Mechanics in the laboratory

Turn a measured trajectory or force into a test of a mechanical law. Derive the observable, propagate it numerically and determine which parameters the experiment can resolve.

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A bead moves inside a vertical circular guide after receiving a known speed at the bottom. Predict whether it turns around, separates from the guide or completes the circle. A normal-force measurement can distinguish these outcomes before a simple top-height energy test can.

Mechanics in the laboratory · 1. Measure gravity with a pendulum of finite amplitude

A metre-long pendulum is an appealing clock because its small-angle period depends only on length and gravity. A precise clock can nevertheless give a precisely wrong value of gravity. Derive the full period, trace the systematic error to an identifiable approximation and design a measurement that distinguishes a length error from a release-angle error.

Define the measured length and time

Use a point bob on a massless, inextensible string attached to a fixed pivot. Let L be the distance from pivot to the bob's centre of mass, and θ the signed departure from the downward vertical. For a rigid extended bob, rotational inertia about its own centre adds to mL^2; a length measured to the bottom of the bob is neither of these dynamical lengths. Measure the elapsed time between successive crossings of a photogate in the same direction. Crossings in opposite directions give half of the period in a symmetric oscillation. A stationary turning point is a poor timing marker because the position error there produces a large time error.

At rest at the release angle θ_0, the energy is $mgL(1-\cos\theta_0)$. At a later angle the speed is $L|\dot\theta|$. Equating kinetic plus potential energy to the release value gives

$$\dot\theta^2 = (2g/L)(\cos\theta - \cos\theta_0).$$

Only a quarter cycle is needed. Inverting the angular speed and integrating from zero to θ_0 produces a square-root singularity at release. The period is finite since integrating the inverse square root over a small interval gives a square root.

Resolve the turning point before approximating

Write $k = \sin(\theta_0/2)$ and substitute $\sin(\theta/2) = k\sin u$. The vanishing square root cancels a factor $\cos u$ from the differential. One obtains the complete elliptic integral of the first kind with modulus k.

$$T = 4\sqrt{\frac{L}{g}}\,K(k),$$

$$K(k) = \int_0^{\pi/2} \frac{du}{\sqrt{1-k^2\sin^2 u}}.$$

Expanding its integrand yields

$$K(k) = (\pi/2)(1 + k^2/4 + 9k^4/64 + \cdots).$$

Since $k \simeq \theta_0/2$ at small amplitude, the first fractional period correction is $\theta_0^2/16$ with angle in radians. It is positive. At larger amplitude the tangential component of gravity rises more slowly with angle than

the linear restoring law predicts. The pendulum spends longer near the turns. Passing degrees directly into the quadratic correction would give nonsense even if the timer and recorded angle were perfect.

Infer the bias in gravity

Suppose the true acceleration is $9.81\,\mathrm{m\,s^{-2}}$, the length is 1.000 m and the release angle is 30 degrees. The small-angle period is about 2.0061 s and the finite-amplitude period about 2.0410 s. Interpreting the longer measured period by the linear formula would give about $9.48\,\mathrm{m\,s^{-2}}$. The fractional gravity error is about twice the fractional period error with opposite sign because gravity is inferred from T^{-2}. Timing a hundred periods reduces random timing error but leaves this systematic shift.

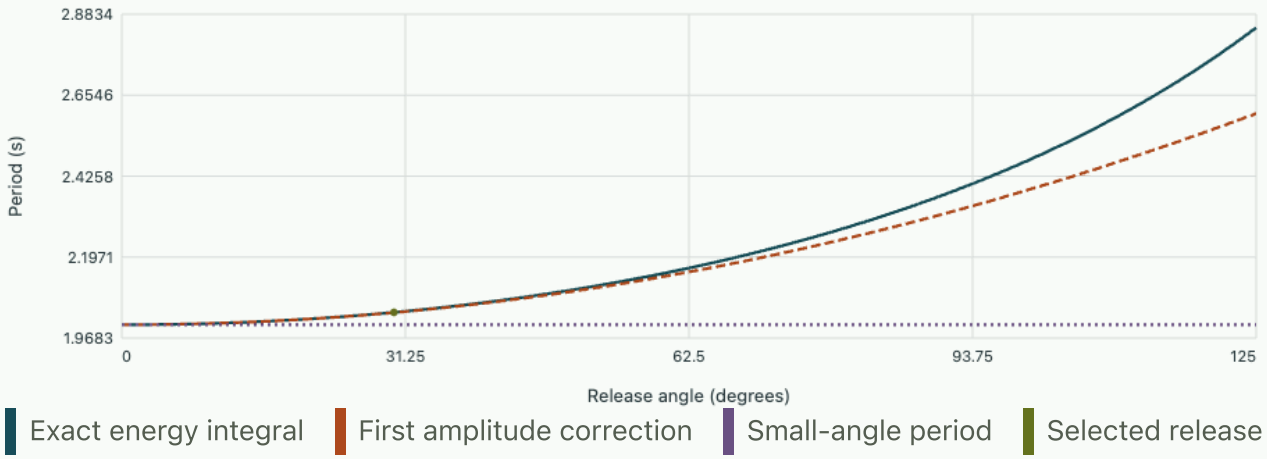
The numerical curve evaluates the complete elliptic integral. It uses the arithmetic–geometric mean identity $K(k) = \pi/[2\,\mathrm{AGM}(1, \sqrt{1-k^2})]$, so no sampled point sits on the turning point. The single-correction curve truncates the amplitude series after its first nonconstant term. Vary angle and length, comparing both the predicted period and the result a linear analysis would infer. Which change rescales both period curves together? Which changes their relative discrepancy?

