the boundary, or by requiring continuity of $B_n = \mu_0(H_n + M_n)$. Integrate these densities with kernel $1/(4\pi R)$ to obtain ψ; there is no factor of ϵ_0^{-1} because these conventions assign ψ units of amperes. The resulting $\boldsymbol{H}$ must be converted back to $\boldsymbol{B}$, adding $\mu_0\boldsymbol{M}$ inside matter.

For the uniformly magnetised finite cylinder the pole representation puts densities $+M$ and $-M$ on the two end faces, none on the sides. Earlier the bound-current representation placed current on the curved side and none on the ends. They predict the same physical field. A single uniformly loaded disk of pole density σ_m at $z = 0$ has on-axis scalar potential $\psi = \sigma_m[\sqrt{z^2 + a^2} - |z|]/2$, obtained by integrating $2\pi s\,ds/[4\pi\sqrt{s^2 + z^2}]$. Its derivative gives

$$H_z = (\sigma_m/2)[\operatorname{sgn}(z) - z/\sqrt{z^2 + a^2}].$$

At the disk plane the two one-sided fields jump by σ_m, precisely the normal-H jump demanded by the magnetisation.

At the midpoint of the cylinder add the two oppositely charged disks. Their axial fields both point opposite the magnetisation and total $H_z = -M[1 - l/\sqrt{a^2 + l^2}]$. Adding M yields the same B_z as the side-current integral. At $l = 2a$, this internal demagnetising field is $-0.1056M$; for a very short cylinder $l \ll a$ it approaches $-M$. A uniformly specified magnetisation in a thin disc therefore

produces only a small normal B at its centre despite the large M and H terms that nearly cancel. In practice, a soft magnetic material in such a shape may choose a nonuniform magnetisation or domains. The calculation describes the field of the imposed distribution; it does not establish that the distribution is a stable material state.

The 'poles' are convenient mathematical source terms, not isolated magnetic charges: adding the effective volume and surface poles over a finite bounded sample gives

$$-\int \nabla \cdot \boldsymbol{M}\, dV + \oint \boldsymbol{M} \cdot \boldsymbol{n}\, dA = 0.$$

North and south effective poles may be far apart, but a surrounding closed Gaussian surface still has zero net magnetic flux. If a model predicts a $1/r^2$ radial field of one sign through an entire distant sphere around a fully included bar magnet, test whether the opposite face or volume contribution has been accidentally dropped.

Infer the field of a spherical void by subtracting a known sample

Superposition can calculate a local cavity field before introducing a detailed microscopic model. Imagine a large body with uniform imposed magnetisation $\boldsymbol{M}$ near an interior point, and remove a small spherical piece. The magnetic field after removal is the original macroscopic field minus the contribution of the excised sphere, provided the surrounding magnetisation stays unchanged. A uniformly magnetised sphere in otherwise empty space alone has uniform internal $H = -M/3$, as can be derived from its pole distribution $\sigma_m = M\cos\theta$. At the sphere centre $\boldsymbol{H}$ follows by integrating the field from surface elements $-\sigma_m \hat{\boldsymbol{r}}\, dA/(4\pi a^2)$. The z-component involves the spherical average $\langle \cos^2\theta \rangle = 1/3$; transverse contributions vanish by symmetry. The unique dipolar harmonic extends that central value uniformly throughout the sphere.

The sphere's own internal B is therefore $2\mu_0 M/3$. Immediately after its removal the field in the empty cavity is

$$B_{\mathrm{cav}} = B_{\mathrm{bulk}} - 2\mu_0 M/3 = \mu_0(H_{\mathrm{bulk}} + M/3).$$

This assumes the sphere is small relative to the variation scale of the pre-existing magnetisation and far from the sample's external boundary on its own size scale. A different cavity shape changes the subtraction factor, and a physical bore can rearrange the surrounding magnetisation or admit demagnetising fields from its ends. Local-field interpretations depend on probe geometry as well as material properties.

As a numerical illustration, suppose a permanent or pinned surrounding material has local $M = 1500\,\mathrm{A/m}$ and $H_{\mathrm{bulk}} = 400\,\mathrm{A/m}$, aligned. The macroscopic bulk field is $\mu_0(1900) = 2.39\,\mathrm{mT}$. A small spherical empty cavity under the stated frozen-distribution approximation has $B_{\mathrm{cav}} = \mu_0(900) = 1.13\,\mathrm{mT}$. Both readings can be correct for the same initial material. A Hall element inserted into a drilled hole measures the field in that hole; labelling it 'the internal material field' without modelling the probe cavity can cause a substantial discrepancy. Small hole volume does not by itself

make the local cavity field correction small, since area and inverse-square field factors cancel in the spherical surface integral.

For a final comparison, cut a very thin flat slit perpendicular to M instead of a sphere, idealising its lateral dimensions as large relative to its thickness. The facing effective pole sheets produce a strong additional field within the slit. Normal B stays continuous across its faces, so the vacuum reading tends towards the original normal B_{bulk} when edge effects and magnetisation changes are negligible. A long narrow cavity parallel to the magnetisation instead tends towards continuity of tangential H on its long sides. Comparing the sphere and slit is an operational way to understand why B and H are different macroscopic variables rather than two symbols for one ambiguous 'magnetic intensity'.

Read the current implied by a magnetisation

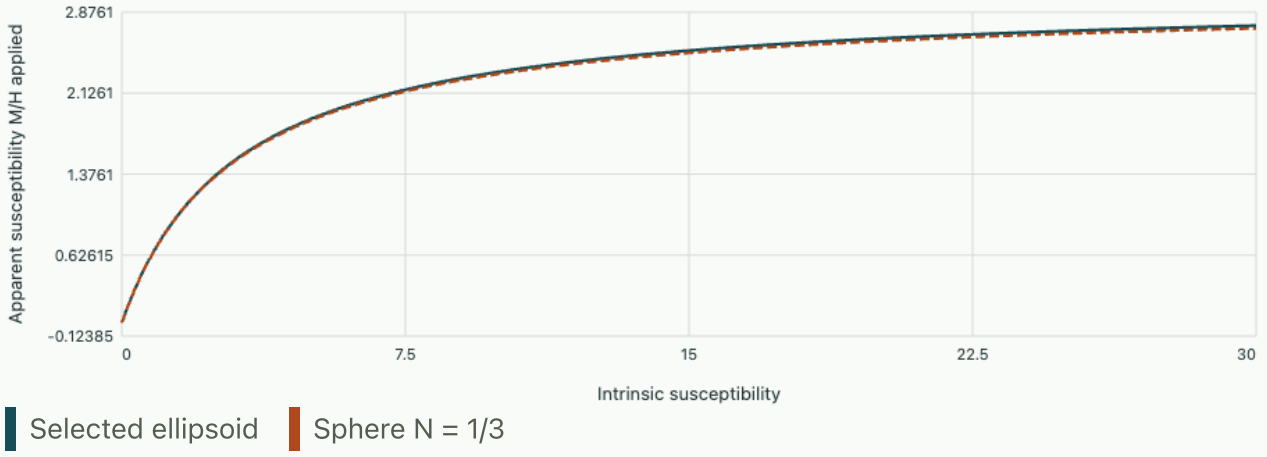
Change the magnetisation and compare interior and exterior fields at the surface. Check which tangential or normal field condition is relevant.

Open this experiment online

Try the calculation

Demagnetising factor N **0.33**

Intrinsic susceptibility **9**



At the selected susceptibility the apparent value is 2.27; the internal H field is 25.2% of the applied H field.

Reconstruct an invisible current distribution in a magnetised film

An extended magnetic film fills the slab $-a < z < a$. A fabrication process leaves a tangential magnetisation $M_x(z)\hat{x}$. Let the magnetisation vanish outside. There is no free current in this calculation, and there is no applied field at infinity. A field measurement above the centre of a very wide

film can then give nearly zero even when the magnetisation inside varies substantially. Work out both the bound currents and the local magnetic field before deciding what an external zero reading means.

Inside the material the bound volume current is

$$J_{\mathrm{b},y} = dM_x/dz.$$

The upper face has outward normal $+\hat{z}$, so its bound sheet current is $-M_x(a^-)\hat{y}$. On the lower face the outward normal reverses and the sheet current is $+M_x(-a^+)\hat{y}$. Integrating the volume current through the thickness gives $M_x(a^-) - M_x(-a^+)$. Thus the combined sheet and volume currents sum to zero per unit length across the entire film. This cancellation is independent of how much structure the magnetisation has inside the slab. An alternative derivation extends M_x by zero beyond the faces and differentiates that piecewise function as a distribution. The jumps generate both surface terms with their proper signs.

For an explicit depth profile take $M_x = M_0(1 + z/a)$. There is a uniform bulk current M_0/a, no lower surface current, and an upper surface current $-2M_0$. Planar symmetry allows only $B_x(z)$. Ampère's law gives

$$dB_x/dz = \mu_0 J_y.$$

Integrating from below the film, where the field vanishes, gives $B_x = \mu_0 M_0(1 + z/a)$ inside. At the upper face the field jumps back to zero. The result also follows directly from $H_x = B_x/\mu_0 - M_x$: its tangential component is continuous wherever there is no free surface current, and there is no volume source for its variation. In this geometry $H_x = 0$ throughout.

Take $a = 0.20\,\mathrm{mm}$ and $M_0 = 2.0 \times 10^4\,\mathrm{A/m}$. The interior current density is $1.0 \times 10^8\,\mathrm{A/m^2}$. That large number represents microscopic circulating moments after coarse graining. It does not imply a transport current passing through external electrical contacts. The peak interior field is approximately $50.3\,\mathrm{mT}$ just below the upper face, and the average is about $25.1\,\mathrm{mT}$. An exterior sensor over the central region of the infinite idealisation reads zero at any height. With a finite lateral extent, currents must close around the film edges. The resulting fringe signal contains shape information and becomes measurable as the probe approaches the rim.

Rotate the magnetisation through ninety degrees so that it now points in the z direction while its magnitude keeps the same depth profile. Each local curl vanishes. Each face has parallel magnetisation and normal, giving zero bound sheet current. Consequently the solution has zero B everywhere in the ideal infinite geometry. Within the film $H_z = -M_z$. In the magnetic-pole construction the two surface pole densities and the bulk density $-dM_z/dz$ provide precisely that auxiliary field. These two slab examples illustrate a real inverse problem: very different internal magnetic states can produce the same exterior measurement. Knowing the material orientation or imposing a changed external field supplies additional information.

Can two depth profiles with the same average magnetisation be distinguished by an ideal measurement of the tangential magnetic flux through the film thickness?

▼ **Work through the solution**

Choose a rectangular sensing surface with its normal along x, its height equal to the film thickness, and its length w along y. The ideal internal flux is $w\mu_0 \int_{-a}^{a} M_x(z)\,dz$. This measurement determines one weighted integral. A constant M_0 and the linear example above give identical results even though the constant profile has only surface currents. A depth-selective measurement would weight the profiles differently. For example, integrating over just the upper half gives $\mu_0 waM_0$ in the constant case and $3\mu_0 waM_0/2$ in the linear case. Actual pickup loops must close outside the sample, so include their full geometry when inferring an internal flux.

Suppose the linear profile relaxes uniformly in amplitude after an aligning field is removed. What information would the integrated voltage of a calibrated pickup provide?

▼ **Work through the solution**

If the pickup links N turns with the preceding full-thickness flux and M_0 decays from its initial value to zero, the time-integrated emf has magnitude $2N\mu_0 waM_0(0)$. Faraday's law supplies this integral independently of whether the decay is exponential or has several relaxation times. A finite integration window that ends before zero magnetisation underestimates the initial amplitude unless the residual flux is measured. Leakage from the changing applied alignment field can also drive the loop directly. A reference winding of the same area, measured without the specimen, allows that part of the transient to be estimated.

David Tong, Electromagnetism in Matter, on macroscopic magnetisation and bound currents.

Tutorial on reconstructing material sources

In the nonuniform cylinder let $M_\varphi = M_0(s/a)^3$ instead. Obtain the bulk current density, the side sheet, the total bound current through the full cross section and the interior field.

▼ **Work through the solution**

$$J_{b,z} = s^{-1}\partial_s(M_0 s^4/a^3) = 4M_0 s^2/a^3.$$

At the wall $K_{b,z} = -M_0$. Integration gives $2\pi aM_0$ in the bulk and its opposite on the wall. For an interior circle the enclosed bulk current is $2\pi M_0 s^4/a^3$, giving $B_\varphi = \mu_0 M_0 (s/a)^3$. Both total cancellation and dimensional consistency check the result.

Two adjoining media have tangential magnetisations $400\hat{\boldsymbol{x}}$ and $100\hat{\boldsymbol{x}}\ \mathrm{A/m}$. The normal from 1 to 2 is $+\hat{\boldsymbol{z}}$. Find the net bound sheet current and the associated tangential jump of $\boldsymbol{B}$ if no free surface current is present.

▼ **Work through the solution**

The equivalent sheet is

$$300\hat{\boldsymbol{x}} \times \hat{\boldsymbol{z}} = -300\hat{\boldsymbol{y}}\,\mathrm{A/m}.$$

Tangential H_x is continuous, so

$$B_{2x} - B_{1x} = \mu_0(M_{2x} - M_{1x}) = -0.377\,\mathrm{mT}.$$

Crossing this difference with the chosen normal gives $\mu_0 \boldsymbol{K}_b$, matching the microscopic Ampère law.

A long solenoidal sample has known M only to 5 percent. Can a single Hall reading at the centre establish that its magnetisation is uniform over the entire volume to that accuracy?

▼ **Work through the solution**

No. The reading is one geometry-weighted integral of the source. Reversing magnetisation can remove a stable external offset, and measurements at several axial positions can test the uniform-cylinder shape, but the single central datum cannot uniquely determine a spatially resolved source. Agreement of a fitted constant at one point supplies no independent uniformity test.

Magnetism, induction and field dynamics · 5. Interfaces, demagnetising fields and the work of magnetisation

Two samples made of the same material can respond very differently to an externally applied field if one is a long rod and the other is a thin disc. The induced moment depends on the internal field, which the magnetised sample helps create. The same distinction reappears when one tries to infer a force from stored magnetic energy.

Introduce the auxiliary field by removing a known current

With magnetisation present the total steady current is $\boldsymbol{J}_f + \nabla \times \boldsymbol{M}$, where the curl includes any sheet contribution if $\boldsymbol{M}$ is extended distributionally. Define $\boldsymbol{H} = \boldsymbol{B}/\mu_0 - \boldsymbol{M}$, giving $\nabla \times \boldsymbol{H} = \boldsymbol{J}_f$. A small ribbon crossing an interface with normal $\boldsymbol{n}$ directed from 1 to 2 gives $\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{K}_f$, and a thin pillbox gives

$$\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0.$$

The boundary conditions distinguish the field whose circulation counts free current from the field whose flux has no sources.

If both media are linear, isotropic and local,

$$\boldsymbol{B}_i = \mu_i \boldsymbol{H}_i.$$

In the absence of free surface current H_t and B_n are continuous. For angles measured from the normal, $\tan\theta = B_t/B_n$ therefore gives $\tan\theta_2/\tan\theta_1 = \mu_2/\mu_1$ when the nonzero components permit the ratio. For instance, going from vacuum into a medium with $\mu_r = 5$, let $B_{1n} = 2\,\mathrm{mT}$ and $B_{1t} = 1\,\mathrm{mT}$. The new components are 2 and $5\,\mathrm{mT}$, tilting the local field from 26.6° to 68.2°. A mounted probe sensitive only to the normal component would register no jump. This local argument says which components can jump; it does not solve the complete field of a finite piece of material.

Solve a sphere to find its internal field

Put a uniform isotropic sphere of radius a and permeability $\mu_r\mu_0$ in an originally uniform external $\boldsymbol{H}_0 = H_0\hat{\boldsymbol{z}}$. There is no free current in or near it, so locally $\boldsymbol{H} = -\nabla\psi$. In each uniform domain $\nabla \cdot \boldsymbol{B} = 0$ implies Laplace's equation for ψ. Spherical symmetry of the geometry and the linear far-field potential select the angular harmonic $\cos\theta$. The finite interior solution is $-Cr\cos\theta$; the exterior is $-H_0 r\cos\theta + D\cos\theta/r^2$. Tangential H agrees at the surface when ψ is continuous up to a chosen common constant. Normal B enforces the second equation:

$$C = H_0 - D/a^3,$$

$$\mu_r C = H_0 + 2D/a^3,$$

$$C = \frac{3H_0}{\mu_r + 2}.$$

Inside, $H_{\mathrm{int}} = C$ is uniform. Multiplying by susceptibility $\chi = \mu_r - 1$ gives

$$M = \chi C.$$

Rearranging yields the memorable self-consistency relation $H_{\mathrm{int}} = H_0 - M/3$. The factor $1/3$ came from solving both boundary conditions for a sphere and is not a universal material property. The dipolar exterior carries moment $m = 4\pi D = (4\pi a^3/3)M$, as expected from summing the uniform magnetisation. An ellipsoid with a principal axis parallel to the drive can likewise have uniform magnetisation and $H_{\mathrm{int}} = H_0 - N_d M$ with a geometric demagnetising factor; an arbitrary block usually has spatial variations near edges.

Take $\mu_r = 10$ and $H_0 = 200\,\mathrm{A/m}$. The sphere has $H_{\mathrm{int}} = 50\,\mathrm{A/m}$, $M = 450\,\mathrm{A/m}$ and $B_{\mathrm{int}} = 0.628\,\mathrm{mT}$. A naive use of $M = 9H_0$ would give four times the actual moment. A long narrow specimen parallel to the drive, with very small N_d, comes much closer to that latter value if the material remains linear. For the sphere $\mu_r \to \infty$ gives $B_{\mathrm{int}} \to 3\mu_0 H_0$ rather than infinity; meanwhile $H_{\mathrm{int}} \to 0$. These two trends are compatible because B/H is growing. Calling either simply a 'field enhancement' without specifying the measured field obscures the limiting result.

Derive the incremental energy for a reversible medium

Imagine increasing the current in a winding slowly while holding all material boundaries fixed. Faraday's law makes the terminal voltage, apart from resistance, $d\lambda/dt$, where λ is the flux linkage. The source delivers incremental energy $I\,d\lambda$. In a uniform toroidal core of mean path length ℓ, area A, and N turns,

$$H = NI/\ell,\; \lambda = NBA.$$

Therefore $I\,d\lambda = (A\ell)H\,dB$. In a lossless quasistatic medium the magnetic energy density accumulated along a reversible constitutive path is $w(B) = \int_0^B H(B')\,dB'$. When $B = \mu H$ with constant μ, this reduces to $BH/2$. The corresponding coenergy $w'(H) = \int_0^H B(H')\,dH'$ satisfies $w + w' = BH$ for a single-valued curve through the origin.

As a nonlinear example, use $H = \alpha B + \beta B^3$ with $\alpha = 10^3\,\mathrm{A/(m\,T)}$ and $\beta = 4.0 \times 10^3\,\mathrm{A/(m\,T^3)}$ over the measured range. At $B = 0.50$ T, $H = 1000\,\mathrm{A/m}$. The integrated energy is

$$w = \alpha B^2/2 + \beta B^4/4 = 187.5\,\mathrm{J/m^3};$$

half the endpoint product gives $250\,\mathrm{J/m^3}$ and misses the curvature. Coenergy is $312.5\,\mathrm{J/m^3}$, compensating the difference. The differential permeability is $dB/dH = (\alpha + 3\beta B^2)^{-1} = 2.50 \times 10^{-4}\,\mathrm{H/m}$, whereas the ratio $B/H = 5.00 \times 10^{-4}\,\mathrm{H/m}$. A small alternating signal about a DC operating point samples the differential value. These numbers are properties of the model curve and should not be mistaken for a universal ferromagnetic law.

Add an air gap and localise the energy

Give a toroidal magnetic circuit a thin gap of length g, neglecting fringing, leakage and flux crowding. For equal cross sections the same normal B passes through core and gap. Ampère's law gives

$$NI = H_c \ell_c + B g/\mu_0.$$

For linear μ_r, this yields

$$B = \mu_0 NI/(\ell_c/\mu_r + g).$$

With $\ell_c = 0.20\,\mathrm{m},\ g = 1.0\,\mathrm{mm},\ \mu_r = 1000,\ NI = 100$ A-turns the result is 0.1047 T. The gap consumes 83.3 A-turns; the much longer core consumes 16.7. Vary the two lengths separately to see why high permeability cannot remove the gap contribution.

Their energy ratio follows from $B^2/(2\mu)$ times volume, giving $U_g/U_c = g\mu_r/\ell_c = 5.0$. For area $A = 1.0\,\mathrm{cm}^2$, the gap stores $0.436\,\mathrm{mJ}$ and the core $0.0873\,\mathrm{mJ}$. The simple force scale at one nearly uniform normal gap face is

$$B^2 A/(2\mu_0) = 0.436\,\mathrm{N}.$$

Decreasing g at fixed current changes the total flux, so differentiating only the instantaneous gap energy while holding everything else implicitly fixed need not reproduce the mechanical force. One may use a vacuum stress surface or a coenergy derivative at prescribed current. Both methods require a clear statement about what is held fixed during the imagined displacement.

Vary the gap and the permeability independently

Distinguish reversible storage from a measured loop

Real ferromagnetic samples can have history-dependent magnetisation. If B and H trace a closed hysteresis loop, the integral $\oint H\,dB$ measures energy per volume delivered in one quasistatic cycle, with a positive sign for a dissipative cycle under passive conventions. A single-valued energy density of B alone cannot reproduce this nonzero path integral. At finite frequency conductive eddy losses and relaxation can broaden a measured loop, so a loop measured at one rate alone cannot isolate a purely quasistatic hysteresis loss.

For an approximately parallelogram-shaped cycle spanning switching fields $\pm 120\,\mathrm{A/m}$ and switching induction swing $0.80\,\mathrm{T}$ from lower to upper branch, its approximate enclosed area is $240 \times 0.80 = 192\,\mathrm{J/m^3}$ per cycle. At 50 Hz a $12\,\mathrm{cm}^3$ sample dissipates approximately $0.115\,\mathrm{W}$. The stated swing here is the full separation, eliminating a common factor-of-two ambiguity. A power meter on the terminals also sees the winding's I^2R and any energy leaving the apparatus; a separate resistance measurement and a frequency sweep help isolate the material contribution.

Reconstruct an anisotropic interface from boundary data

At an interface with a single crystal the relation between $\boldsymbol{H}$ and $\boldsymbol{B}$ may be tensorial. The formulas for angle refraction derived earlier assume scalar permeability; the underlying boundary conditions do not. Take a planar interface with normal $+\hat{\boldsymbol{z}}$, vacuum below and a linear reciprocal crystal above. Choose units such that its relative permeability in the x, z plane is the symmetric matrix with entries $\mu_{xx} = 4, \mu_{xz} = \mu_{zx} = 1, \mu_{zz} = 2$. Its positive determinant and positive diagonal make the associated magnetic energy positive. Suppose the vacuum drive near the boundary has $H_{1x} = 100\,\mathrm{A/m}$ and $B_{1z}/\mu_0 = 200\,\mathrm{A/m}$, with no y component and no free sheet current. These are local interface data, not a complete boundary condition for a finite crystal.

Continuity of tangential H gives $H_{2x} = 100$. Continuity of normal B gives $100 + 2H_{2z} = 200$, hence $H_{2z} = 50\,\mathrm{A/m}$. The crystal's tangential flux density then follows from $B_{2x}/\mu_0 = 4(100) + 50 = 450\,\mathrm{A/m}$; $B_{2z}/\mu_0 = 200$ by construction. In the crystal $\boldsymbol{H}$ and $\boldsymbol{B}$ point at different angles to the normal, with tangent ratios 2 and 2.25 respectively. The local reversible magnetic energy density is

$$\boldsymbol{H} \cdot \boldsymbol{B}/2 = \mu_0[100(450) + 50(200)]/2 = 0.0346\,\mathrm{J/m^3}.$$

Replacing the tensor by a scalar fitted to the normal component would give the wrong tangent field and energy even though that fitted scalar reproduces one measurement by design.

For a general symmetric matrix, the cross energy in $w(\boldsymbol{H}) = \boldsymbol{H} \cdot \boldsymbol{\mu}\boldsymbol{H}/2$ involves both off-diagonal elements. Its derivative with respect to one component yields the corresponding component of $\boldsymbol{B}$. One can test reciprocity by preparing two small field increments about the same reversible operating point and interchanging their directions. The mixed incremental energy should agree. Large hysteresis loops, saturation along an easy axis or externally biased nonreciprocal media require more information before equating the two cross responses. The scalar material tables often used for a gapped iron core therefore encode a particular direction and operating regime.

Determine a demagnetising factor or an intrinsic susceptibility

For an ellipsoidal sample along a principal axis, combine $H_{\mathrm{int}} = H_0 - N_d M$ with $M = \chi H_{\mathrm{int}}$. Define apparent susceptibility $\chi_{\mathrm{app}} = M/H_0$. Rearrangement gives

$$1/\chi_{\mathrm{app}} = 1/\chi + N_d.$$

This linear relation suggests a comparative measurement that can separate material and shape without having a probe inside either sample. Prepare geometrically distinct ellipsoids from one linear isotropic material, measure their moments along specified axes in the same calibrated uniform field, and divide by their respective volumes. If their demagnetising factors are independently known and the material history has not changed, the quantities $1/\chi_{\mathrm{app}} - N_d$ should agree.

Take intrinsic $\chi = 20$. A sphere with $N_d = 1/3$ gives $\chi_{\mathrm{app}} = 2.609$; a slender ellipsoid with $N_d = 0.050$ gives $\chi_{\mathrm{app}} = 10.0$. One might attribute their nearly fourfold moment ratio to different

compositions if the shapes were omitted. Suppose the measured slender-sample value is 9.0 instead. Its inferred intrinsic susceptibility would be $1/(1/9 - 0.05) = 16.36$. Comparing the sphere's inferred value of 20 tests whether the discrepancy exceeds shape, volume and magnetometer uncertainties. The inversion is especially ill-conditioned near the large-χ apparent saturation $1/N_d$; small errors in a heavily demagnetised sample then produce large errors in the recovered intrinsic response.

The sensitivity can be quantified without introducing a fitting convention. Differentiate $\chi = \chi_{\mathrm{app}}/(1 - N_d\chi_{\mathrm{app}})$, giving $d\chi/d\chi_{\mathrm{app}} = 1/(1 - N_d\chi_{\mathrm{app}})^2$. At the preceding spherical point this derivative is nearly 59, so an absolute error 0.02 in the apparent value shifts the recovered intrinsic result by about 1.18. For the slender sample the derivative is only four at the stated point. The large spherical amplification follows from subtraction of nearly equal quantities, not poor electrical calibration. An elongated specimen may therefore yield a more informative intrinsic measurement even if a sphere is easier to machine and align.

Check a material explanation against mechanical and electrical work

Imagine measuring a single coil fitted with an adjustable magnetic insert. At a particular position the current ramp up from zero to 0.50 A produces linkage $\lambda(I) = aI + bI^3$ with $a = 0.040\,\mathrm{H}$, $b = -0.020\,\mathrm{H/A^2}$ over this restricted range. Differential inductance is $a + 3bI^2$ and stays positive here. Stored energy is $\int I\,d\lambda = aI^2/2 + 3bI^4/4$, whereas coenergy is $\int \lambda dI = aI^2/2 + bI^4/4$. At the stated end current they are 4.06 and 4.69 mJ. Their sum 8.75 mJ equals $I\lambda$. A measured terminal waveform can integrate the first expression directly by accumulating $I(v - RI)dt$; independently plotting λ versus I allows the second by planimetric integration. These two areas have different shapes on a nonlinear constitutive curve.