Open this experiment online

Try the calculation

Release angle (degrees) 30

Pendulum length (m) 1



Period 2.041 s. Small-angle analysis would infer g = 9.4772 m/s².

Design a two-release check

For the same pendulum the ratio $T(\theta_a)/T(\theta_b) = K(k_a)/K(k_b)$ cancels the unknown length and local gravity. Measure at two amplitudes before deciding whether to recalibrate the length. If the ratio agrees but both absolute times differ from prediction, check the effective length, clock rate and model of the

bob. If the ratio changes during a decaying swing, record initial and final amplitudes for the timed sequence. The period associated with a long sequence is an average over its slowly changing amplitude. Substituting the initial angle throughout would overcorrect the measurement.

Tutorial

At small amplitude, what release angle gives a fractional period bias of approximately one part in a thousand?

▼ **Work through the solution**

Set $\theta_0^2/16 = 0.001$. This gives 0.1265 rad, about 7.25 degrees. The gravity bias from an uncorrected measurement would be approximately two parts in a thousand. A tolerance on one quantity is not automatically the same tolerance on the other.

A photogate records successive interruptions every 1.02 seconds for the metre pendulum. A calculation gives nearly four times local gravity. Identify one possible error in the observable.

▼ **Work through the solution**

The gate may have counted both directions and returned a half period. Doubling that interval gives about 2.04 s. Because the inferred acceleration is inversely proportional to the square of the period, the missing factor of two gives a factor of four.

Why does reducing the grid spacing in an integration of the linear equation fail to recover the finite-amplitude curve?

▼ **Work through the solution**

Integration error vanishes as the step is refined, but the restoring torque in that equation is $mgL\theta$. The true torque is $mgL\sin\theta$. A converged numerical solution can therefore provide independent evidence of an inadequate physical approximation.

For the equation of motion, phase portrait and dissipation, see the MIT treatment of the simple pendulum.

Mechanics in the laboratory · 2. Measure inertia and gravity with two suspension axes

A rigid pendulum can oscillate from either of two knife edges. If the periods agree at distinct distances from the centre of mass, the spacing of the edges determines gravity. Derive the cancellation, identify a second way equal periods can arise and find why a poorly chosen geometry magnifies timing errors.

Torque about an offset pivot

Take a rigid body of mass m with parallel suspension axes perpendicular to its plane of oscillation. Its moment of inertia about the centre of mass is $I_C = m\rho^2$; ρ is the radius of gyration. For a pivot a distance $a > 0$ from the centre, the parallel-axis theorem gives

$$I_a = m(\rho^2 + a^2).$$

Gravity exerts torque $-mga\sin\theta$. In the small-angle limit,

$$T_a^2 = \frac{4\pi^2}{g}\,\frac{\rho^2 + a^2}{a},$$

$$L_a = a + \frac{\rho^2}{a}.$$

Here L_a is the length of a point pendulum with the same period. It can lie beyond the physical centre of mass. A tap applied at this distance from a pivot gives the centre of percussion when the ideal instantaneous pivot reaction is to vanish. The same combination of mass distribution and geometry appears in an impulsive and in an oscillatory measurement because both depend on angular momentum about that pivot.

Match the periods and keep both algebraic branches

Put the second knife edge on the opposite side of the centre at distance b. Equal periods require

$$a + \rho^2/a = b + \rho^2/b.$$

Rearrange before dividing by $a - b$:

$$(a - b)\left(1 - \frac{\rho^2}{ab}\right) = 0.$$

Two equal distances always give equal periods. That branch provides no information about ρ. If the distances are distinct, equality implies

$$\rho^2 = ab.$$

Both equivalent lengths then equal $a + b$, the directly measurable edge separation. For $a = 0.200\,\mathrm{m},\ b = 0.450\,\mathrm{m},\ \rho = 0.300\,\mathrm{m}$, the equivalent length is 0.650 m and both periods at standard laboratory gravity are about 1.62 s. A small value of $a - b$ is an experimental warning sign, even if the observed equality is exceptionally clean.

Background · Can unequal release amplitudes imitate a mismatch between pivots?

A rigid pendulum about either knife edge obeys the same sine nonlinearity as a simple pendulum. To leading order in its release angle θ in radians, the measured period exceeds the small-amplitude period by the fraction $\theta^2/16$. At matched small-amplitude periods, measurements with amplitudes θ_a, θ_b therefore give

$$\frac{T_a - T_b}{T_0} \simeq \frac{\theta_a^2 - \theta_b^2}{16}.$$

Releasing at 6° about the first edge and 2° about the other creates a fractional mismatch of about 6.1×10^{-4}. For a two-second period this is 1.2 ms. Record the two amplitudes along with the timing data and reduce both periods to the same limit before moving the trim mass. Otherwise the trim can compensate a preparation error.

Extract gravity when the two measured periods differ

Multiply the first period equation by a and the second by b. Subtract to eliminate ρ^2. The exact small-angle result for unequal distances is

$$g = 4\pi^2 \frac{a^2 - b^2}{aT_a^2 - bT_b^2}.$$

At matched periods this reduces to the simple expression in terms of the edge spacing. The general formula involves a difference in its denominator. If a and b nearly coincide, the denominator is small and two independent timing errors subtract with poor relative precision. This is a property of the inverse measurement, even if computing each individual period is perfectly well conditioned.

Once g is known, use $\rho^2 = gaT_a^2/(4\pi^2) - a^2$ to recover the body inertia per unit mass. A negative value may indicate a wrong distance measured to the centre or a timing observable that is actually a half period. Verify the result using the second pivot and recover a separately predicted period after shifting a suspension point. A single successful cancellation is weaker than reproducing an additional measurement.

Move one knife edge in the experiment

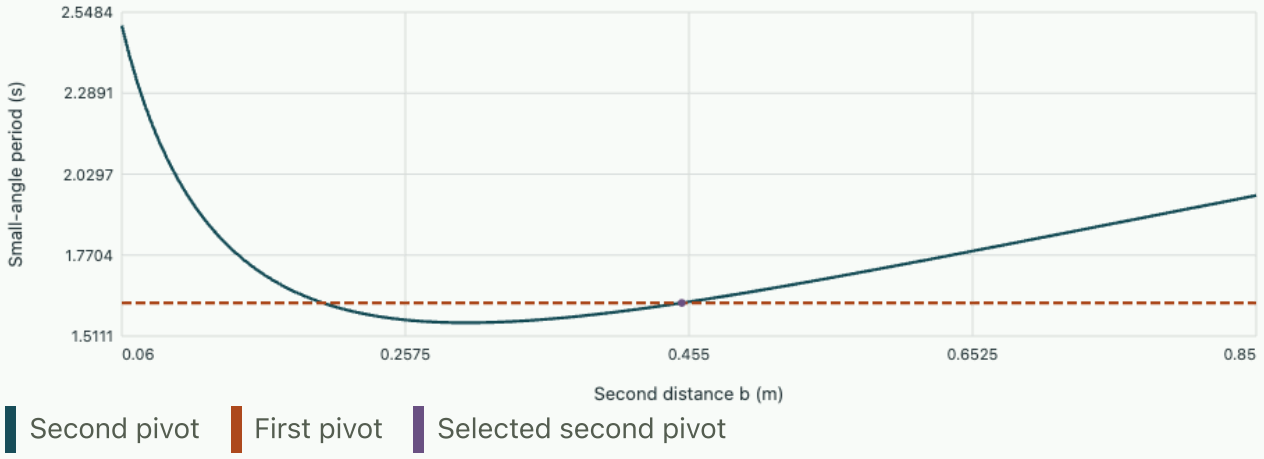
Hold $a = 0.200\,\mathrm{m}$ fixed and vary both the radius of gyration and the second distance. The graph plots the period against the possible second distance. Before inspecting intersections, predict where the second match should lie as ρ grows. The dependence on distance has a minimum at $b = \rho$ because effective length comprises a growing and a decreasing term.

Open this experiment online

Try the calculation

Radius of gyration (m) **0.3**

Second pivot distance (m) **0.45**



First period 1.6173 s; second 1.6173 s. The other matching distance is 0.45 m.

With a finite oscillation amplitude, the physical pendulum has the same elliptic correction as a point pendulum of length L_a, provided the body is rigid and the frictionless pivot stays fixed. Matching different amplitudes can bias the cancellation. Measuring both at the same small angle removes the leading difference, and extrapolating both to zero amplitude provides a check. Rotational compliance of the knife edge and a shifting centre of mass require a different model.

Tutorial

For $a = 0.240$ m and $\rho = 0.300$ m, where should the second distinct match occur and what length should be used in $g = 4\pi^2 L/T^2$?

▼ **Work through the solution**

The second branch gives

$$b = 0.0900/0.240 = 0.375\,\mathrm{m}.$$

Because the pivots straddle the centre of mass the equivalent length is 0.615 m.

Both pivots lie on the same side of the centre, at the same distances a, b as in the previous question. What changes in the cancellation?

▼ **Work through the solution**

The periods depend on the magnitudes of the offsets, so equal periods still imply $\rho^2 = ab$ on the distinct branch. The sum $a + b$ still gives the equivalent length, but the physical ruler distance between pivots is $|a - b|$. The direct measurement shortcut relies on the chosen geometry.

Why does setting both axes exactly at the same distance make even noise-free measurements insufficient to determine both g and ρ?

▼ **Work through the solution**

The two equations are then identical. For every positive choice of inertia radius the first period determines a corresponding value of gravity. Improving timing precision cannot provide the missing independent equation.

Connect the offset-pivot calculation to the rigid-body inertia chapter.

Mechanics in the laboratory · 3. Distinguish a physical precession from numerical drift

A bound orbit circulates many times about its centre. A small change in the angle of successive closest approaches could be physical or could be accumulated phase error. Construct an exactly solvable family of precessing orbits that lets the two effects be compared without assuming a reference trajectory from another code.

Replace time by the measured polar angle

A particle of mass m moves in a static central potential $V(r) = -k/r + \alpha/(2r^2)$ with $k > 0$. The radial force is $-k/r^2 + \alpha/r^3$. Angular momentum $L = mr^2\dot{\phi}$ is constant. Put $u = 1/r$ and differentiate with respect to ϕ, denoting this derivative by a prime. The identities $\dot{r} = -(L/m)u'$ and $\ddot{r} = -(L^2/m^2)u^2u''$ follow by the chain rule. Subtracting the centripetal term $r\dot{\phi}^2 = (L^2/m^2)u^3$ from radial acceleration gives Binet's equation.

$$u'' + (1+\delta)u = \frac{mk}{L^2},$$

$$\delta = \frac{m\alpha}{L^2}.$$

The sign of the perturbation matters. Positive α gives a repulsive short-range correction, increasing radial frequency per unit angle. A mildly negative correction decreases it. This family is a test potential; a measured nonzero advance in nature need not have this particular cause.