If the insert moves, coenergy at each position can be obtained by repeating the reversible $\lambda(I)$ scan. Its spatial difference divided by travel distance estimates the average force for fixed end current. Simply using half the product of the end current and linkage would predict neither energy correctly in this example. A hysteretic insert follows different curves on raising and lowering the current, so a measured mean force or cycle loss must state the prepared branch. A force predicted by one reversible branch can be checked with a load cell at the same position and current; an electrical scan at another bias is not enough to test the mechanical boundary condition.

Put the magnetic energy on the correct side of a gap

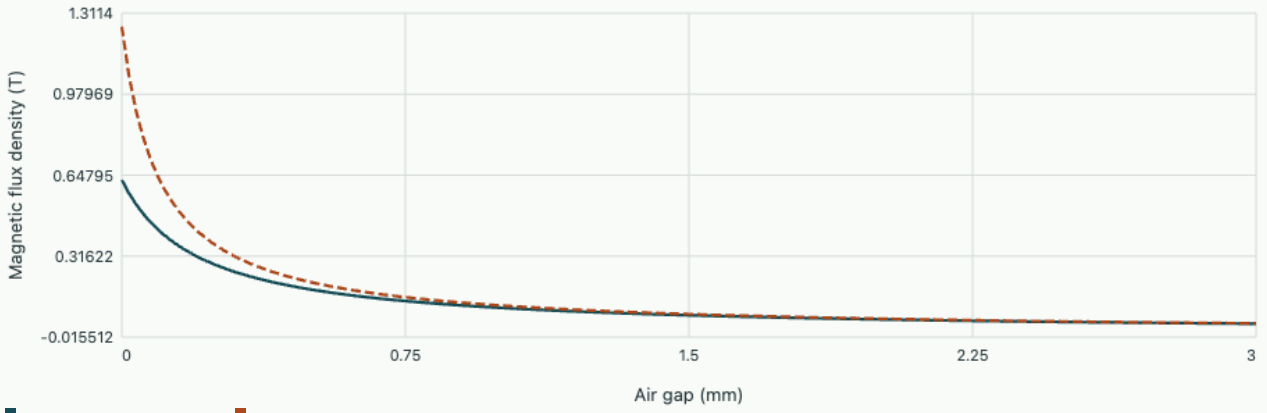
Change the reluctance of a narrow air gap and inspect which section controls flux. Check the assumptions needed before replacing the complete field with a circuit.

Open this experiment online

Try the calculation

Relative permeability **1000**

Magnetomotive force NI (A turns) **100**



Core and gap Core with twice the permeability

At 1 mm the field is 0.1047 T. Equal core and gap reluctances occur at 0.200 mm.

Derive the directional permeability of a stack of thin layers

A material assembled from alternating sheets can respond differently to fields along and across the sheets even when each constituent is isotropic. Consider many parallel layers of two linear, reversible magnetic materials with absolute permeabilities μ_1 and μ_2. Their volume fractions in a representative period are f and $1-f$. Start far from the outer edges. Suppose the imposed field varies only on lengths much longer than a period, and take the frequency low enough that eddy currents do not change the field distribution. These conditions make a comparison with homogeneous bulk measurements meaningful.

First apply a field tangent to the layers. The tangential field H_t is continuous at each interface because no free sheet current flows there. Each material has a different magnetic induction $B_{it} = \mu_i H_t$. Averaging the induction over one period and dividing by the shared field gives

$$\mu_{\parallel} = f\mu_1 + (1-f)\mu_2.$$

This is an arithmetic mean. A heavily permeable layer can carry a large fraction of the tangential flux through only part of the period. No condition requires equal tangential induction on the two sides of the internal interface.

Next drive the magnetic flux through the stack in the normal direction. Normal induction is continuous because magnetic flux lines cannot end at a layer. The normal field in each layer is B_n/μ_i. Averaging the field at fixed common induction now gives a harmonic mean,

$$\frac{1}{\mu_{\perp}} = \frac{f}{\mu_1} + \frac{1-f}{\mu_2}.$$

A small fraction of a weakly permeable spacer can account for most of the magnetomotive drop across a stack. One can derive the same result by adding the magnetic reluctances of a column of common area. That circuit calculation works here because the interface conditions guarantee a common induction. Tilting the applied field creates both normal and tangential components and requires both averages at once. In axes attached to the layers,

$$\langle B_t \rangle = \mu_{\parallel} \langle H_t \rangle,$$

$$\langle B_n \rangle = \mu_{\perp} \langle H_n \rangle.$$

The averaged field vectors are generally not parallel.

For a stack containing 90% by volume of a material with relative permeability 1000 and 10% of a spacer with relative permeability one, the two effective relative permeabilities are 900.1 and approximately 9.91. Suppose a fixture produces equal average components $H_t = H_n = 10\,\mathrm{A/m}$. The induction components are then about $11.31\,\mathrm{mT}$ and $0.125\,\mathrm{mT}$. Their ratio corresponds to an induction angle only 0.63° above the layers while the average field points at 45°. A probe sensitive to a single component could therefore return an apparently inconsistent permeability when the orientation of an anisotropic specimen is ignored.

Verify the energy before using the homogenised material law. For a purely normal induction the microscopic average energy density is

$$f B_n^2/(2\mu_1) + (1-f) B_n^2/(2\mu_2) = B_n^2/(2\mu_{\perp}).$$

For tangential field the microscopic average is

$$H_t^2 [f\mu_1 + (1-f)\mu_2]/2 = \mu_{\parallel} H_t^2/2.$$

Adding both contributions gives $\langle \mathbf{B} \rangle \cdot \langle \mathbf{H} \rangle / 2$ for this linear model. Directly replacing each constituent field by the spatially averaged field before evaluating its energy would give an incorrect normal contribution. The field concentration in the spacer is a physical effect that survives passage to an effective continuum.

Estimate how precisely the spacer fraction must be known if a low-frequency normal-induction calibration is to be reproducible.

▼ Work through the solution

Write $s = 1 - f$ and $\mu_{\perp,r} = [(1-s)/1000 + s]^{-1}$. Differentiating its logarithm gives $d\ln\mu_{\perp,r}/ds = -0.999/[0.001 + 0.999 s]$, about -9.90 at $s = 0.10$. Increasing the spacer volume fraction by one absolute percentage point therefore reduces normal permeability by roughly ten percent to first order. Direct substitution at $s = 0.11$ gives 9.02, or a reduction of about nine percent, showing the size of the linearisation error for that relatively large change. The same change reduces the tangential relative permeability from 900.1 to 890.1, only about one percent.

How does local saturation change the normal and tangential experiments?

▼ **Work through the solution**

In the normal test the induction is shared. A high-permeability layer reaches a specified saturation induction at approximately the same common induction one measures outside a suitably shaped test column. In the tangential test the field is shared and the high-permeability phase can have a much larger local induction than the reported average. For an infinitesimal variation around a bias point, repeat the interface argument with the local differential permeabilities dB_i/dH_i. The differential effective response obeys the same arithmetic or reciprocal averaging rules under the one-dimensional hypotheses. A finite path that traverses a hysteresis loop needs its history and dissipation as well; no single constant differential permeability specifies its loop area.

David Tong, Electromagnetism in Matter, on constitutive laws and material interfaces.

Tutorial on geometry and constitutive response

A spherical sample has apparent susceptibility $M/H_0 = 2.4$. Infer its intrinsic linear susceptibility and the internal field fraction. Can an apparent value of 3.5 arise in this model with positive intrinsic susceptibility?

▼ **Work through the solution**

For a sphere $M/H_0 = \chi/(1+\chi/3)$. Inversion gives

$$\chi = 2.4/(1-0.8) = 12,$$

$$H_{\mathrm{int}}/H_0 = 1/5.$$

Positive finite χ gives apparent values less than three. A measured larger result would call for checking shape, units, background or the linear homogeneous assumption.

An unknown material is placed in a long gapped circuit. At a given operating point twice the current gives only 1.6 times the old induction. Can a fixed linear permeability with unchanged geometry explain it?

▼ **Work through the solution**

A fixed linear reluctance at fixed geometry gives exactly proportional induction. One can infer $H_c = (NI - Bg/\mu_0)/\ell_c$ at each point and compare the results with a measured constitutive curve. Field-dependent permeability, leakage, fringing or a changing physical gap would each need evaluation before calling the effect saturation.

An interface has no sheet current, the normal field happens to vanish on both sides, and the tangential magnetic flux density differs by a factor of eight. What material ratio follows if both media are scalar and linear? Does this require a discontinuity in H_t?

▼ **Work through the solution**

Continuity of H_t gives $B_{2t}/B_{1t} = \mu_2/\mu_1 = 8$ if the tangential drive is nonzero. No tangential H jump is needed. The magnetisation jump can supply an equivalent bound surface current producing the jump in B_t.

Magnetism, induction and field dynamics · 6. Motional emf and transformer emf in moving circuits

Two laboratory demonstrations produce the same voltmeter trace. In one, a stationary loop sees a changing magnetic field. In the other, a conductor moves through a stationary field. The charges experience different local forces, yet under identifiable conditions both readings follow from the rate of change of magnetic flux.

Define the measured emf through a material loop

Consider a thin conducting loop that can translate or deform. A point of the wire moves with velocity $\boldsymbol{u}(\boldsymbol{r}, t)$. In the laboratory a conducting charge also has drift velocity along the wire, and the magnetic force associated with that parallel drift has zero projection along a slender wire element. The tangential Lorentz force per charge, integrated in the direction of current convention, is therefore

$$\mathcal{E} = \oint_{C(t)} (\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell}.$$

It is a closed-path force per charge at one laboratory time. A voltmeter attached across an opening samples a specific pair of terminals and lead path. To compare it with a predicted closed-loop emf, include the instrumentation leads and consider their field exposure.

For a stationary contour, Faraday's law applies directly:

$$\oint_{C} \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\int_{S} \partial_t \boldsymbol{B} \cdot d\boldsymbol{a}.$$

For a moving contour one must also differentiate the surface. Over a short time each boundary segment sweeps an oriented strip whose area vector is $\boldsymbol{u}\, dt \times d\boldsymbol{\ell}$ for the increment to the enclosed surface in the compatible orientation. Contracting with $\boldsymbol{B}$ gives $-(\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell}\, dt$. Interior choices of how to move the spanning surface change nothing when $\nabla \cdot \boldsymbol{B} = 0$ and all surfaces stay in the physical domain. Thus

$$\frac{d}{dt} \int_{S(t)} \boldsymbol{B} \cdot d\boldsymbol{a} = \int_{S(t)} \partial_t \boldsymbol{B} \cdot d\boldsymbol{a} - \oint_{C(t)} (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell},$$

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

The derivative is of the flux through the actual circuit. The partial derivative samples the field change at a fixed spatial point. Confusing the two would predict that moving a loop into a static field can never generate a signal. Conversely, inserting just $B\ell u$ whenever a conductor moves would incorrectly predict a signal for a rigid entire loop translating through an exactly uniform static field.

Follow a sliding bar and the energy feeding the current

Take parallel rails extending along x separated by length L in the y-direction, joined at their left ends, with a conducting bar making the right edge at $x(t)$. A uniform $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ permeates the rails. Choose counterclockwise circuit orientation as viewed from $+z$; on the right edge $d\boldsymbol{\ell}$ points $+y$. For $\dot{x} = u > 0$, $\boldsymbol{u} \times \boldsymbol{B} = -uB\hat{\boldsymbol{y}}$, giving $\mathcal{E} = -BLu$. The area flux is BLx with the chosen orientation, yielding the same result. In a purely resistive loop $I = \mathcal{E}/R$ with this signed convention.

With $B = 0.50\,\mathrm{T},\ L = 0.20\,\mathrm{m},\ u = 2.0\,\mathrm{m/s}, R = 0.25\,\Omega$, the emf magnitude is $0.200\,\mathrm{V}$, current $0.800\,\mathrm{A}$, and magnetic drag magnitude $BLI = 0.0800\,\mathrm{N}$. An external pull supplying constant speed delivers $0.160\,\mathrm{W}$, exactly I^2R. The magnetic Lorentz force on each individual charge is perpendicular to its total velocity and does no direct work on that charge. The moving conductor, its constraints and the electric field transfer mechanical energy from the pull to the circuit. Treating the magnetic force on the bar as directly heating a stationary isolated charge would miss the mechanism.

Allow a $40\,\mathrm{g}$ bar to coast after release. Neglecting mechanical friction and electrical inductance gives

$$m\dot{u} = -(BL)^2 u/R.$$

Its decay time is

$$mR/(BL)^2 = 1.00\,\mathrm{s};$$

after one second the speed falls to $0.736\,\mathrm{m/s}$. Integrating heat from launch to that moment gives

$$mu_0^2(1 - e^{-2})/2 = 0.0692\,\mathrm{J}.$$

An open bar still develops a charge-separation voltage if its terminals are measured with a high-impedance meter, but without sustained closed current the ideal damping disappears.

Compare the open and closed moving-bar traces

Change the field while changing the area

Use the same rails in a slowly varying uniform field $B(t)$. Differentiating flux gives

$$\mathcal{E} = -L[x\dot{B} + B\dot{x}].$$

The first contribution is transformer emf from a circulating electric field and the second is the motional contribution. If $L = 0.20\,\mathrm{m},\ x = 0.40\,\mathrm{m}, \dot{x} = 0.10\,\mathrm{m/s}, B = 0.50\,\mathrm{T}$, the motional term is -0.0100 V. A field ramp of $-0.125\,\mathrm{T/s}$ contributes $+0.0100$ V so net emf vanishes at that instant. Yet the local electric and motional forces need not vanish on every part of the loop. This null experiment tests whether a purported expression for the emf includes both mechanisms with the correct sign.

Keep the area fixed instead, but rotate a rigid N-turn coil of area A in a uniform field such that its normal makes angle ωt with $\boldsymbol{B}$. The flux linkage is $NBA\cos(\omega t)$, giving $\mathcal{E} = NBA\omega\sin(\omega t)$. For

$N = 100$, $A = 4.0 \times 10^{-4}\,\mathrm{m}^2$, $B = 0.050\,\mathrm{T}$, a rotation at 10 cycles/s gives peak 0.1257 V and rms 0.0889 V. The flux itself is largest when emf is zero and crosses zero when emf magnitude is largest. If the wiring rotates with the loop until brushes interrupt or reverse a contact, the terminal waveform depends on those connections as well as this geometrical flux. A slip ring and a commutator do not measure the same oriented winding throughout a mechanical cycle.

Calculate a rotating disc without inventing a moving loop

A conducting disc of radius a spins about its axis with angular speed Ω in a steady uniform axial field B. A high-resistance stationary voltmeter contacts the centre and the rim with sliding brushes. A point of the metal has $\boldsymbol{u} = \Omega s \hat{\boldsymbol{\varphi}}$, so

$$\boldsymbol{u} \times \boldsymbol{B} = \Omega s B \hat{\boldsymbol{s}}.$$

Positive charge is pushed radially outward until an electrostatic field counterbalances this force in open circuit. Thus $E_s = -\Omega s B$, and the rim potential relative to the centre is $+\Omega B a^2/2$.

With $a = 0.080\,\mathrm{m}$, $B = 0.20\,\mathrm{T}$ and 300 revolutions/min,

$$\Omega = 10\pi\,\mathrm{s}^{-1}.$$

The voltage is $20.1\,\mathrm{mV}$. The lab shape traced by a radial line and a stationary external lead can appear fixed; taking the time derivative of flux through that fixed geometrical contour alone would give zero. A radial line drawn on the disc rotates while the brushes continue to contact different material points, so the chosen fixed path is not one closed material contour carried along with the conductor velocity. Evaluating the local Lorentz force inside each moving and stationary element is the unambiguous starting point for sliding-contact arrangements. The terminals' open-circuit potential follows from the balancing electric field, which is compatible with zero electrostatic circulation on a fixed spatial loop.

If a resistive load is connected, radial currents in the disc experience an azimuthal magnetic force opposite the drive. For a current element carrying total radial current I at radius s, the torque increment has magnitude $sIB\,ds$ after integrating around the annulus. The whole torque is $IBa^2/2$; multiplying by Ω gives $I\mathcal{E}$. This independent energy calculation recovers the same half, which would be lost by using the rim speed for every radial point. In a real disc both contact resistances and the spreading resistance near a small central contact influence the load current.

Retain the electrical transient in the sliding-bar experiment

The purely resistive moving-bar calculation assumed that current adjusts on a time short compared with any change in speed. To test this assumption connect a fixed external inductor L_c in the return branch, large relative to the small variation of the rails' own inductance over the travel of interest. Let i denote the positive current that produces a force opposing positive bar velocity u, and put $k = BL$ for the field–length product. With the circuit closed at $t = 0$, the coupled equations become

$$m\dot{u} = -ki,\; L_c\dot{i} + Ri = ku.$$

These signs make the mechanical loss term and electrical source term cancel when their equations are multiplied by u and i. Thus $d(mu^2/2 + L_c i^2/2)/dt = -Ri^2$. At the instant of closure with $i = 0$, the bar starts with zero magnetic braking force even though it has nonzero speed and induced emf.

Eliminate $i = -m\dot{u}/k$ to obtain

$$mL_c\ddot{u} + mR\dot{u} + k^2 u = 0.$$

The two roots are $[-R/L_c \pm \sqrt{(R/L_c)^2 - 4k^2/(mL_c)}]/2$. In the limit L_c/R very short, one fast root fills in the initial current and the slow root approaches $-k^2/(mR)$, recovering the original decay. An electrical inductance can also make the idealised system underdamped. In that case magnetic energy temporarily stored in L_c can cause a velocity reversal, despite the total coupled energy decreasing monotonically through resistance. Observing a reversal alone would not imply that the resistor supplied energy.

Choose $m = 0.040\,\mathrm{kg}$, $k = 0.10\,\mathrm{T\,m}$, $R = 0.10\,\Omega$, $L_c = 0.040\,\mathrm{H}$. The characteristic equation is $s^2 + 2.50s + 6.25 = 0$, giving roots $-1.25 \pm 2.165i\,\mathrm{s^{-1}}$. With $u(0) = 2.0\,\mathrm{m/s}$, $i(0) = 0$, one obtains

$$u(t) = 2e^{-1.25t}[\cos(2.165t) + (1.25/2.165)\sin(2.165t)].$$

Initially $\dot{i}(0) = ku(0)/L_c = 5.0\,\mathrm{A/s}$ and $\ddot{u}(0) = -k\dot{i}(0)/m = -12.5\,\mathrm{m/s^3}$. The first zero of u satisfies $\tan(2.165t) = -2.165/1.25$ on the second-quadrant branch, giving $t \simeq 0.967\,\mathrm{s}$. The approximation of fixed k and nearly fixed total inductance requires enough rail length and a sufficiently uniform field for this motion.

A practical experiment need not wait for a reversal. Record both current and speed at matching times and integrate Ri^2 over the record. The decrease of mechanical energy alone will temporarily exceed the accumulated heat while the inductor is charging. Subtracting the measured increase $L_c i^2/2$ should close the budget. If it does not, evaluate mechanical friction, position-dependent inductance or an external drive before ascribing the discrepancy to a violation of Lenz's law. A lightweight probe loop placed next to the return inductor would register a changing magnetic field during this transfer, independently verifying storage between motion and heat.

Measure flux across a moving boundary without assuming a uniform field

Take a rectangular loop of height H spanning the interval $[x_0, x_0 + w]$ in a static field $B_z(x) = B_p \exp[-x^2/(2d^2)]$. Translate it along $+x$ at speed u. The flux is $H\int_{x_0}^{x_0+w} B_z(x)dx$; differentiating the moving endpoints gives $\mathcal{E} = Hu[B_z(x_0) - B_z(x_0 + w)]$ for positive z orientation. This formula directly illustrates what a translating induction pickup measures. It returns a difference of

the field at its two transverse conductor edges rather than the field at its centre. At the instant the loop is centred on the symmetric peak the flux is largest, the two edges see equal fields, and the measured voltage is zero.

For $w = d = 20\,\mathrm{mm}$, $H = 15\,\mathrm{mm}$, $u = 0.30\,\mathrm{m/s}$, $B_p = 0.040\,\mathrm{T}$, suppose the trailing edge is at $x_0 = 0$. Then the leading edge sees $B_p e^{-1/2}$; the emf is

$$H u B_p (1 - e^{-1/2}) = 70.8\,\mu\mathrm{V}.$$

When the leading edge had just reached $x = 0$, the same magnitude occurred with the opposite sign. Integrating the entire voltage pulse from far left to far right gives minus the change in initial and final flux, hence zero despite a nonzero positive lobe and a nonzero negative lobe. A DC offset in acquisition would spoil that cancellation and can imitate a change of background flux.

When $w \ll d$, a Taylor expansion about the centre $x_c = x_0 + w/2$ gives

$$\mathcal{E} \simeq -H u w\, dB_z/dx|_{x_c},$$

with the first odd correction containing $w^3 B_z'''/24$. When $w \gg d$, the two edges traverse the field at well-separated times and give separate pulses approximating its profile with opposite signs. These two regimes can be distinguished by changing only the loop width. A finite voltmeter bandwidth can also smooth the pulses and push their apparent extrema, so a timing fit must include the measurement response if it is comparable with the transit time d/u. An isolated open bar, by contrast, measures the difference in radial charge equilibrium and lacks the cancelling second edge. This accounts for apparently conflicting interpretations of 'measuring the field by moving a wire' in demonstrations with different connections.

Close the power balance of a moving conductor

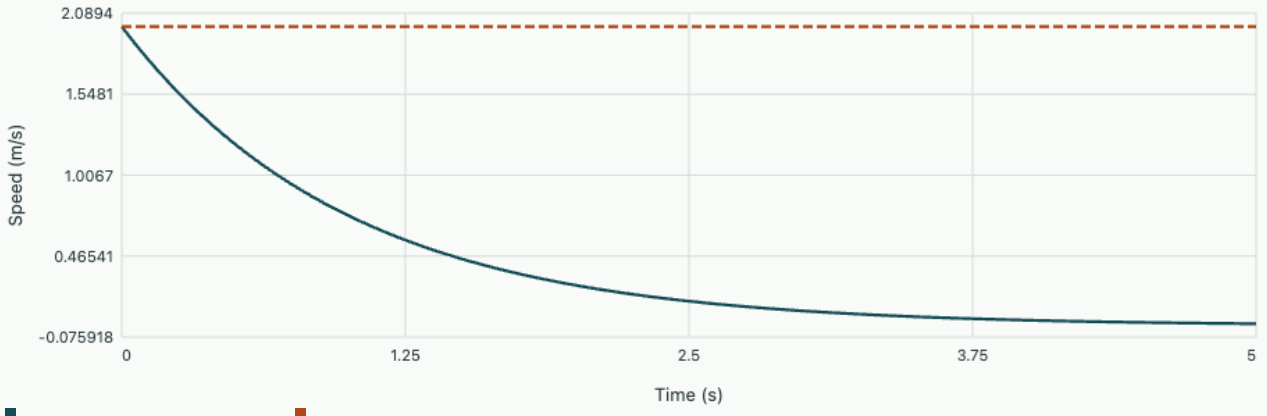
Change circuit resistance and magnetic field. Predict how quickly the bar slows, then relate the lost kinetic energy to the resistive power.

Open this experiment online

Try the calculation

Magnetic field (T) **0.5**

Circuit resistance (Ω) **0.25**



Decay time 1.000 s. At one decay time the speed is 0.736 m/s and 86.5% of the original kinetic energy has gone into Joule heat.

Let a freely moving loop brake itself at a field boundary

A rigid rectangular wire loop of extent w along x and height h along y slides on frictionless guides through a region of uniform vertical magnetic induction $B\hat{z}$. The magnetic region begins at $x = 0$ and ends at $x = L$, with $L > w$. Take its edges sufficiently sharp compared with the loop width, and suppose the electrical relaxation time of the loop is short compared with the time to cross an edge. The ring then acts like a resistance R. Its mass m and initial speed u_0 determine whether it can cross each boundary at all. Holding it at constant speed would describe a separate experiment in which a drive supplies mechanical work.

While the front edge is inside and the rear edge is outside, the enclosed flux grows as Bhx, where x is the distance the front edge has entered. Faraday's law gives an emf of magnitude Bhu, and the circulating current has magnitude Bhu/R. The horizontal force on the immersed vertical side opposes the motion. The two horizontal sides carry equal and opposite vertical force components and supply no longitudinal drive. The braking force therefore has magnitude $(Bh)^2 u/R$. Its direction can also be obtained without a sign convention for current by requiring the mechanical power loss to equal $I^2 R$.

Write $k = (Bh)^2/R$ and $\tau = m/k$. Newton's equation reads $m\,du/dt = -ku$, so $u(t) = u_0 e^{-t/\tau}$. At a known position, a particularly simple expression follows by using $du/dt = u\,du/dx$. As long as the speed is positive one factor of u cancels and $du/dx = -1/\tau$. The loop loses the fixed speed increment w/τ in traversing the first edge, regardless of its initial speed, provided it reaches the end of that traversal. If $u_0\tau < w$, the penetration tends to $u_0\tau$ while the speed tends to zero. It approaches its stopping position asymptotically in this ideal model.

Once both vertical sides are in the same uniform field, changing the position leaves the flux unchanged. The loop coasts through this region at the speed it retained after entry. When it reaches the far boundary the enclosed flux decreases and the induced current reverses. The remaining immersed vertical side changes too. These two sign changes leave the net longitudinal force pointing backwards. A loop that reaches the far edge and crosses it completely loses a second speed increment w/τ. The signs of the current at the two edges therefore distinguish the direction of flux change, while both edges remove kinetic energy.

For $h = 0.10\,\mathrm{m}$, $w = 0.20\,\mathrm{m}$, $B = 0.50\,\mathrm{T}$, $R = 0.10\,\Omega$ and $m = 0.020\,\mathrm{kg}$, the damping coefficient is $0.025\,\mathrm{kg/s}$ and the time constant $0.80\,\mathrm{s}$. Released at $0.80\,\mathrm{m/s}$, the loop drops to $0.55\,\mathrm{m/s}$ after entry and $0.30\,\mathrm{m/s}$ after exit. The durations of the two braking intervals are respectively $0.80\ln(0.80/0.55) = 0.300\,\mathrm{s}$ and $0.80\ln(0.55/0.30) = 0.485\,\mathrm{s}$. They differ because the second identical field boundary is crossed more slowly. The total heat is

$$m(0.80^2 - 0.30^2)/2 = 5.50\,\mathrm{mJ}.$$

A laboratory replaces the free loop by a motor that drags it at $0.80\,\mathrm{m/s}$ throughout. What mechanical signature distinguishes the two experiments?

▼ **Work through the solution**

A constant-speed drive must balance $ku = 0.020\,\mathrm{N}$ during each partially immersed interval. The associated drive power is $0.016\,\mathrm{W}$. Each interval covers $0.20\,\mathrm{m}$, giving $4.0\,\mathrm{mJ}$ of heat and mechanical work per edge. The two intervals take equally long and the two current pulses have the same magnitude. In the freely moving test the current drops during each pulse and the second pulse starts at a lower magnitude. Agreeing values of B, R and geometry do not fix the energy loss unless the mechanical boundary conditions are also specified.

Under what conditions would a voltmeter or current sensor register a significant signal in the nominally uniform middle section?