Read apses from a harmonic equation

For $1 + \delta > 0$, write $\nu = \sqrt{1+\delta}$, $u_c = mk/[L^2(1+\delta)]$ and impose $u(0) = u_c(1+e)$, $u'(0) = 0$ with $0 < e < 1$. Then $u = u_c[1 + e\cos(\nu\phi)]$. The radial minimum corresponds to a maximum of u. Its next occurrence is at angle $2\pi/\nu$, so the change of line of apsides per radial period is

$$\Delta\omega = 2\pi\left(\frac{1}{\sqrt{1+\delta}} - 1\right).$$

For $\delta = 0.04$, successive pericentres rotate backwards by approximately 7.0 degrees. At $\delta = 0$ the Kepler ellipse closes. For a strictly circular orbit the algebraic radial frequency has a well-defined small-oscillation limit, but the direction of pericentre of the observed circle has no unique value. A nearly circular orbit therefore has a genuine experimental difficulty even when the integration is accurate.

Compute the bias of a discrete orbit exactly

Shift $v = u - u_c$. On a uniform angular grid of spacing h, the centred second difference gives

$$v_{j+1} = 2v_j - v_{j-1} - (h\nu)^2 v_j.$$

Insert the trial $v_j = e^{ij\psi}$ and simplify:

$$\cos\psi = 1 - \tfrac{1}{2}(h\nu)^2,$$

$$\nu_h = \frac{2}{h}\arcsin(h\nu/2).$$

The numerical mode oscillates if $h\nu < 2$. Inside that range it is too fast, since the arcsine exceeds its argument. A bounded orbit can thus accumulate a negative apsidal error. For an unperturbed Kepler case and a 30-degree grid, the numerical angular frequency is roughly 1.012. Treating this as a physical result would infer a retrograde precession of roughly four degrees per revolution. Refining h halves the angular step and quarters the leading error, because $\nu_h = \nu + \nu^3 h^2/24 + \cdots$.

The experiment plots radial phase after multiple nominal revolutions and reports the implied apsidal angles. Vary the perturbation at fixed step first, then halve the step at fixed perturbation. Distinguish the discrepancy that persists in the continuum from the discrepancy that can be extrapolated away. Starting all curves at pericentre makes the phase convention identical.

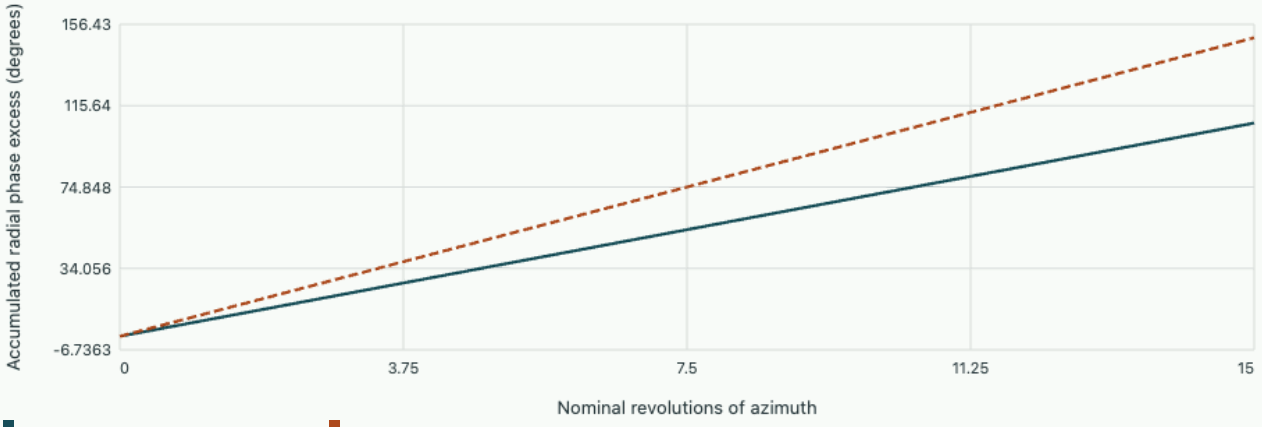
Hold the force exactly Newtonian. Compare the accumulated numerical phase on a 30° grid and 6° grid. The exact radial phase stays at zero in both runs. Use the exported values to estimate how the phase error scales when the angular step is divided by five.

Open this experiment online

Try the calculation

Relative force correction δ **0.04**

Angular grid step (degrees) **24**



Apsidal advance per radial period: exact -6.991°, numerical -9.7105°.

Tutorial

A measured radial period is 364 degrees of azimuthal advance. In this family, which sign of α is indicated and what is δ?

▼ Work through the solution

The radial cycle takes more than 360 degrees, so the radial frequency per unit angle is below one. The inferred correction is attractive,

$$\delta = (360/364)^2 - 1 \simeq -0.02186.$$

A simulation reports advances of –1.20, –0.60 and –0.45 degrees at successive halvings of the angular step. What leading continuum estimate follows?

▼ Work through the solution

With second-order phase bias, the final difference 0.15 equals three times the remaining magnitude of that bias. Adding 0.05 to –0.45 gives –0.40 degrees. The earlier pair gives the same estimate. An additional refinement tests the assumed scaling.

Can a long observation of a closed numerically integrated orbit rule out a time-step error in the orbital period?

▼ Work through the solution

No. Closure depends on a ratio of radial to azimuthal phase advances. Both may be biased in the same proportion. Independently time successive crossings and compare with an analytic period or a finer integration. Different observables can hide different errors.