▼ **Work through the solution**

With a finite loop inductance the current present at the end of entry cannot vanish instantaneously. It decays on the electrical time $\mathcal{L}/R$ while dissipating previously stored magnetic energy. A nonuniform field also produces an emf whenever the leading and trailing sides sample different inductions, even if both are inside the geometric footprint of the magnet. Finally, a change of the applied field in time drives a transformer emf over the whole loop. Varying the speed, holding the loop stationary, and measuring the field along the track discriminate these mechanisms because their time and position dependence differs.

David Tong, Electrodynamics, on induction and the energy of magnetic circuits.

Tutorial on signs, loops and measurement leads

A rectangular closed loop of width w in the motion direction and height h translates with speed u in $B_z(x) = B_0 + \alpha x$. Predict its signed emf for positive orientation and evaluate at $w = 0.10\,\mathrm{m},\ h = 0.040\,\mathrm{m}, \alpha = 0.30\,\mathrm{T/m}, u = 1.5\,\mathrm{m/s}$.

▼ **Work through the solution**

The flux is $h\int_{x_0}^{x_0+w}(B_0+\alpha x)dx$. Differentiating the two moving endpoints gives

$$hu[B(x_0+w) - B(x_0)] = hw\alpha u.$$

The signed emf is $-1.80\,\mathrm{mV}$. Uniform B_0 cancels between equal opposite moving edges, verifying the uniform-field translation limit.

Two stationary voltmeters are attached to the same pair of terminals but their leads enclose additional area $0.015\,\mathrm{m}^2$ between the two measurement paths. A normal field grows at $0.40\,\mathrm{T/s}$. What discrepancy can occur?

▼ **Work through the solution**

Concatenate the first measurement path with the reverse of the second. It forms a closed stationary loop with induced emf of magnitude $A|\dot{B}| = 6.0\,\mathrm{mV}$. The discrepancy's sign is fixed by the chosen paths and orientation. Keeping outgoing and returning meter leads together reduces the area and makes a lumped terminal voltage a better approximation.

A conducting ring expands radially so $a(t)$ grows while the applied uniform axial field falls as $B \propto a^{-2}$. Does the vanishing net emf imply zero local induced electric field?

▼ **Work through the solution**

The flux $B\pi a^2$ is constant, so a material ring has zero net Lorentz-force circulation. Faraday's law on a stationary circle at its instantaneous radius still gives

$$E_\varphi = -a\dot{B}/2.$$

The moving metal supplies $(\boldsymbol{u}\times\boldsymbol{B})_\varphi = -\dot{a}B$; these terms cancel for the stated time dependence. A stationary charge and a charge bound to the moving ring therefore probe different local forces.

Magnetism, induction and field dynamics · 7. Flux linkage, coupled coils and magnetic coenergy

A winding links flux through each turn. In two windings the current in either one can change the voltage in the other. A matrix of flux linkages describes this coupling, but predicting power or mechanical force requires specifying the current directions and the control conditions.

Measure a linkage rather than a field at one point

Choose a positive traversal direction for every turn of winding i. Define $\lambda_i = \sum_{\text{turns}} \int \boldsymbol{B} \cdot d\boldsymbol{a}$ with normals consistent with traversal. If every turn sees equivalent flux, this is $N_i \Phi_i$; a distributed winding can have significantly different flux per turn. A passive terminal reference takes current entering the positive terminal. For a stationary winding with series resistance R_i, its terminal voltage then obeys $v_i = R_i I_i + d\lambda_i/dt$. The opposing internal induced emf is $-d\lambda_i/dt$. These statements use different definitions of voltage direction, so the opposite signs are consistent.

To measure a differential inductance at fixed geometry, one may superimpose a small sinusoidal current on a DC bias, measure its voltage quadrature, and infer $d\lambda/dI$. A large slow ramp instead measures finite changes in linkage by integrating $v - RI$ over time. If the winding warms during the ramp, using a cold resistance biases the flux integral. At sufficiently high frequency winding capacitance, frequency-dependent conductor resistance and nonuniform flux can invalidate the simple two-terminal model. Entering the source's physical boundary and frequency regime is part of interpreting any inferred inductance.

Derive reciprocity and energy from the field equations

For a linear reciprocal material with fixed geometry and no remanent background, write $\lambda_i = \sum_j L_{ij} I_j$. In free space the Neumann double line integral for distinct slender loops is unchanged when loop indices are exchanged, giving $L_{12} = L_{21}$. For a broader proof take field solutions $(\boldsymbol{H}_i, \boldsymbol{B}_i, \boldsymbol{A}_i)$ for unit currents in windings i. The linked flux in winding j is $\int \boldsymbol{A}_i \cdot \boldsymbol{J}_j \, dV$, where $\boldsymbol{J}_j$ is its unit-current density. Integrate $\boldsymbol{A}_i \cdot (\nabla \times \boldsymbol{H}_j)$ by parts. With appropriate enclosing conditions the surface term vanishes and the integral is $\int \boldsymbol{B}_i \cdot \boldsymbol{H}_j dV$. Since the permeability tensor is symmetric, exchanging i, j gives the same value. Reciprocity depends on the constitutive law and boundary conditions.

At fixed mechanical coordinates the electrical work put into the field is $dW = \sum I_i \, d\lambda_i$ after subtracting losses. Ramp all currents up in fixed proportion from zero and use linearity. The instantaneous integrand scales linearly with fractional ramp amplitude, giving half its endpoint product:

$$W = \tfrac{1}{2} \sum_i I_i \lambda_i = \tfrac{1}{2} L_1 I_1^2 + M I_1 I_2 + \tfrac{1}{2} L_2 I_2^2, \qquad |M| \le \sqrt{L_1 L_2}.$$

The inequality follows from requiring nonnegative energy for every current vector. At fixed I_2, minimise the quadratic with respect to I_1; at $I_1 = -M I_2 / L_1$ energy is $(L_2 - M^2/L_1) I_2^2/2$. Thus no passive linear pair can have coupling magnitude greater than the geometric mean of its self-inductances. Equality corresponds to a combination of currents producing zero linked-field energy in the ideal

model, as for perfect coupling without leakage. In an actual device leakage, parasitics and measurement uncertainty matter close to that limit.

Background · Can zero mutual flux coincide with maximum torque?

Consider a small coil that rotates in the nearly uniform field of a fixed coil. For a suitable angle origin the linked flux is proportional to the projected area, so $M(\theta) = M_0 \cos\theta$. Hold both currents fixed with ideal supplies. In a linear medium the angle-dependent part of the magnetic coenergy is $M(\theta) I_1 I_2$. Its derivative gives the torque.

$$\tau_\theta = I_1 I_2 \frac{dM}{d\theta} = -M_0 I_1 I_2 \sin\theta.$$

Precisely at a right angle the instantaneous mutual linkage is zero while its angular derivative has maximum magnitude. A freely rotating current loop therefore feels its largest aligning torque there. Measuring linkage at one orientation cannot supply this force. Measuring it at two nearby angles estimates the derivative that enters the mechanical response.

Reverse a connection and include a shorted secondary

Consider $L_1 = 9.0\,\mathrm{mH}$, $L_2 = 4.0\,\mathrm{mH}$, $M = 4.8\,\mathrm{mH}$, so coupling coefficient $k = M/\sqrt{L_1 L_2} = 0.80$. With currents $I_1 = 0.40$ A and $I_2 = -0.20$ A, the flux linkages are $2.64\,\mathrm{mWb}$ and $1.12\,\mathrm{mWb}$ using the signed references. The energy is

$$(0.40 \times 2.64 - 0.20 \times 1.12)/2 = 0.416\,\mathrm{mJ}.$$

Reversing only the second physical winding while keeping the named terminal references changes the mutual sign but leaves the diagonal self terms positive.

Connect the coils in series. When the same signed current flows in both, the total linkage along the series path gives

$$L_{\mathrm{aid}} = L_1 + L_2 + 2M = 22.6\,\mathrm{mH}.$$

With opposite reference directions the result is

$$L_{\mathrm{oppose}} = L_1 + L_2 - 2M = 3.4\,\mathrm{mH}.$$

The difference of the measured series inductances divided by four estimates the signed mutual coupling for the chosen assignment. Both inferred inductances are nonnegative. A negative value obtained from a model suggests incorrect signs, units or incompatible measurements.

Now close winding two through an ideal lossless short while holding geometry fixed. With zero initial secondary linkage the voltage condition $d\lambda_2/dt = 0$ implies $I_2 = -(M/L_2) I_1$, hence

$\lambda_1 = (L_1 - M^2/L_2) I_1$. In the example the input inductance becomes $3.24\,\mathrm{mH}$, smaller than the open-secondary value of 9.0. Its reduction is not the same calculation as series opposition; the secondary current is determined by a flux constraint rather than imposed as $-I_1$. With finite secondary resistance the DC steady secondary current eventually returns to zero. The notion of a perfectly shorted, flux-conserving secondary is an ideal time-scale limit.

See the secondary's losses from the primary

For sinusoidal current use peak phasors with $e^{i\omega t}$. Let the full secondary resistance be R_2 and close it without an applied voltage. Solve $0 = (R_2 + i\omega L_2)\widetilde{I}_2 + i\omega M \widetilde{I}_1$, then substitute into the primary equation. The input impedance becomes

$$Z_{\mathrm{in}} = R_1 + i\omega L_1 + \frac{\omega^2 M^2}{R_2 + i\omega L_2}.$$

The additional real part $\omega^2 M^2 R_2 / [R_2^2 + (\omega L_2)^2]$ is positive and represents power dissipated in winding two. Its imaginary part is negative and reduces the primary reactive voltage. In the high-frequency limit within the lumped model the reactive term tends towards $i\omega(L_1 - M^2/L_2)$. In the low-frequency limit the induced coupling vanishes quadratically in the real part. A sense of physical scale is obtained by choosing $\omega = 250\,\mathrm{s}^{-1}$ and $R_2 = 1.0\,\Omega$. Then

$$\omega L_2 = 1.0\,\Omega, \omega M = 1.20\,\Omega.$$

The added impedance is $1.44/(1+i) = 0.72 - 0.72i\,\Omega$, so a primary with negligible resistance sees $0.72 + 1.53i\,\Omega$.

With primary amplitude 1.0 A, the secondary amplitude is

$$1.20/\sqrt{2} = 0.849\,\mathrm{A}.$$

Its mean Joule loss is

$$0.849^2 (1.0)/2 = 0.360\,\mathrm{W}.$$

Independently the primary's reflected resistance draws $1.0^2(0.72)/2 = 0.360$ W. This agreement tests the sign of the induced secondary current and the correct use of peak versus rms amplitudes. Reversing the secondary orientation flips its voltage reference and its current phase by π, but the reflected impedance depends on M^2 and is unchanged.

Let geometry move while current is maintained

If a mechanical coordinate x changes, terminal electrical work can go into both field energy and mechanical output. For a reversible device $dW_{\mathrm{field}} = \sum I_i d\lambda_i - F_x dx$. Introduce coenergy

$W' = \sum I_i \lambda_i - W_{\text{field}}$; the product rule gives

$$dW' = \sum \lambda_i dI_i + F_x dx.$$

Therefore force at fixed currents is $F_x = (\partial W'/\partial x)_{\boldsymbol{I}}$. At fixed linkages instead,

$$F_x = -(\partial W_{\text{field}}/\partial x)_{\boldsymbol{\lambda}}.$$

A linear reversible system has $W' = W_{\text{field}}$ at each state, but their natural independent variables differ. Differentiating the same algebraic expression under an inappropriate constraint reverses a force prediction.

Suppose translating one coil increases M at $0.20\,\mathrm{H/m}$, while its self-inductances are approximately constant. For maintained $I_1 = 0.40\,\mathrm{A}$, $I_2 = 0.30\,\mathrm{A}$, $F_x = I_1 I_2\, dM/dx = 0.024\,\mathrm{N}$ in the direction of increasing coupling. If x changes by 1.0 mm slowly, the field energy rises $24\,\mu\mathrm{J}$. Source work from both windings is twice this value because both linkages change. The remaining $24\,\mu\mathrm{J}$ goes to mechanical output. With currents of opposite sign the force reverses. The mechanism is the same as that behind an electromechanical transducer, but the sign is easier to see in a reciprocal two-port than in a one-coil moving-core drawing.

Check force by differentiating magnetic coenergy

Split the resonances of two coupled electrical oscillators

Close each winding through its own identical capacitor C, remove resistance initially, and take identical self-inductances L. Let q_1, q_2 be capacitor charges in orientations for which $I_i = \dot{q}_i$ and the passive capacitor voltage drop is q_i/C. The two source-free loop equations become

$$L\ddot{q}_1 + M\ddot{q}_2 + q_1/C = 0,$$
$$M\ddot{q}_1 + L\ddot{q}_2 + q_2/C = 0.$$

Adding and subtracting decouple the symmetric coordinate $q_1 + q_2$ and antisymmetric coordinate $q_1 - q_2$. Their angular frequencies are $1/\sqrt{(L+M)C}$ and $1/\sqrt{(L-M)C}$ respectively. Thus the pair measures $L + M$ and $L - M$ independently of any assumption that current remains confined to the driven winding.

For $L = 10.0\,\mathrm{mH}$, $M = 2.0\,\mathrm{mH}$ and $C = 1.0\,\mu\mathrm{F}$, the uncoupled frequency is 1592 Hz. The symmetric mode lies at 1453 Hz, the antisymmetric at 1779 Hz. A sensor observing one coil after an initial excitation may see both: with $q_1(0) = q_0, q_2(0) = 0$ and both currents zero, $q_1 = q_0[\cos(\omega_s t) + \cos(\omega_a t)]/2$, while $q_2 = q_0[\cos(\omega_s t) - \cos(\omega_a t)]/2$. The envelope of the first can almost vanish as energy appears in the second. Energy has not been externally driven during this interval. The coupled magnetic cross term is part of the complete energy, so simply adding the two uncoupled winding energies at arbitrary instants can give an apparent oscillation of the total.

Inverse measurement is possible. If the two frequencies are f_s, f_a, eliminate unknown C to get $M/L = (f_a^2 - f_s^2)/(f_a^2 + f_s^2)$ for the stated same-sign winding labels. If frequencies alone are known but one cannot tell which mode carries same-sign current, one obtains the magnitude of the coupling. Measuring the relative current phase identifies the assignment. Unequal capacitances, unequal self-inductances or appreciable damping change the eigenvalue problem, so a frequency splitting should not be identified with $2M$ or with a material coupling independent of the circuit model. A loss measurement can differentiate genuine normal-mode splitting from two overlapping modes broadened by resistance.

Recover transformer relations and identify the leakage mode

An ideal high-permeability closed core links all turns by a common flux Φ, so

$$\lambda_1 = N_1\Phi,\ \lambda_2 = \pm N_2\Phi.$$

Its terminal induced voltage ratio follows on differentiation, $v_2/v_1 = \pm N_2/N_1$ when resistive drops and parasitic voltages are negligible. The relation is between oriented time derivatives, not a rule about the magnitude of any static potential across a disconnected winding. A DC voltage held across a resistanceless winding would ramp the core flux indefinitely in this ideal model; a real core saturates or winding resistance establishes a different state. The harmonic transformer approximation operates within a finite flux and frequency range.

For a linear core with reluctance $\mathcal{R}$, $\Phi = (N_1 I_1 + N_2 I_2)/\mathcal{R}$ with consistent orientations. Thus $L_1 = N_1^2/\mathcal{R},\ L_2 = N_2^2/\mathcal{R},\ M = N_1 N_2/\mathcal{R}$. The inductance matrix has a zero eigenvalue since the current combination $N_1 I_1 + N_2 I_2 = 0$ creates no core flux. Real leakage breaks this exact cancellation. Adding small series leakage inductances corresponds to storing additional positive energy outside the common flux region. The shorted-secondary input test is particularly sensitive to this residual storage because the coupled part largely cancels.

Take an 80-turn primary and 20-turn secondary on the same linear core. With 4.0 V rms across the magnetising branch at 2.0 kHz, sinusoidal flux amplitude follows from $V_{\mathrm{rms}} = N_1\omega\Phi_{\mathrm{peak}}/\sqrt{2}$, yielding $5.63\,\mu\mathrm{Wb}$. A $2.0\,\mathrm{cm}^2$ core then sees 0.0281 T peak. The open secondary gives 1.0 V rms. A $5.0\,\Omega$ resistive secondary load would draw 0.20 A rms and dissipate 0.20 W in the ideal lossless-transformer limit. The load-related primary current is then 0.050 A rms, corresponding to a reflected resistance of $4.0/0.050 = 80\,\Omega$, as given by $(N_1/N_2)^2 R_L$. An additional reactive magnetising current is needed for finite $\mathcal{R}$; a primary ammeter can consequently read more than the reflected load component even with negligible winding power loss.

Halving drive frequency at unchanged voltage doubles peak flux. This can increase saturation and harmonic distortion even though a naive turns-ratio calculation still predicts the same rms open-circuit voltage. Repeating the experiment with unchanged voltage-to-frequency ratio instead holds the ideal peak flux fixed. These different protocols distinguish a failure of flux linearity from simple winding loss. They also show why mutual inductance as a constant scalar is a model tied to a current, geometry and frequency range. Its measurement should specify the secondary termination, DC bias and whether the observed winding responds to essentially all the changing flux.

Infer force from linked flux

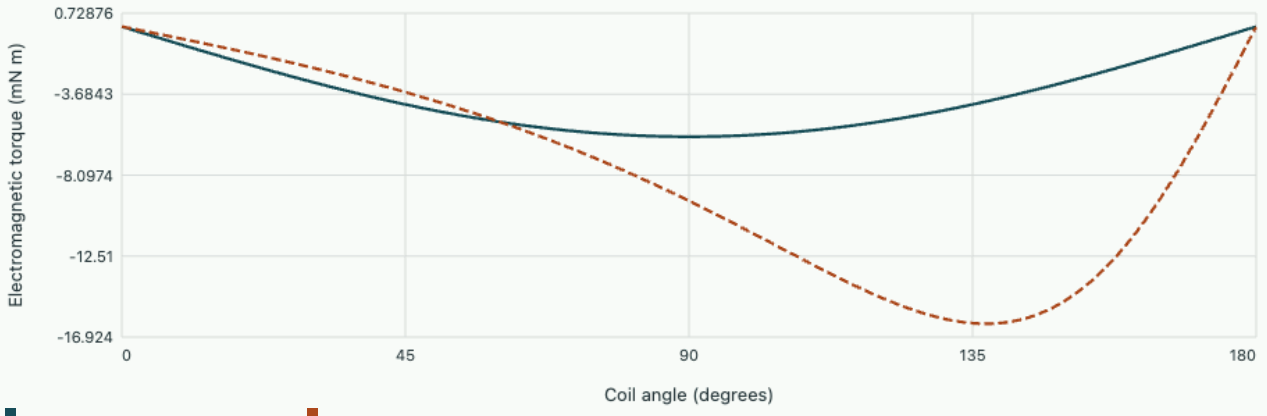
Change the coil geometry and current while checking which electrical variable is held fixed in the mechanical derivative.

Open this experiment online

Try the calculation

First current at preparation (A) **2**

Second current at preparation (A) **1**



At 60° both preparations give -5.196 mN m. At 90° the current-controlled and flux-controlled torques are -6 and -9.5 mN m.

Compute the reaction of a ring that conserves its flux linkage

A primary coil is held at current I_1 by an ideal supply. Along its axis a second closed coil can move with coordinate x. The two self-inductances L_1, L_2 may be treated as constants, while their mutual inductance $M(x)$ increases as the centres approach. Choose both winding directions so that aiding positive currents produce positive M. Imagine first that resistance in the secondary is negligible over the entire experiment. Faraday's circuit equation then conserves its total linkage $\lambda_2 = L_2 I_2 + M(x) I_1$. The conserved quantity is fixed by the preparation of the ring, so an unreported initial field can change even the sign of the force.

Begin with zero secondary linkage. As the two coils approach at fixed current I_1, positive linkage from the primary grows and the secondary current must become negative. Its value is

$$I_2 = -M I_1 / L_2.$$

For two linear windings the magnetic coenergy with the instantaneous currents independently specified is $W' = (L_1 I_1^2 + 2 M I_1 I_2 + L_2 I_2^2)/2$. The mechanical force is the derivative with respect

to x at fixed winding currents. Take that partial derivative first to get $F_x = I_1 I_2 M'(x)$, and then insert the actual secondary current. At zero trapped linkage this gives

$$F_x = -\frac{I_1^2}{L_2} M(x) M'(x).$$

For an approach coordinate that increases toward stronger coupling, the sign is negative. Magnetic induction supplies a restoring tendency against entry into the high-coupling region. The measured primary linkage is $[L_1 - M^2/L_2]I_1$, which gives the same terminal inductance previously found from a short circuit at sufficiently high frequency in the linear model. With zero trapped linkage the stored energy also reduces to $I_1^2[L_1 - M^2/L_2]/2$. For fixed current and moving coils the supply exchanges energy. Its work $I_1\, d\lambda_1$ must be retained when using a field-energy calculation to deduce mechanical force.

For example, let $L_1 = 20\,\mathrm{mH}$, $L_2 = 5.0\,\mathrm{mH}$, $M = 3.0\,\mathrm{mH}$, $M' = 0.12\,\mathrm{H/m}$, and $I_1 = 0.50\,\mathrm{A}$. The secondary current is $-0.30\,\mathrm{A}$, the effective primary inductance is $18.2\,\mathrm{mH}$, and the force is $-0.018\,\mathrm{N}$. The coupling coefficient $M/\sqrt{L_1 L_2} = 0.30$ satisfies positive energy. A small approach of $1.0\,\mathrm{mm}$, over which this local gradient varies negligibly, gives approximately $-18\,\mu\mathrm{J}$ of work by the magnetic force. The external apparatus must do positive work to maintain a slow approach.

Now prepare a positive trapped linkage $3.0\,\mathrm{mWb}$, with all the other numerical data unchanged. The secondary current is $(\lambda_2 - M I_1)/L_2 = +0.30\,\mathrm{A}$, so the same geometry attracts with $+0.018\,\mathrm{N}$. This reversal comes from the initial state of the secondary. The threshold $M I_1 = \lambda_2$ gives zero secondary current and hence zero force at that point, although the primary still stores field energy. For a permanently closed ideal conductor the trapped linkage must be specified separately. Applying a steady-field circuit equation with zero resistance and zero terminal voltage does not determine its current by itself.

What changes if the secondary has a small measured resistance and the motion stops immediately after the first force reading?

▼ **Work through the solution**

Once the coils and the primary current are stationary, mutual linkage from the primary stops changing. The secondary equation becomes

$$L_2\, dI_2/dt + R_2 I_2 = 0.$$

The current and the magnetic force decay together with time constant L_2/R_2. With $R_2 = 0.10\,\Omega$ that time is $50\,\mathrm{ms}$, and the initially measured force has fallen to about 37% after one time constant. A slowly moving ordinary ring can lose much of its induced current during motion. A fast passage can resemble the conserved-linkage limit even when a later static reading shows no induced force.

A second student substitutes a nonzero conserved linkage into the stored field energy and differentiates at fixed I_1. Why does this prescription miss the extra trapped-current force?

▼ **Work through the solution**

Substitution puts the field energy in the form $W = (L_1 - M^2/L_2)I_1^2/2 + \lambda_2^2/(2L_2)$. At fixed I_1, λ_2 it has no term linear in trapped linkage. The source work does. The primary linkage is $\lambda_1 = (L_1 - M^2/L_2)I_1 + M\lambda_2/L_2$. At fixed primary current, subtract dW from $I_1 d\lambda_1$ and divide by dx. The resulting mechanical force is $I_1(\lambda_2 - MI_1)M'/L_2$, precisely the independently evaluated mutual force. An energy derivative determines a force only after the controlled variables and any energy supplied at electrical terminals have been identified.

David Tong, Electrodynamics, on mutual inductance and field energy.

Tutorial on electrical and mechanical constraints

A winding pair measures 35 mH in series aiding and 11 mH in series opposing. The first self-inductance is 16 mH. Infer the second, mutual coupling, and primary inductance with the ideal secondary shorted.

▼ **Work through the solution**

Adding the readings gives $2(L_1 + L_2) = 46$ mH, so $L_2 = 7$ mH. Their difference gives $4M = 24$ mH, hence $M = 6$ mH and $|k| = 6/\sqrt{112} = 0.567$. The shorted-secondary primary value is

$$16 - 36/7 = 10.86\,\mathrm{mH}.$$

For a single ideal coil, L increases from 2.0 to 3.0 mH while a source holds current at 0.50 A. Find the change of magnetic energy, source work and mechanical output. Compare to a separate isolated lossless coil with the same initial current and conserved linkage.

▼ **Work through the solution**

At fixed current the stored energy rises by $(0.5)^2(0.001)/2 = 0.125$ mJ. Source work is $I^2\Delta L = 0.250$ mJ, leaving 0.125 mJ mechanical. For the isolated coil $\lambda = 1.00$ mWb stays fixed and energy falls from 0.250 to 0.167 mJ. The released 0.0833 mJ supplies mechanical output without source work.

A sinusoidal test infers a negative reflected resistance in a pair of ordinary passive stationary windings with a passive termination. Which steps would you check before interpreting it physically?

▼ **Work through the solution**

Verify primary and secondary voltage/current reference directions and the distinction between a passive port voltage and the internally induced emf. Then check calibration phase, instrument delay, any active load, and the subtracted winding resistance. The passive model's reflected real part is explicitly nonnegative for $R_2 \geq 0$, regardless of the sign of M.

Magnetism, induction and field dynamics · 8. How a magnetic field relaxes into a finite conductor

A conducting sheet initially shields its interior from an applied magnetic step. Given time, a normal resistive metal admits a static tangential magnetic field. The same equations give a finite slab a discrete relaxation spectrum and a periodic drive a complex penetration depth. These are different initial and boundary conditions for one diffusion operator.

Identify the time derivative lost in the approximation

Let the conductor be stationary, Ohmic, uniform and isotropic with conductivity σ, permeability μ and permittivity ϵ. In the full Maxwell equations both $\sigma \boldsymbol{E}$ and $\epsilon \partial_t \boldsymbol{E}$ contribute to the curl of $\boldsymbol{H}$. At a characteristic angular frequency ω, the ratio of the latter to the former is $\omega\epsilon/\sigma$. For a good conductor at low frequencies this is very small. Neglect displacement current only in that constitutive region; a vacuum gap or a capacitor elsewhere can still carry significant displacement current. Ampère then gives

$$\boldsymbol{E} = (\nabla \times \boldsymbol{B})/(\mu\sigma).$$

Substitution into Faraday and use of $\nabla \cdot \boldsymbol{B} = 0$ yield

$$\partial_t \boldsymbol{B} = D\nabla^2 \boldsymbol{B},$$

$$D = 1/(\mu\sigma).$$

The diffusion time for scale a is of order a^2/D. Increasing conductivity reduces D, because smaller electric fields can sustain stronger opposing currents. In an ideal mathematical conductor with $\sigma \to \infty$, an initial flux can be frozen in this model. That is a different preparation from a superconductor expelling a field in equilibrium. Magnetic diffusion alone contains neither a thermodynamic superconducting transition nor London screening. At sufficiently short times the dropped displacement current also matters: an ideal step is an approximation to a finite-risetime drive, and the diffusion equation's nonzero Gaussian tails at arbitrarily large distances should not be used to infer a superluminal signal.