The midpoint experiment shows the converse sign of phase error for another bounded integrator.

Mechanics in the laboratory · 4. Which track reaches a lower point first?

A bead is released from rest and slides without friction along a track between two points. A straight track is shortest, yet another path can deliver the bead sooner. Derive the travel-time functional, solve its boundary problem and compare actual times while changing the horizontal distance independently of the vertical drop.

Choose the coordinate appearing in the speed law

Let y increase downwards from the release point, x increase along the horizontal and g be uniform. A small frictionless bead conserves mechanical energy, so its speed at depth y is $\sqrt{2gy}$. For a track expressed as $y(x)$ the time element is line element divided by speed. Hence

$$T[y] = \frac{1}{\sqrt{2g}} \int_0^X \sqrt{\frac{1+y'^2}{y}}\, dx,$$

$$y(0) = 0,$$

$$y(X) = Y.$$

A straight line has distance $\sqrt{X^2+Y^2}$ and component of acceleration $gY/\sqrt{X^2+Y^2}$, giving $T_{\text{line}} = \sqrt{2(X^2+Y^2)/(gY)}$. The advantage of a steeper early path is to build speed at the part of the trajectory where the bead would otherwise remain slow. A path with a longer distance can take less time because distances at different depths cost different amounts of time.

Use a first integral to remove a differential equation

Write the integrand excluding its constant multiplier as $F = \sqrt{1+y'^2}/\sqrt{y}$. Since it contains no explicit x, the Euler–Lagrange equation implies $F - y'F_{y'} =$ constant. Differentiating this expression with respect to x shows the cancellation explicitly:

$$dF/dx = F_y y' + F_{y'} y'';$$

the derivative of the second term removes $F_{y'}y''$ and leaves $y'(F_y - dF_{y'}/dx) = 0$. Substituting the integrand gives $y(1+y'^2) = 2a$ for some positive length a.

Set $y = a(1-\cos u)$. On the descending part $y' = \cot(u/2)$. Integrating $dx = \tan(u/2)dy$ yields

$$x = a(u - \sin u).$$

The signed derivative continues through zero at $u = \pi$ and produces an ascending segment for $\pi < u < 2\pi$. A track may pass below the final point, then climb towards it while spending speed acquired earlier.

Solve for the actual endpoint and clock reading

Divide the endpoint coordinates to eliminate a. The angle u_f solves $(u_f - \sin u_f)/(1 - \cos u_f) = X/Y$ on the first arch. The ratio starts at zero and increases without bound; bisection gives a robust root for positive finite X/Y. Recover $a = Y/(1 - \cos u_f)$. Along the cycloid $ds = 2a \sin(u/2) du$, while $v = 2\sqrt{ga}\sin(u/2)$. Their ratio is finite even at the cusp where both vanish:

$$T_{\mathrm{cyc}} = u_f \sqrt{a/g}.$$

For $Y = 1$ m and $X = \pi/2$ m, $u_f = \pi$ and $a = 0.5$ m. The cycloid time is approximately 0.709 s and the straight track needs about 0.841 s. The endpoint is the bottom of this particular cycloid. Increasing the horizontal offset beyond $\pi Y/2$ makes the first-arch solution turn upwards before it arrives.

Compare two feasible experiments

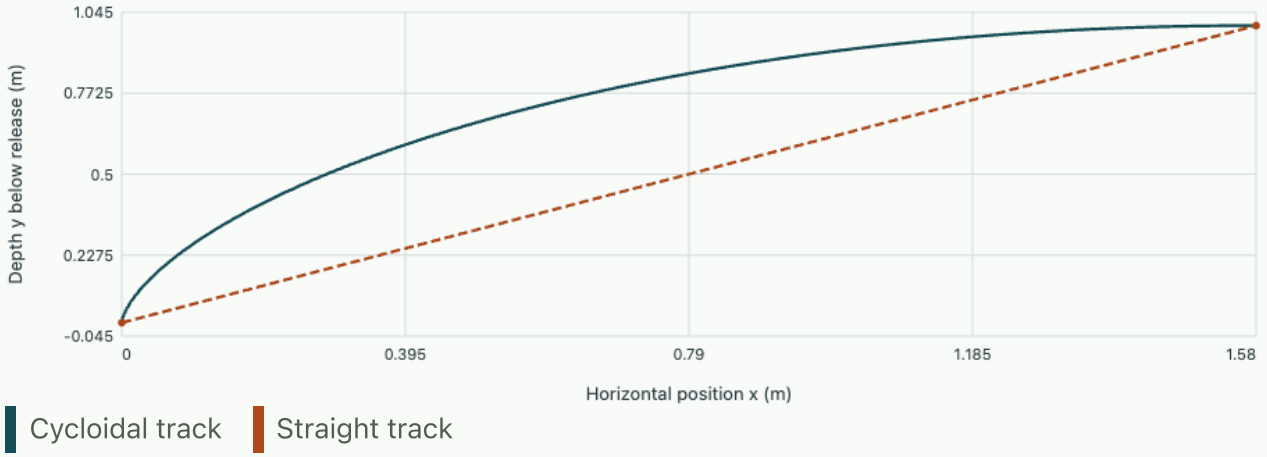
The graph plots downward depth against horizontal position. Vary endpoint distance and depth, then predict whether the fastest path has an ascending segment. Read the two calculated times. A physical rolling sphere differs from the sliding particle: rotational kinetic energy multiplies the time by a constant factor if the body rolls without slipping with fixed inertia ratio, whereas frictional loss can alter the optimum shape itself. A track with a prescribed minimum bend radius would also define a different variational problem.