Obtain a transient similarity profile

Take a half-space $x > 0$ with field $\boldsymbol{B} = B_z(x,t)\hat{\boldsymbol{z}}$; the surface is driven to B_s at $t = 0^+$, while initially the interior has zero field. Scaling x by $\sqrt{Dt}$ suggests $B_z/B_s = F(\eta)$, $\eta = x/(2\sqrt{Dt})$. Compute $\partial_t F = -\eta F'/(2t)$ and $D\partial_x^2 F = F''/(4t)$; hence

$$F'' + 2\eta F' = 0.$$

Its first integral is

$$F' = Ce^{-\eta^2}.$$

Set $F(0)=1, F(\infty)=0$ and use the Gaussian integral to obtain

$$F(\eta)=\operatorname{erfc}\eta.$$

The associated current $J_y=-(1/\mu)\partial_x B_z$ is positive for this sign of surface field and has a Gaussian depth profile.

$$J_y(x,t)=\frac{B_s}{\mu\sqrt{\pi D t}}e^{-x^2/(4Dt)},$$

$$\int_0^\infty J_y\,dx=\frac{B_s}{\mu}.$$

The current integral remains fixed as the profile spreads, even though the surface current density falls as $t^{-1/2}$. A finite conductivity does not imply uniform current at every instant. For $\sigma=5.8\times10^7\,\mathrm{S/m}$ and $\mu=\mu_0$, $D=0.01372\,\mathrm{m^2/s}$. The point reaching half the surface field is

$$x=2\operatorname{erfc}^{-1}(1/2)\sqrt{Dt}=0.9539\sqrt{Dt}.$$

At $0.10\,\mathrm{ms}$ it is $1.12\,\mathrm{mm}$ deep. At $0.40\,\mathrm{ms}$, four times the time, it is twice as deep. Measuring several half-amplitude positions against time lets one fit D without calibrating the absolute magnetic field strength.

Satisfy both faces of a finite slab

Let the conductor occupy $-a<x<a$. Impose the same sudden tangential field B_s at both faces. At long time the unique steady solution with these equal boundaries is uniform B_s. Subtract this final value by defining the deficit $u=1-B_z/B_s$. It satisfies homogeneous boundary values $u(\pm a,t)=0$ and begins at one inside. Even symmetry permits cosine modes $\cos(k_m x)$ with $k_m a=(m+1/2)\pi$. Separating time gives the multiplier $e^{-Dk_m^2 t}$. Orthogonality on $(-a,a)$ gives squared norm a for each cosine and projection of the constant equal to $2\sin(k_m a)/(a k_m)$. Thus

$$\frac{B_z(x,t)}{B_s}=1-\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\cos\frac{(2m+1)\pi x}{2a}\exp\!\left[-(2m+1)^2 t/\tau_0\right],$$

$$\tau_0=\frac{4a^2}{\pi^2 D}.$$

The initial boundary jump is real in the idealisation; the series reproduces the interior initial value in the usual Fourier sense, while the imposed boundary value applies for $t>0$. Later times damp the higher spatial harmonics most rapidly. At the centre and $t=\tau_0$, the first term predicts

$$B/B_s=1-(4/\pi)e^{-1}=0.5316.$$

The next correction is $4e^{-9}/(3\pi)=0.0000524$, so this is already accurate to within 6×10^{-5}. At very early times many terms are needed because their cancellations encode a nearly field-free interior. A

half-space similarity expression near each face can be more efficient there.

For a $4.0\,\mathrm{mm}$-thick copper slab $a = 2.0\,\mathrm{mm}$, giving $\tau_0 = 0.118\,\mathrm{ms}$. At late times the centre deficit $1 - B/B_s$ approaches a single exponential with log slope $-1/\tau_0$. Reaching 90 percent at the centre requires

$$t \simeq \tau_0 \ln(4/(0.1\pi)) = 2.54\tau_0 = 0.300\,\mathrm{ms}.$$

The initial square-root penetration law cannot simply be extrapolated into this final regime, since the penetration fronts from both sides interact. Doubling full thickness at constant material quadruples the long relaxation time.

Compare a periodic solution and a surface impedance

Instead drive the half-space sinusoidally and adopt $e^{i\omega t}$ peak phasors. Solving $i\omega\widetilde{B} = D\,d^2\widetilde{B}/dx^2$ gives $\widetilde{B} = B_s e^{-(1+i)x/\delta}$ with the root $\delta = \sqrt{2D/\omega}$ that decays towards infinity. The envelope falls as $e^{-x/\delta}$, while phase lags the boundary by x/δ. The electrical surface field follows by differentiating, yielding $\widetilde{E}_y = (1+i)\widetilde{H}_z/(\sigma\delta)$. The surface impedance in these orientations is $Z_s = (1+i)/(\sigma\delta)$. Its positive real part measures dissipation, and the equal quadrature term records energy storage with this convention.

At $f = 1.0\,\mathrm{kHz}$ in the copper model, $\delta = 2.09\,\mathrm{mm}$ and $\mathrm{Re}\,Z_s = 8.25\,\mu\Omega$. A tangential surface peak field $H_s = 1000\,\mathrm{A/m}$ gives mean incoming power density $\mathrm{Re}\,Z_s|H_s|^2/2 = 4.13\,\mathrm{W/m^2}$. The same number follows from volume Joule heating. Integrate $\sigma|\widetilde{E}_y|^2/2$ over depth; the squared envelope $e^{-2x/\delta}$ integrates to $\delta/2$. An experiment monitoring just the in-phase field can find a zero at $x = \pi\delta/2$ while the full harmonic magnitude is still $e^{-\pi/2} = 0.208$ of the surface amplitude.

Compare a half-space step and a sinusoidal depth profile using the same material

Use the solution to design a convergence test

A mesh across the finite slab can step $B_j^{n+1} = B_j^n + r(B_{j+1}^n - 2B_j^n + B_{j-1}^n)$, with $r = D\Delta t/h^2$. A Fourier mode e^{ikjh} is multiplied by $1 - 4r\sin^2(kh/2)$, so forward Euler is stable for $0 \le r \le 1/2$. Above that bound a high-frequency grid mode can grow despite the physical diffusion being dissipative. For $h = 0.20\,\mathrm{mm}$ in the copper model, $\Delta t \le 1.46\,\mu\mathrm{s}$. Halving the mesh spacing demands quartering this explicit time step. An implicit scheme can overcome the same stability restriction but still needs small enough steps to resolve the measured transient.

Set both face values, initialise the interior once and record the centre at $t = \tau_0$. Compare first against the series, then against a second mesh and time step. A mistaken use of full thickness in τ_0 would shift the inferred time by a factor four even if the code has converged perfectly. Independent checks include reflection symmetry about the slab centre, zero gradient at that centre, a bounded response between zero and B_s for the continuous problem, and ohmic heating that is nonnegative at every point. The

latter can catch sign mistakes in an energy budget even though squaring J hides the sign of a local measured current.

Compare surface and centre in a periodically driven finite slab

At a fixed angular frequency a finite slab with equal tangential field on both faces has a different profile from a half-space. Write the phasor equation $d^2\widetilde{B}/dx^2 = k^2\widetilde{B}$, $k = (1+i)/\delta$ for temporal dependence $e^{i\omega t}$, and impose $\widetilde{B}(\pm a) = B_s$. Even symmetry eliminates $\sinh kx$, giving $\widetilde{B}(x) = B_s \cosh(kx)/\cosh(ka)$. The centre amplitude is $1/|\cosh[(1+i)a/\delta]|$ of the applied field, and its phase lag is the argument of that denominator. Both incoming penetration profiles must be combined as complex amplitudes satisfying boundary conditions; adding two decaying real envelopes is not a valid phasor solution.

The low-frequency limit has $ka \ll 1$, yielding $B(0)/B_s \simeq 1 - (ka)^2/2 = 1 - i(a/\delta)^2$. The leading correction is a phase lag, while the magnitude change starts at higher order. Thus a sensitive phase detector can pick up eddy-current effects before a large change of amplitude is visible. At $a/\delta = 1$, $\cosh(1+i) = \cosh 1 \cos 1 + i \sinh 1 \sin 1 \simeq 0.834 + 0.989i$, giving centre amplitude 0.773 and lag 0.870 radians. An isolated half-space viewed one skin depth in from a single face would have amplitude 0.368 and lag 1.00 radian. These are genuinely different field boundary conditions, not two definitions of skin depth.

The phasor of the current varies as $\sinh(kx)$ and is odd. Current on one half of the slab runs opposite to that on the other; the net integrated current across both halves vanishes for equal exterior fields. That cancellation does not suppress losses, because local Joule heating is proportional to $|J|^2$. In the thin-slab limit expand Faraday directly. If $B_z(t)$ is nearly uniform,

$$\partial_x E_y = -\dot{B}_z.$$

Symmetry about the centre gives

$$E_y \simeq -x\dot{B}_z.$$

For a sinusoidal peak field B_s, the mean volumetric loss averaged across thickness is $\sigma\omega^2 B_s^2 a^2/6$. The a^2 scaling at fixed total magnetic volume explains why dividing a core into electrically insulated thin laminations can reduce classical eddy losses when the applied field lies along the lamination faces. The thin-slab approximation, isolation between laminae and field orientation are essential parts of that conclusion.

For a material with $\sigma = 2.0 \times 10^6\,\mathrm{S/m}$, half-thickness $0.10\,\mathrm{mm}$, sinusoidal peak field 0.10 T and frequency 50 Hz, the formula gives $3.29\,\mathrm{W/m^3}$. If effective permeability were $100\mu_0$, the skin depth at that frequency would be about $5.03\,\mathrm{mm}$, so the thin-slab ordering is internally consistent for these model numbers. Hysteresis or anomalous dynamic losses can still dominate a measured sample. Halving full lamination thickness at fixed material volume should quarter only this classical contribution, not every loss mechanism in the test fixture.

A thin conducting cylindrical shell gives a single time constant

A different finite system exposes how geometry changes a diffusion transient. Consider a long cylindrical shell of radius a and wall thickness $w \ll a$ surrounding vacuum. Assume the skin depth in the wall exceeds w on the important time scales, so current is approximately uniform across the wall, and the solenoid producing an approximately uniform external axial field is sufficiently long. An azimuthal sheet current K_φ produces an additional axial field inside and negligible external field in the long-cylinder approximation. The boundary condition is

$$B_{\mathrm{in}} = B_{\mathrm{ext}} + \mu_0 K_\varphi.$$

Faraday's law around the shell gives

$$2\pi a E_\varphi = -\pi a^2 \dot{B}_{\mathrm{in}}.$$

The sheet obeys $K_\varphi = \sigma w E_\varphi$. Combining the three relations gives

$$B_{\mathrm{in}} + \tau_s \dot{B}_{\mathrm{in}} = B_{\mathrm{ext}},$$

$$\tau_s = \mu_0 \sigma w a / 2.$$

For a step from zero to B_s the field inside grows as $B_s(1 - e^{-t/\tau_s})$. For a sinusoid the interior phasor transfer is $1/(1 + i\omega\tau_s)$. This model has a single time constant instead of a slab's infinite relaxation spectrum because the wall thickness has been collapsed into one azimuthal sheet-current degree of freedom; magnetic energy is stored mainly throughout the bore. Take copper with $a = 0.030\,\mathrm{m}, w = 0.10\,\mathrm{mm}$ to obtain $\tau_s = 0.109\,\mathrm{ms}$. At its half-power frequency $f = 1/(2\pi\tau_s) \simeq 1.46\,\mathrm{kHz}$, the copper skin depth is about 1.73 mm, comfortably larger than the specified wall. At 100 times this frequency the uniform-wall assumption is much weaker.

This result also suggests a discriminating experiment. Compare shells made of the same material with fixed wall thickness and different bore radius. The leading screening time grows linearly with radius. Compare solid sheets of different normal penetration thickness in the earlier slab experiment and their time grows with thickness squared. Either system can display a late exponential, so an observed straight tail in a semilog trace does not alone reveal which diffusion path limits the response. The thickness and radius dependences, together with the phase response and the known current paths, provide independent evidence. A longitudinal insulating slit in the cylindrical shell interrupts its closed eddy-current path, substantially altering the ideal shielding despite leaving most of the metal present.

Distinguish propagation from diffusion

Change conductivity or drive frequency and inspect how the field attenuates with depth. Compare its amplitude decay and phase lag.

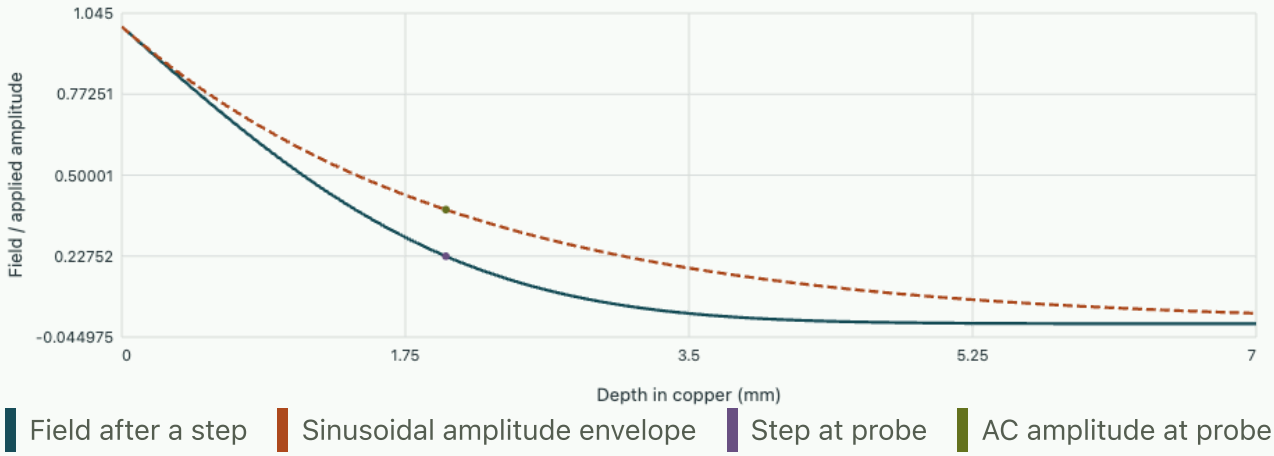
Open this experiment online

Try the calculation

Time after field step (ms) **0.1**

Sinusoidal drive frequency (kHz) **1**

Probe depth (mm) **2**



Half-step depth 1.117 mm. AC skin depth 2.09 mm. At the probe the step fraction is 0.2273, AC amplitude fraction 0.384 and AC phase lag 0.957 rad.

Account for every joule admitted by a magnetic step

A conducting half-space initially has zero internal field. At $t = 0$ a source clamps the surface induction to $B_s\hat{z}$. Let $x > 0$ measure inward distance and $D = 1/(\mu\sigma)$. The previously derived field profile is $B_z = B_s\,\mathrm{erfc}[x/(2\sqrt{Dt})]$. An observer sees an ever-deepening field front even though the surface induction subsequently remains constant. Someone must provide both the magnetic field energy and the Joule heat generated by the induced current. The appropriate bookkeeping is Poynting's theorem, with positive surface power defined into the conductor.

In this geometry Ampère's law gives

$$J_y = -(1/\mu)\partial_x B_z.$$

Differentiating the error function produces a Gaussian current density $J_y = B_s[\mu\sqrt{\pi Dt}]^{-1}\exp[-x^2/(4Dt)]$. Ohm's law sets $E_y = J_y/\sigma$. At the surface the Poynting vector $E_y H_z\hat{x}$ points into the material and has magnitude

$$\frac{P_{\mathrm{in}}}{A} = \frac{B_s^2}{\mu}\sqrt{\frac{D}{\pi t}}.$$

The initial mathematical step creates an integrable square-root singularity in the surface power. Holding B_s fixed still requires the source to deliver power for $t > 0$. Faraday's law relates the surface

electric field to the changing flux below the surface. Its value falls with time because the depth derivative of the induction spreads and weakens. A physical source with a finite rise time smooths the initial singularity and sets the earliest time for which the ideal step is a useful approximation.

To calculate heat production, square the full depth-dependent current and integrate J_y^2/σ from zero to infinity. The squared Gaussian has exponent $-x^2/(2Dt)$, and its integral on the half-line is $\sqrt{\pi Dt/2}$. The result is

$$P_{\mathrm{heat}}/A = (B_s^2/\mu)\sqrt{D/(2\pi t)}.$$

Consequently the heat fraction of instantaneous input is $1/\sqrt{2}$, about 70.7%. The rest, $1 - 1/\sqrt{2}$, supplies an increasing store of magnetic field energy. Both powers decay as $t^{-1/2}$; their integrals up to a finite time increase as $\sqrt{t}$. These fractions belong to the step-driven semi-infinite problem. They do not describe steady sinusoidal loss or a finite specimen after the diffusion front traverses it.

With $B_s = 10\,\mathrm{mT}$, copper conductivity $5.8 \times 10^7\,\mathrm{S/m}$, and permeability close to μ_0, the diffusion coefficient is $0.0137\,\mathrm{m^2/s}$. At $100\,\mu\mathrm{s}$, the incoming power is about $526\,\mathrm{W/m^2}$. Joule heating accounts for $372\,\mathrm{W/m^2}$, and field energy is increasing at about $154\,\mathrm{W/m^2}$. Integration from the ideal step to this time gives $2tP_{\mathrm{in}}(t)/A = 0.105\,\mathrm{J/m^2}$, split into approximately $0.0744\,\mathrm{J/m^2}$ of heat and $0.0308\,\mathrm{J/m^2}$ of stored field energy. At that time $2\sqrt{Dt}$ is about $2.34\,\mathrm{mm}$. Applying these numbers to a foil much thinner than a millimetre would use the half-space solution beyond its range.

A finite slab of thickness d is driven from both faces by the same step. How much magnetic energy is eventually stored, and why must a decaying input replace the half-space power law?

▼ **Work through the solution**

At long times, Ampère's law allows a uniform induction B_s with zero current and the prescribed values on both faces. Its stored magnetic energy is $dB_s^2/(2\mu)$ per unit footprint, independent of conductivity. With $d = 5.0\,\mathrm{mm}$ and the parameters above this is about $0.199\,\mathrm{J/m^2}$. At early times each face behaves approximately as an independent half-space. Once their diffusion profiles overlap, extended use of that approximation would predict unlimited energy accumulation in a finite volume with bounded field magnitude. The finite-slab modes replace the late algebraic law by exponential decay of the current and the surface electric field.

What is the total heat in a finite linear slab after full penetration if both surface inductions are clamped to the same ideal step?

▼ **Work through the solution**

Integrate the inward surface Poynting power over all time. With symmetry about the midplane, Faraday's law on the upper half relates the magnitude of the surface electric field to $d[\int_0^{d/2} B\,dx]/dt$. The surface magnetic field after the step is the constant B_s/μ. Including both faces gives total input energy per area $B_s^2 d/\mu$. Half remains stored, so half has been dissipated as heat, or about $0.199\,\mathrm{J/m^2}$ in the example. The integrated fraction one half over the complete finite transient differs from the

early-time half-space fraction. This energy identity provides a stringent check of a discretised slab calculation because it tests the surface flux, the volume heating and their time integration together.

Hermann Haus and James Melcher, Magnetoquasistatic Relaxation and Diffusion, including experimental demonstrations.

Tutorial on extracting material parameters

The late centre deficit of a slab of full thickness 6.0 mm falls by a factor e in $0.60\,\mathrm{ms}$. Find D, and estimate σ if $\mu = 2\mu_0$.

▼ **Work through the solution**

Using $a = 3.0\,\mathrm{mm}$ gives

$$D = 4a^2/(\pi^2\tau_0) = 0.00608\,\mathrm{m}^2/\mathrm{s}.$$

Then

$$\sigma = 1/(\mu D) = 6.55\times10^7\,\mathrm{S/m}.$$

A transient alone determines $\mu\sigma$; the independent permeability assumption is essential.

At fixed frequency the signal amplitude at one depth is 10 percent of the surface amplitude in a periodic half-space experiment. Find the phase lag there and predict the new amplitude when frequency drops to one quarter.

▼ **Work through the solution**

The depth is $x = \delta\ln 10$, so the lag is 2.303 radians. Quartering frequency doubles the skin depth; the ratio becomes

$$e^{-(\ln 10)/2} = 0.3162.$$

Treating the original 10 percent as an intensity ratio instead of a field ratio would produce a different inferred depth.

A step field applied at $t = 0$ is removed again at $t = T$. Obtain the field after removal and explain why it need not vanish immediately in the interior.

▼ **Work through the solution**

By linearity superpose a positive step and a delayed negative step. For $t > T$ the normalised half-space profile is $\mathrm{erfc}[x/(2\sqrt{Dt})] - \mathrm{erfc}[x/(2\sqrt{D(t-T)})]$. It vanishes at the driven boundary but

remains positive at interior points where the later disturbance has not substantially diffused. The induced currents subsequently relax and dissipate energy.

Magnetism, induction and field dynamics · 9. Displacement current and the charging capacitor

A wire feeds a capacitor whose dielectric blocks the transport of mobile charges between its plates. A magnetic probe near the gap still responds to charging. Resolving which surface a circulation law should use leads to displacement current, an electric field correction during a ramp, and an energy-flow test around the rim.

Start from a contradiction involving two surfaces

Draw a circle around the conductor leading to one plate. One spanning surface crosses the metal wire and records free current I. Another bulges into the gap and misses the metal entirely. A single circulation $\oint \boldsymbol{H}\cdot d\boldsymbol{\ell}$ cannot equal two distinct free-current integrals for the same contour. This difficulty is already visible locally. If $\nabla\times\boldsymbol{H}=\boldsymbol{J}_f$ held at every time, its divergence would require $\nabla\cdot\boldsymbol{J}_f=0$, but charge conservation near a plate gives

$$\nabla\cdot\boldsymbol{J}_f=-\partial_t\rho_f.$$

Now impose Gauss's law $\nabla\cdot\boldsymbol{D}=\rho_f$. Differentiating gives

$$\nabla\cdot(\partial_t\boldsymbol{D})=\partial_t\rho_f.$$

Therefore $\boldsymbol{J}_f+\partial_t\boldsymbol{D}$ is divergence free. Choosing it in Maxwell's circulation law respects both the steady limit and surface independence. Conservation alone determines only the required divergence of the additional term; the full Maxwell law also specifies its transverse content and is tested by electromagnetic waves. A displacement current density has units $\mathrm{A/m^2}$. Its flux can exist in a vacuum region in which no conduction charge crosses the integration surface.

$$\oint_C \boldsymbol{H}\cdot d\boldsymbol{\ell}=\int_S(\boldsymbol{J}_f+\partial_t\boldsymbol{D})\cdot d\boldsymbol{a}.$$

Subtract the fluxes through two surfaces with the same boundary. They form a closed surface when the second orientation is reversed. The difference vanishes by the divergence theorem applied to the sum. If one uses only the conduction term, the missing difference is the rate at which free charge accumulates in the closed volume. This explains why a DC magnetostatic approximation works in a settled resistor and fails for an insulating gap being charged through the same wire.

Resolve the magnetic field across the plate area

Model two circular conducting plates of radius a separated by $d\ll a$, with a linear dielectric filling the gap. Ignore edge fringing for interior points, use symmetric feeding, and take the charging time long compared with the propagation time across the plates. Then $D_z\simeq Q(t)/(\pi a^2)$. The total displacement current is $I=\dot{Q}$, but a centred Amperian circle at $s<a$ samples only fraction s^2/a^2 of that area. Cylindrical symmetry gives

$$H_\varphi(s) \simeq \frac{Is}{2\pi a^2} \quad (s < a),$$

$$H_\varphi(s) \simeq \frac{I}{2\pi s} \quad (s > a)$$

for a symmetric idealisation with negligible other return paths within the chosen circle. The second expression is a useful intermediate description near the capacitor before the distant return or fringing geometry intervenes. At the rim the formulas match. At $s = 0$ the field vanishes, so assigning the full terminal current to every small circle would produce a fictitious on-axis singularity.

Take $a = 0.050\,\mathrm{m}, d = 1.0\,\mathrm{mm}, I = 10\,\mathrm{mA}$, nonmagnetic dielectric. At $s = a/2$ the field is $H_\varphi = 0.0159\,\mathrm{A/m}$ and $B_\varphi = 20.0\,\mathrm{nT}$. At the rim $B_\varphi = 40.0\,\mathrm{nT}$. The enclosed displacement current at half radius is $2.50\,\mathrm{mA}$, a quarter of terminal current. Doubling current doubles both readings; doubling plate radius at the same fractional measurement radius halves them. For vacuum the voltage rises at $\dot{V} = Id/(\epsilon_0 \pi a^2) = 1.44 \times 10^8\,\mathrm{V/s}$, consistent with the ordinary lumped $I = C\dot{V}$.

Distinguish material and vacuum contributions

In matter $\partial_t \boldsymbol{D} = \epsilon_0 \partial_t \boldsymbol{E} + \partial_t \boldsymbol{P}$. In a transparent ideal linear dielectric with relative permittivity four, impose the same field ramp as in an empty capacitor. The required free current is four times as large, with three quarters of the total changing displacement supplied by polarisation current. If instead the external supply maintains the same current, $\dot{D}$ is unchanged, the field ramp is four times smaller, and the interior magnetic field calculated above is unchanged. These statements compare different control conditions. Supplying an equivalent charge-free background with the same prescribed voltage waveform is a different experiment from supplying it with the same current.

Add weak leakage $\boldsymbol{J}_f = \sigma \boldsymbol{E}$ in the dielectric. Under a harmonic spatially uniform drive the terminal current per area is $(\sigma + i\omega\epsilon)\widetilde{E}$ with $e^{i\omega t}$. In phase with the voltage there is real conduction, and in quadrature there is changing polarisation and vacuum electric displacement. For area $A = 0.010\,\mathrm{m}^2$, thickness $d = 0.002\,\mathrm{m}$, $\epsilon_r = 5, \sigma = 10^{-8}\,\mathrm{S/m}$, the static parallel model has $C = 221\,\mathrm{pF}$, conductance $G = 50.0\,\mathrm{nS}$, and crossover $\omega = G/C = \sigma/\epsilon = 226\,\mathrm{s}^{-1}$ or $f = 36.0\,\mathrm{Hz}$. At that frequency equal in-phase and quadrature currents do not imply equal dissipated powers; only the resistive component draws nonzero average power in the ideal nondispersive model.

Ask how energy gets into the gap

For $\boldsymbol{E} = E_z \hat{\boldsymbol{z}}$ and $\boldsymbol{H} = H_\varphi \hat{\boldsymbol{\varphi}}$, the Poynting vector is $-E_z H_\varphi \hat{\boldsymbol{s}}$. During charging positive E_z and rising positive D_z give energy flow radially inward. Just inside the cylindrical edge the surface area through which this flow enters is $2\pi a d$, and $H_\varphi(a) = I/(2\pi a)$. Thus incoming electromagnetic power is

$$E_z H_\varphi(a) 2\pi a d = VI,$$

which is $d(CV^2/2)/dt$ at constant capacitance in this approximation. Energy enters the dielectric through field flow, although the mobile charges in the plates arrive through conductors. During discharge the sign reverses.

At $V = 100$ V and $I = 10\,\mathrm{mA}$ the instantaneous rate is 1.0 W. At half radius the local inward flux density is half its edge value, but the local cylindrical area is also half as large. Their product is one quarter, exactly the fraction of stored electric volume inside half radius. The radial Poynting profile therefore agrees with local energy conservation for every coaxial subvolume. When driving becomes fast enough that substantial magnetic energy also accumulates or radiative effects matter, the simple uniform-field model no longer exhausts this balance.