Open this experiment online

Try the calculation

Horizontal distance X (m) **1.58**

Vertical drop Y (m) **1**



Travel time: cycloid 0.71133 s, straight 0.84429 s. Final cycloid parameter 3.1508 rad.

Tutorial

Keep track shape geometrically similar and double every length. By which factor does the travel time change?

▼ Work through the solution

The parameter angle is fixed by a length ratio and a doubles. The time grows by $\sqrt{2}$. The straight track time obeys the same scaling. A measured different exponent would suggest another dimensional scale in the apparatus.

Why might a numerical quadrature applied directly in x at the first sample point report infinite time?

▼ Work through the solution

The release speed vanishes and the track has a vertical tangent. The local time integral remains finite. The cycloid parametrisation cancels both vanishing factors analytically. Evaluating a singular endpoint as an ordinary sample in a trapezoidal rule is an integration error.

How can two paths be compared if a common release mechanism imparts a small initial speed v_0?

▼ Work through the solution

Energy now gives

$$v(y) = \sqrt{v_0^2 + 2gy}.$$

Recompute the time functional using that speed and the actual initial condition. Subtracting an arbitrary time offset from both runs cannot undo a depth-dependent modification of travel speed.

Revisit the origin of stationary action and essential boundary data in the moving-constraint chapter.

Mechanics in the laboratory · 5. When can shaking a support stabilise an upside-down pendulum?

A rapidly oscillating pivot can stabilise a pendulum near its normally unstable upright orientation. Develop the slowly varying effective potential, test its assumptions with the full time-dependent equation and compare the observable that each calculation predicts.

Start in the moving frame

Measure angle θ from the downward vertical. Let the upward support displacement be $y_p = a\cos\Omega t$. The effective downward acceleration in the translating frame is $g + \ddot{y}_p = g - a\Omega^2\cos\Omega t$. The exact angular equation under these assumptions is

$$\ddot{\theta} + \left(\frac{g}{L} - \frac{a\Omega^2}{L}\cos\Omega t\right)\sin\theta = 0.$$

A simple time average of the coefficient would leave the ordinary pendulum. That would miss the correlation between the rapidly oscillating force and the rapidly oscillating part of the angle. The pivot can do work even though its mean position is fixed. The energy of the pendulum alone is therefore not generally conserved.

Eliminate the fast motion

Write $\theta = \Theta + \xi$, with slowly varying Θ and a small fast correction ξ. In one forcing cycle treat $\sin\Theta$ as constant. Integrating $\ddot{\xi} = (a\Omega^2/L)\sin\Theta\cos\Omega t$ twice for its oscillatory part gives

$$\xi = -(a/L)\sin\Theta\cos\Omega t.$$

Inserting $\sin(\Theta+\xi) \simeq \sin\Theta + \xi\cos\Theta$ into the exact equation creates a product of two cosines with nonzero mean. Retaining the leading averaged terms yields

$$\ddot{\Theta} = -\frac{g}{L}\sin\Theta - \frac{a^2\Omega^2}{2L^2}\sin\Theta\cos\Theta,$$

$$\frac{U_{\text{eff}}}{mL^2} = -\frac{g}{L}\cos\Theta + \frac{a^2\Omega^2}{4L^2}\sin^2\Theta.$$

Two approximations entered. The displacement ratio a/L must keep micromotion small and the drive must be fast compared with the slow angular motion. A large acceleration $a\Omega^2$ can coexist with a small displacement a; acceleration alone is not the stability criterion.

An observable threshold

Expand about $\Theta = \pi + \eta$. The averaged restoring coefficient is

$$\omega_{\mathrm{up}}^2 = a^2\Omega^2/(2L^2) - g/L.$$

A local minimum appears if $a\Omega > \sqrt{2gL}$. For $L = 0.15$ m and a 60 Hz vertical vibration the threshold amplitude is approximately 4.55 mm. At 6 mm the averaged small-oscillation frequency is about 1.11 Hz. A motion sensor attached to the bob will also see the 60 Hz micromotion. Comparing an unfiltered peak with the slow prediction can make the theory seem wrong even when the averaged phase portrait is accurate.

Test the slow approximation against a full integration

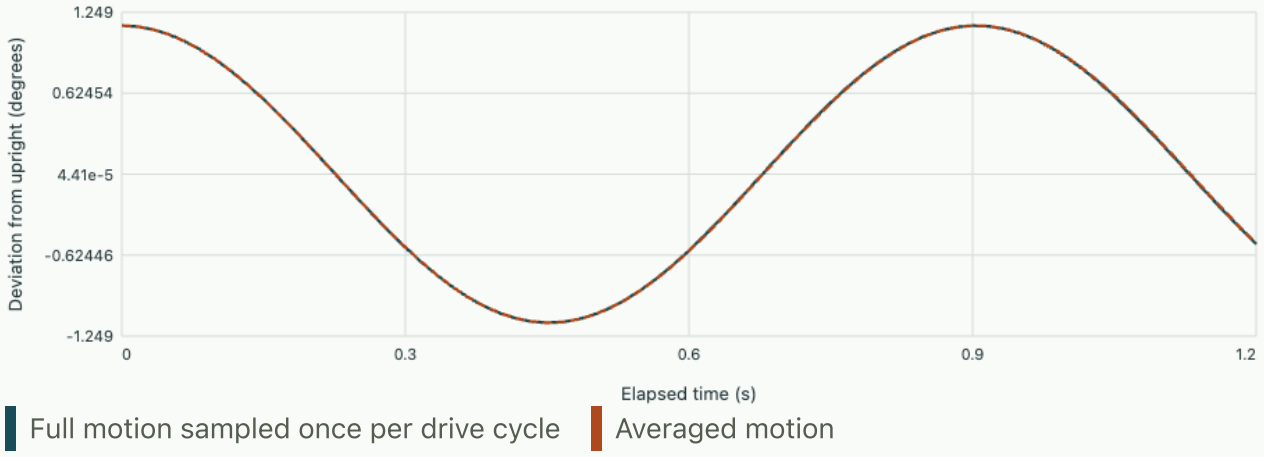
Release the bob slightly off the top and change the drive frequency or amplitude. The plot samples the exact forced equation once per forcing period at a fixed phase and compares it with the averaged equation evaluated on the same time grid. The observable is angular deviation from the top, with full turns retained. Small bounded oscillations depend on initial displacement as well as the local curvature; outside the linear range a pendulum can leave the top even when the curvature is positive. Very near the threshold a short record is poor evidence for stability because both a small growth rate and a small oscillation frequency produce slow motion.

Open this experiment online

Try the calculation

Vertical amplitude (mm) **6**

Drive frequency (Hz) **60**



Threshold amplitude 4.5506 mm. Averaged upright curvature 48.298 s⁻²; oscillation frequency 1.1061 Hz.

Tutorial

Two drives have the same product $a\Omega$ but one uses twice the frequency. What is identical in the averaged model and what differs in the full motion?

▼ **Work through the solution**

The leading effective potential is unchanged. The higher-frequency drive has half the displacement amplitude and therefore smaller leading angular micromotion. Its separation of time scales is typically better.

Is exactly upright motion evidence that the drive has stabilised the pendulum?

▼ **Work through the solution**

At $\theta = \pi, \dot{\theta} = 0$ the sine vanishes for all time. This exact trajectory exists even below threshold. Release with controlled positive and negative tilts and assess whether nearby trajectories remain close.

If the drive is switched off after one second, can the effective potential still be used to predict a permanently stable upright position?

▼ **Work through the solution**

Once the pivot stops, only the ordinary gravitational potential remains. Integrate the undriven equation using the angle and angular velocity at switch-off. The phase of the rapid motion can affect that initial state.

Compare slow stabilisation here with parametric growth near twice a mode frequency.

Mechanics in the laboratory · 6. When does a smooth circular guide lose contact?

A bead moves inside a vertical circular guide after receiving a known speed at the bottom. Predict whether it turns around, separates from the guide or completes the circle. A normal-force measurement can distinguish these outcomes before a simple top-height energy test can.

Check the constraint force after solving for speed

Use $\theta = 0$ at the bottom, increasing along the rising side. Let R be the radius of the bead centre's path, v_0 its bottom speed and $u = v_0^2/(gR)$. The smooth guide does no work. The rise in gravitational energy $mgR(1 - \cos\theta)$ therefore gives

$$v^2/(gR) = u - 2 + 2\cos\theta.$$

This relation applies only while the bead follows the circle and the computed speed is real.