Move the integration radius and inspect the field and inward power

Check the first spatial correction to a uniform electric field

A strictly time-varying azimuthal magnetic field also enters Faraday's equation in the gap, forcing a small radial dependence of E_z. For an axisymmetric z-directed field with negligible z-variation, $(\nabla \times \boldsymbol{E})_\varphi = -\partial_s E_z = -\partial_t B_\varphi$, hence $\partial_s E_z = \partial_t B_\varphi$. Insert the leading Ampère solution $B_\varphi \simeq \mu\epsilon s\,\dot{E}_z(0,t)/2$ and integrate from zero to s. The first correction is $E_z(s,t) - E_z(0,t) \simeq \mu\epsilon s^2 \ddot{E}_z(0,t)/4$. For a pure harmonic drive it scales as $-(ks)^2 E_z(0,t)/4$, with $k = \omega\sqrt{\mu\epsilon}$. This expansion is valid when $ka \ll 1$; it also predicts no correction from a strictly constant-slope ramp after initial transients have settled.

In an idealised uniform-height cylindrical model the harmonic radial wave equation is

$$s^{-1}\partial_s(s\partial_s E_z) + k^2 E_z = 0.$$

Regularity selects $E_z \propto J_0(ks)$, whose series starts with $1 - (ks)^2/4$. This independent result validates both sign and factor in the iterative argument. For a $5\,\mathrm{cm}$ vacuum radius at $10\,\mathrm{MHz}$, $ka = 0.0105$, giving an edge-to-centre relative difference only 2.75×10^{-5}. At $1\,\mathrm{GHz}$, ka is about 1.05 and the uniform premise itself requires a distributed treatment including leads and boundaries. Small geometrical thickness by itself does not ensure instantaneous behaviour across a broad plate.

Track where the conduction current flows on the plate

The uniform-gap approximation says how much displacement current passes a circle. Current in the metal plate is distributed differently. Put a feed contact at the centre of a thin circular top plate and inject free current $I(t)$ there. Assume that the plate charge density $\sigma_f = Q/(\pi a^2)$ is approximately uniform away from the contact and edge. On the metal surface let the radial sheet current be $K_s(s)$, positive outward. For a disk of radius s, the input current minus the outward sheet flux must equal the rate at which that disk acquires charge. This gives $I - 2\pi s K_s = I s^2/a^2$, hence

$$K_s(s) = \frac{I}{2\pi s}\left(1 - \frac{s^2}{a^2}\right), \qquad 0 < s < a.$$

At $s = a/2$ the outward sheet current across the circle is $0.75I$, whereas the displacement current through its interior is $0.25I$. At the rim the radial sheet current goes to zero because there is no continuation beyond the ideal metal edge. Near $s = 0$ the point-contact idealisation gives $K_s \propto 1/s$, revealing its own range of validity. A finite feed contact rounds this behaviour; finite conductivity and inductance can change the assumed uniform charging during a rapid ramp. The estimated currents on the metal and in the insulator add consistently without a conduction electron having to traverse the dielectric.

With $a = 50\,\mathrm{mm}$, $s = 25\,\mathrm{mm}$ and $I = 10\,\mathrm{mA}$, the radial sheet current at that point is $0.0477\,\mathrm{A/m}$. A ring segment spanning 30° at that radius carries $K_s s(\pi/6) = 0.625\,\mathrm{mA}$ radially outwards across its arc. The corresponding sector has collected only one twelfth of the enclosed displacement current, $0.208\,\mathrm{mA}$. Summing the two contributions over all twelve sectors restores the full terminal current. Measuring a single thin lead and assigning its whole current to a small patch of plate would give a wrong local Joule-loss density even though a terminal ammeter remained accurate.

For a plate with sheet resistance $R_\square$, the resistive dissipation per area under this low-frequency distribution is $R_\square K_s^2$. Integrate over $b < s < a$ to cut off the finite contact at radius b. The radial spreading contribution is

$$P_{\mathrm{sheet}} = \frac{R_\square I^2}{2\pi}\left[\ln\frac{a}{b} - \frac{3}{4} + \frac{b^2}{a^2} - \frac{b^4}{4a^4}\right].$$

The logarithm comes from the high current concentration near the feed. For $a/b = 10$, the bracket is 1.563 and the effective resistance P/I^2 of this plate region is $0.249R_\square$. This is a weak-loss estimate: if the voltage drop across the plate becomes comparable to the capacitor voltage, the current and charge distributions must instead be solved jointly. It is nevertheless an instructive correction to an apparently straightforward 'lossless capacitor' measurement, in which dissipation may actually be concentrated in the contacts or electrode sheets.

Recover an unknown dielectric response from a controlled ramp

Use a parallel-plate cell of area A and separation d. Its measured small-signal admittance is $Y = I/V$. In the low-frequency geometrical limit $Y = (A/d)(\sigma + i\omega\epsilon)$ for simple frequency-independent real ϵ, σ and phasor convention $e^{i\omega t}$. Suppose $A = 0.0020\,\mathrm{m}^2, d = 0.50\,\mathrm{mm}$, $f = 10\,\mathrm{kHz}$, and after subtracting lead capacitance the measured admittance is $0.80 + i8.90\,\mu\mathrm{S}$. The capacitance is $8.90 \times 10^{-6}/(2\pi 10^4) = 142\,\mathrm{pF}$, giving $\epsilon_r = Cd/(A\epsilon_0) \simeq 4.00$. Treating the real component as DC conduction gives

$$\sigma = Gd/A = 2.0 \times 10^{-7}\,\mathrm{S/m}.$$

A terminal measurement at a single frequency alone cannot confirm that interpretation.

A delayed polarisation can produce the same real terminal component without any mobile charge crossing the entire sample. For example take a single orientational relaxation $\tau\dot{P} + P = \epsilon_0\Delta\chi E$. Then $\widetilde{P} = \epsilon_0\Delta\chi\widetilde{E}/(1 + i\omega\tau)$, and the associated current is $i\omega\widetilde{P}$. Its real coefficient is

$\epsilon_0 \Delta\chi\omega^2\tau/[1+(\omega\tau)^2]$, positive in this passive model. The quadrature coefficient is $\epsilon_0 \Delta\chi\omega/[1+(\omega\tau)^2]$. A signal analyser would assign the first part to an equivalent conductance. Its strong frequency dependence distinguishes this process from a constant leakage σ if a sufficient frequency range is measured.

A step up in voltage is another way to separate hypotheses. An instantaneous ideal capacitance responds during the step only, a Debye polarisation adds a decaying current over its characteristic τ, and a true Ohmic path continues to carry current at constant final voltage. In a cell whose polarisation term has $\Delta C = 50\,\mathrm{pF}$, step magnitude $1.0\,\mathrm{V}$ and $\tau = 2.0\,\mathrm{ms}$, the delayed contribution starts at $25\,\mathrm{nA}$ and integrates to $50\,\mathrm{pC}$. The initial current in a measured trace will also contain the finite-risetime vacuum and fast-electronic capacitance currents. Integrating after a separately modelled step response is more robust than inferring charge from a single noisy peak. A steady $0.80\,\mu\mathrm{A}$ residual under 1.0 V would support the constant-conductance interpretation for the above cell; decay toward zero would point to polarisation, instrument offsets or a more complex model.

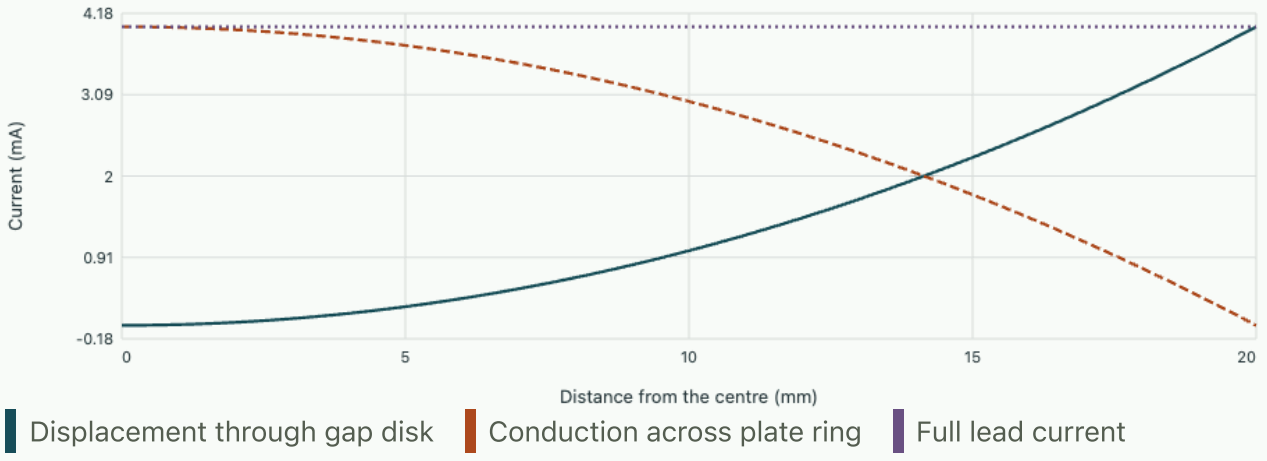
Count displacement current in a charging capacitor

Move the integration surface from wire to dielectric. Check which term in the current balance changes and which loop integral stays the same.

Open this experiment online

Try the calculation

Plate radius (mm)	20
Charging current (mA)	4
Measurement radius / plate radius	0.5



At r = 10 mm the gap encloses 1 mA and the radial plate current is 3 mA. The local azimuthal magnetic field is 20 nT.

Determine the charge that accumulates inside a layered capacitor

Stack two plane dielectric layers in series between extended parallel electrodes. Material 1 has thickness d_1, permittivity ϵ_1 and nonzero conductivity σ_1, with corresponding quantities for material 2. A source suddenly applies a fixed voltage V, and the stack is initially uncharged. The Ohmic current density alone will generally differ between the two layers early in the transient. A conduction-current mismatch deposits free charge on their internal interface. The total current including displacement remains continuous because equal currents must pass through successive cross sections of this one-dimensional stack.

Choose the field positive from electrode 1 toward electrode 2. Voltage imposes $d_1E_1 + d_2E_2 = V$. For positive times after the voltage step, current continuity imposes $\sigma_1E_1 + \epsilon_1\dot{E}_1 = \sigma_2E_2 + \epsilon_2\dot{E}_2$. Substituting $E_2 = (V - d_1E_1)/d_2$ and collecting terms gives

$$(\epsilon_1d_2 + \epsilon_2d_1)\dot{E}_1 + (\sigma_1d_2 + \sigma_2d_1)E_1 = \sigma_2V.$$

The coefficient of the derivative divided by the coefficient of E_1 is a positive relaxation time τ. The initial split of voltage follows from the absence of pre-existing free interfacial charge, which requires

$$\epsilon_1E_1(0^+) = \epsilon_2E_2(0^+).$$

This is the capacitive voltage division of the instantaneous step. At long times, the two steady conduction currents must match, so $\sigma_1E_1(\infty) = \sigma_2E_2(\infty)$. This is the resistive voltage division. At intermediate times each field relaxes exponentially from its initial value to its own final value with the same τ, while their thickness-weighted sum remains exactly fixed by the supply.

To see the signs in an example take $d_1 = d_2 = 1.0\,\mathrm{mm}$, $V = 10\,\mathrm{V}$, $\epsilon_1 = 2\epsilon_0$, $\epsilon_2 = 4\epsilon_0$,

$$\sigma_1 = 1.0 \times 10^{-8}\,\mathrm{S/m},$$

$$\sigma_2 = 4.0 \times 10^{-8}\,\mathrm{S/m}.$$

The fields just after charging are 6.67 and $3.33\,\mathrm{kV/m}$, respectively. At long times they are 8.00 and $2.00\,\mathrm{kV/m}$. The relaxation time is

$$6\epsilon_0/(5.0 \times 10^{-8}\,\mathrm{S/m}) = 1.06\,\mathrm{ms}.$$

Increasing a conductivity at fixed permittivity shortens this time because free charge can move faster to redistribute the internal electric field.

The normal directed from material 1 into material 2 gives a free interface charge $q_s = \epsilon_2E_2 - \epsilon_1E_1$ per unit area. Its final value is $-7.08 \times 10^{-8}\,\mathrm{C/m^2}$. The initial conduction current into the interface from layer 1 is $6.67 \times 10^{-5}\,\mathrm{A/m^2}$, while $1.33 \times 10^{-4}\,\mathrm{A/m^2}$ leaves through layer 2. Thus our convention predicts decreasing, negative interface charge. Direct differentiation gives $\dot{q}_s = J_{\mathrm{cond},1} - J_{\mathrm{cond},2}$, precisely the same sign. Measuring the total electrode current alone does not tell the experimenter that the current inside each layer is all conductive.

Find the total through-current just after the initial ideal charging impulse, and compare it with the long-time value.

▼ **Work through the solution**

Immediately after the impulse, layer 1 has $\dot{E}_1 = (8000 - 6667)/(1.06\,\mathrm{ms})$, about $1.25 \times 10^{6}\,\mathrm{V/(m\,s)}$. Its displacement contribution is therefore about $2.22 \times 10^{-5}\,\mathrm{A/m^2}$. Adding layer 1 conduction gives $8.89 \times 10^{-5}\,\mathrm{A/m^2}$. In layer 2 the derivative has the opposite sign because the layer thicknesses match. The negative displacement contribution subtracts about $4.44 \times 10^{-5}\,\mathrm{A/m^2}$ from its larger conduction current and gives the same total. At long times only the matched conduction current $8.0 \times 10^{-5}\,\mathrm{A/m^2}$ remains. A real pulse generator gives a finite initial charging burst whose duration should be separated from the material relaxation when fitting data.

Can this sample have nonzero conductivity and still produce no slowly accumulating free interface charge?

▼ **Work through the solution**

It can if $\sigma_1/\epsilon_1 = \sigma_2/\epsilon_2$. In that case the initial capacitive field division also satisfies the steady conduction condition, and the two equilibrium values in the exponential coincide. The voltage step can still deliver an initial charging current and sustain later resistive heating. An absence of the interfacial relaxation feature does not by itself imply perfectly insulating layers. Conversely, a measured slow response from a stack need not indicate slow molecular reorientation in either homogeneous constituent. A change of layer thickness or replacement with a homogeneous control can separate that possibility from charge accumulation at an internal interface.

David Tong, Electrodynamics, on displacement current and field propagation.

Tutorial on two spanning surfaces

A gas-filled capacitor has $C = 35\,\mathrm{pF}$ and rms drive $12\,\mathrm{V}$ at 20 kHz. Determine rms displacement current and the fraction surrounded by a centred circle at radius $0.7a$.

▼ **Work through the solution**

$$I_{\mathrm{rms}} = 2\pi f C V_{\mathrm{rms}} = 52.8\,\mu\mathrm{A}.$$

The fractional area is $0.7^2 = 0.49$, so the smaller circle encloses $25.9\,\mu\mathrm{A}$ in the uniform idealisation. Rms and peak conventions must be used consistently throughout the calculation.

An isolated capacitor initially carries $Q_0 > 0$ on its positive plate. With weak dielectric conductivity and no external leads attached, derive its voltage decay time. What happens to the net current crossing an internal spanning surface?

▼ **Work through the solution**

The parallel conductance $G = \sigma A/d$ drains the free charge, so $C\dot{V} + GV = 0$ and $V = V_0 e^{-t/\tau}$ with $\tau = C/G = \epsilon/\sigma$. Conduction current and displacement current through the gap have opposite signs and equal magnitudes in the uniform model. Their sum vanishes even though Joule heat is produced by the decreasing stored electric energy.

With the second plate connected through a distant return wire, a large probe loop encircles the feed and return together. Should the $1/s$ exterior gap formula be extrapolated to this loop?

▼ **Work through the solution**

The derivation assumed only the charging current was enclosed and the surrounding geometry retained the required symmetry. Including the return changes the enclosed free-current balance and often destroys cylindrical symmetry. Choose a new surface and include its complete total-current flux before predicting any circulation. The local field need not vanish just because the complete circulation is zero.

Magnetism, induction and field dynamics · 10. Energy flow and mechanical force from vacuum fields

An external support holds a charged capacitor plate or a current-carrying coil in place. One can infer the support force from fields just outside the body, without summing forces on every microscopic charge. A second surface integral tracks the energy feeding the same apparatus. Both methods start with Maxwell's equations, but they transport different conserved quantities.

Derive a local energy balance before choosing the surface

In vacuum write $\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$ and $\nabla \times \boldsymbol{B}/\mu_0 = \boldsymbol{J} + \epsilon_0 \partial_t \boldsymbol{E}$. Dot the second equation with $\boldsymbol{E}$ and the first with $\boldsymbol{B}/\mu_0$. The vector identity $\nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \nabla \times \boldsymbol{E} - \boldsymbol{E} \cdot \nabla \times \boldsymbol{B}$ converts their difference into

$$\partial_t u + \nabla \cdot \boldsymbol{S} = -\boldsymbol{J} \cdot \boldsymbol{E},$$

$$u = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0},$$

$$\boldsymbol{S} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}.$$

Integrate over a volume held fixed in the chosen frame. Outward power $\oint \boldsymbol{S} \cdot d\boldsymbol{a} > 0$ represents energy leaving; positive $\int \boldsymbol{J} \cdot \boldsymbol{E}\, dV$ represents electromagnetic work on matter. They do not have to agree instantaneously: their sum plus the rate of increase of contained field energy is zero with these sign conventions. In matter, one can include the total microscopic currents inside the vacuum-form balance, or use an appropriate macroscopic constitutive energy description. A vacuum integration surface enclosing the whole body is particularly useful for a force calculation because it avoids assigning momentum to material polarisation separately.

Consider a long straight resistor with radius a, length L, uniform current along $+z$ and voltage drop V in that direction. Immediately outside its curved face the tangential electric field is V/L under the ideal uniform model, and $H_\varphi = I/(2\pi a)$. Their product makes a radial inward $\boldsymbol{S}$. Integrating over $2\pi a L$ gives VI, equal to the internal $I^2 R$ in steady state. At $V = 4.0\,\mathrm{V}$, $I = 0.20\,\mathrm{A}$, $a = 1.0\,\mathrm{mm}$, $L = 0.10\,\mathrm{m}$, the ideal side flux density is $1273\,\mathrm{W/m^2}$ and the total is $0.80\,\mathrm{W}$. A directed charge drift down the resistor does not imply that its supply of energy travels entirely down the same path.

Convert the Lorentz force into field derivatives

The microscopic force density is

$$\boldsymbol{f} = \rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}.$$

Eliminate charge with $\rho = \epsilon_0 \nabla \cdot \boldsymbol{E}$ and current with Ampère–Maxwell. The current term contains $-\epsilon_0 \partial_t \boldsymbol{E} \times \boldsymbol{B}$. Add and subtract the missing piece to make $-\partial_t(\epsilon_0 \boldsymbol{E} \times \boldsymbol{B})$; replace the introduced

$\boldsymbol{E} \times \partial_t \boldsymbol{B}$ by $-\boldsymbol{E} \times \nabla \times \boldsymbol{E}$ using Faraday. The spatial terms regroup with $(\boldsymbol{F} \cdot \nabla)\boldsymbol{F} = \nabla(F^2/2) - \boldsymbol{F} \times (\nabla \times \boldsymbol{F})$. The result in Cartesian components is

$$f_i = \sum_j \partial_j T_{ij} - \partial_t g_i,$$

$$T_{ij} = \epsilon_0 (E_i E_j - \tfrac{1}{2}\delta_{ij} E^2) + \mu_0^{-1}(B_i B_j - \tfrac{1}{2}\delta_{ij} B^2),$$

$$\boldsymbol{g} = \epsilon_0 \boldsymbol{E} \times \boldsymbol{B} = \boldsymbol{S}/c^2.$$

For a body enclosed by a fixed vacuum surface S, $F_i = \oint T_{ij} n_j \, dA - d(\int g_i dV)/dt$ if the enclosed Lorentz force is entirely on that body. The traction $\boldsymbol{T} \cdot \boldsymbol{n}$ has units of pressure; the field momentum inside has units of $\mathrm{kg\,m/s}$. The tensor convention used here makes $-T_{ij}$ the outward flux density of i-momentum along j in the conservation equation. Texts that call this outward flux the stress can print the opposite sign. Test the convention on a simple physical force before interpreting a long component sum.

Obtain the attractive pressure on a conducting plate

Take a small vacuum enclosure straddling the surface of an ideal conductor in electrostatic equilibrium. Inside it $\boldsymbol{E} = 0$; just outside $\boldsymbol{E} = E_n \boldsymbol{n}$. Tangential field vanishes. The traction from the exterior face is $\epsilon_0 E_n^2 \boldsymbol{n}/2$; the other thin faces give no limiting contribution. For opposed charged plates this force draws each plate towards the facing gap. Equivalently, a surface charge $\sigma = \epsilon_0 E_n$ experiences the average of the one-sided fields,

$$\boldsymbol{E}_{\text{acting}} = E_n \boldsymbol{n}/2.$$

Using the outside field alone in $\sigma \boldsymbol{E}$ would count the sheet's own field and give twice the measured force.

With area $0.010\,\mathrm{m^2}$, air gap $0.50\,\mathrm{mm}$ and voltage $250\,\mathrm{V}$, $E = 5.0 \times 10^5\,\mathrm{V/m}$, pressure is $1.11\,\mathrm{Pa}$ and force magnitude $0.0111\,\mathrm{N}$. Holding voltage fixed while doubling the gap quarters both pressure and force. If the source is disconnected first, the initial free charge stays fixed and so does the pressure in the ideal large-plate limit. These two protocols agree at the starting configuration and immediately diverge when a plate moves. Neglecting the source when differentiating the fixed-voltage energy $\frac{1}{2}CV^2$ makes a wrong force sign because C changes during the displacement.

Compare force during fixed-charge and fixed-voltage separation

Resolve magnetic tension and side pressure on a surface

For a pure magnetic field of magnitude B along z, the vacuum stress tensor is diagonal with entries $(-B^2/2\mu_0, -B^2/2\mu_0, +B^2/2\mu_0)$. An element whose outward normal is parallel to the field feels positive normal traction in that direction in the tensor convention, while a normal perpendicular to the field gives negative normal traction. For a magnet pole face with a nearly normal gap field and

negligible fringe fields elsewhere, integrating the appropriate vacuum surface gives an attraction across the gap of magnitude $AB^2/(2\mu_0)$. For $A = 2.0\,\mathrm{cm}^2$ and $B = 0.20$ T this is $3.18\,\mathrm{N}$. The equal and opposite reaction belongs to the rest of the closed apparatus.

An oblique field produces shear. At a plane with $\boldsymbol{n} = \hat{\boldsymbol{z}}$, $\boldsymbol{B} = B_x\hat{\boldsymbol{x}} + B_z\hat{\boldsymbol{z}}$, one obtains

$$t_x = B_x B_z/\mu_0,$$

$$t_z = (B_z^2 - B_x^2)/(2\mu_0).$$

For $B_x = 0.020$ T, $B_z = 0.050$ T the respective values are 796 Pa and 836 Pa. The sign of the shear reverses when only one component reverses, unlike the pure normal pressure. A single scalar value $B^2/(2\mu_0)$ cannot describe a general loaded face. The surface here is a calculation tool inside vacuum; a proposed net force on a body must still include all the other faces of the enclosure.

Test a radiation-pressure experiment using both integrals

A plane electromagnetic wave in vacuum has transverse electric and magnetic fields with $E = cB$, so its instantaneous energy density is $\epsilon_0 E^2$. The Poynting magnitude is cu, and the momentum density magnitude is u/c. A normally incident pulse of energy U carries momentum U/c towards an absorbing surface. If fully reflected along the reverse direction it changes the field momentum by $2U/c$, giving twice the impulse to the mirror. For a continuous beam of average intensity I on a perfectly absorbing target, pressure is I/c; on a perfect reflector it is $2I/c$. At $I = 600\,\mathrm{W/m^2}$ these are 2.00 and $4.00\,\mu\mathrm{Pa}$. The lower pressure scale relative to the static capacitor example reflects an energy divided by a very large propagation speed.

A pulse striking a surface need not produce its final impulse from the stress term alone at every intermediate time. If the integration box includes part of an incoming or outgoing pulse, $\int \boldsymbol{g}\,dV$ changes. Select a large time interval over which no pulse is present inside at its beginning or end. Then the integrated field-momentum derivative vanishes, and the mechanical impulse equals the time integral of surface traction. Alternatively use a smaller interval and calculate both terms explicitly. Independence of the final impulse from the box location and time window, when all terms are included, is a strong numerical test. It detects spurious forces from artificial absorbing boundaries or inconsistent subtraction of the incoming background.

A standing wave tests energy and momentum at the same surface

A mirror terminating a one-dimensional vacuum beam provides a case in which the cycle-averaged power entering an ideal mirror vanishes but the mechanical force remains finite. Let E_i be the incident peak amplitude polarised along y, and $B_i = E_i/c$ in the appropriate z direction. At an ideal conducting plane the reflected tangential electric field cancels the incident field. The corresponding reflected magnetic field adds there because reversing propagation reverses the sign between E and B. The total magnetic peak is $2E_i/c$, while the total tangential electric field on the surface is zero. Hence the

normal instantaneous Poynting vector at the mirror vanishes. There is no energy deposition in a stationary lossless perfect conductor.

The vacuum magnetic stress on the mirror surface still has magnitude $B^2/(2\mu_0)$ at each instant. Averaging its sinusoidal square gives $\langle p\rangle = (2E_i/c)^2/(4\mu_0) = \epsilon_0 E_i^2 = 2I_i/c$, since the incident mean intensity is $\epsilon_0 c E_i^2/2$. A reflector can therefore experience twice the absorbing radiation pressure while dissipating no power. This does not imply a free mechanical power supply: holding a mirror fixed means its speed and the mechanical power $\boldsymbol{F}\cdot\boldsymbol{u}$ are zero even when its support carries a continuous force. If the mirror moves significantly, the Doppler shift and work on it require an altered reflected field.

Away from the mirror the incident and reflected waves store electric and magnetic energy in spatially alternating regions. At a node of the electric field the magnetic field can be maximal. One quarter wavelength away there is an electric antinode and a magnetic node. The electric field there reaches its temporal maximum a quarter period after the magnetic maximum at the first position. The instantaneous normal Poynting vector generally oscillates and its cycle mean is zero in the lossless standing wave. A detector resolving only the cycle-averaged flux could incorrectly infer that the cavity is empty. A magnetic loop and an electric dipole probe at the same location would give complementary indications of stored energy. They have different coupling and must be calibrated before their raw voltages are compared.

Now consider partial reflection with intensity reflectance $\mathcal{R}$, transmittance $\mathcal{T}$ and absorptance $\mathcal{A} = 1-\mathcal{R}-\mathcal{T}$, with vacuum incident and exit regions and no additional scattering channels. At normal incidence the net incident-minus-outgoing momentum flow gives $\langle p\rangle = (1+\mathcal{R}-\mathcal{T})I_i/c = (2\mathcal{R}+\mathcal{A})I_i/c$, while absorbed heating per area is $\mathcal{A}I_i$. If $I_i = 300\,\mathrm{W/m^2}, \mathcal{R}=0.60, \mathcal{T}=0.30$, the material absorbs $30\,\mathrm{W/m^2}$ and the pressure is $1.30\,\mu\mathrm{Pa}$. Measuring absorbed power alone would account for only $0.10\,\mu\mathrm{Pa}$; the reversed momentum of the reflected fraction supplies the remainder. If transmitted light emerges at an angle or into a liquid, momentum components and material forces must be included with a surface appropriate to that experiment.

Compare stress and virtual work on a finite electrostatic source

A conductor held at zero potential occupies the lower half-space. Place a point charge q a height a above it. The image method gives the exterior field by adding an opposite charge at $-a$; the conductor's interior field vanishes. Immediately above the conductor, the normal electric field has magnitude $E_n(s) = |q|a/[2\pi\epsilon_0(s^2+a^2)^{3/2}]$ at radial distance s. Its square gives a nonuniform surface attraction $\epsilon_0 E_n^2/2$. Integrating it over rings on the full plane checks the force calculation without applying a singular self-field to the point charge.