With inward radial direction positive, force balance is

$$N - mg\cos\theta = mv^2/R.$$

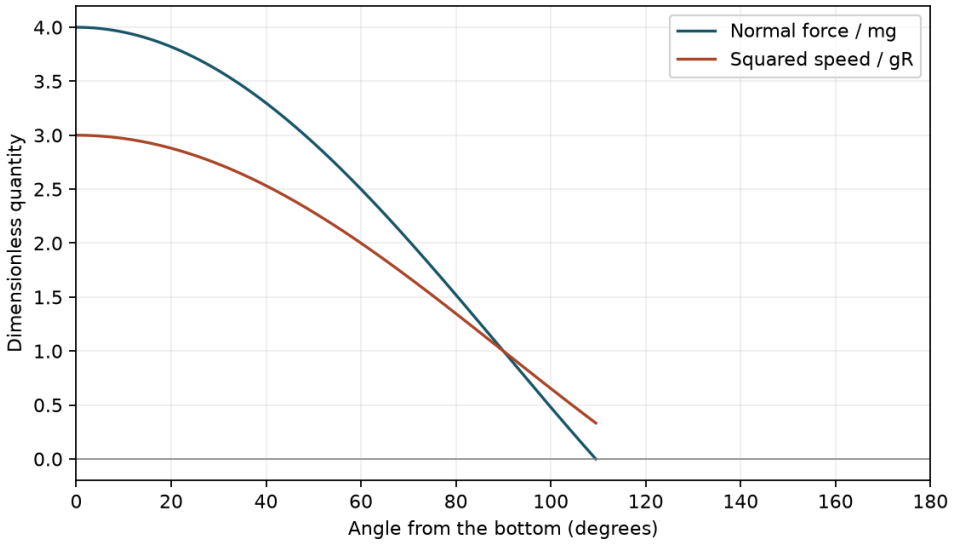
Substitution gives

$$\frac{N}{mg} = u - 2 + 3\cos\theta.$$

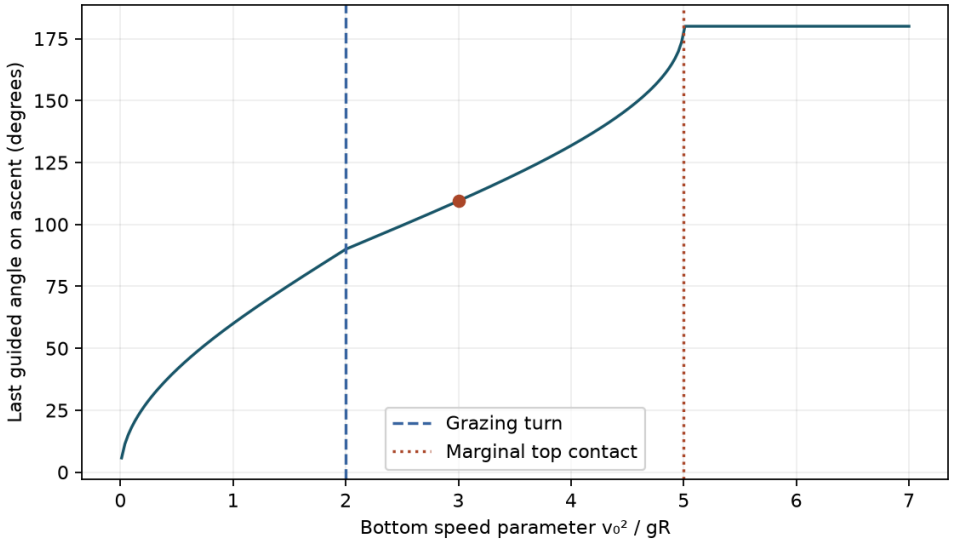
A guide underneath the bead can push but cannot pull. For $u < 2$ the speed first reaches zero while the contact force is still positive. For $2 < u < 5$ the force reaches zero with positive speed at $\cos\theta = (2 - u)/3$. Above that point circular motion would require a negative normal force. For $u > 5$ the bead can remain in contact throughout the circle. At the boundaries $u = 2$ gives a zero-speed, zero-normal grazing turn at the side and $u = 5$ gives marginal contact at the top. A captive slotted carriage or a bead threaded onto a wire has a different constraint and can transmit force of either sign.

Distinguish an energetically accessible top from a feasible trajectory

Let $R = 0.25$ m and choose $v_0 = \sqrt{3gR} = 2.71$ m/s. The guide loses contact when $\cos\theta = -1/3$, or at 109.5 degrees from the bottom. The remaining speed there is $\sqrt{gR/3} = 0.904$ m/s. Raising the initial energy so that $u = 4$ pays the gravitational energy cost to reach the top, but a continuously guided motion still fails the nonnegative force condition earlier. The mechanical-energy and force-balance tests answer different questions.



The first quantity to reach zero determines which constraint fails. The part after contact loss requires a different trajectory.



The horizontal branch records completion of the ascent, including marginal contact at its threshold. It does not mean that the bead stops at the top.

Exercise · find the lower turning point

For $u = 1$, determine the turning angle and force there.

▼ **Work through the solution**

Zero speed requires $\cos\theta = 1 - u/2 = 1/2$, hence $\theta = 60^\circ$. The contact force then balances the outward radial component of gravity, $N = mg\cos\theta = mg/2$. It remains compressive through the turn.

Exercise · size a classroom loop

A bottom speed of 2.0 m/s is available. What is the largest ideal path radius for continuous contact through the top?

▼ **Work through the solution**

The limiting condition is $v_0^2 = 5gR$, so $R = 4/(5 \cdot 9.81) = 0.0815$ m. This is the radius of the bead centre. Rolling friction, rotational kinetic energy or a different channel radius would alter this calculation.

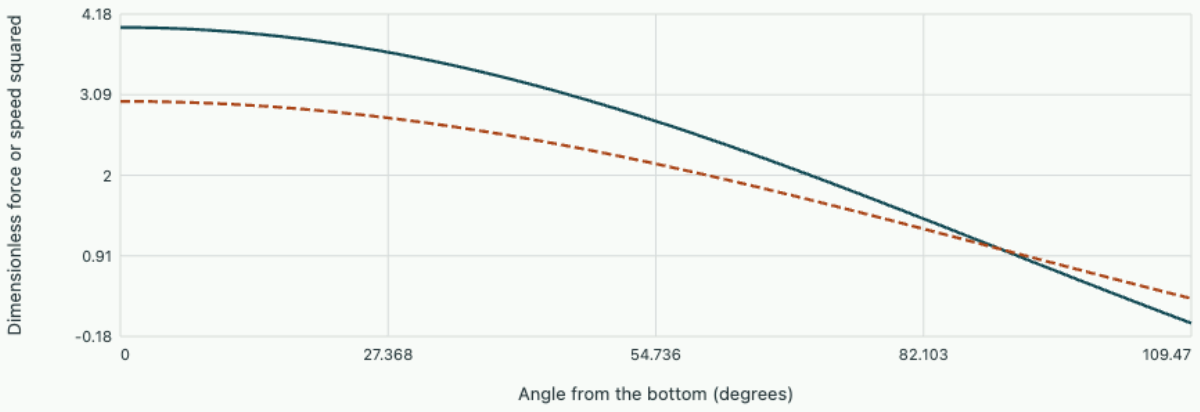
Try it · Guide contact

Change the inputs and compare the recalculated curves with the preceding derivation and baseline figure. The axes give the units; the numerical readout states the model assumptions and any limiting cases. Reset the inputs to return to the reference setting.

Open this live calculation

Try the calculation

Bottom speed squared / gR 3



Loses contact at 109.47°. Curves stop at the first physical limit of the unilateral guide.

Download the corresponding editable notebook. The browser recalculates the reduced equations directly; the notebook exposes the full calculation and its checks.

Open editable notebook Download .ipynb