Specifically, the total attractive force on the conductor has magnitude $q^2a^2/(4\pi\epsilon_0)\int_0^\infty s(s^2+a^2)^{-3}ds$. Substituting $v = s^2+a^2$ gives $1/(4a^4)$ for the integral. The net force is $q^2/(16\pi\epsilon_0 a^2)$, exactly the Coulomb force between the real charge and the mathematical image. For $q = 1.0\,\mathrm{nC}, a = 20\,\mathrm{mm}$, this is $5.62\,\mu\mathrm{N}$. The normal pressure is concentrated near the charge's projection even though the induced surface charge distribution extends indefinitely.

Integrating only out to radius b captures a force fraction $1 - a^4/(a^2 + b^2)^2$. Already at $b = a$ one collects 75 percent of the force.

The electrostatic mechanical potential for the real charge relative to the grounded conductor is $-q^2/(16\pi\epsilon_0 a)$, obtained by integrating the attractive force with respect to a. A tempting use of the real-charge–image pair energy $-q^2/(8\pi\epsilon_0 a)$ doubles that answer because the image is an induced representation and moves as the real charge moves. The correct assembly work is half $q\phi_{\text{induced}}$ for linear grounded boundary response: as the real charge is built up from zero, its induced potential grows linearly with it. This factor shares its origin with the half in electrostatic field energy, while the force on a test charge already present is the full $q\boldsymbol{E}_{\text{induced}}$. The surface pressure, image force and derivative of mechanical potential therefore agree when each calculation counts the experiment's degrees of freedom correctly.

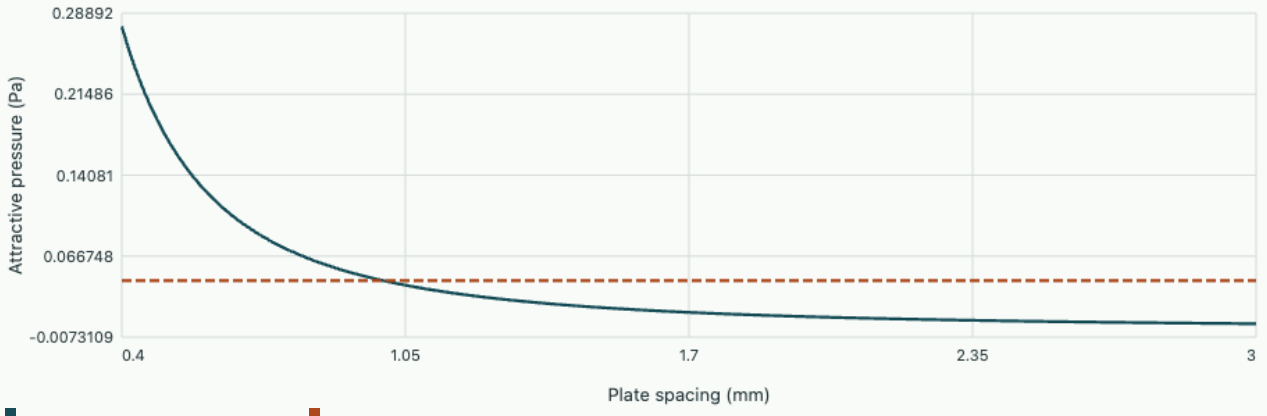
Compare force from stress with energy gradients

Change the capacitor conditions and derive the force independently from field pressure. Work out which external source does work during the displacement.

Open this experiment online

Try the calculation

Reference voltage at 1 mm (V) **100**



At 2 mm the voltage supply gives 0.01107 Pa. With the original charge trapped, the pressure stays at 0.04427 Pa.

Derive the compression of a straight current column

Consider a long nonmagnetic round wire or an ideal cylindrical conducting liquid column of radius a, carrying a uniform axial current I. End regions and any imposed external magnetic field are well

removed from the section of interest. For a solid wire the stresses needed to maintain equilibrium depend on elastic properties. A simple fluid model allows us to calculate a scalar pressure difference directly from the magnetic body force. This example distinguishes the total force on a body from substantial internal stresses whose vector sum vanishes by symmetry.

In cylindrical coordinates the current density is

$$J_z = I/(\pi a^2).$$

Applying Ampère's law to an interior circle of radius s encloses the fraction s^2/a^2 of the current and gives

$$B_\phi(s) = \mu_0 I s/(2\pi a^2).$$

The radial Lorentz force density is $f_s = -J_z B_\phi$ because $\hat{z} \times \hat{\phi} = -\hat{s}$. Consequently the magnetic force on a small volume is inward everywhere except on the axis, where the local field vanishes. A static fluid satisfies $0 = -dp/ds + f_s$, so its radial pressure gradient must be negative. Integrating from the axis outwards gives

$$p(s) - p(a) = \frac{\mu_0 I^2}{4\pi^2 a^4}(a^2 - s^2),$$

$$p(0) - p(a) = \frac{B_\phi(a)^2}{\mu_0}.$$

Pressure at the centre must exceed pressure at the rim to balance the inward force. There is no factor one half in this centre-to-rim result. The expression $B^2/(2\mu_0)$ often described as a magnetic pressure is a local Maxwell surface stress in vacuum. Here the field permeates the current-bearing material, and the interior magnetic force is distributed continuously through its volume. The stress on a single outer surface is insufficient to read off the interior fluid pressure profile. Likewise, integrating the radial vector force around a full circular cross section yields zero net translational force, while the scalar pressure difference remains finite.

With $I = 20\,\mathrm{A}$ and $a = 1.0\,\mathrm{mm}$, the edge induction is $4.0\,\mathrm{mT}$, giving a centre-to-rim pressure difference of $12.7\,\mathrm{Pa}$. Doubling the current quadruples the pressure difference; doubling the radius at fixed current reduces it to one quarter. Now consider a thin cylindrical current sheet of the same radius carrying the same total current. Its interior field vanishes, and the exterior field just outside the sheet equals the former edge field. The surface current density $I/(2\pi a)$ experiences the average of the fields on its two sides. The resulting inward surface force has magnitude $B_{\mathrm{out}}^2/(2\mu_0) = 6.37\,\mathrm{Pa}$. The fields outside the two sources agree, yet their internal force distributions differ.

A practical test can exploit the square-law dependence of stress. If $I(t) = I_{\mathrm{pk}} \cos \omega t$, and the magnetic profile follows the current quasistatically with negligible skin effect, the Lorentz force varies as $\cos^2 \omega t = [1 + \cos(2\omega t)]/2$. A pressure or strain sensor can detect a steady signal and a component at twice the current frequency. The actual displacement follows the mechanical response and may resonate or shift in phase. An added external dc field perpendicular to the current gives a force at the current frequency, distinguishable by its angular pattern as well as its frequency.

If the conductor has the same externally measured field at its surface, how could one check whether the uniform-current pressure profile is applicable?

▼ **Work through the solution**

Estimate the skin depth at the highest significant driving frequency and compare it with the radius. A skin depth comfortably larger than the radius favours a nearly uniform current in a homogeneous conductor far from nearby returns. A much smaller skin depth confines the current to the surface. A measurement of ac resistance or an interior-field measurement in a suitable analogue provides an independent check. A dc current can still be distributed unevenly in a conductor with nonuniform conductivity. Surface field alone measures total enclosed current and cannot distinguish these internal distributions.

A fluid column has an external mechanical pressure p_{ext} and a uniform volume current all the way to the boundary. Must one add $B(a)^2/(2\mu_0)$ as an extra fluid pressure jump at the rim?

▼ **Work through the solution**

In the ideal nonmagnetic volume-current example no singular surface current is present. The tangential induction is continuous just across the rim, so the normal magnetic Maxwell stresses immediately outside and inside are equal there. Ignoring surface tension, other surface forces and fluid motion, the fluid pressure at the rim matches the external mechanical pressure. The inward volume force then builds the pressure toward the centre. In a model with a genuine surface current, the fields on the two sides differ and the magnetic stress jump is real. Checking which current distribution is assumed prevents counting an exterior magnetic traction twice.

David Tong, Electromagnetism and Relativity, on the energy–momentum tensor.

Tutorial on mechanical checks

A lossless test mirror intercepts a pulse of energy $3.0\,\mathrm{mJ}$ at normal incidence. It reflects 75 percent and transmits the remainder in the same direction in vacuum. Calculate its impulse and contrast the result if the remaining quarter is absorbed instead.

▼ **Work through the solution**

Initial momentum is U/c. The outgoing net momentum is $(0.25-0.75)U/c=-0.50U/c$, so impulse is

$$1.50U/c=1.50\times10^{-11}\,\mathrm{N\,s}.$$

If the unreflected portion is absorbed, the outgoing momentum is $-0.75U/c$ and impulse rises to

$$1.75U/c=1.75\times10^{-11}\,\mathrm{N\,s}.$$

Absorption changes the energy budget and transfers its own share of incident momentum.

The fields at the opposite sides of an ideal current sheet are $+B_0\hat{\boldsymbol{x}}$ and $-B_0\hat{\boldsymbol{x}}$. Is its own field sufficient to exert a net transverse force on the sheet? What changes in a uniform additional field also along x?

▼ Work through the solution

The two self-field magnitudes are equal, so their opposite vacuum-face tractions cancel. An external field changes the total magnitudes differently: the pressure difference is $[(B_{\text{ext}}+B_0)^2-(B_{\text{ext}}-B_0)^2]/(2\mu_0)=2B_{\text{ext}}B_0/\mu_0$ in magnitude with orientation set by the actual sheet current. Since $|K|=2B_0/\mu_0$, this equals $|\boldsymbol{K}\times\boldsymbol{B}_{\text{ext}}|$.

A simulation reports an outward energy flux of 1.2 W and positive work on matter of 0.30 W in a closed fixed region with no other sources there. What must happen to its contained field energy over a 2.0 ms interval if the rates stay constant?

▼ Work through the solution

The rate is $-1.2-0.30=-1.50\,\text{W}$, so stored energy must fall by $3.0\,\text{mJ}$. A steady-state claim for the same numbers would violate Poynting's balance unless an additional input or change of sign were identified.

Magnetism, induction and field dynamics · 11. Field momentum, momentum exchange and the quasistatic limits

A system can have static electric and magnetic fields with nonzero local electromagnetic momentum density. A circuit can also be electrically small while either induction or displacement current remains indispensable. Both puzzles become clearer when one follows the relevant conservation law and compares the scales of the discarded terms.

Interpret a local momentum density without invoking radiation

In vacuum $\boldsymbol{g} = \epsilon_0 \boldsymbol{E} \times \boldsymbol{B} = \boldsymbol{S}/c^2$. A plane wave gives an especially simple example because its momentum and energy travel in the same direction at c. The formula itself also applies to near fields, standing fields and stationary crossed fields. A large local $\boldsymbol{g}$ does not imply outward radiation from the entire system: the integral can vanish by symmetry, and neighbouring Poynting vectors can form a circulating pattern. The force law over a fixed region uses the time derivative of the volume integral as well as the Maxwell stress over its boundary.

Consider idealised crossed uniform fields $\boldsymbol{E} = 2.0 \times 10^5 \hat{\boldsymbol{x}}\,\mathrm{V/m}$ and $\boldsymbol{B} = 0.20 \hat{\boldsymbol{z}}\,\mathrm{T}$. The local density is $-3.54 \times 10^{-7} \hat{\boldsymbol{y}}\,\mathrm{N\,s/m^3}$. A $1.0\,\mathrm{litre}$ counted overlap region contributes $-3.54 \times 10^{-10} \hat{\boldsymbol{y}}\,\mathrm{N\,s}$. Treat this number as the integral over that region, not automatically the momentum of the full apparatus: the fringing fields, magnets, sources, support and return currents have not been included. The integrated momentum balance will determine which part changes when a supply is switched off.

Store angular momentum in a coaxial electric field

Take two very long coaxial nonconducting cylindrical shells of radii a, b carrying opposite free charges per length $+\lambda, -\lambda$. Ignore end fields in a central region. For $a < s < b$, the electric field is $\lambda/(2\pi\epsilon_0 s)\hat{\boldsymbol{s}}$; it is negligible elsewhere in the ideal symmetric model. A coaxial long solenoid of radius R with $a < R < b$ creates $B_z = B_0$ for $s < R$ and negligible field outside. In the overlap annulus $\boldsymbol{g} = -\lambda B_0/(2\pi s)\hat{\boldsymbol{\varphi}}$. Integration over angle cancels total linear momentum, yet the z-component of angular momentum density $s g_\varphi$ has the same negative sign everywhere.

$$\frac{L_{\mathrm{field},z}}{\ell} = \int_a^R s g_\varphi \, 2\pi s \, ds = -\frac{\lambda B_0}{2}(R^2 - a^2).$$

The units are angular momentum per length, $\mathrm{kg\,m/s}$. With $\lambda = 2.0\,\mathrm{nC/m}$, $B_0 = 0.10\,\mathrm{T}$, $a = 1.0\,\mathrm{cm}$, $R = 3.0\,\mathrm{cm}$, the field contribution is $-8.0 \times 10^{-14}\,\mathrm{kg\,m/s}$. Its small size makes the thought experiment cleaner than the mechanical measurement, yet the sign can be tested without an ambiguous assignment of a local outward momentum beam.

Now reduce B_0 slowly to zero with the shells free to rotate. Faraday's law gives $E_\varphi = -s\dot{B}_0/2$ for $s < R$, and $E_\varphi = -R^2\dot{B}_0/(2s)$ outside the solenoid. The torque per unit length on a charged shell is $s\lambda_{\mathrm{shell}} E_\varphi$. Time integration gives the inner shell $+\lambda a^2 B_0/2$ and the outer shell $-\lambda R^2 B_0/2$. The

combined mechanical angular impulse equals the initial field angular momentum. There was no need to posit a magnetic force on the initially stationary shell charges; the induced azimuthal electric field provides the torque. The calculation assumes negligible friction, enough length for end effects to be small, and shell speeds too low to alter the imposed fields appreciably.

Compare retardation with the strength of two different couplings

Let the apparatus have size L, characteristic variation time T, and typical field magnitudes E_*, B_*. A light-travel time estimate $\eta = L/(cT)$ tests the importance of retardation in vacuum. Faraday's equation compares the induction electric scale LB_*/T against E_*; define $r_F = LB_*/(TE_*)$. Ampère's displacement term compared with a characteristic magnetic curl B_*/L gives

$$r_D = LE_*/(c^2TB_*).$$

Multiplication yields

$$r_F r_D = \eta^2.$$

These are scale estimates, not equalities asserting that every component has its characteristic derivative. Symmetry, very small local fields, and boundary layers can change a local ordering.

When $\eta \ll 1$, at least one of the two ratios is small for the chosen representative scales. In an electroquasistatic description electric fields mainly come from charge distributions, with magnetic induction's correction to their circulation small ($r_F \ll 1$). One can retain displacement current when determining a magnetic field from a charging current. In a magnetoquasistatic description inductive electric fields can dominate, while displacement current contributes negligibly to the magnetic curl ($r_D \ll 1$ in the region of interest). Circuit windings and good conducting bodies often meet this latter condition. Simply setting every time derivative to zero would suppress the very changing flux or charging measurement motivating the approximation.

Carry out two estimates on a physically small apparatus

A parallel-plate vacuum capacitor of radius L with harmonic voltage has $B_\varphi(L) \simeq \mu_0\epsilon_0 L\omega E/2$. Therefore

$$cB/E \simeq \omega L/(2c).$$

With $L = 0.050\,\mathrm{m}$ and $f = 100\,\mathrm{kHz}$, $\eta = \omega L/c = 1.05 \times 10^{-4}$, taking $T = 1/\omega$. Then $r_F \simeq \eta^2/2 = 5.5 \times 10^{-9}$, while $r_D \simeq 2$. The factor two reflects the cylindrical curl geometry; the order-one conclusion is what matters. A scalar-potential calculation of the dominant electric field is excellent here. Dropping displacement current when calculating the small magnetic field in the vacuum gap would give the wrong leading magnetic result.

For a long solenoid with time-dependent uniform interior B_z, Faraday's law gives

$$E_\varphi(L) \simeq L\omega B/2.$$

At the same size and frequency $E/(cB) \simeq \eta/2$, so $r_D \simeq \eta^2/2$ and $r_F \simeq 2$. Induction is now the leading mechanism creating the observed azimuthal electric field. Nevertheless the apparatus is just as electrically small as the capacitor. Comparing the species of source and the requested observable guides the choice of reduced equations. A single physical circuit can contain a capacitor gap suited to the first analysis and an inductive winding suited to the second.

Preserve causality when an approximation has an instantaneous kernel

The static Coulomb and Biot–Savart integrals use source values at one labelled time. In a slowly changing near zone, evaluating such integrals with instantaneous data may be the leading term of a retarded solution. For a localised source at separation R, source variation enters as $F(t - R/c) = F(t) - (R/c)\dot{F}(t) + \cdots$. A small $L/(cT)$ must apply to the relevant source-to-observer distance and the fastest features actually retained. A step with a nanosecond leading edge can violate the expansion over a metre even if its repetition rate is only one kilohertz. A distant measurement of the radiation of a small antenna cannot inherit the near-zone approximation merely because the antenna itself is small.

For a sinusoidal test $R = 0.30\,\mathrm{m}$ and $f = 10\,\mathrm{MHz}$, the vacuum delay is 1.00 ns and the phase shift is $2\pi f R/c = 0.0629$ rad, or 3.60°. If the measurement tolerates only 1° phase error, this apparently modest delay is already too large. At $R = 0.030\,\mathrm{m}$ the phase error is 0.36°. In a cable the signal speed depends on materials and geometry and can be lower than c. Finally, leading contributions to an observable can cancel by symmetry; then even a small induction or radiation correction can be the first nonzero effect. Estimate the error against the specific signal being measured, not just the largest field anywhere in the apparatus.

Track the angular impulse while a magnetic flux changes

Compute the momentum carried by steady power in a coaxial line

Between perfect concentric conducting cylinders with radii a, b, a TEM line carrying a DC potential difference V and current I has radial electric field $E_s = V/[s\ln(b/a)]$ and azimuthal magnetic field $H_\varphi = I/(2\pi s)$. Use an insulated nondispersive vacuum region and suppose the return carries the opposite current. The Poynting vector is axial and has magnitude $VI/[2\pi s^2 \ln(b/a)]$. Integrating over its annular cross section gives VI because $\int_a^b ds/s = \ln(b/a)$. The magnetic and electric fingerprints are time independent after a DC state is established, yet power flows from the source to a load continuously. Looking only for a time derivative in the local fields would miss this steady energy transport.

Momentum per unit line length in the vacuum field between the conductors is VI/c^2 along the line. At $V = 50\,\mathrm{V}, I = 0.20\,\mathrm{A}$, the power is 10 W and the field momentum per metre is $1.11 \times 10^{-16}\,\mathrm{kg/s}$. In a two-metre section the total field momentum is $2.23 \times 10^{-16}\,\mathrm{kg\,m/s}$. If that section starts or stops transmitting power, changing field momentum belongs in the momentum budget along with stresses at both ends and forces on conductors. A distributed steady system with nonzero power flow can have stationary material when its supports and internal stresses carry the corresponding mechanical momentum balance. One should not infer a centre-of-mass acceleration of the complete apparatus from the field integrand in an arbitrarily selected subvolume.

Compare a matched travelling TEM wave with an imposed DC line state. In vacuum the characteristic impedance is

$$Z_c = (Z_0/2\pi)\ln(b/a);$$

for a pure forward wave $V = Z_c I$. Then magnetic and electric field energies per length are equal and the energy velocity is c. A DC source and independently specified load may have any real ratio V/I compatible with their circuit, so stored electric and magnetic energies need not be equal even though axial power is again VI. With line coefficients L' and C', energy per length is

$$U' = (L'I^2 + C'V^2)/2.$$

Because $L'C' = 1/c^2$, the elementary inequality $L'I^2 + C'V^2 \geq 2|VI|/c$ gives

$$|P|/U' \leq c.$$

Equality holds for $V/I = \pm Z_c$. This is a check on units and a clean example where energy flow and a single pure propagating wave are not interchangeable descriptions.

For a passive matched line with a finite voltage pulse, the electromagnetic momentum of its vacuum portion is U/c in the propagation direction. Two equal counterpropagating pulses have the same local energy sum as two separate pulses but zero net momentum when their integrals agree. During overlap one must combine the electric fields and magnetic fields first. In one transverse dimension the cross terms cancel in the axial Poynting vector and in the sum of the two field energies, yielding the same directional decomposition found from the characteristic fields. Local electric energy alone can show strong interference, so comparing a local electric-field map to the *total* energy sum would manufacture a contradiction where none exists.

Quantify the low-frequency limit of an open line

A lumped open cable looks capacitive at low frequency even though its fields obey the full wave equations. Solve the lossless one-dimensional voltage equation for a line of length ℓ with zero load current at the far end. With $e^{i\omega t}$, one may take $\widetilde{V}(x) = V_L \cos[k(\ell - x)]$, $\widetilde{I}(x) = i(V_L/Z_c)\sin[k(\ell - x)]$, as follows from the lossless telegrapher equations. Therefore $Y_{\mathrm{in}} = i\tan(k\ell)/Z_c$, where $k = \omega/v$. Since $k/Z_c = \omega C'$, the leading term is $i\omega C'\ell$, just the static

capacitance per unit length times length. The omitted fractional admittance correction begins at $(k\ell)^2/3$. This is an error estimate for a specified terminal measurement, stricter or weaker than a generic 'small in wavelengths' rule depending on the tolerance.

For $Z_c = 50\,\Omega, \ell = 0.50\,\mathrm{m}, v = 2.0 \times 10^8\,\mathrm{m/s}$, $C' = 1/(vZ_c) = 100\,\mathrm{pF/m}$, so the static input capacitance is 50 pF. At 10 MHz, $k\ell = 0.1571$, giving an apparent capacitance multiplier $\tan(0.1571)/0.1571 = 1.0083$. A one-percent measurement can barely tolerate the lumped approximation. At 50 MHz the same length gives $k\ell = \pi/4$ and multiplier $4/\pi = 1.273$, even though the physical cable geometry and static capacitance have not changed. At a quarter wavelength the ideal open line presents a short input; resolving that resonance is inherently a distributed problem.

The calculation also illustrates why the correct comparison depends on the observable. Travelling from end to end incurs a phase $k\ell$, a first-order effect, while the open input admittance differs from its electroquasistatic approximation only at second order. A line can therefore meet a terminal capacitance tolerance yet fail a tight timing requirement for independently instrumented endpoints. A good test records both reflected waveform and terminal admittance, specifies the same termination for the two models, and checks the leading discrepancy against a frequency sweep. A fit that allows a different effective capacitance at every frequency can reproduce almost any smooth input curve over a narrow interval while concealing the first distributed correction.

Track field momentum during a field change

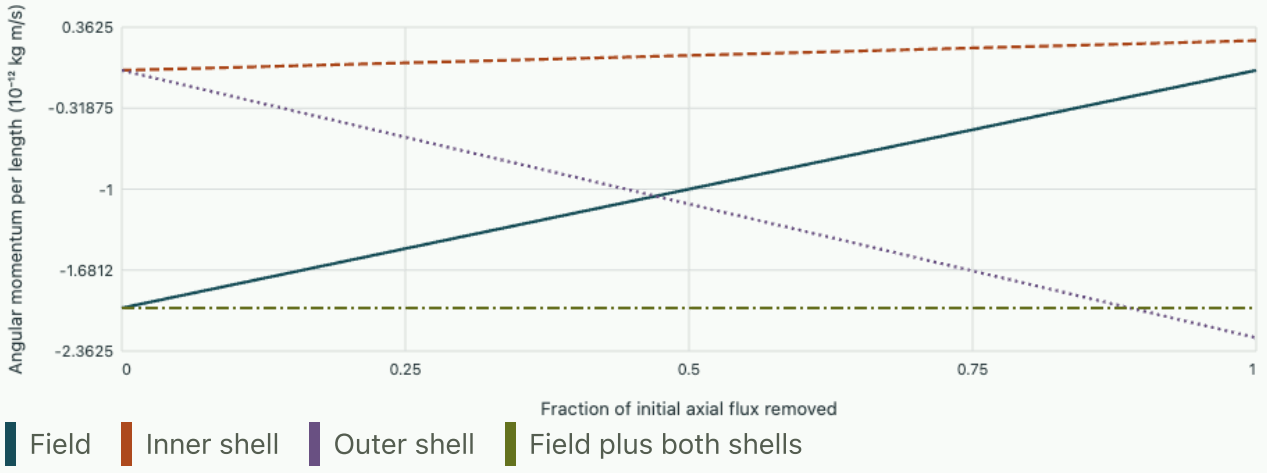
Vary the electric and magnetic fields. Check how a change of field momentum enters a complete mechanical impulse balance.

Open this experiment online

Try the calculation

Outer radius b (mm) **30**

Initial field B (T) **0.25**



After switch-off the inner shell carries 0.25 and the outer shell -2.25 in plotted units. Their sum equals the initial field value -2.

Determine the rest frame of a pair of light pulses

Two short pulses propagate in opposite directions through vacuum and do not overlap at the instant we examine them. A right-moving pulse has energy U_+ and momentum $+U_+/c$. The left-moving pulse has energy U_- and momentum $-U_-/c$. These relations follow by integrating the local densities of a plane travelling wave over its packet. For a broad pulse whose diffraction is negligible one can use them to arbitrary accuracy. Add the energies and the signed momenta before forming the relativistic invariant of the complete two-pulse system. It is

$$U^2 - c^2 P_x^2 = (U_+ + U_-)^2 - (U_+ - U_-)^2 = 4U_+U_-.$$

Each ideal one-direction pulse separately has zero invariant mass. The pair has a nonzero invariant mass whenever both energies are positive, because part of its total energy is associated with motion of the packets in opposite directions. A frame exists where the two momenta cancel. Its speed relative to the measuring laboratory is $v = c^2 P_x / U = c(U_+ - U_-)/(U_+ + U_-)$. The centre of energy moves at this speed even though each individual pulse continues to propagate at c. Making one pulse arbitrarily weak sends the centre-of-energy speed toward the speed of the stronger pulse, leaving no finite rest frame in the ideal single-pulse limit.

For $U_+ = 3.0\,\mathrm{mJ}$ and $U_- = 1.0\,\mathrm{mJ}$ the combined energy is $4.0\,\mathrm{mJ}$ and its momentum is

$$2.0\,\mathrm{mJ}/c = 6.67 \times 10^{-12}\,\mathrm{kg\,m/s}.$$

The centre-of-energy velocity is $0.50c$. The rest energy is $\sqrt{12}\,\mathrm{mJ} = 3.46\,\mathrm{mJ}$, corresponding to a mass of about $3.85 \times 10^{-20}\,\mathrm{kg}$. Transform each pulse into that frame as a check. A boost at half the speed of light multiplies the right-moving energy by $\gamma(1 - 0.5) = 1/\sqrt{3}$ and the left-moving energy by $\gamma(1 + 0.5) = \sqrt{3}$. Both then have $1.73\,\mathrm{mJ}$, and their momenta cancel directly.

The same bound has a local form for arbitrary electromagnetic fields. Start with $u = \epsilon_0(E^2 + c^2B^2)/2$ and $\mathbf{g} = \epsilon_0\mathbf{E} \times \mathbf{B}$. The identity $|\mathbf{E} \times \mathbf{B}|^2 = E^2B^2 - (\mathbf{E} \cdot \mathbf{B})^2$ gives

$$u^2 - c^2|\mathbf{g}|^2 = \frac{\epsilon_0^2}{4}\left[(E^2 - c^2B^2)^2 + 4c^2(\mathbf{E} \cdot \mathbf{B})^2\right] \geq 0.$$

Thus the magnitude of the Poynting flux never exceeds cu in vacuum. Equality for a nonzero local field requires perpendicular electric and magnetic fields with $E = cB$, as in a plane travelling wave. Static crossed fields can have a nonzero Poynting vector, yet generally fail to meet that equality. This distinction matters in quasistatic devices. A computation that finds a circulating flux near charges and currents has not, by that observation alone, demonstrated outgoing radiation. To find emitted power measure the net outward energy flux through a surrounding surface at distances where the radiation field dominates.

Put the two ideal pulses on opposite sides of a stationary perfectly absorbing plate, illuminating it simultaneously. Is the impulse on the plate fixed by the total energy alone?

▼ **Work through the solution**

If the pulses are absorbed and other support impulses are negligible during delivery, the mechanical plus internal momentum of the plate changes by the algebraic sum $P_x = (U_+ - U_-)/c$. Equal incident energies give zero net impulse, even though the plate gains their combined energy. If the two pulse energies differ, reversing the beams reverses the impulse without changing total energy. A subsequent thermal re-emission event introduces additional carried momentum depending on its angular distribution. To compare a complete measured impulse with the initial incoming momentum one must specify the time window and include any significant outgoing flux.

Could a stationary detector sampling both fields at one point determine the invariant mass of an arbitrary radiation packet from the local invariant alone?

▼ **Work through the solution**

The complete energy and momentum are spatial integrals over a common time slice. Squaring their sums produces cross terms between contributions at different locations and directions. In the nonoverlapping example the local invariant expression vanishes wherever a single travelling pulse is present. The invariant of the combined packets is nevertheless positive. A spatially resolving detector or sufficient independent knowledge of the directions and integrated energies is needed to reconstruct the total. The example also explains why one cannot integrate the square root of the local expression and treat that as the invariant mass of an extended electromagnetic system.

David Tong, Electromagnetism and Relativity, on electromagnetic momentum and angular momentum.

Tutorial on choosing what to retain

In the shell calculation, increase the outer shell radius b while keeping $b > R$. Does its final angular impulse change? What happens to the induced tangential impulse per unit charge?

▼ **Work through the solution**

The linked magnetic flux at the outer shell is fixed by R. The time-integrated electric field is proportional to $1/b$, so the tangential impulse per charge decreases. The moment arm b cancels this factor. The outer shell's angular impulse per length is unchanged. The field angular momentum also stays unchanged because its overlap region ends at R.

A measured source has $L = 0.10\,\mathrm{m}$, $\omega = 10^{6}\,\mathrm{s}^{-1}$, and $E_*/(cB_*) = 0.002$. Estimate both induction ratios.

▼ **Work through the solution**

$$\eta = \omega L/c = 3.34 \times 10^{-4}.$$

Hence

$$r_F = \eta/(E_*/cB_*) = 0.167,$$
$$r_D = \eta(E_*/cB_*) = 6.7 \times 10^{-7}.$$

A magnetoquasistatic magnetic-curl approximation is very strong at these scales. A ten-percent electric-circulation accuracy requirement would call for retaining the induction contribution when reconstructing electric fields.

A tiny loop has negligible local retardation, yet its measured signal several kilometres away arrives after an unmistakable propagation delay. Have the near and far measurements contradicted each other?

▼ **Work through the solution**

The small-loop current distribution may be approximated as coherent across the source. This constrains the source size, not the propagation distance to a far detector. A retarded radiation calculation can use a small-source multipole approximation while retaining the full travel delay to the observer.

Magnetism, induction and field dynamics · 12. Posing an electromagnetic evolution problem

A solver cannot evolve 'a pulse in a box' from geometry and a time step alone. It needs material laws, compatible initial fields, a charge-conserving source, and enough boundary data to specify what enters the domain. The same bookkeeping is required when interpreting a real transient measurement.

Separate the evolution equations from the constraints

In vacuum with prescribed sources the evolution system can be written $\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}$ and $\partial_t \boldsymbol{E} = c^2 \nabla \times \boldsymbol{B} - \boldsymbol{J}/\epsilon_0$. The divergence conditions $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$ constrain the initial data. Differentiating the first in time gives zero from the divergence of a curl. Differentiating the second constraint residual gives $-[\nabla \cdot \boldsymbol{J} + \partial_t \rho]/\epsilon_0$. Compatible charge evolution therefore preserves an initially satisfied Gauss constraint. An arbitrarily chosen charge movie and unrelated current movie can violate it before a single boundary reflection occurs.

A numerical solver based only on the two curl equations does not automatically repair bad initial data. In an initial periodic vacuum box a uniform charge density of one sign is inadmissible because the integral of a periodic divergence is zero. One can prescribe a neutralising background as part of a separately justified model, choose net-neutral initial sources, or choose open boundaries that permit net electric flux. Silently discarding the zero Fourier mode of the charge would solve a different problem. Similarly, preserving a constant divergence error exactly during time integration shows consistency of the discrete curls with the chosen derivative, not correctness of the initial physical state.

Construct a right-running pulse from two initial fields

In one-dimensional vacuum choose $\boldsymbol{E} = E_y(x,t)\hat{\boldsymbol{y}},\ \boldsymbol{B} = B_z(x,t)\hat{\boldsymbol{z}}$. Then

$$\partial_t E_y = -c^2 \partial_x B_z,\ \partial_t B_z = -\partial_x E_y.$$

Define $w_+ = E_y + cB_z,\ w_- = E_y - cB_z$. Addition and subtraction give

$$\partial_t w_+ + c\partial_x w_+ = 0,$$
$$\partial_t w_- - c\partial_x w_- = 0.$$

Thus w_+ travels towards $+x$ and w_- towards $-x$. The initial conditions $E_y = E_0 e^{-(x-x_0)^2/(2s^2)}$, $B_z = E_y/c$ specify a right-running packet. Prescribing the same electric shape but $B_z = 0$ launches two oppositely directed electric pulses each of half the original height. Measuring only the initial electric waveform would fail to distinguish the two preparations. At $E_0 = 60\ \mathrm{V/m}$, the peak magnetic field for the first preparation is $0.200\ \mu\mathrm{T}$. An observer 0.90 m farther right sees its peak 3.00 ns later in vacuum.

The component energies provide another independent comparison. Initialising only E_y contains half as much total energy as initialising E_y and its matched B_z. At late times when the two pulses no longer overlap, each half-amplitude pulse has one quarter of the energy of the full matched pulse; the sum is

half, as required. In the overlap region it is better to use the characteristic combinations themselves. In these units $u = \epsilon_0(w_+^2 + w_-^2)/4$ and $S_x = \epsilon_0 c(w_+^2 - w_-^2)/4$ for the transverse fields. Equal oppositely running amplitudes can store energy with zero net flux.

Specify only incoming characteristic information at an open end

In $0 < x < L$, w_+ enters at the left end and w_- at the right. Values travelling out are determined by the interior solution. For a pure outgoing plane wave at the right boundary prescribe $w_- = 0$, equivalently $E_y - cB_z = 0$, while allowing w_+ to exit. At a perfectly conducting fixed wall the tangential electric field vanishes, so there $w_- = -w_+$, reflecting the electric amplitude with a sign reversal. A boundary with a real voltage-like reflection coefficient r gives $w_- = r w_+$ in this one-dimensional equal-medium case. The time-integrated returned energy fraction is r^2. Specifying both E_y and B_z independently at all times generally overdetermines a wave boundary and can amount to fixing both incident and reflected waves inconsistently.

Suppose $L = 1.20\,\mathrm{m},\ x_0 = 0.30\,\mathrm{m}$. The centre of a right-running pulse reaches the conducting end at $3.00\,\mathrm{ns}$. A detector at $x = 0.60\,\mathrm{m}$ first sees the incident centre after $1.00\,\mathrm{ns}$, then the reflected centre after $(0.90 + 0.60)\mathrm{m}/c = 5.00\,\mathrm{ns}$. The two electric traces have opposite signs, while their magnetic contributions at the wall add. Swapping the conducting end for a matched boundary removes the reflected peak within this exact one-dimensional model. A local planar absorbing condition on a curved three-dimensional boundary may still reflect oblique waves; the exact statement depended on one normal mode and its wave impedance.

Derive uniqueness from energy

Take two solutions with the same sources, initial data and perfectly conducting closed boundaries in vacuum. Their difference satisfies homogeneous Maxwell equations with zero initial fields and zero tangential electric field at the wall. Poynting's theorem for the difference says $dU_\Delta/dt = -\oint (\boldsymbol{E}_\Delta \times \boldsymbol{H}_\Delta)\cdot \boldsymbol{n}\, dA = 0$ because the flux normal to a conducting wall vanishes. Since U_Δ is a sum of nonnegative squared fields and initially zero, it stays zero. Thus the interior fields agree. A passive outgoing boundary can only remove this difference energy, so the same conclusion holds given no incoming difference signal. This proof shows why specifying incoming boundary data and initial fields correctly is more fundamental than choosing a particular potential gauge.

The argument also supplies a stability target for computation. With no drive and a perfectly reflecting cavity the discrete total energy should remain bounded, and with an absorbing passive end the loss of field energy should match outgoing flux. An unphysical sign in the wall update can inject energy, producing a growing solution even when the interior scheme is stable. Fixing the numerical grid spacing alone cannot cure an active boundary condition. At a material interface Maxwell jump conditions must be applied to the appropriate tangential or normal fields, since setting all Cartesian components continuous would generally violate flux or polarisation physics.

Discretise space and time without destroying wave kinematics

For the one-dimensional system use E_j^n at integer time and integer position, with $B_{j+1/2}^{n+1/2}$ at half steps in both. Integrating Faraday on a small rectangular loop gives

$$B_{j+1/2}^{n+1/2} = B_{j+1/2}^{n-1/2} - (\Delta t/h)(E_{j+1}^n - E_j^n).$$

Ampère then gives $E_j^{n+1} = E_j^n - c^2(\Delta t/h)(B_{j+1/2}^{n+1/2} - B_{j-1/2}^{n+1/2})$ without current. For material sources include the appropriate time-centred current. The staggered layout places discrete curls where the updated component lives. In several dimensions a compatible divergence of the curl vanishes by cancellation of oriented face-edge contributions.

Substitution of waves $e^{i(kjh-\omega n\Delta t)}$ with the matching half-step phases yields $\sin(\omega\Delta t/2) = (c\Delta t/h)\sin(kh/2)$ on the positive-frequency branch. Stability on a one-dimensional uniform grid requires $c\Delta t/h \le 1$. For $h = 5.0\,\mathrm{mm}$, the upper vacuum time step is $16.7\,\mathrm{ps}$. A cubic three-dimensional mesh imposes the more restrictive $c\Delta t/h \le 1/\sqrt{3}$, about $9.63\,\mathrm{ps}$. In one dimension the equality case transports compatible resolved plane waves at the correct speed on this grid. Numerical dispersion generally returns at smaller Courant number or for diagonal propagation in multiple dimensions. Sample a pulse across enough sites that phase and dispersion errors lie below the intended time-delay uncertainty.

A careful initialisation also respects the half time step. Assign $B^{1/2}$ from the analytic field at that shifted time or use a consistent half-step update. Assigning B^0 directly where $B^{1/2}$ is required can create a spurious left-running component in what should have been a one-way pulse. Examine w_- reconstructed at common space and time before attributing a tiny backward peak to the material. For a real calculation, refine both space and time, compare detector arrival times as well as peak amplitudes, and integrate energy flux over complete transit windows. Solver convergence, charge compatibility, and a pulse moving in the intended direction are independent tests.

Compare a matched return and a reflected pulse on a line

Match a travelling pulse at a material boundary

A single homogeneous characteristic impedance was built into the one-dimensional end conditions. Replace the boundary by an interface between lossless homogeneous nondispersive media with positive real impedances Z_1, Z_2. At normal incidence from the left, write electric amplitudes E_i, E_r, E_t and choose the magnetic sign so a right-running wave has $H = E/Z$ and a left-running one $H = -E/Z$. With no free sheet current and no singular electric field at the interface, tangential continuity gives

$$E_i + E_r = E_t,$$
$$(E_i - E_r)/Z_1 = E_t/Z_2.$$

Solving these two equations yields

$$r = \frac{Z_2 - Z_1}{Z_2 + Z_1},$$

$$t = \frac{2Z_2}{Z_2 + Z_1},$$

$$\mathcal{R} = |r|^2,$$

$$\mathcal{T} = \frac{Z_1}{Z_2}|t|^2.$$

The impedance factor in transmitted power is required even when electric continuity makes $t = 1 + r$. With $Z_1 = 300\,\Omega$, $Z_2 = 100\,\Omega$, the electric reflection is $-1/2$ and transmission $+1/2$. The reflected energy is one quarter, while transmitted energy is three quarters. Mistaking both energy fractions for one quarter merely because the two electric amplitudes have equal magnitudes leaves half the input pulse energy unexplained. At $Z_2 \to 0$ the coefficients approach an electric short: the tangential electric field vanishes, even though the magnetic field doubles in the incident medium. Matching the impedances gives zero reflection even if material wave speeds differ, since those speeds depend on the product $\mu\epsilon$ whereas impedance depends on their ratio.

Inject a short pulse with known area and width a sufficient distance to the left that it initially lies entirely in medium one. If the media are nondispersive and impedance independent of frequency across the pulse band, each electric waveform copies the incident temporal shape multiplied by its amplitude coefficient. A transmitted wave travelling at a different speed has a different spatial width for the same temporal width, so comparing stored energy by squaring peak fields and comparing equal geometrical intervals can easily mislead. A fixed detector measuring $\int (EH)\,dt$ across a complete pulse directly measures transmitted energy per area. In a simulation integrate forward and backward flux at separate planes, include any energy still between the planes, and check that the field boundary gives the same answer when the detector positions are changed.

Manufacture a source that obeys charge continuity exactly

A verification case with prescribed charge and current can test different components from a source-free travelling wave. Consider a one-dimensional periodic interval with period $2\pi/k$. Choose a longitudinal electric field $E_x(x,t) = E_a \sin(kx)\cos(\omega t)$ and zero magnetic field. Gauss's law fixes $\rho = \epsilon_0 k E_a \cos(kx)\cos(\omega t)$. Ampère–Maxwell requires

$$J_x = -\epsilon_0 \partial_t E_x = \epsilon_0 \omega E_a \sin(kx)\sin(\omega t).$$

Faraday gives no magnetic-field change because a purely x-directed function of x has zero curl. Differentiating the two source expressions verifies $\partial_t \rho + \partial_x J_x = 0$, and the periodic spatial mean of the charge vanishes. This is a prescribed-source test of the equations, not a claim that an unspecified material spontaneously supplies those currents.

For period $0.20\,\mathrm{m}$, $E_a = 100\,\mathrm{V/m}$, and $f = 2.0\,\mathrm{MHz}$,

$$k = 31.42\,\mathrm{m}^{-1}.$$

The charge-density amplitude is $27.8\,\mathrm{nC/m^3}$ and the free-current-density amplitude $0.0111\,\mathrm{A/m^2}$. At $t = 0$ the longitudinal electric energy is maximal and current vanishes. One quarter period later the electric field and charge are zero while the source current is maximal at the spatial electric-field antinodes. At every point where E, J are nonzero their product is $-\partial_t(\epsilon_0 E^2/2)$; the prescribed external charges exchange energy with the field without a magnetic Poynting flux. A discrete divergence residual and an independently measured field-energy change can therefore check the signs and amplitudes of the source implementation.

Prescribing those continuous formulas sampled separately on a staggered mesh does not necessarily produce exact *discrete* continuity. If the numerical current is sampled at half time and half space, integrate the cell continuity equation to update ρ from numerical boundary currents. Alternatively use the corresponding discrete sinusoidal differentiation factors $2\sin(kh/2)/h$ and $2\sin(\omega\Delta t/2)/\Delta t$ when manufacturing exact discrete data. Comparing the continuous and discrete constructions separates a model truncation error that converges with mesh refinement from a genuinely nonconserving source injection. A code can look excellent on transverse vacuum pulses and still fail this test because those pulses never challenge its charge handling.

Decide what a late echo means

A narrow pulse incident on a material slab can return signals from both interfaces. Separate the two in time when possible. If the slab thickness is d and its propagation speed v_2, the round-trip delay inside it is $2d/v_2$. Later echoes spaced by that delay suggest repeated internal reflections. Changing the outer computational boundary location while keeping the material and detector fixed provides a different test: an artificial wall echo moves in time by twice the boundary displacement divided by exterior speed. A genuine material echo keeps its material-defined delay. A calibrated single detector trace and these two perturbations can thus identify which part of the model causes a late pulse.

For $d = 0.30\,\mathrm{m}$ and $v_2 = 1.5 \times 10^8\,\mathrm{m/s}$ the material echo spacing is $4.0\,\mathrm{ns}$. Moving an external vacuum boundary outward by $0.45\,\mathrm{m}$ would delay a returning boundary echo by $3.0\,\mathrm{ns}$. If the incident pulse lasts 8 ns the echoes overlap strongly and individual peaks are difficult to assign. Broadening the bandwidth with a shorter source or fitting the full complex frequency response is then more informative than selecting the largest local maximum. Any comparison must include the source bandwidth and detector response because a perfectly converged field solution can still predict a different voltage from a finite-bandwidth probe.

Check a discretised field evolution against its phase

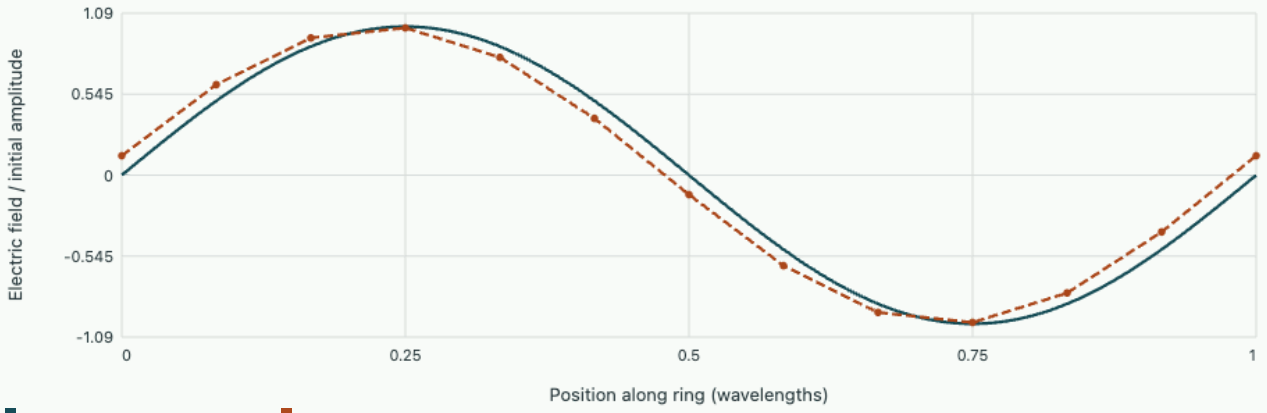
Keep the equations and initial mode fixed while refining the grid. Examine numerical phase delay alongside the physical propagation law.

Open this experiment online

Try the calculation

Points per wavelength 12

Courant number 0.8



75 steps, 5 physical periods. Phase speed ratio 0.99582. Accumulated phase lag 7.5244°.

Set a cavity ringing from specified initial fields

A one-dimensional lossless cavity consists of perfectly conducting parallel planes at $x = 0$ and $x = L$. Use a polarisation E_y, B_z and ignore lateral edges. With no sources inside, the incoming characteristic at each wall is set by the condition $E_y = 0$. Instead of injecting a pulse during the calculation, prescribe an initial field $E_y(x, 0) = E_0 \sin(\pi x / L)$ and $B_z(x, 0) = 0$. These data obey the tangential electric wall condition. They also obey the divergence constraints because the spatial variation is along x while the fields are transverse. Initial data that satisfy a wave equation but violate either of these checks would model a different physical preparation.

Let $k = \pi / L$ and try $E_y(x, t) = E_0 \sin(kx) F(t)$. The wave equation gives

$$F'' + c^2 k^2 F = 0.$$

The initial magnetic field has no spatial gradient and the source-free Maxwell–Ampère law therefore sets $F'(0) = 0$. It follows that $F = \cos \omega t$, with $\omega = ck$. Use Faraday's first-order equation to find the sign and phase of B_z, obtaining

$$E_y = E_0 \sin(kx) \cos \omega t,$$

$$B_z = -\frac{E_0}{c} \cos(kx) \sin \omega t.$$

For a first-order time solver this pair gives a stronger check than the electric standing-wave frequency alone. A code with an incorrect sign in one curl can reproduce an electric snapshot but will transfer energy incorrectly on the next advance. The signs of the magnetic fields at the two ends are opposite.

Their amplitudes peak one quarter period after the initial electric maximum. Perfectly conducting walls carry the necessary surface currents. The standing solution has no Poynting flux through them because the tangential electric field at each wall vanishes.

Per unit transverse area the total initial energy is $\epsilon_0 E_0^2 L/4$, since the integral of $\sin^2(kx)$ over the cavity is $L/2$. At a general time the electric share is that total multiplied by $\cos^2 \omega t$, and the magnetic share is the same total multiplied by $\sin^2 \omega t$. The integral of their sum is constant, although energy continually flows within the cavity. For $L = 0.30\,\mathrm{m}$ the fundamental frequency is $c/(2L) = 499.65\,\mathrm{MHz}$ and its quarter period is about $0.500\,\mathrm{ns}$. If $E_0 = 100\,\mathrm{V/m}$, the stored energy is $6.64\,\mathrm{nJ/m^2}$ and the peak induction is $0.334\,\mu\mathrm{T}$.

The experiment can also reveal the discrete wave speed. A staggered one-dimensional mesh with spacing $h = L/10$ and Courant number $C = c\Delta t/h = 0.5$ obeys $\sin(\omega_{\mathrm{num}}\Delta t/2) = C\sin(kh/2)$. The predicted frequency ratio is $2\arcsin[C\sin(kh/2)]/(Ckh)$, approximately 0.9969 here. Measuring too short a time interval to distinguish this phase lag from sampling uncertainty would give little information about spatial accuracy. Track many periods, use consistent time-centred field estimates, and halve h at fixed C. At these small kh, the leading phase error should decrease by roughly a factor of four.

A student prepares a mixture $E_y(x,0) = E_0[\sin(\pi x/L) + 0.30\sin(3\pi x/L)]$, still with initial $B_z = 0$. What should a midplane electric probe observe, and what fraction of the initial total energy is in the third mode?

▼ **Work through the solution**

At the midplane the fundamental sine equals $+1$, while the third sine equals -1. The probe therefore reads $E_0[\cos(\omega t) - 0.30\cos(3\omega t)]$. Integrated over the full cavity, cross terms vanish because sine modes of different integer index are orthogonal. The third mode's energy fraction is

$$0.30^2/(1 + 0.30^2) = 0.0826.$$

A single spatial probe samples mode amplitudes times their local shape and cannot identify an energy fraction directly. At the midplane all even modes have a node and would escape detection even if they contained substantial energy.

A solver allows an arbitrary spatially uniform constant to be added to B_z while leaving the computed oscillating electric field unchanged. Is this necessarily a bug?

▼ **Work through the solution**

A constant transverse magnetic induction has zero curl and divergence in the ideal interior. It can be added as a static bias if the physical initial magnetic field and the exterior wall-current preparation permit it. In the test posed above the initial magnetic field was explicitly specified as zero, so a later spontaneously generated offset would be a numerical error. Monitoring the spatial mean of B_z provides a direct test: integrating Faraday's law over the interval gives its rate of change proportional to the

difference of the wall electric fields, which is zero. This check tracks an initial-data constraint invisible to a comparison of electric oscillation frequencies alone.

Hermann Haus and James Melcher, Electrodynamic Fields, on wave propagation and standing waves.

Tutorial on initial and boundary compatibility

A vacuum pulse has at its centre $E_y = 100\,\mathrm{V/m}$ and $cB_z = 60\,\mathrm{V/m}$. If both fields have the same spatial shape, infer the two travelling electric amplitudes and the ratio of the energies associated with the two directions.

▼ **Work through the solution**

The characteristics are $w_+ = 160$ and $w_- = 40\,\mathrm{V/m}$. Since $E = (w_+ + w_-)/2$, the component electric amplitudes are 80 and 20 V/m. Their separate directional energy ratio is $160^2/40^2 = 16$, assuming identical envelopes. The total local flux initially points right although neither characteristic is zero.

A numerical field in a periodic box starts with $\nabla \cdot \boldsymbol{B} = 10^{-4}$ in some cells. The code uses a discrete divergence of its discrete curl that is identically zero and carries that pattern unchanged. Does reaching a small time-step tolerance guarantee physically acceptable initial conditions?

▼ **Work through the solution**

No. The curl evolution preserves its initially supplied divergence. A compatible projection or constrained initialisation must remove the error in the correct discrete divergence norm, subject to boundary conditions and any physical mean field. Refining the time integration of inconsistent initial data leaves inconsistency.

A detector sees an amplitude echo 0.15 of an incident Gaussian pulse after reflection at a passive boundary. What is the reflected energy fraction and how much of the input energy remains unaccounted for if measured transmitted energy is 92 percent?

▼ **Work through the solution**

Same-medium, same-shape electric amplitudes give $0.15^2 = 0.0225$ reflected energy. With 0.92 transmitted, the remainder is $1 - 0.0225 - 0.92 = 0.0575$ or 5.75 percent. It may include absorption, energy still in the structure, lateral flux or measurement error. A completed energy audit distinguishes these possibilities.

Magnetism, induction and field dynamics · 13. Why does a Helmholtz pair have a useful uniform region?

A magnetometer averages the field over a finite sample. Derive the coil spacing that suppresses curvature and predict how rapidly the remaining calibration error grows away from the midpoint.

Cancel a spatial derivative with geometry

For an ideal circular loop, Biot–Savart gives the axial field $B(z) = \mu_0 N I R^2/[2(R^2+z^2)^{3/2}]$. It follows by adding equal axial components of the current elements around the circumference. Place two identical loops at $\pm s/2$, carrying currents in the same sense. Superposition and mirror symmetry remove every odd derivative at the midpoint.

Differentiate the single-loop shape $f(u) = (R^2+u^2)^{-3/2}$ twice. The result is $3(4u^2-R^2)(R^2+u^2)^{-7/2}$. The second derivative of the pair changes sign at $s = R$. At this spacing both the linear and quadratic spatial errors vanish. Expanding the even sum one order farther gives

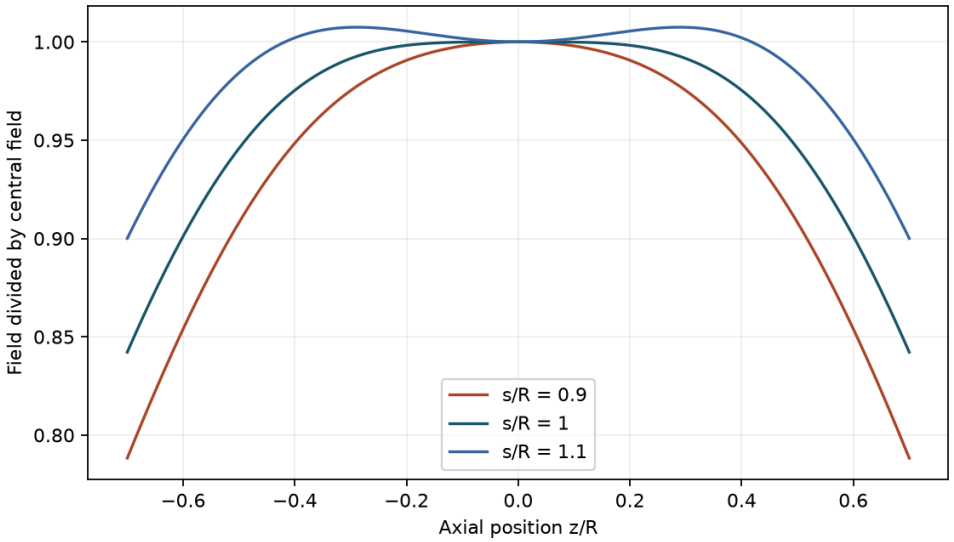
$$\frac{B(z)}{B(0)} = 1 - \frac{144}{125}\left(\frac{z}{R}\right)^4 + O((z/R)^6).$$

Flatness concerns a region of space. A high field value at the centre alone cannot establish that a sample extending on both sides was excited uniformly.

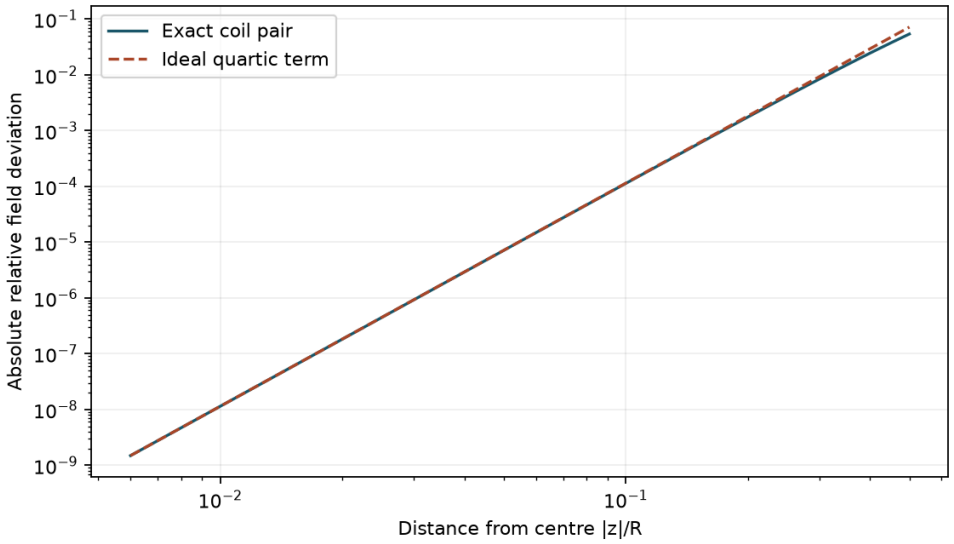
Set the acceptance criterion before choosing a coil

If the allowed pointwise fractional variation is 10^{-3}, the asymptotic half-length of the uniform region is $R(125\times10^{-3}/144)^{1/4} \simeq 0.172R$. With a 100 mm radius this suggests 17 mm on either side. Compare that estimate with the exact field in the notebook. Small spacing errors restore the quadratic term. The accuracy of the coil fixture can therefore matter more than the current-meter resolution when calibrating an extended sample. At 100 ampere-turns per coil and $R = 0.100$ m, the central field is $(4/5)^{3/2}\mu_0 NI/R = 0.899$ mT. A point 10 mm off centre experiences a leading relative deviation of 1.152×10^{-4}, about 0.104 µT in absolute field. A Hall probe with resolution 1 µT would struggle to distinguish a single such deviation. The pointwise specification and the spatial resolution of a verification scan therefore belong in the same calculation.

For a uniformly sensitive sample extending from $-L$ to L, average the quartic term instead of substituting the outermost coordinate. Since the average of z^4 is $L^4/5$, the leading bias of its mean response is a fifth of the edge deviation. A sample that fills the nominal tolerance region can report an apparently excellent mean field while a process sensitive to the edge already operates at the specified limit.



Offset the separation by a tenth of the radius in each direction. The signs of the resulting calibration errors reverse close to the centre.



At ideal spacing the error falls sixteenfold when the sample half-length halves. Move the spacing off its design value to recover quadratic scaling.

Exercise · compare two sample lengths

For an ideal pair, predict the leading error at $z/R = 0.1$ and 0.2.

▼ **Work through the solution**

The quartic terms give 1.152×10^{-4} and 1.8432×10^{-3}. Higher powers change the latter more strongly. Comparing the values on a linear vertical axis can conceal the rapid improvement near the centre.

Exercise · vary the winding current

Will doubling both currents double the length of the region satisfying a relative uniformity tolerance?

▼ **Work through the solution**

Both $B(z)$ and $B(0)$ scale with current, so their ratio and the accepted length stay fixed. An absolute field-error criterion would give a different answer.

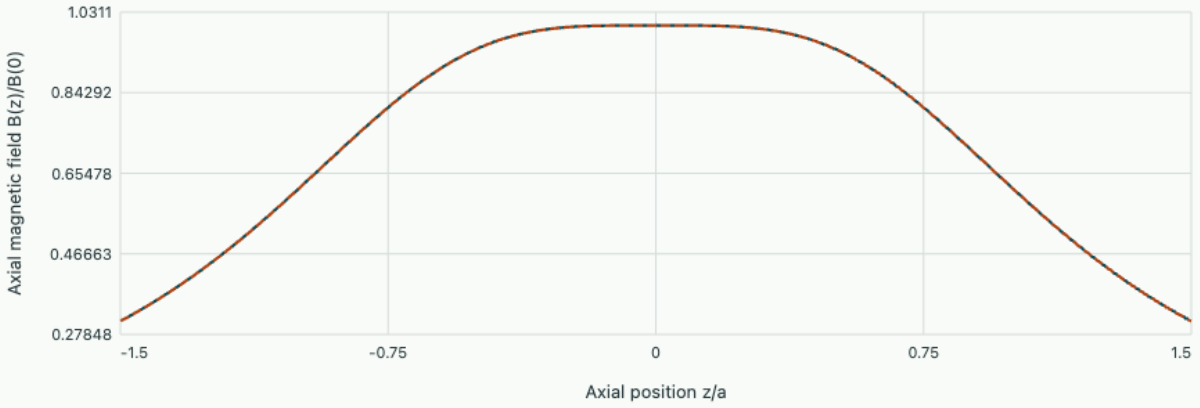
Try it · Em coils

Explore the same physical relation with adjustable inputs. This independent parameter study has its own displayed reference settings; compare its assumptions and units before matching the static example.

Open this live calculation

Try the calculation

Coil separation / radius	**1**
Axial test position / radius	**0.25**



At z/a = 0.25 the selected field differs from the central field by -0.42%. The sign distinguishes a central peak from a central dip close to the origin.

Download the corresponding editable notebook. The browser recalculates the reduced equations directly; the notebook exposes the full calculation and its checks.

Open editable notebook Download .ipynb

Magnetism, induction and field dynamics · 14. From atomic motion to magnetic susceptibility

Calculate the response of a bound orbit and of an ensemble of existing moments, then follow how interactions permit spontaneous magnetisation.

Perturb an orbit at fixed restoring potential

Use a classical bound electron as a model for the induced orbital response. The field is weak enough that the binding potential remains the dominant influence. A circular orbit of radius r, angular speed ω_0 and electron mass m has an inward restoring force $mr\omega_0^2$. Switch on a uniform axial field B slowly. The magnetic force changes the radial balance, and the induced electric field during switching changes the angular momentum. At first order, with the unperturbed radius used in the magnetic correction, the frequency shift has magnitude eB/(2m). The sign of the induced magnetic moment opposes the applied field for either sense of the original orbit.

$$\Delta\omega = \frac{eB}{2m},$$

$$\Delta m_z = -\frac{e^2 r^2 B}{4m}.$$

The factor one half in the frequency can be recovered by inserting $\omega=\omega_0+\delta\omega$ in the radial balance and keeping $2\omega_0\delta\omega$. The magnetic term is proportional to $\omega_0 B$. The orbit's magnetic moment follows from current times area, with the negative charge determining its direction. For an isotropic shell, the squared radius transverse to the field averages to two thirds of the full squared radius.

$$\chi_{\mathrm{dia}} = -\frac{\mu_0 N e^2}{6m} \sum_j \langle r_j^2 \rangle.$$

Here N is the atom number density and the sum runs over the bound electrons in an atom. The susceptibility is negative. This elementary closed-shell result is largely independent of temperature if the atomic size stays fixed. It describes bound orbital response, and its microscopic model should be specified when comparing with conduction-electron magnetism or superconducting screening.

> **Recall · Current direction and the magnetic moment of an orbit**
>
> A charge q circulating with period $2\pi/\omega$ produces current $q\omega/(2\pi)$. Multiplication by the oriented orbit area yields $m=q\omega r^2/2$. Its mechanical angular momentum is $L=mr^2\omega$. Therefore the orbital ratio is q/(2m). Intrinsic spin needs its own gyromagnetic factor and a quantum state description.

Align pre-existing moments statistically

A permanent magnetic moment μ has energy $-\mu\cdot B$. For freely rotating classical moments the orientation integral is identical to the permanent electric dipole calculation, with $x=\mu B/(kBT)$. The

density of moments N gives M=NμL(x). In weak fields B≈μ₀H for a dilute sample, and expansion yields Curie's law.

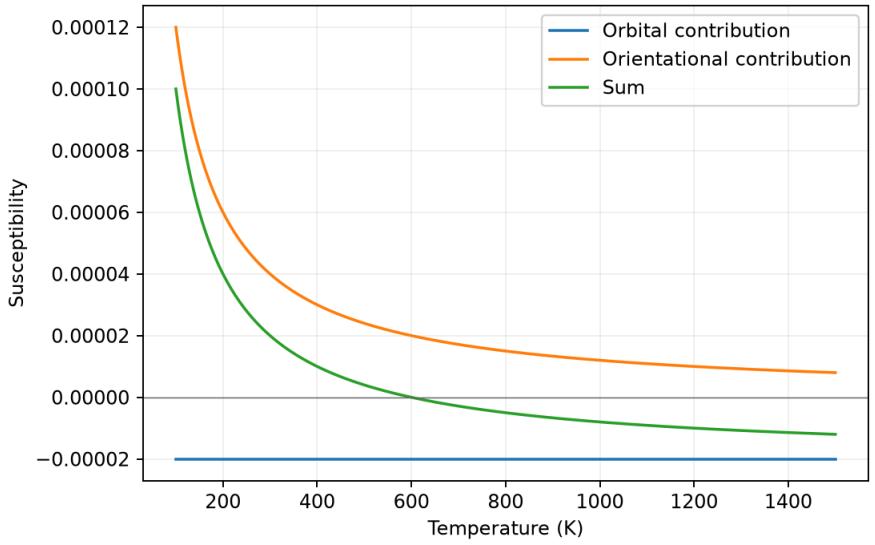
$$\chi_{\mathrm{para}} = \frac{\mu_0 N \mu^2}{3 k_B T}.$$

For quantum angular momentum magnitude J, label the magnetic-moment projections by m=−J,…,J. For an electron with positive g this m is the negative of its angular-momentum projection. The magnetic energy is −gμB mB and the partition function is a finite sum. Expanding the exponential at weak field uses Σm=0 and Σm²=(2J+1)J(J+1)/3.

$$Z = \sum_{m=-J}^{J} e^{g\mu_B m B/(k_B T)},$$

$$\chi = \frac{\mu_0 N (g\mu_B)^2 J(J+1)}{3 k_B T}.$$

For J=1/2 the exact moment is M=N(gμB/2)tanh[gμBB/(2kBT)]. It approaches saturation as temperature falls at fixed field. With μ=μB, T=10 K and B=1 T the argument is 0.06717. A spin-half ensemble is only about 6.71 percent aligned. The classical freely rotating model gives a different weak-field coefficient because its accessible orientations differ.



The two weak-field contributions in exercise 1 have different temperature dependence. Their sum changes sign at 600 K.

Try it · Curie dia

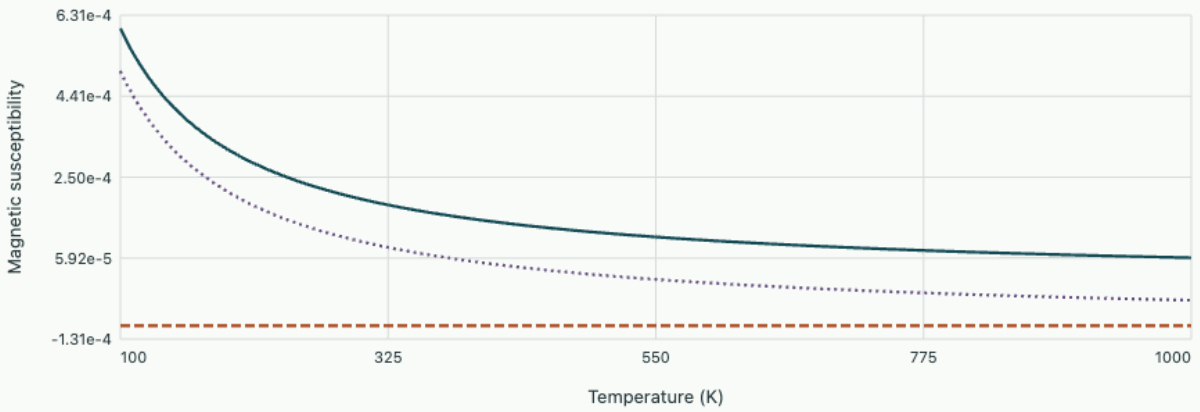
Change the inputs and compare the recalculated curves with the preceding derivation and baseline figure. The axes give the units; the numerical readout states the model assumptions and any limiting cases. Reset the inputs to return to the reference setting.

Open this live calculation

Try the calculation

Curie coefficient C (K) **0.06**

Diamagnetic susceptibility magnitude **1.00e-4**



Curie contribution · Diamagnetic contribution · Sum

Sign change at T = 600 K within the independent-dipole, weak-field model. An ordering transition would invalidate extrapolation of this law.

Allow moments to change each other's effective field

Represent interactions at mean-field level by H_eff=H+λM. Above the transition, insert this in M=(C/T)H_eff and solve the linear equation.

$$\chi = \frac{C}{T - C\lambda} = \frac{C}{T - T_c}.$$

The denominator identifies where the zero-magnetisation solution ceases to be stable. A spin-half model at zero external field gives m=tanh[(Tc/T)m] for reduced magnetisation m=M/Ms. Expand tanh to cubic order near Tc. Besides m=0, the nonzero branch satisfies m²≈3(Tc/T−1)/(Tc/T)³. This predicts a square-root onset within this approximation.

Remember · A mean-field transition and a measured hysteresis loop answer different questions

The equilibrium equation predicts possible uniform order. Hysteresis requires a history-dependent route through metastable configurations, often involving domain nucleation and motion. A multi-domain specimen can have large local magnetisation and small net moment. A fitted Curie–Weiss temperature alone does not determine coercivity.

A continuum model makes the domain scale explicit. For an easy-axis magnet use energy density $A(\theta')^2+K\sin^2\theta$ in a one-dimensional wall. Translation invariance gives the first integral $A(\theta')^2=K\sin^2\theta$ for a wall joining $\theta=0$ and π. Integration yields $\tan(\theta/2)=\exp[(x-x_0)/\delta]$, with $\delta=\sqrt{(A/K)}$. Substitution in the energy integral gives wall energy per area $4\sqrt{(AK)}$. Exchange penalises rapid variation, whereas anisotropy penalises spending distance away from the easy directions. Magnetostatic energy and sample shape determine whether forming walls lowers the total energy.

Use temperature and shape to discriminate models

1. A sample has $\chi=-2\times10^{-5}+0.012\ \mathrm{K/T}$. Determine the temperature where the net weak-field response changes sign.

▼ **Work through the solution**

Set both terms equal in magnitude. $T=600$ K. At 300 K $\chi=+2\times10^{-5}$, while at 1200 K $\chi=-1\times10^{-5}$. This inference assumes a fixed density and no transition over the interval.

---

2. For $A=10^{-11}$ J/m and $K=10^{4}$ J/m³ find the wall length scale and energy. Predict the effect of quadrupling K.

▼ **Work through the solution**

$\delta=31.6$ nm and wall energy 1.265 mJ/m². Quadrupling K halves δ and doubles the wall energy. The full conventional wall width may include a factor π, so specify the definition when comparing measurements.

---

3. Derive the measured susceptibility of a uniformly magnetised ellipsoid with demagnetising factor N_d and intrinsic χ.

▼ **Work through the solution**

Use $H_{int}=H_{applied}-NdM$ and $M=\chi H_{int}$. Thus $M/H_{applied}=\chi/(1+Nd\chi)$. A strong measured temperature dependence may be suppressed by specimen geometry. Inverting the expression requires the same axis and shape convention used to determine N_d.

Magnetism, induction and field dynamics · 15. The electric field of a rotating magnet

Use charge balance and a surface matching calculation to predict a voltage around a rotating magnetic body, then close the mechanical and electrical power accounts.

Specify the moving material and the electrical constraint

Consider an electrically isolated conducting sphere of radius a, with prescribed uniform axial magnetisation M, rotating slowly at angular speed Ω about that axis. Work to first order in Ωa/c. Neglect deformation and the field produced by the small convective electric current. The magnetostatic interior field is uniform, $B_0=2\mu_0 M/3$. Uniform rotation about the magnetisation axis leaves that magnetic field stationary in the laboratory.

At a material point with cylindrical radius s the velocity is Ωs eφ. For a highly conducting body in stationary open circuit, mobile charges rearrange until the local motional force vanishes. This gives a laboratory electric field even though the magnetic field is steady.

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}_0=0,$$

$$E_s=-\Omega B_0 s,$$

$$V_{\mathrm{in}}=\frac{\Omega B_0}{2}s^2+C.$$

Gauss's law then gives the total volume charge density $-2\varepsilon_0\Omega B_0$. Its volume integral is $-8\pi\varepsilon_0\Omega B_0 a^3/3$. Since the sphere was neutral, the surface must carry the opposite total charge. In a magnetic material the assignment between free and bound charge needs a constitutive convention. The equations here determine the total charge that produces E.

> **Recall · A moving conductor need not be equipotential in the laboratory**
>
> Static equipotential reasoning assumes charges are at rest in the chosen frame. A charge carried with a rotating conductor feels q(E+v×B). In open circuit it is this total force that vanishes. Stationary voltmeter leads and brushes therefore matter when defining the measured voltage.

Solve the exterior problem using neutrality

At r=a, $\sin^2\theta=2[1-P_2(\cos\theta)]/3$. The surface potential is therefore a constant plus a quadrupole. A neutral isolated sphere has no exterior monopole, so the constant spherical harmonic must vanish when V tends to zero at infinity. This fixes $C=-\Omega B_0 a^2/3$ and determines the exterior field uniquely.

$$V_{\mathrm{out}}=-\frac{\Omega B_0 a^5}{3r^3}P_2(\cos\theta).$$

Differentiate radially on each side. The interior radial field at the surface is $-\Omega B_0 a \sin^2\theta$, while the exterior radial field is $-\Omega B_0 a P_2$. The normal jump gives the total surface charge.

$$\sigma(\theta) = \epsilon_0 \Omega B_0 a \left[\frac{2}{3} - \frac{5}{3} P_2(\cos\theta)\right].$$

Integrating the constant term over the sphere cancels the volume charge exactly, since the angular integral of P_2 vanishes. The tangential derivative of the continuous potential also matches. These are independent checks on the sign and coefficient.

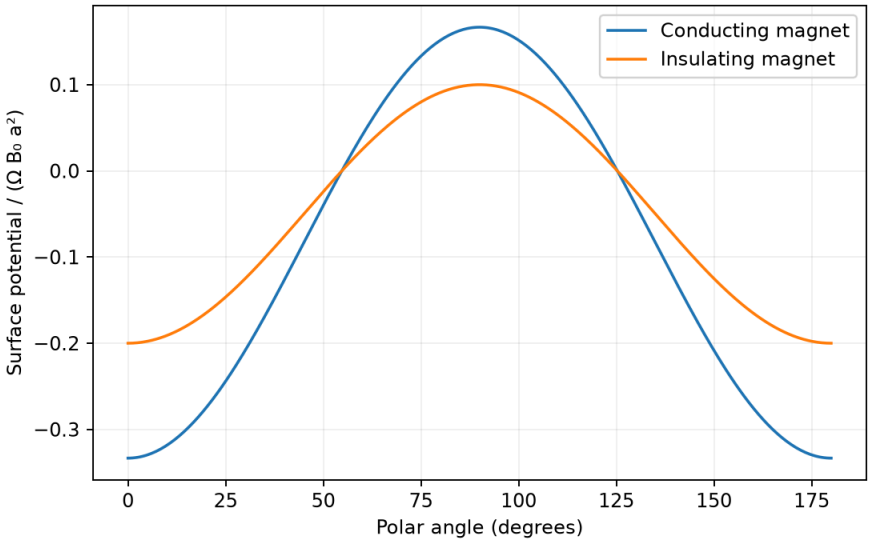
For $a=0.02$ m, $B_0=0.30$ T and $\Omega=100$ s^{-1}, the potential difference between equator and pole is $\Omega B_0 a^2/2=6.0$ mV. A laboratory voltmeter with stationary leads and sliding contacts measures that difference in the ideal open-circuit model. Grounding the sphere through a stationary brush at a specified latitude fixes the potential at that contact. This changes the constant harmonic and can supply a net charge.

Track the different constraint in an insulator

An insulating magnet cannot freely redistribute charge until $\mathbf{E}+\mathbf{v}\times\mathbf{B}$ vanishes. To leading order, a moving magnetisation contributes electric polarisation $\mathbf{P}=\mathbf{v}\times\mathbf{M}/c^2$. For axial rotation this points radially outward with magnitude $\Omega M s/c^2$. Its bound volume charge is $-2\Omega M/c^2$, and its bound surface charge is $\Omega M a \sin^2\theta/c^2$. Integrating them cancels the net charge. Solve Poisson's equation with these specified sources to determine the field. A particular interior potential is $\Omega M r^2/(3\varepsilon_0 c^2)$. Write the remaining interior quadrupole as $D\, r^2 P_2$ and the exterior one as $B_2 P_2/r^3$. Continuity of potential gives $B_2 = D\, a^5$. The quadrupole part of the radial field jump is $5 D a P_2$. The prescribed surface charge has quadrupole part $-2\Omega M a P_2/(3c^2)$, so $5Da = -2\Omega M a/(3\varepsilon_0 c^2)$. Solving for D and substituting B_2 gives the exterior potential below.

$$V_{\text{out,ins}} = -\frac{2\Omega M a^5}{15\epsilon_0 c^2 r^3} P_2(\cos\theta).$$

For the same M and Ω, the exterior quadrupole is three fifths of the conducting result. The distinction is a physical charge-mobility constraint. Rotation of the magnetic pattern alone cannot decide the voltage. The derivation assumes the prescribed magnetisation and ignores additional electric susceptibility of the insulating material.



The exterior quadrupole evaluated at the surface for identical magnetisation and rotation. Electrical mobility changes the amplitude.

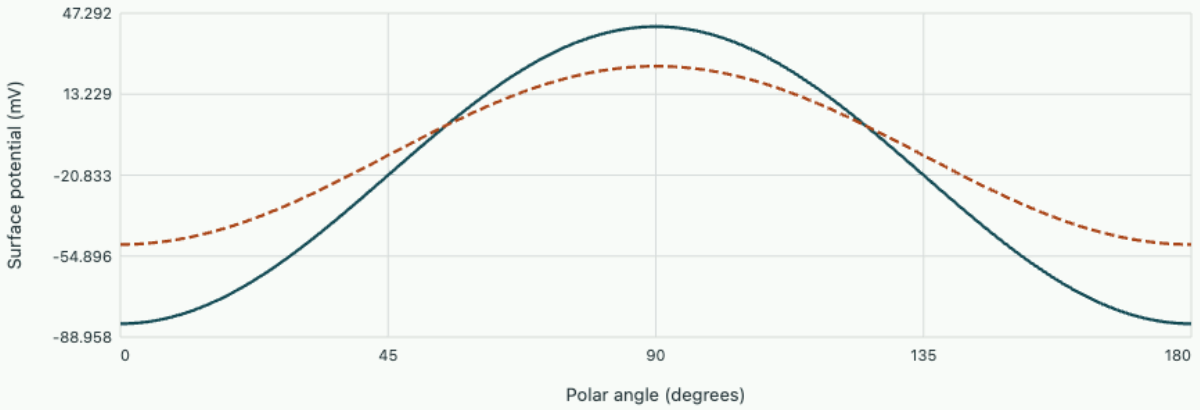
Try it · Rotating magnet

Change the inputs and compare the recalculated curves with the preceding derivation and baseline figure. The axes give the units; the numerical readout states the model assumptions and any limiting cases. Reset the inputs to return to the reference setting.

Open this live calculation

Try the calculation

Rotation speed (rad/s)	100
Radius (cm)	5
Internal induction (T)	1



Conducting sphere Insulating sphere, same magnetisation

Exterior quadrupole evaluated at r = a. Potentials are measured relative to infinity; B0 = 2μ0M/3. Insulating/conducting quadrupole ratio is 3/5.

Account for a working generator

Connecting a load permits a current and changes the open-circuit field. In the related conducting-disc model with imposed uniform B, an outward radial current I experiences a braking torque $d\tau_z = -IBs\,ds$ for positive axial B. Integration from axis to rim gives the braking torque $-IBa^2/2$. The driver supplies the opposite torque. Multiplying that driving torque by Ω yields mechanical input power $I\Omega Ba^2/2 = I\mathcal{E}$. The driver supplies the electrical power.

Remember · The magnetic force on each charge is perpendicular to its full velocity

Write a carrier velocity as the material velocity plus drift. The magnetic force transfers momentum between moving lattice and drifting carriers. Its dot product with the full carrier velocity is zero. Mechanical constraints, electric fields and collisions perform the work appearing in the machine-level energy balance. Follow both carrier and body when resolving that balance.

Test a hypothetical magnetic source consistently

In source-free vacuum define F=E+icB. The curl equations combine into i∂tF=c curl F and div F=0. Multiplying F by a constant phase rotates the electric and magnetic fields into one another while preserving the source-free equations. Ordinary electric charges select which combination couples to matter.

A symmetric hypothetical source theory can be written by assigning a magnetic charge the same units as electric charge. Then div(cB)=ρm/ε₀ and curl E=−∂tB−Jm/(ε₀c), while the electric equations retain their usual SI form. Taking a divergence of the modified Faraday equation gives ∂tρm+div Jm=0. The magnetic source convention has to be declared before using a force or energy formula. No magnetic monopole is assumed in the preceding sphere calculation.

Exercises on the observed voltage

1. Reverse Ω while leaving the magnetisation fixed. Which measured quantities change sign?

▼ **Work through the solution**

V, E and the leading induced electric charges change sign. The leading magnetostatic B stays fixed. Reversing both Ω and M leaves the electric signal unchanged.

---

2. A sensor observes the quadrupole at r=2a and at r=4a along the axis. Find the potential and field ratios.

▼ **Work through the solution**

The potential varies as r^{-3}, so the farther reading is one eighth. The radial field varies as r^{-4}, so its magnitude is one sixteenth. A monopole contamination would decay more slowly and spoil these ratios.

---

3. In the disc model let I=0.20 A and use the numerical values above. Calculate torque and power.

▼ **Work through the solution**

τ=12 µN m. Mechanical power is 1.2 mW, equal to I times 6.0 mV. Contact and resistive losses must be included before comparing with a real load.